Superasymptotic and hyperasymptotic approximation to the OPE

Xabier Lobregat

IFAE-BIST

03/07/2019

Work in collaboration with César Ayala and Antonio Pineda
Phys.Rev.D99,074019
arXiv:1902.07736[hep-th]

Institut de Física d'Altes Energies
EXCELENCIA SEVERO OCHOA
BIST Barcelona Institute of Science and Technology
Universitat Autònoma de Barcelona

Outline of the talk

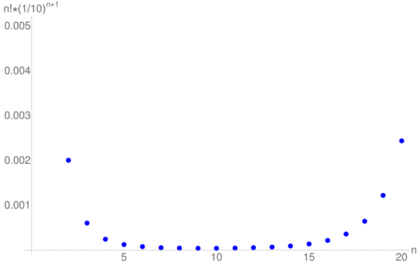
- Context and motivation
  - Divergent series in QCD
  - The OPE and condensates
  - The principal value Borel sum
- Method 1
  - The QCD static potential in the large β_0 approximation
- Conclusions

Divergent perturbative expansions

- Perturbative QCD yields power series in α for observable quantities

$$\sum_{n=0}^{\infty} a_n \alpha^{n+1} \tag{1}$$

- These series are expected to be divergent



- Nonetheless they are expected to be asymptotic to the true value

Superasymptotic series

- We are forced to truncate the series at some order

$$\sum_{n=0}^{N} a_n \alpha^{n+1} \tag{2}$$

- It is then natural to ask oneself what is the optimal truncation order $N+1$
- *Superasymptotic* series
- For example the Stieltjes function has the asymptotic expansion

$$S(\alpha) = \alpha \int_0^\infty dt\, e^{-t} \frac{1}{1+\alpha t} \qquad S(\alpha) \sim \sum_{n=0}^{\infty} (-1)^n n! \alpha^{n+1} \tag{3}$$

$$S(\alpha)_N \equiv \sum_{n=0}^{N_{min}} (-1)^n n! \alpha^{n+1} \qquad N_{min} \sim \frac{1}{\alpha} \tag{4}$$

- We find $S(1/10) = 0.0915633$ and $S_N(1/10) = 0.0915819$

The OPE and condensates

- In Phys.Rev.Lett.113,092001 Bali et al. considered the OPE-s of the average plaquette and the binding energy of a B meson in the heavy quark mass limit

$$\langle P\rangle_{MC} = \sum_{n=0}^{\infty} p_n \alpha^{n+1} + \frac{\pi^2}{36} C_G(\alpha) a^4 \langle \mathcal{O}_G\rangle + \mathcal{O}(a^6) \tag{5}$$

$$E_{MC}(\alpha) = \frac{1}{a}\sum_{n=0}^{\infty} c_n \alpha^{n+1} + \overline{\Lambda} + \mathcal{O}(a\Lambda_{QCD}^2) \tag{6}$$

- The authors truncated the series around the numerically minimal term and used them to obtain the gluon condensate and $\overline{\Lambda}$

$$\langle \mathcal{O}_G\rangle = \frac{36 C_G^{-1}(\alpha)}{\pi^2 a^4}\left(\langle P\rangle_{MC} - \sum_{n=0}^{N} p_n \alpha^{n+1}\right) + \mathcal{O}(a^2\Lambda_{QCD}^2) \tag{7}$$

$$\overline{\Lambda} = E_{MC}(\alpha) - \frac{1}{a}\sum_{n=0}^{N} c_n \alpha^{n+1} + \mathcal{O}(a\Lambda_{QCD}^2) \tag{8}$$

Ambiguities in truncation

- Truncation is inherently ambiguous. No absolute rule that sets the exact truncation order
- Truncated sums have a dependence on the renormalization scale and scheme
- Coming back to the condensates of the previous slide, these ambiguities will also see themselves reflected in the condensates

$$\langle \mathcal{O}_G \rangle_N = \frac{36 C_G^{-1}(\alpha)}{\pi^2 a^4} \left(\langle P \rangle_{MC} - \sum_{n=0}^{N} p_n \alpha^{n+1} \right) + \mathcal{O}(a^2 \Lambda_{QCD}^2) \tag{9}$$

$$\overline{\Lambda}_N = E_{MC}(\alpha) - \frac{1}{a} \sum_{n=0}^{N} c_n \alpha^{n+1} + \mathcal{O}(a \Lambda_{QCD}^2) \tag{10}$$

- The goal of our work is to estimate condensates of the OPE subtracting perturbative expansions to observables, but instead of truncating the series we will explore Borel summation

$$\langle \mathcal{O}_G \rangle_{PV} = \frac{36 C_G^{-1}(\alpha)}{\pi^2 a^4} \left(\langle P \rangle_{MC} - \left[\sum_{n=0}^{\infty} p_n \alpha^{n+1} \right]_{PV} \right) + \mathcal{O}(a^2 \Lambda_{QCD}^2) \tag{11}$$

$$\overline{\Lambda}_{PV} = E_{MC}(\alpha) - \frac{1}{a} \left[\sum_{n=0}^{\infty} c_n \alpha^{n+1} \right]_{PV} + \mathcal{O}(a \Lambda_{QCD}^2) \tag{12}$$

Borel summation

- From a divergent series we can construct a Borel transform

$$R \sim \sum_{n=0}^{\infty} a_n \alpha^{n+1} \tag{13}$$

$$\hat{R} = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n \tag{14}$$

- The Borel transform $\hat{R}$ has a finite radius of convergence. Then one considers its analytic continuation and a Laplace transform of it

$$\int_0^{\infty} dt\, e^{-t/\alpha} \hat{R} \tag{15}$$

- There are singularities in the integration path of the Laplace transform
- Contour deformation needed to avoid them. In this work I will consider the principal value (PV) prescription

$$PV \int_0^{\infty} dt\, e^{-t/\alpha} \hat{R} \tag{16}$$

PV vs truncation

- It can be seen that the PV Borel sum is renormalization scale and scheme independent as opposed to truncation schemes
- One huge disadvantage, in principle we need to know *all* the coefficients of a perturbative expansion
- On the contrary, truncation can always be carried out
- Quite interestingly, it is possible to relate truncated sums of perturbative series with their principal value Borel sums
  - Method 1 which uses the so called Dingle's theory of terminants
  - Method 2 uses results from Phys.Rev.D69,125006
- Due to time constraints and because it's more promising I will only talk about method 1

Method 1

- Based on Dingle's theory of terminants

$$PV = \sum_{n=0}^{N} a_n \alpha^{n+1} + T + \sum_{n=N+1}^{N'} (a_n - a_{n;asymp})\alpha^{n+1} + T' + \dots \quad (17)$$

- Terminant related to leading divergence of the coefficients

$$a_{n;asympt} = A^n(-1)^n(n+k)! \to T_1 \equiv \frac{(-1)^{N+1}}{\alpha^k} A^{N+1} \int_0^\infty dt\, e^{-t/\alpha} \frac{t^{N+1+k}}{1+At} \quad (18)$$

$$a_{n;asympt} = A^n(n+k)! \to T_2 \equiv \frac{1}{\alpha^k} A^{N+1} PV \int_0^\infty dt\, e^{-t/\alpha} \frac{t^{N+1+k}}{1-At} \quad (19)$$

- If N is such that we have truncated at the minimal order the terminant has a power expansion in α modulated by an exponentially suppressed term
- For the static potential in the large β_0 approximation the leading terminant will be

$$e^{\frac{-2\pi}{\beta_0\alpha}}\left(\#_0 + \#_1\alpha + \#_2\alpha^2 + \dots\right) \quad (20)$$

- The subleading terminant will be

$$e^{\frac{-6\pi}{\beta_0\alpha}}\left(\#'_0 + \#'_1\alpha + \#'_2\alpha^2 + \dots\right) \quad (21)$$

- An expansion where we add exponentially suppressed terms to a superasymptotically truncated series is called a *hyperasymptotic* expansion

Method 1

- In QCD in general for a dimensionless observable we will have

$$B[\mathcal{O}_d^X](t(u)) = Z_{\mathcal{O}_d}^X \frac{\mu^d}{Q^d} \frac{1}{(1-\frac{2u}{d})^{1+db}} \left\{ 1 + \frac{db}{n+db} b_1 + \frac{db(db-1)}{(n+db)(n+db-1)} w_2 + \dots \right\} \tag{22}$$

- Then the leading singularity of the coefficients will be

$$a_{n;asympt} = Z_{\mathcal{O}_d}^X \frac{\mu^d}{Q^d} \frac{\Gamma(1+db+n)}{\Gamma(1+db)} \left\{ 1 + \frac{db}{n+db} b_1 + \frac{db(db-1)}{(n+db)(n+db-1)} w_2 + \dots \right\} \left(\frac{\beta_0}{2\pi d} \right)^n \tag{23}$$

- Defining

$$\Delta\Omega(db) \equiv Z_{\mathcal{O}_d}^X \frac{\mu^d}{Q^d} \frac{1}{\Gamma(1+db)} \left(\frac{\beta_0}{2\pi d} \right)^{N+1} \alpha_X^{N+2}(\mu) PV \int_0^\infty dx\, e^{-x} \frac{x^{db+N+1}}{1 - x\frac{\beta_0 \alpha_X(\mu)}{2\pi d}} \tag{24}$$

the terminant Ω is

$$\Omega = \Delta\Omega(db) + b_1 \Delta\Omega(db-1) + w_2 \Delta\Omega(db-2) + \dots \tag{25}$$

- We truncate the series at $\alpha_X^{N+1}(\mu)$ around the optimal order

$$N = \frac{d2\pi}{\beta_0 \alpha_X(\mu)} (1 - c\alpha_X(\mu)) \tag{26}$$

which allows us to write the $\alpha_X \sim 0$ expansion for the terminant

$$\Omega = \sqrt{\alpha_X(\mu)} K_X^{(P)} \frac{\mu^d}{Q^d} e^{-\frac{d2\pi}{\beta_0 \alpha_X(\mu)}} \left(\frac{\beta_0 \alpha_X(\mu)}{4\pi} \right)^{-db} \left(1 + \bar{K}_{X,1}^{(P)} \alpha_X(\mu) + \bar{K}_{X,2}^{(P)} \alpha_X^2(\mu) + \mathcal{O}(\alpha_X^3(\mu)) \right) \tag{27}$$

where

$$K_X^{(P)} = \frac{-Z_{O_d}^X}{\Gamma(1+bd)} \left(\frac{2\pi d}{\beta_0} \right)^{bd+1} \left(\frac{\beta_0}{4\pi} \right)^{bd} \left(\frac{\beta_0}{d} \right)^{1/2} \left[-\eta_c + \frac{1}{3} \right] \qquad \eta_c \equiv -bd + \frac{2\pi d}{\beta_0} c - 1 \tag{28}$$

$$\bar{K}_{X,1}^{(P)} = \frac{\beta_0/(\pi d)}{-\eta_c + \frac{1}{3}} \left[-b_1 (bd) \left(\frac{1}{2}\eta_c + \frac{1}{3} \right) - \frac{1}{12}\eta_c^3 + \frac{1}{24}\eta_c - \frac{1}{1080} \right] \tag{29}$$

$$\bar{K}_{X,2}^{(P)} = \frac{\beta_0^2/(\pi d)^2}{-\eta_c + \frac{1}{3}} \left[-w_2(bd-1)bd \left(\frac{1}{4}\eta_c + \frac{5}{12} \right) + b_1 bd \left(-\frac{1}{24}\eta_c^3 - \frac{1}{8}\eta_c^2 - \frac{5}{48}\eta_c - \frac{23}{1080} \right) - \frac{1}{160}\eta_c^5 - \frac{1}{96}\eta_c^4 + \frac{1}{144}\eta_c^3 + \frac{1}{96}\eta_c^2 - \frac{1}{640}\eta_c - \frac{25}{24192} \right] \tag{30}$$

Method 1

- Thus the principal value Borel sum of the observable can be written

$$S_{PV}(Q) = S_P(Q;\mu) + \sqrt{\alpha_X(\mu)} K_X^{(P)} \frac{\mu^d}{Q^d} e^{-\frac{d2\pi}{\beta_0 \alpha_X(\mu)}} \left(\frac{\beta_0 \alpha_X(\mu)}{4\pi}\right)^{-db} \times \left(1 + \bar{K}_{X,1}^{(P)} \alpha_X(\mu) + \bar{K}_{X,2}^{(P)} \alpha_X^2(\mu) + \mathcal{O}(\alpha_X^3(\mu))\right) + \dots \tag{31}$$

- In the large β_0 approximation things simplify. For the static potential we would have

$$\Omega = Z_V^X \mu r \frac{4\pi}{\beta_0} 2^{N+1} PV \int_0^\infty du\, e^{\frac{-4\pi u}{\beta_0 \alpha_X(\mu)}} \frac{u^{N+1}}{1-2u} \qquad Z_V = -2\frac{C_F}{\pi} e^{-\frac{c_X}{2}} \tag{32}$$

  where c_X parametrizes the renormalization scheme in the large β_0 approximation. Thus the hyperasymptotic expansion for the PV Borel sum can be written like this

$$V_{PV} = V_P + \frac{1}{r}\Omega + \sum_{n=N+1}^{3N} (v_n - v_{n;asymp}) \alpha_X^{n+1}(\mu) + \frac{1}{r}\Omega' + \dots \tag{33}$$

$$\Omega \sim r\Lambda_{QCD} \qquad \Omega' \sim r^3 \Lambda_{QCD}^3 \tag{34}$$

- We will test this method for the static potential in the larget β_0 approximation where we have a lot of analytic control

$$B[V](t(u)) = \frac{-C_F}{\pi^{1/2} r} e^{-c_X u} \left(\frac{\mu^2 r^2}{4}\right)^u \frac{\Gamma(1/2-u)}{\Gamma(1+u)} \qquad u = \frac{\beta_0}{4\pi} t \tag{35}$$

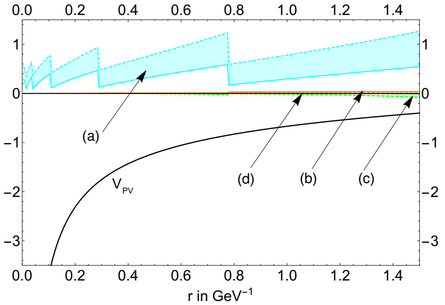
  and we can compute the PV Borel sum exactly

$$V_{PV} = PV \int_0^\infty dt\, e^{-t/\alpha_X(\mu)} B[V](t) \tag{36}$$

The static potential in the large β_0 approximation

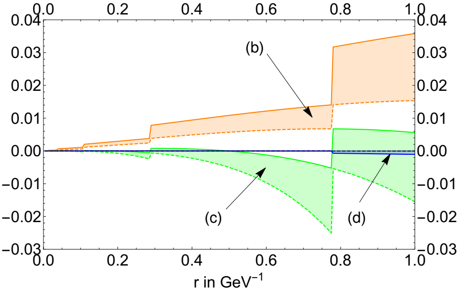
- I consider the static potential in the large β_0 approximation in the $\overline{MS}$ scheme and the lattice scheme
- I fix $\mu = 1/r$ and I consider two truncation points, the N that makes c smallest and still positive, and one order further which will make c the smallest posible in absolute value and negative
$$N = \frac{2\pi}{\beta_0 \alpha_X(1/r)}(1 - c\alpha_X(1/r)) \tag{37}$$
- We will see that the more terms we include in the hyperasymptotic expansion either of these two truncation points will converge to the same result

$\overline{MS}$ $Nf = 3$



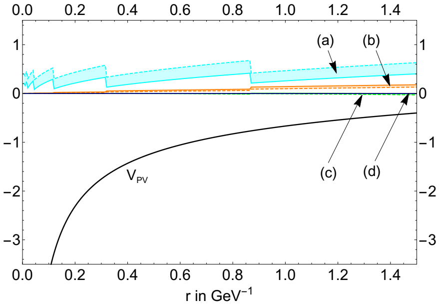
- V_{PV} (black line)
- (a) $V_{PV} - V_P$ (cyan)
- (b) $V_{PV} - V_P - \frac{1}{r}\Omega_V$ (orange)
- (c) $V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N+1}^{3N}(v_n - v_n^{(as)})\alpha^{n+1}$ (green)
- (d) $V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N+1}^{3N}(v_n - v_n^{(as)})\alpha^{n+1} - \frac{1}{r}\Omega'_V$ (blue)

$\overline{MS}$ $Nf = 3$ Zoom



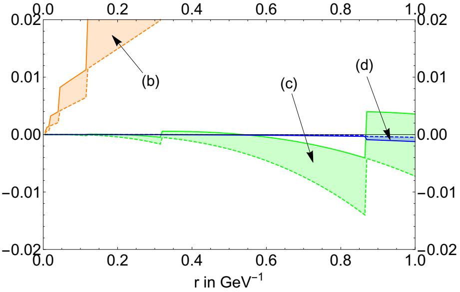
- V_{PV} (black line)
- (a) $V_{PV} - V_P$ (cyan)
- (b) $V_{PV} - V_P - \frac{1}{r}\Omega_V$ (orange)
- (c) $V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N+1}^{3N}(v_n - v_n^{(as)})\alpha^{n+1}$ (green)
- (d) $V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N+1}^{3N}(v_n - v_n^{(as)})\alpha^{n+1} - \frac{1}{r}\Omega'_V$ (blue)

Lattice $Nf = 3$



- V_{PV} (black line)
- (a) $V_{PV} - V_P$ (cyan)
- (b) $V_{PV} - V_P - \frac{1}{r}\Omega_V$ (orange)
- (c) $V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N+1}^{3N}(v_n - v_n^{(as)})\alpha^{n+1}$ (green)
- (d) $V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N+1}^{3N}(v_n - v_n^{(as)})\alpha^{n+1} - \frac{1}{r}\Omega_V'$ (blue)

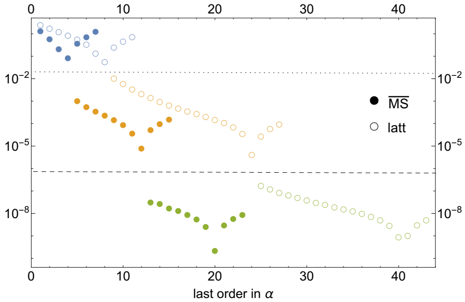
Lattice $Nf = 3$ Zoom



- V_{PV} (black line)
- (a) $V_{PV} - V_P$ (cyan)
- (b) $V_{PV} - V_P - \frac{1}{r}\Omega_V$ (orange)
- (c) $V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N+1}^{3N}(v_n - v_n^{(as)})\alpha^{n+1}$ (green)
- (d) $V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N+1}^{3N}(v_n - v_n^{(as)})\alpha^{n+1} - \frac{1}{r}\Omega'_V$ (blue)

Comparing $\overline{MS}$ and lattice

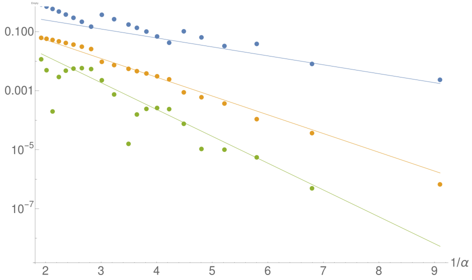
For $Nf = 3$ and $r = 0.1 GeV^{-1}$



- $V_{PV} - \sum_{n=0} v_n \alpha^{n+1}$ (blue)
- $V_{PV} - \sum_{n=0}^{N} v_n \alpha^{n+1} - \frac{1}{r}\Omega_V - \sum_{n=N+1}(v_n - v_n^{(as)})\alpha^{n+1}$ (orange)
- $V_{PV} - \sum_{n=0}^{N} v_n \alpha^{n+1} - \frac{1}{r}\Omega_V - \sum_{n=N+1}^{3N}(v_n - v_n^{(as)})\alpha^{n+1} - \frac{1}{r}\Omega'_V - \sum_{n=3N+1}(v_n - v_n^{(as)} - v_n^{(as;2)})\alpha^{n+1}$ (green)

Order of magnitude in the hyperasymptotic expansion

For $\overline{MS}$ with $Nf = 3$



- $r\left(V_{PV} - \sum_{n=0}^{N} v_n \alpha^{n+1}\right)$ vs $r\Lambda_{QCD}$ (blue)
- $r\left(V_{PV} - \sum_{n=0}^{N} v_n \alpha^{n+1} - \frac{1}{r}\Omega_V\right)$ vs $(r\Lambda_{QCD})^{1+\log 3}$ (orange)
- $r\left(V_{PV} - \sum_{n=0}^{N} v_n \alpha^{n+1} - \frac{1}{r}\Omega_V - \sum_{n=N+1}^{3N}(v_n - v_n^{(as)})\alpha^{n+1}\right)$ vs $r^3\Lambda_{QCD}^3$ (green)

Conclusions

- We have seen a method to hyperasymptotically approximate the PV Borel sum of an observable
- We only need knowledge of the exact coefficients until the perturbative series starts to diverge, and the structure of the singularities in the Borel plane
- We have checked that the methods work for the static potential in the large β_0 approximation
- We mean to apply the same method to real QCD observables and to use them to obtain condensates from the OPE