Unimodular Pure dS Supergravity

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based on work done in collaboration with:
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Making contact between supergravity and nature

- **Problem:** (unbroken) supersymmetry doesn’t allow de Sitter solutions (i.e. positive cosmological constant).
Easy to see from algebra

- Attempt to embed (A)dS algebra

\[ [P_\mu, P_\nu] = s \frac{1}{4L^2} M_{\mu\nu}, \quad \text{with} \quad s = \begin{cases} -1 & \text{dS} \\ +1 & \text{AdS} \end{cases} \]

into SUSY algebra:

\[ [P_\mu, Q_\alpha] = \frac{1}{4L} (\gamma_\mu Q)_\alpha \]

\[ \{Q_\alpha, Q_\beta\} = -\frac{1}{2} (\gamma^\mu)_{\alpha\beta} P_\mu - \frac{1}{8L} (\gamma^{\mu\nu})_{\alpha\beta} M_{\mu\nu} \]

then Jacobi identity \([P_\mu, P_\nu, Q] = 0\) fixes

\[ s = +1 \]
Making contact between supergravity and nature

• **Problem:** (unbroken) supersymmetry doesn’t allow de Sitter solutions (i.e. positive cosmological constant).

• **Solution:** SUSY is broken on dS solution.

• Requires the introduction of the *goldstino*, on which supersymmetry is realised non-linearly.

• Very difficult to achieve in pure supergravity theory.
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Constrained superfields

- Would like to incorporate the goldstino into a superfield (or supermultiplet), such that we can make full use of the machinery of SUSY/sugra, make contact with $\mathcal{N} > 1$, etc.
- Proposed solution is that the goldstino is part of a nilpotent superfield $X$ [Rocek '78, Lindstrom, Rocek '79, Samuel, Wess '83, Casalbuoni, De Curtis, Dominici, Feruglio, Gatto '89, Komargodski, Seiberg '09, Kuzenko, Tyler '11]

\[
X = \varphi + \sqrt{2} \theta G + \theta^2 F^X
\]

\[
X^2 = 0 \quad \Rightarrow \quad \varphi = \frac{G^2}{2F^X}
\]
Constrained superfields

- Couple to supergravity via superpotential

\[ \mathcal{W} = a(X^0)^3 + b(X^0)^2X \]

with \( a \) and \( b \) arbitrary constants and \( X^0 \) is the compensator. This gives solutions with cosmological constant and gravitino mass

\[ \Lambda \propto |b|^2 - |a|^2, \quad m_{\psi} \propto a \]

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Stueckelberg: gentle introduction

- Start with a Lagrangian with an explicitly broken symmetry:

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{m^2}{2} A_\mu A^\mu \]

where the gauge field \( A_\mu \) lacks the usual gauge symmetry \( \delta A_\mu = \partial_\mu \alpha \) due to the mass term. This can be restored by introducing an extra scalar field \( s \), and writing

\[ A_\mu = \tilde{A}_\mu + \partial_\mu s, \quad \begin{cases} 
\delta \tilde{A}_\mu = \partial_\mu \alpha \\
\delta s = -\alpha
\end{cases} \]

and plugging back into the Lagrangian

\[ \mathcal{L} = -\frac{1}{4} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} - \frac{m^2}{2} \partial_\mu s \partial^\mu s - \frac{m^2}{2} \left( \tilde{A}_\mu \tilde{A}^\mu + 2 A_\mu \partial^\mu s \right) \]

- The procedure can be interpreted as performing a gauge transformation on the field and promoting the parameter to a field (the Stueckelberg field).
Unimodular gravity

- Original formulation of unimodular gravity

\[ S = \int d^4x \left[ \sqrt{-g} R - 2\Lambda (\sqrt{-g} - \epsilon_0) \right] \]

where \( \Lambda \) is a Lagrange multiplier field which imposes the unimodularity condition

\[ \sqrt{-g} = \epsilon_0 \]

with \( \epsilon_0 \) being a constant (which could be set to 1 if we want).

- The Einstein Equation also imposes:

\[ \partial_\mu \Lambda = 0 \]

and hence we interpret the v.e.v. of \( \Lambda \) as the cosmological constant.
Original formulation of unimodular gravity

\[ S = \int d^4x \left[ \sqrt{-g} R - 2\Lambda \left( \sqrt{-g} - \epsilon_0 \right) \right] \]

This version of the Lagrangian is not invariant under the full group of diffeomorphisms, but only under a subgroup of transformations called TDiffs, whose parameter satisfies \( \nabla_\mu \xi^\mu = 0 \). It is obvious that the problematic term is

\[ 2 \int d^4x \Lambda \epsilon_0 \]

where \( \Lambda \) transforms as a scalar.
Unimodular gravity with Stueckelberg

• One can restore the full general coordinate invariance through the Stueckelberg trick of introducing an extra field which transforms appropriately. We do this by performing a g.c.t. $x^\mu \to y^\mu(x)$ on the problematic term

$$2 \int d^4 x \Lambda(x) \epsilon_0 \to 2 \int d^4 y \Lambda'(y) \epsilon_0 = 2 \int d^4 x |J| \Lambda(x) \epsilon_0$$

where $|J| = \text{Det} \left( \frac{\partial y^\mu}{\partial x^\alpha} \right)$. We promote $y^\mu$ to be 4 functions of space-time $y^\mu \to \phi^\mu(x)$ and the new action is

$$S_{St} = \int d^4 x \left[ \sqrt{-g} R - 2\Lambda \left( \sqrt{-g} - \text{Det} \left( \frac{\partial \phi}{\partial x} \right) \epsilon_0 \right) \right]$$

which is now invariant under the full group of g.c.t.’s, provided the $\phi$’s transforms as scalars:

$$\phi'^\mu(y(x)) = \phi^\mu(x)$$
Unimodular gravity with Stueckelberg

- We performed the Stueckelberg trick by acting with a finite transformation on the coordinates. However, this will not always be practical, and one can alternatively restore the diffeomorphism invariance by performing an infinitessimal transformation either in the passive form:

  \[ x^\mu \rightarrow x^\mu + \xi^\mu + \frac{1}{2} \xi^\rho \partial_\rho \xi^\mu + \ldots \]

  or in the active form, where we vary the scalar field \( \Lambda \):

  \[ \delta \Lambda = -\xi^\rho \partial_\rho \Lambda - \frac{1}{2} \xi^\rho \partial_\rho \xi^\mu \partial_\mu \Lambda + \ldots \]

  and promoting the parameters to fields: \( \xi^\mu \rightarrow s^\mu \). The Lagrangian can then be constructed order by order in the Stueckelberg field \( s^\mu \)

  \[ \mathcal{L} = \mathcal{L}_0 + \Lambda \left[ 1 + \partial_\mu s^\mu + \frac{1}{2} s^\rho \partial_\rho \partial_\mu s^\mu + \frac{1}{2} \left( \partial_\mu s^\mu \right) \left( \partial_\rho s^\rho \right) + \ldots \right] \]
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Chiral superspace

- We define a *chiral density* superfield $\Delta$ via its transformation property

$$\delta \Delta = -\partial_M \left[ (-1)^M \eta^M \Delta \right]$$

where

$$\eta^\mu (\epsilon) = \Theta^\beta y^\mu_{1\beta} (\epsilon) + \Theta^2 y^\mu_2 (\epsilon)$$

$$\eta^\alpha (\epsilon) = \epsilon^\alpha + \Theta^\beta \Gamma^\alpha_{1\beta} (\epsilon) + \Theta^2 \Gamma^\alpha_2 (\epsilon)$$

with

$$y^\mu_{1\alpha} (\epsilon) = 2i (\sigma^\mu \bar{\epsilon})_\alpha$$

$$y^\mu_2 (\epsilon) = \bar{\psi}_\nu \bar{\sigma}^\mu \sigma^\nu \bar{\epsilon}$$

$$\Gamma^\alpha_{1\beta} (\epsilon) = -i (\sigma^\mu \bar{\epsilon})_\beta \psi^\alpha_\mu$$

$$\Gamma^\alpha_2 (\epsilon) = -i \omega^\alpha_{\mu \beta} (\sigma^\mu \bar{\epsilon})_\beta + \frac{1}{3} M^* \epsilon^\alpha$$

$$- \frac{1}{2} \psi^\alpha_\nu \left( \bar{\psi}_\mu \bar{\sigma}^\nu \sigma^\mu \bar{\epsilon} \right) + \frac{1}{6} b_\mu (\epsilon \sigma^\mu \bar{\epsilon})^\alpha$$

- It naturally generalises scalar densities (such as $\sqrt{-g}$).
Chiral superspace

• We define a *chiral density* superfield $\Delta$ via its transformation property

$$\delta \Delta = -\partial_M \left[ (-1)^M \eta^M \Delta \right]$$

• It naturally generalises scalar densities (such as $\sqrt{-g}$).

• The product of a chiral density with a chiral superfield is a chiral density

$$\delta \Delta \Phi = -\partial_M \left[ (-1)^M \eta^M \Delta \right] \Phi - \Delta \eta^M \partial_M \Phi$$

$$= -\partial_M \left[ (-1)^M \eta^M \Delta \Phi \right]$$

• It can be used to construct invariant actions.
Chiral superspace

- The pure supergravity action in chiral superspace \((X^M = (x^\mu, \Theta^\alpha))\) is

\[
S = \int d^4x d^2\Theta \mathcal{E} R + h.c.
\]

with \(R\) a chiral superfield with components

\[
R| = -\frac{1}{6} M
\]

\[
\mathcal{D}_a R| = -\frac{1}{6} \left( \sigma^\alpha \bar{\sigma}^\beta \psi_{\alpha \beta} + i b^\alpha \psi_\alpha - i \sigma^\alpha \bar{\psi}_\alpha M \right)_a
\]

\[
\mathcal{D}^2 R| = -\frac{1}{3} e^\mu_a e^\nu_b R_{\mu \nu}^{ab} + \frac{2}{3} i \bar{\psi}^\mu \bar{\sigma}^\nu \psi_{\mu \nu} + ...
\]
The pure supergravity action in chiral superspace \((X^M = (x^\mu, \Theta^\alpha))\) is

\[
S = \int d^4x d^2\Theta \mathcal{E} R + h.c.
\]

with \(\mathcal{E}\) a chiral density with components

\[
\mathcal{E} = a + \sqrt{2} \Theta \rho + \Theta \Theta f, \quad \text{with}
\]

\[
a = \frac{1}{2} e
\]

\[
\rho = \frac{i\sqrt{2}}{4} e \sigma^\mu \bar{\psi}_\mu
\]

\[
f = -\frac{1}{2} e M^* - \frac{1}{8} e \bar{\psi}_\mu \left( \bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu \right) \bar{\psi}_\nu
\]
Unimodular Supergravity

- We define the unimodular supergravity action to be:

\[
S = \int d^4x d^2\Theta [\mathcal{E} R - 2\Lambda (\mathcal{E} - \mathcal{E}_0)] + h.c.
\]

where \(\Lambda\) is now a lagrange multiplier chiral superfield, and we define

\[
\mathcal{E}_0 = \epsilon_0 - \frac{i}{2}m\Theta^2
\]

with \(\epsilon_0, m\) some real constants. Varying over \(\Lambda\), we get

\[
\mathcal{E} = \mathcal{E}_0,
\]

which is the SUSY analogue of the unimodularity condition. The action is not invariant under full SUSY+diffeo, but subset preserving condition above.
The Super-Stueckelberg trick

- We perform a transformation $X^M \rightarrow Y^M(X)$ on the problematic term

$$2 \int d^4x d^2\Theta \Lambda(X) \mathcal{E}_0 \rightarrow 2 \int d^4y d^2\Gamma \Lambda'(Y) \mathcal{E}_0 = 2 \int d^4x d^2\Theta |sJ| \Lambda(X) \mathcal{E}_0$$

where $|sJ| = Ber \left( \frac{\partial Y^M}{\partial X^N} \right)$. Crucially, unlike in the gravity case, $Y^M$ will not be a general superfunction of $X^M$, but will depend on the diffeomorphism and SUSY parameters in a very particular way.
The Super-Stueckelberg trick - perturbatively

- At linear order in the parameters, we have the usual transformation
  \[ \delta^{(1)} X^M = \Xi^M \]
  with
  \[ \Xi^M = (\xi^\mu + \eta^\mu(\epsilon), \eta^\alpha(\epsilon)) \]

- At second order, we have
  \[ \delta^{(2)} X^M = \frac{1}{2} \Xi^R \partial_R \Xi^M + \frac{1}{2} \delta^{(s)} \left( \Xi^M \right) \]

where the first term is the usual one at second order, while the second one takes into account the fact that the objects \( \eta^\mu, \eta^\alpha \) appearing in the SUSY transformations of the coordinates are not arbitrary functions, but depend on the supergravity fields \( e_\mu, \psi_\mu, b_\mu, M \).
The Super-Stueckelberg trick

We can thus proceed, order by order in the transformation parameters, and write:

\[ X^M \rightarrow X^M + \delta^{(1)}X^M + \delta^{(2)}X^M + ... \equiv Y^M(X, \Xi), \]

Again, we promote the parameters to fields to get:

\[
\begin{align*}
\xi^\mu & \rightarrow \phi^\mu \\
\epsilon & \rightarrow \zeta 
\end{align*}
\]

\[ \Xi^M = (\xi^\mu + \eta^\mu(\epsilon), \eta^\alpha(\epsilon)) \rightarrow \varphi^M = (\phi^\mu + \eta^\mu(\zeta), \eta^\alpha(\zeta)) \]

\[ Y^M(X, \Xi) \rightarrow \Phi^M(X, \varphi) \]

At this point, we can construct the Lagrangian:

\[ S = \int d^4x d^2\Theta \left[ \mathcal{E} R - 2\Lambda \left( \mathcal{E} - \text{Ber} \left( \frac{\partial \Phi^M}{\partial X^N} \right) \mathcal{E}_0 \right) \right] + h.c. \]
The Stueckelberg superfield

- The superfield

\[ S = \text{Ber} \left( \frac{\partial \Phi^M}{\partial X^N} \right) \equiv S_0 + \sqrt{2} \Theta S_1 + S_2 \Theta^2 \]

is a chiral density superfield, thus allowing us to restore SUSY to the action.

- Its components \( S_0, S_1, S_2 \) transform \textit{linearly}, but they are functions of the Stueckelberg fields \( f^\mu, \zeta \), which transform \textit{non-linearly} and of the sugra fields \( e_\mu, \psi_\mu, b_\mu, M \).
Stueckelberg action in components

In components, the action will be (setting $\epsilon_0 = \frac{1}{2}$)

$$S = \frac{1}{2\kappa^2} \int \left[ \sqrt{-g} \left[ R - \frac{2}{3} M^* M + \frac{2}{3} b^\mu b_\mu + \epsilon^{\mu\nu\rho\sigma} \left( \bar{\psi}_\mu \tilde{\sigma}_\nu \tilde{D}_\rho \psi_\sigma - \psi_\mu \sigma_\nu \tilde{D}_\rho \bar{\psi}_\sigma \right) \right] ight.$$

$$+ \frac{1}{2} \sqrt{-g} \left[ -2\Lambda_2 + \sqrt{2} i \Lambda_1 \sigma^\mu \bar{\psi}_\mu + 2\Lambda_0 \left( \bar{\psi}_\mu \tilde{\sigma}^{\mu\nu} \bar{\psi}_\nu + M^* \right) \right] + h.c.$$

$$+ \left[ \Lambda_0 \left( \partial_\mu y_2^\mu - m - m \partial_\mu f^\mu \right) - \Lambda_1^a \left( \frac{\sqrt{2}}{2} \partial_\mu y_2^\mu + \sqrt{2} \Gamma_{a2} - \sqrt{2} m \zeta_a \right) \right.$$

$$+ \Lambda_2 \left( 1 + \partial_\mu f^\mu - \Gamma^m_{m1} \right) \right] + h.c. + O^2$$

with:

$$y_{a1}^\mu = 2i \left( \sigma^\mu \zeta \right)_a \ , \ y_2^\mu = \bar{\psi}_\nu \tilde{\sigma}^{\mu\nu} \bar{\psi}_\zeta \ , \ \Gamma_{a1}^m = -i \left( \sigma^\mu \zeta \right)_a \psi_\mu^m \ , \ \Gamma_2^m = -i \omega_\mu^{mn} \left( \sigma^{\mu \zeta} \right)_n + \frac{1}{3} M^* \zeta^m$$
The Stueckelberg fields will transform as:

\[ \delta f^\mu = -\xi^\mu + \frac{1}{2} [\zeta^r \tilde{y}^\mu_{r1} - \epsilon^r y^\mu_{r1}] + ... \]

\[ \delta \zeta^m = -\epsilon^m + \frac{1}{2} f^\rho \partial_\rho \epsilon^m + ... \]

Then \( S \) transforms as required for a chiral density superfield

\[ \delta S = -\partial_M \left[ \eta^M S(-1)^M \right] \]

We have restored full SUSY invariance to the action!
E.o.M and deSitter solution

- Write e.o.m. for all our fields (supergravity multiplet components, Λ components and Stueckelberg superfields $f^\mu$ and $\zeta$) we see that our theory admits solutions with c.c. of arbitrary sign:

$$\Lambda \propto \langle \Lambda_2 \rangle - m^2$$

$$m_\psi \propto m$$
• Extend to non-perturbative constructions, via conformal construction.
• Consequences for proposed solutions to c.c. problem.
• Relation to constrained superfields.
• Relation to brane constructions for goldstino.