Fractal Structure of Yang-Mills Fields

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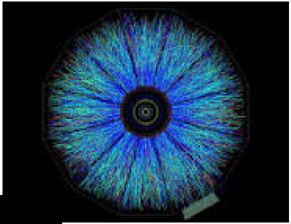
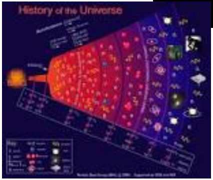
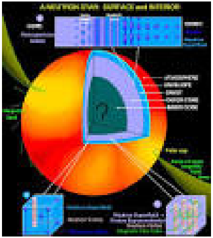
Based on: A.Deppman., E.M., D.P.Menezes, arXiv:1908.08799[hep-th].

Other references: E.M. D.P.Menezes, A.Deppman, Physica A421 (2015) 15; A.Deppman, PRD93 (2016) 054001; A.Deppman, E.M., D.P.Menezes, T.Frederico, Entropy 20 (2018) 9, 633; A.Deppman, E.M., MDPI Physics 1 (2019) 1 103; E.Andrade II et al. arXiv:1906.08301[nucl-th].

QGP: QCD and its applications







Thermodynamical approach

R. Hagedorn: thermodynamical approach to High Energy Collisions

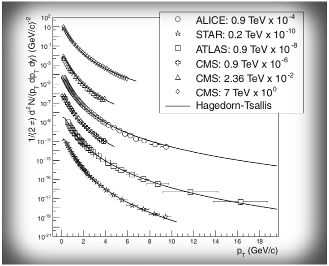
exponential distributions of energy and momentum

exponential hadron mass spectrum

Hadron Resonance Gas models, conf,/deconf. phase-transition

➡ but disagrees from experimental data

➡ when using Tsallis statistics $\longrightarrow$ power-law distribution $\longrightarrow$ $\longrightarrow$ the agreement is perfect



Tsallis Statistics and QCD Thermodynamics

- Tsallis statistics constitutes a generalization of Boltzmann-Gibbs (BG) statistics, under the assumption that the entropy of the system is non-additive. For two independent systems A and B

$$S_{A+B} = S_A + S_B + (1-q) S_A S_B \,,$$

where the entropic index q measures the degree of non-extensivity **[C.Tsallis, J.Stat.Phys. 52 '98]**.

- Grand-canonical partition function for a non-extensive ideal quantum gas is **[EM, A.Deppman, C.P.Menezes, Physica A421 '15]**

$$\log \Xi_q(V,T,\mu) = -\xi V \int \frac{d^3p}{(2\pi)^3} \sum_{r=\pm} \Theta(rx) \log_q^{(-r)} \left(\frac{e_q^{(r)}(x) - \xi}{e_q^{(r)}(x)} \right) \,,$$

where

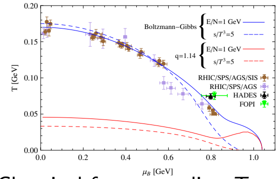
$$e_q^{(\pm)}(x) = [1 \pm (q-1)x]^{\pm 1/(q-1)} \,, \quad \log_q^{(\pm)}(x) = \pm (x^{\pm(q-1)} - 1)/(q-1) \,,$$

and $x = \beta(E_p - \mu)$, the particle energy is $E_p = \sqrt{p^2 + m^2}$, with m being the mass and μ the chemical potential, $\xi = \pm 1$ for bosons and fermions respectively, and Θ is the step function.

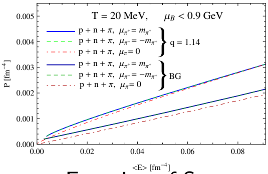
Tsallis Statistics and QCD Thermodynamics

- $e_q^{(\pm)}(x) \underset{q\to 1}{\longrightarrow} \exp(x)$ and $\log_q^{(\pm)}(x) \underset{q\to 1}{\longrightarrow} \log(x)$ ➡ This result reduces to the BG statistics in the limit $q \to 1$.
- The thermodynamics of Quantum Chromodynamics (QCD) in the confined phase can be studied within the Hadron Resonance Gas approach ➡ Assumption: physical observables in this phase admit a representation in terms of hadronic states which are treated as non-interacting and point-like particles **[Hagedorn, Lec.Not.Phys.221 '85]**.
- Partition function given by

$$\log \Xi_q(V,T,\{\mu\}) = \sum_{i\in\text{hadrons}} \log \Xi_q(V,T,\mu_i)\,,$$



Chemical freeze-out line $T = T(\mu_B)$.



Equation of State.

Tsallis Statistics and Thermofractals

- Emergence of the non-extensive behavior has been attributed to different causes: 1) long-range interactions, correlations and memory effects; 2) temperature fluctuations; 3) finite size of the system **[L.Borland, PLA 245 '98]**.
- We will study a natural derivation of non-extensive statistics in terms of Thermofractals.
- Thermofractals ≡ Systems in thermodynamical equilibrium presenting the following properties **[A.Deppman, PRD93 '16]**:

1. Total energy is given by:

$$U = F + E\,,$$

   where $F \equiv$ kinetic energy, and $E \equiv$ internal energy of N constituent subsystems, so that $E = \sum_{i=1}^{N} \varepsilon_i^{(1)}$.

2. Constituent particles are thermofractals: distribution $P_{\mathrm{TF}}(E)$ is self-similar or self-affine ➞ at level n of the subsystem hierarchy $P_{\mathrm{TF}(n)}(E)$ is equal to the distribution in the other levels.

Tsallis Statistics and Thermofractals

- The energy distribution according to BG statistics is given by

$$P_{\rm BG}(U)dU = A \exp(-U/kT)dU \,.$$

- In thermofractals ➡ phase space must include momentum degrees of freedom of free particles as well as internal degrees of freedom. According to property 2 of self-similar thermofractals

$$P_{\rm TF(0)}(U)dU = A' \underbrace{F^{\frac{3N}{2}-1} \exp\left(-\frac{\alpha F}{kT}\right) dF}_{\text{Momentum d.o.f.}} \underbrace{\left[P_{\rm TF(1)}(\varepsilon)\right]^{\nu} d\varepsilon}_{\text{internal d.o.f.}} \,,$$

with $\alpha = 1 + \frac{\varepsilon}{kT}$ and $\frac{\varepsilon}{kT} = \frac{E}{F}$, and $\nu \equiv$ exponent to be determined.

- By imposing self-similarity

$$P_{\rm TF(0)}(U) \propto P_{\rm TF(1)}(\varepsilon)$$

one finds: $P_{\rm TF}(\varepsilon) = A\left[1+\frac{\varepsilon}{kT}\right]^{-\frac{3N}{2}\frac{1}{1-\nu}}$ ➡ $P_{\mathrm{TF}(n)}(\varepsilon) = A_{(n)}\, e_q\left[-\frac{\varepsilon}{k\tau}\right]$

➡ The distribution of thermofractals then obeys Tsallis statistics with $q-1 = \frac{2}{3N}(1-\nu)$.

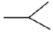
Scale Invariance and Diagrammatic Representation

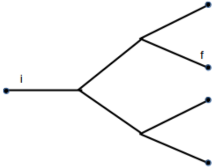
- Thermofractals are scale invariant ➞ this should be accomplished with the scale invariance of the distribution of kinetic and internal energy. Then

$$\frac{F^{(0)}}{T^{(0)}} = \frac{F^{(n)}}{T^{(n)}} \quad \Longrightarrow \quad \lambda_n := \frac{E^{(n)}}{E^{(0)}} \left(\frac{1}{N}\right)^{\frac{n}{1-D}} ,$$

where D is the fractal dimension.

- From the thermofractal structure one can obtain the fractal dimension of hadrons, resulting in $D = 0.69$ **[A.Deppman PRD93 '16]**.
- Diagrammatic representation of the probability densities of thermofractals that can facilitate calculations of the partition function and other quantities:

	$\int_{-\infty}^{\infty} d^3\pi\, \epsilon^{-3/2} e^{-f}$
	$(2\pi)^{-3/2} \prod_{i=1}^{N'} \delta\left(f - \sum_{j=1}^{N'} f_j\right)$
	$\int_0^{\infty} A\, kT\, e^{-\epsilon} \left[\tilde{P}(\varepsilon)\right]^{\nu} d\epsilon$



Left: Basic diagrams and their mathematical expressions.
Right: Tree graphs representing different levels of a thermofractal.

Callan-Symanzik equation

- The vertex function of thermofractals can be written in the form

$$\Gamma(E,\varepsilon,T) \propto (kT)^{-(1-D)} g \left[\prod_{i=1}^{N'} \left(2\pi \frac{E_i}{kT_i} \right)^{-3/2} \right] [P_{\rm TF}(\varepsilon_i)]^{\nu} \,.$$

- Then one can derive the Callan-Symanzik equation for thermofractals, which writes

$$\left[M\frac{\partial}{\partial M} + \sum_{i=1}^{N'} \beta_i \frac{\partial}{\partial m_i} + \beta_g \frac{\partial}{\partial \bar{g}} + \gamma \right] \Gamma = 0 \,,$$

where $m_i \equiv E_i$ is the thermofractal mass, which is identified with the thermofractal internal energy,

$$\beta_i = M\frac{\partial m_i}{\partial M} \,, \qquad \beta_{\bar{g}} = M\frac{\partial \bar{g}}{\partial M} \,,$$

and we have defined the effective coupling

$$\bar{g}(m,\varepsilon,t) = g \prod_{i=1}^{N'} \left[P_{\rm TF}\left(\frac{m(p_i)e^{t/d}}{M_0} \right) \right]^{\nu/2} \,, \qquad t := -d\log(M^2/M_0^2) \,.$$

Renormalization of gauge fields

- QCD thermodynamics can be described by Tsallis statistics
- Thermofractals obey Tsallis statistics.
- Question: Is it possible a thermofractal description of Yang-Mills theory?
- Yang-Mills theory $\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + i\bar{\psi}_j \gamma_\mu D^\mu_{ij} \Psi_j$ is renormalizable:

$$\Gamma(p, m, g) = \lambda^{-D} \Gamma(p, \mu, \bar{g})$$

F. Dyson, PR 75 (1949) 1736

M. Gell-Mann and F.E. Low, PR 95 (1954) 1300

- Renormalization group equation:

$$\left[M \frac{\partial}{\partial M} + \beta_g \frac{\partial}{\partial \bar{g}} + \gamma \right] \Gamma = 0$$

Callan-Symanzik Equation

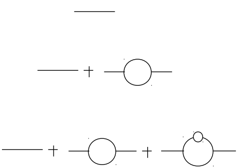
C.G. Callan Jr., PRD 2 (1970) 1541

K. Symanzik, Comm. Math. Phys. 18 (1970) 227

n=0

n=1

n=2



Effective coupling constant $\bar{g}$

Effective mass μ

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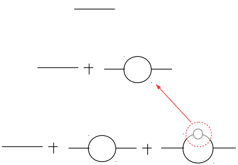


Effective coupling constant $\bar{g}$

Effective mass μ

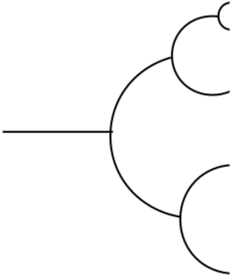
Scaling properties of YM fields →

loop in higher order is identical.

to a diagram in lower order.

Multiparticle production

Example of complex graphs in multiparticle production:

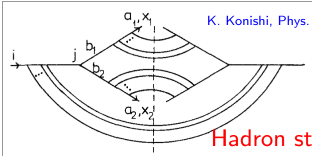


Summing up all diagrams → ideal gas of particles with different masses

R. Dashen, S. Ma and H. J. Bernstein, Phys. Rev. 187 (1969) 345

R. Venugopalan and M. Prakash, Nucl. Phys. A546 (1992) 718

Particle production is complex, not chaotic



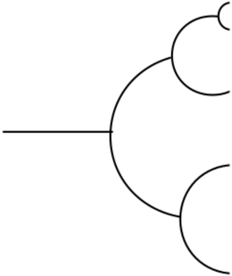
K. Konishi, Phys. Scr. 19 (1979) 195

Hadron structure is complex

Too many complex graphs to be considered.
Calculations limited to first leading orders or Lattice QCD.

Including fractal structure in YM fields

At any scale: ideal gas of particles with different masses.



Time evolution of initial partonic state

$|\Psi\rangle \equiv |\Psi(t)\rangle = e^{-iHt} |\Psi_o\rangle$

$|\Psi\rangle = \sum_{\{n\}} \langle \Psi_n | \Psi \rangle |\Psi_n\rangle$

$|\Psi_n\rangle \equiv$ state with n interactions in vertex function

$|\Psi_n\rangle = \sum_n e^{iH_o t_n} g e^{iH_o t_{n-1}} g \dots e^{iH_1 t_1} g |\Psi_o\rangle$

$|\Psi_n\rangle = \sum_N \langle \psi_N | \Psi_n \rangle |\psi_N\rangle$

$|\psi_N\rangle = \mathcal{S} |\gamma_1, m_1, p_1, \dots, \gamma_N, m_N, p_N\rangle$

$|\psi_N\rangle \equiv$ state with well defined number of partons.

N partons. It is like a state of a quantum gas of particles with different masses.

$\tilde{N} \equiv$ number of particles created or annihilated at each iteration

In Yang-Mills field theory ⟶ $\tilde{N} = 2$

Including fractal structure in YM fields

Probability to find a state with one parton with mass between m_o and $m_o + dm_o$ and momentum between p_o and $p_o + dp_o$:

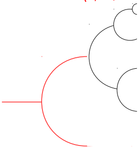
$$\langle \gamma_o, m_o, p_o, \ldots | \Psi(t) \rangle = \sum_n \sum_N \langle \Psi_n | \Psi(t) \rangle \langle \psi_N | \Psi_n \rangle \langle \gamma_o, m_o, p_o, \ldots | \psi_N \rangle$$

$$\langle \Psi_n | \Psi(t) \rangle = G^n P(E) dE, \qquad \langle \psi_N | \Psi_n \rangle = C_N(n) \text{ with } \sum_n C_N(n) = 1$$

$$\langle \gamma_o, m_o, p_o, \ldots | \psi_N \rangle \to f(p_j) d^4 p_j \equiv \text{Computed statistically}$$

$$f(p_j) d^4 p_j = d^4 p_j \frac{1}{8\pi} \frac{\Gamma(4N)}{\Gamma(4(N-1))} E^{-4} \left(1 - \frac{p_j^0}{E}\right)^{4N-5} .$$

$$\langle \gamma_o, m_o, \ldots | \Psi(t) \rangle = \sum_n \sum_N G^n \left(\frac{N}{n\tilde{N}}\right)^4 \left(1 - \frac{\varepsilon_j}{M}\right)^{4N-5} d^4 \left(\frac{p}{M}\right) P(E) dE .$$



Parent parton is also a parton $\to P(E) \propto \tilde{P}(p_o)$.

Self-symmetry in gauge fields!

It can be shown that $P(\mu)$ must be such that:

$$P(\varepsilon) = G^n [1 - (q-1) \frac{\varepsilon}{\lambda}]^{\frac{1}{q-1}}$$

$$\frac{\varepsilon}{\lambda} = \frac{p_j^o}{E}$$

A.Deppman, PRD (2016)

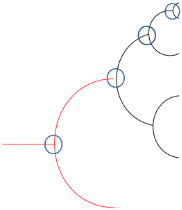
A.Deppman, EM, D.P.Menezes, T.Frederico, Entropy 20 (2018) 633

Non extensivity in gauge field theory

$$P(\varepsilon) = G^n[1-(q-1)\frac{\varepsilon}{\lambda}]^{\frac{1}{q-1}}$$ A.Deppman, PRD (2016)

Tsallis q-exponential function → Tsallis Statistics

q is related to the number of internal degrees of freedom in the fractal structure

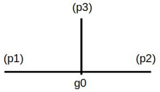


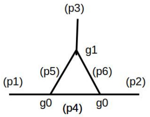
Suggest that at each vertex, momentum and effective masses are determined by the same scaled distribution

$$\bar{g} = \prod_{i=1}^{\tilde{N}} G\left[1-(q-1)\frac{\varepsilon_i}{k\tau}\right]^{\frac{1}{q-1}}$$

Calculation of q from gauge field parameters

First order calculation of γ and Γ was performed for YM-theory and QCD. We can then compare what is obtained with our ansatz.

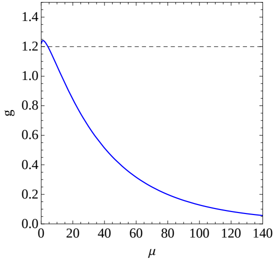


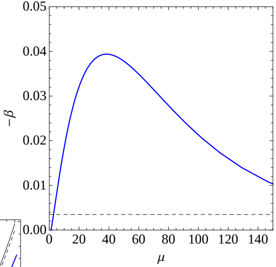


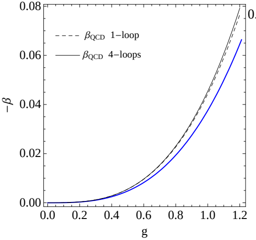
- From our ansatz: $\beta_g = \mu \frac{\partial \bar{g}}{\partial \mu} = -\frac{1}{16\pi^2} \frac{1}{q-1} g^{\tilde{N}+1}$ with $\tilde{N} = 2$
- From QCD: $\beta_g = -\frac{1}{16\pi^2} \left[\frac{11}{3} c_1 - \frac{4}{3} c_2\right] g^3$ and: $d - \gamma = \left[\frac{11}{3} c_1 - \frac{4}{3} c_2\right]$

➞ And finally we get $q = 1.14$

Effective coupling and beta function







Comparison with experiments

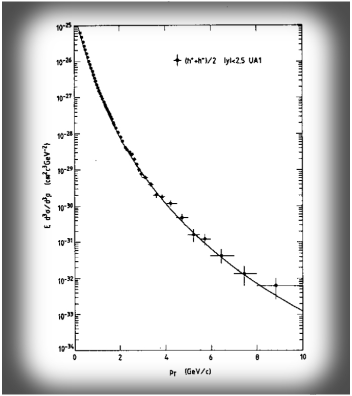
Extended Hagedorn theory to non extensive statistics: A.Deppman, Physica A 391 '12

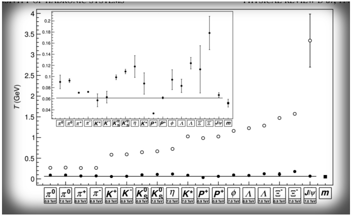
use of Tsallis factor: $P(\varepsilon) = A[1 + (q-1)\frac{\varepsilon}{k\tau}]^{-\frac{1}{q-1}}$

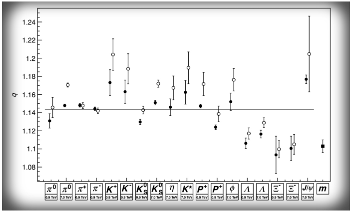
L.Marques, E.Andrade-II, A.Deppman, PRD 87 (2013) 114022

L.Marques, J.Cleymans, A.Deppman, PRD 91 (2015) 054025

Experimental value $q = 1.14 \pm 0.01$





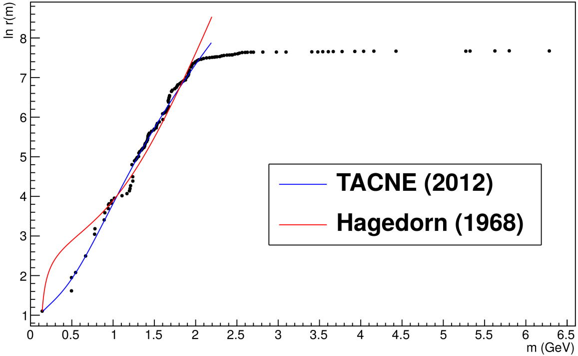


Effective mass spectrum and observed data

$$\rho(m) = \rho_o \left[1 + (q-1)m/M\right]^{1/(q-1)}$$

Obtained in Non Extensive Self-Consistent Thermodynamics,

by a completely different approach A.Deppman, Physica A 391 (2012) 6380



Non-extensive mass spectrum

L.Marques, E.Andrade-II, A.Deppman, PRD 87 (2013) 114022

Applications

High energy collisions:

J. Cleymans; D.J. Worku. Phys. G Nucl. Part. Phys. 2012, 39, 025006
C.-Y. Wong; G. Wilk, G.; Tsallis, C. Phys. Rev. D 2015, 91, 11402
L. Marques, J. Cleymans, and A.Deppman, PRD 91 (2015) 054025

Hadron models:

P.H.G Cardoso; T.N. da Silva; A. Depmman; D.P. Menezes, EPJA 51 (2015) 155

Hadron mass spectrum:

L. Marques; E. Andrade-II; A. Deppman, Phys. Rev. D 2013, 87, 114022

Neutron stars:

D.P. Menezes, A. Deppman, E.M., and L.B. Castro, EPJA 51, (2015) 155

Lattice QCD:

A. Deppman JPG 41 (2014) 055108

Non extensive statistics:

E.M., A. Deppman, D.P. Menezes, Physica A 421 (2015) 15
A. Deppman, Physica A 391 (2012) 6380
A. Deppman, E.M., D.P. Menezes, T. Frederico, (2018) Entropy 20 (2017) 633

Conclusions:

- We have reviewed the non-extensive statistics in the form of Tsallis statistics of a quantum gas at finite T and μ, and applied it to study the EoS and phase diagram of QCD.
- We have investigated the structure of a thermodynamical system presenting fractal properties, and shown that it naturally leads to non-extensive statistics.
- A diagrammatic formulation for practical calculations with the fractal structure was introduced.
- Based on the scale invariance of thermofractals, the Callan-Symanzik equation was obtained ⟶ *'Field theoretical approach'* for thermofractals.
- Scale invariance in gauge fields leads to

  Self-consistency and fractal struture

  Recursive calculations at any order

  Non extensive statistics

  Reconciles Hagedorn's theory with QCD

  Agreement with experimental data

Thank You!