CHIRAL VORTICAL EFFECT AT A FINITE MASS AND UNRUH EFFECT FOR FERMIONS

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Abstract

The effects of acceleration and rotation of the medium in the axial current and the energy-momentum tensor of fermions are investigated. When considering the axial current, higher order corrections in vorticity and acceleration to chiral vortical effect (CVE) were defined, and a general formula for CVE at finite mass on the axis of rotation was justified. When considering the energy-momentum tensor based on the calculation of the fourth-order corrections in acceleration and vorticity, the Unruh effect for fermions is shown. This effect is manifested in the vanishing of the energy-momentum tensor in the laboratory frame at the proper temperature equal to the Unruh temperature $T_U = \frac{a}{2\pi}$. Thus, the Minkowski vacuum corresponds to the Unruh temperature, measured by the comoving observer. It is shown that the angular velocity plays the role of a real chemical potential, while acceleration plays the role of an imaginary one. The appearance of acceleration as an imaginary chemical potential leads to instabilities, which appear in the energy density at a temperature equal to the Unruh temperature. This is another manifestation of Unruh effect. It is shown that these instabilities arise as a result of the existence of singularities at the Fermi distribution on the complex momentum plane. The occurrence of instabilities corresponds to the cross of the poles by the integration contour.

Background

Chiral vortical effect

- The essence of the CVE is the appearance of axial current in a rotating medium, directed along the vorticity. In the chiral limit, the axial current is Ref. [1,2]:

$$j^5_\mu = \left(\frac{T^2}{6} + \frac{\mu^2}{2\pi^2}\right)\omega_\mu . \quad (1)$$

- Chiral effects are closely related to the anomalies of quantum field theory such as axial electromagnetic and gravitational. Eq. (1) corresponds to the particular case of zero mass and takes into account only linear terms in the derivatives of the velocity. We investigated the effects of higher orders, the role of acceleration, as well as the effects of mass.

Unruh effect

- The meaning of the Unruh effect (also called Unruh-Hawking effect) Ref. [3] is that the accelerated observer sees the Minkowski vacuum as a medium filled with particles with a Unruh temperature proportional to the acceleration

$$T_U = \frac{a}{2\pi} .$$

- **Thus, the values of the observables in the laboratory frame should be equal zero when the proper temperature, measured by comoving observer, equals the Unruh temperature Ref. [4].**
- It was shown for scalar particles in Ref. [4]. For fermions, a similar effect was observed in Ref. [5] based on the Wigner function Eq. (3) at a twice the Unruh temperature $2T_U$.

Methods

Zubarev quantum-statistical density operator

- The most fundamental object describing a medium in a state of thermodynamic equilibrium is the density operator, introduced by Zubarev Ref. [6,7]

$$\hat{\rho} = \frac{1}{Z}\exp\left\{-\beta_\mu(x)\hat{P}^\mu + \frac{1}{2}\varpi_{\mu\nu}\hat{J}^{\mu\nu}_x + \zeta\hat{Q}\right\} . \quad (2)$$

- Effects associated with the vorticity and acceleration are described by the term with thermal vorticity tensor $\varpi_{\mu\nu} = -\frac{1}{2}(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu)$. Quantum statistical mean values can be calculated within finite temperature quantum field perturbation theory. In particular, CVE and second order hydrodynamic coefficients in different quantities were calculated in Ref. [7].

Covariant Wigner function for particles with spin

- In Ref. [8] Wigner function was proposed to describe media with fermionic constituents in the state of a local thermodynamic equilibrium

$$X(x,p) = \left(\exp[\beta_\mu p^\mu - \zeta]\exp\left[-\frac{1}{2}\varpi_{\mu\nu}\Sigma^{\mu\nu}\right] + I\right)^{-1} . \quad (3)$$

- It was checked that this Wigner function reproduces correctly known results, with pure rotation (but gives a discrepancy with other methods in case of acceleration Ref. [9, 13]).

CVE at finite mass *(on the basis of Ref. [9,10])*

Parity allows the appearance of 3 types of terms in the third order of perturbation theory

$$\langle \hat{j}^\lambda_5(x)\rangle^{(3)} = A_1\omega^2\omega^\lambda + A_2 a^2\omega^\lambda + A_3(\omega a)a^\lambda .$$

These coefficients were found based on the density operator Eq. (1) both for the case of massive fermions and in the chiral limit:

Massless fermions

$$A_1 \to -\frac{1}{24\pi^2}, \quad A_2 \to -\frac{1}{8\pi^2}, \quad A_3 = 0 ,$$

$$\langle j^5_\mu\rangle = \left(\frac{1}{6}\left[T^2 - \frac{\omega^2}{4\pi^2}\right] + \frac{\mu^2}{2\pi^2} - \frac{a^2}{8\pi^2}\right)\omega_\mu + \mathcal{O}(\varpi^5). \quad (4)$$

In red: new terms in comparison to standard CVE in Eq. (1).

Equality $A_3 = 0$ ensures the conservation of axial current $\partial^\mu\langle j^5_\mu\rangle = 0$ in Eq.(4) (in contrast to axial current calculated based on the Wigner function Eq. (3) (look Ref. [11]), where $A_3 \neq 0$ and $\partial^\mu\langle j^5_\mu\rangle \neq 0$).

Massive fermions

In particular, for the coefficient A_1 we get at finite mass:

$$A_1 = \frac{1}{48\pi^2|\beta|^3}\int_0^\infty dp\Big(n'''_F(E_p-\mu) + n'''_F(E_p+\mu)\Big)p^2 , \quad (5)$$

where n_F is Fermi-Dirac distribution, $E_p = \sqrt{p^2+m^2}$ energy and $n'''_F(E_p) = \frac{d^3}{dE_p^3}n_F(E_p)$.

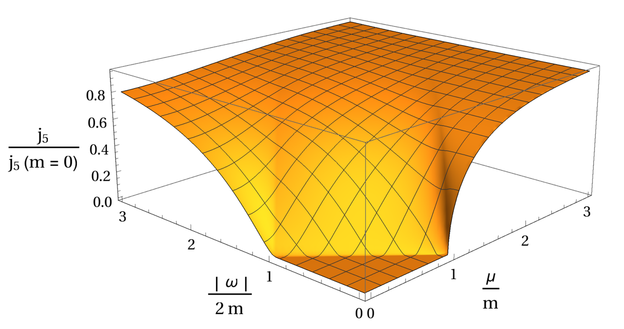
General nonperturbative formula for CVE at finite mass

Comparison with the hydrodynamic coefficients of the first order (Eq. (7.4) in Ref. [7]) and the third order, obtained above Eq. (5), at a finite mass, allows to restore the general nonperturbative formula for axial current (CVE at a finite mass):

$$\langle j^5_\mu\rangle = \int\frac{d^3p}{(2\pi)^3}\Big\{n_F\Big(E_p-\mu-\frac{|\omega|}{2}\Big) - n_F\Big(E_p-\mu+\frac{|\omega|}{2}\Big) + n_F\Big(E_p+\mu-\frac{|\omega|}{2}\Big) - n_F\Big(E_p+\mu+\frac{|\omega|}{2}\Big)\Big\}\frac{\omega_\mu}{|\omega|}, \; |\omega| = \sqrt{-\omega^2} . \quad (6)$$

All the dependence of CVE on mass is accumulated in energy $E_p = \sqrt{p^2+m^2}$.

- Eq. (6) refers to the case $a = 0$ and therefore gives a general expression for the global equilibrium axial current on the axis of rotation.
- Eq. (6) gives expression for CVE at arbitrary large ω and m.
- Eq. (6) exactly coincides with the prediction of the Wigner function Eq. (1) (look Ref. [9,10]).
- Angular velocity enters as a real chemical potential $\mu \to \mu \pm \frac{|\omega|}{2}$.
- In a more general case with both acceleration and vorticity, combination of the form $\mu \to \mu \pm \frac{|\omega|}{2} \pm \frac{i|a|}{2}$ would appear Ref.[10] if the angular velocity is parallel to acceleration (and more complex combination in general case), *but the effects of acceleration are not limited only to chemical potential modification (look, for example Eq. (9)).*



The appearance of the angular velocity as a chemical potential leads to the vanishing of axial current in the limit $T \to 0$ in the region $|\omega| < 2(m - |\mu|)$.

Coincidence with other methods

Let us compare the consequences of Eq. (6) with known results:

1. The first order term in ω in the limit $m \to 0$ reproduces the formula for CVE: $j^5_\mu = \left(\frac{T^2}{6} + \frac{\mu^2}{2\pi^2}\right)\omega_\mu$.
2. The nonperturbative result of Ref. [2] is reproduced in the chiral limit: $\langle j^5_\mu\rangle = \left(\frac{T^2}{6} - \frac{\omega^2}{24\pi^2} + \frac{\mu^2}{2\pi^2}\right)\omega_\mu$.
3. The first correction in mass to CVE, obtained from the calculation with massive propagators in Ref. [12], is exactly reproduced: $\langle j^5_\mu\rangle = \left(\frac{T^2}{6} - \frac{m^2}{4\pi^2}\right)\omega_\mu$.

Conclusion

- **Nonperturbative Eq. [6] satisfies to all known results on the CVE, reproducing them as particular limiting cases.**

Unruh Effect for fermions *(on the basis of Ref. [13])*

According to Ref. [4], the essence of Unruh effect from the point of view of quantum statistical mechanics is that at proper temperature, measured by the comoving observer, equal to the Unruh temperature, different physical quantities in inertial frame are to be equal to 0, which corresponds to Minkowski vacuum in inertial frame. To show this, lets consider Zubarev density operator in the absence of chemical potential and vorticity Ref. [7]

$$\hat{\rho} = \frac{1}{Z}\exp\left\{-\beta_\mu\hat{P}^\mu - \alpha_\mu\hat{K}^\mu_x\right\} .$$

The effects of acceleration are described by a term with a boost operator $\hat{K}_x$.

Quantum-statistical mean values can be found using perturbation theory, following the algorithm of Ref. [7]. In the fourth order of the perturbation theory, parity fixes the structure of the energy-momentum tensor

$$\langle\hat{T}^{\mu\nu}\rangle = (\rho_0 + A_1a^2T^2 + A_2a^4)u^\mu u^\nu - (p_0 + B_1a^2T^2 + B_2a^4)\Delta^{\mu\nu} + (A_3T^2 + A_4a^2)a^\mu a^\nu + \mathcal{O}(a^6) \qquad \Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu , \quad (7)$$

The coefficients up to the second order in Eq. (7) for fermions ρ_0, p_0, A_1, B_1, A_3 were calculated in Ref. [7]. However, in the second order, the perturbation theory fails to reach the vanishing of the observables. In particular, energy density is not zero:

$$\rho^{(2)} = \frac{7\pi^2T^4}{60} + \frac{T^2a^2}{24}\Big|_{T=T_U} \neq 0 .$$

Unlike Eq. (4), here $a^2 = |a|^2 = -a_\mu a^\mu$.

Thus, it is necessary to calculate the terms of higher orders in acceleration, starting with the fourth order term a^4.

4th order terms in energy-momentum tensor

4th order terms were calculated in Ref. [13]: $A_2 = -\frac{17}{960\pi^2}$, $B_2 = \frac{A_2}{3} = -\frac{17}{2880\pi^2}$, $A_4 = 0$.

Taking into account these terms, in particular, the energy density becomes:

$$\rho^{(4)} = \frac{7\pi^2T^4}{60} + \frac{T^2a^2}{24} - \frac{17a^4}{960\pi^2} + \mathcal{O}(a^6) . \quad (8)$$

In red: new term, which was calculated.

Unlike Eq. (4), here $a^2 = |a|^2 = -a_\mu a^\mu$.

The energy density is 0 at proper temperature $T = \frac{a}{2\pi}$, if we take into account the terms of 4th order

$$\rho^{(4)}(T = T_U) = 0 ,$$

which can be obtained as the result of summation $\frac{a^4}{960\pi^2}(7 + 10 - 17) = 0$. The same is true, in general, for the energy-momentum tensor: $\langle\hat{T}^{\mu\nu}\rangle = 0$ at $T = T_U$. Other observables are 0 due to parity and due to the conditions $\mu = \mu_5 = 0$ and $\omega^\mu = 0$.

Thus, the Minkowski vacuum in the laboratory system corresponds to the proper temperature measured by the comoving observer equal to the Unruh temperature $T = \frac{a}{2\pi}$, which is the essence of Unruh effect.

- Confirmation and generalization for the case of fermions of the result Ref. [4].

Intergral representation

Eq. (8) has the form of a polynomial with rather unusual numerical coefficients obtained as a result of calculating quantum correlators. However, it can be shown that these coefficients can be obtained naturally from integrals with Fermi and Bose distributions. One can check that in the region $T > T_U$ the next equality is exactly satisfied

$$\rho = \frac{7\pi^2T^4}{60} + \frac{T^2a^2}{24} - \frac{17a^4}{960\pi^2} = 2\int\frac{d^3p}{(2\pi)^3}\Big(\frac{|\mathbf{p}|+ia}{1+e^{\frac{|\mathbf{p}|}{T}+\frac{ia}{2T}}} + \frac{|\mathbf{p}|-ia}{1+e^{\frac{|\mathbf{p}|}{T}-\frac{ia}{2T}}}\Big) + 4\int\frac{d^3p}{(2\pi)^3}\frac{|\mathbf{p}|}{e^{\frac{2\pi|\mathbf{p}|}{a}}-1} \quad (T > T_U), \quad (9)$$

in red: modifications compared to the result Eq. (10), obtained from the Wigner function.

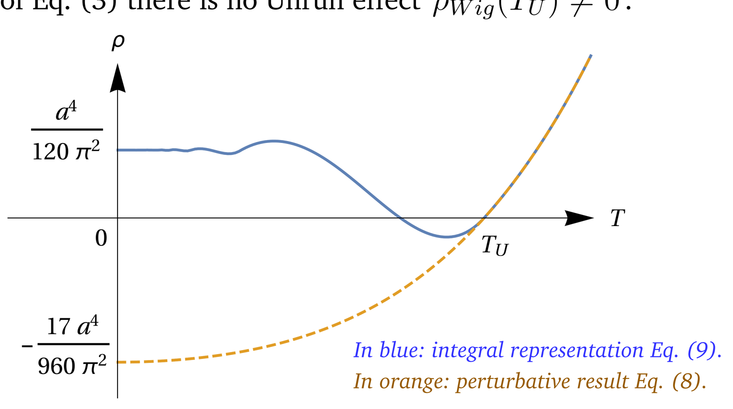
- Acceleration enters as an imaginary chemical potential $\mu \to \mu \pm \frac{ia}{2}$.

Motivation from the Wigner function

The additional motivation for introducing such an integral representation is the result for energy density, obtained using the Wigner function (beyond the Boltzmann approximation)

$$\rho_{Wig} = 2\int\frac{d^3p}{(2\pi)^3}\varepsilon\Big(\frac{1}{1+e^{\frac{\varepsilon}{T}+\frac{ia}{2T}}} + \frac{1}{1+e^{\frac{\varepsilon}{T}-\frac{ia}{2T}}}\Big), \quad (10)$$

where acceleration also appears as imaginary chemical potential $\mu \to \mu \pm \frac{ia}{2}$. However due to approximate nature of Eq. (3) there is no Unruh effect $\rho_{Wig}(T_U) \neq 0$.



Motivation:

- Exact match with the fundamental perturbative result Eq. (8) from the density operator at $T > T_U$.
- Motivation from the Wigner function result Eq. (10).

Energy of rotating and accelerated fermion gas

(on the basis of Ref. [14])

At non-zero values of both ω and a in the energy density ρ the terms of the next form may appear: ω^2 (calculated in Ref.[7]), ω^4, ω^2a^2 and $(\omega a)^2$ in addition to Eq. (8). In the case when the vorticity and acceleration are **parallel** $(\omega a)^2 = \omega^2a^2$, the calculation of quantum corrections based on the Zubarev density operator Eq. (2) leads to the expression

$$\rho = \frac{7\pi^2T^4}{60} + \frac{T^2a^2}{24} - \frac{17a^4}{960\pi^2} + \frac{T^2\omega^2}{24} + \frac{\omega^4}{64\pi^2} + \frac{\omega^2a^2}{32\pi^2} + \mathcal{O}(\varpi^6) . \quad (12)$$

In red: new terms in comparison to Eq. (8).

Unlike Eq. (4), here $a^2 = |a|^2 = -a_\mu a^\mu$ ...

It can be shown, by analogy with Eq. (9), that Eq. (12) exactly coincides with the integral representation:

$$\rho = \int\frac{d^3p}{(2\pi)^3}\Big(\frac{|\mathbf{p}|+ia}{1+e^{\frac{|\mathbf{p}|}{T}-\frac{\omega}{2T}+\frac{ia}{2T}}} + \frac{|\mathbf{p}|+ia}{1+e^{\frac{|\mathbf{p}|}{T}+\frac{\omega}{2T}+\frac{ia}{2T}}} + c.c\Big) + 4\int\frac{d^3p}{(2\pi)^3}\frac{|\mathbf{p}|}{e^{\frac{2\pi|\mathbf{p}|}{a}}-1} . \quad (13)$$

In red: new terms in comparison to Eq. (9).

In the case of a finite mass $m \neq 0$ and zero acceleration $a = 0$, the coefficient of ω^2 was calculated in Ref.[7] and of ω^4 in Ref. [14]. These calculations confirm the validity of the nonperturbative formula for the energy of the rotating gas of massive fermions on the rotation axis:

Eq. (14) can also be derived from the Wigner function Eq.(3).

$$\rho = 2\int\frac{d^3p}{(2\pi)^3}E_p\Big(\frac{1}{1+e^{\frac{E_p}{T}+\frac{\omega}{2T}}} + \frac{1}{1+e^{\frac{E_p}{T}-\frac{\omega}{2T}}}\Big). \quad (14)$$

Here again, like in Eq. (6): $E_p = \sqrt{p^2+m^2}$.

Instability at Unruh temperature *(on the basis of Ref. [14])*

It can be shown that in addition to the vanishing of the energy density at the Unruh temperature (which is the content of the Unruh effect), instabilities also occur in different observables at the Unruh temperature *(in particular, it was previously discussed for axial current in Ref. [10])*. In particular, they manifest themselves in the discontinuity of a second-order derivative of energy density $\frac{\partial^2\rho}{\partial T^2}$. This effect is directly related to the appearance of acceleration as an imaginary chemical potential. To investigate this effect in more details, we use Blankenbecler's method Ref. [15] and represent the energy density in Eq. (9) as contour integrals in the complex plane:

$$\rho = 2\int\frac{d^3p}{(2\pi)^3}\Big(\frac{|\mathbf{p}|+ia}{1+e^{\frac{|\mathbf{p}|}{T}+\frac{ia}{2T}}} + \frac{|\mathbf{p}|-ia}{1+e^{\frac{|\mathbf{p}|}{T}-\frac{ia}{2T}}}\Big) + 4\int\frac{d^3p}{(2\pi)^3}\frac{|\mathbf{p}|}{e^{\frac{2\pi|\mathbf{p}|}{a}}-1} ,$$

$$\Downarrow$$

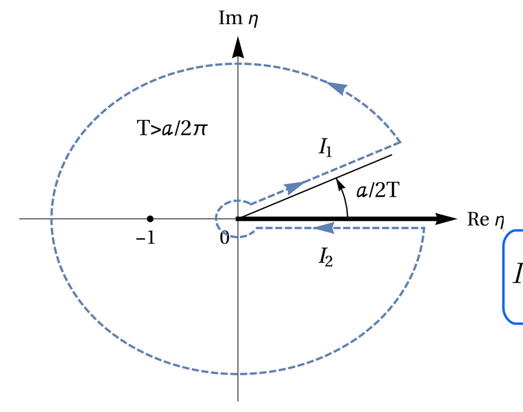
$$I_1 = \int_{I1}\frac{\eta^D}{(\eta+1)^2}f(-iy) , \quad D = \frac{d}{d(-iy)} , \quad y = \frac{a}{2T} ,$$

where the integrand has a pole at $\eta = -1$ connected to the Fermi-distribution pole in the complex momentum plane

$$\frac{1}{e^{p/T}+1} , \; e^{p/T} = -1 ,$$

and a cut along the positive real semi-axis. Function f equals $f(-iy) = \frac{(-iy)^3}{3}$ and $\frac{(-iy)^4}{4}$. It appears from the weight function in Fermi integrals.

Acceleration creates a non-zero angle of inclination of the contour, along which integration takes place:



Case $T > \frac{a}{2\pi}$

Integrals can be found from the residue theorem

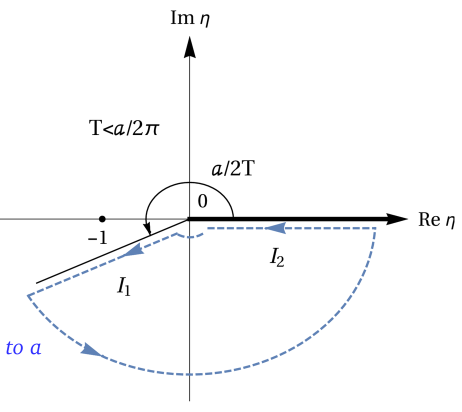
$$I_1 + I_2 = 2\pi i\,\mathrm{Res}_{\eta=-1}\frac{\eta^D}{(\eta+1)^2}f(-iy) ,$$

$$\Downarrow$$

$$I_1 = \int_{I1}\frac{\eta^D}{(\eta+1)^2}f(-iy) = \frac{\pi D}{\sin(\pi D)}f(-iy), \quad (11)$$

$\Downarrow$ *The polynomiality of function f(-iy) leads to a polynomial energy.*

$$\rho = \frac{7\pi^2T^4}{60} + \frac{T^2a^2}{24} - \frac{17a^4}{960\pi^2} .$$



Case $\frac{a}{6\pi} < T < \frac{a}{2\pi}$

When $T = \frac{a}{2\pi}$, the contour of integration crosses the pole $\eta = -1$ - this is the source of instability, manifested in the discontinuity of the second order derivative $\frac{\partial^2\rho}{\partial T^2}$:

$$I_1 + I_2 = 0 ,$$

$$\Downarrow$$

$$\rho = \frac{127\pi^2T^4}{60} - \frac{11T^2a^2}{24} - \frac{17a^4}{960\pi^2} - \pi T^3a + \frac{Ta^3}{4\pi} .$$

In general, instabilities occur every time the contour crosses a pole. Thus, there is a series of discontinuities at $T = T_U/(2k+1)$, $k = 0, 1..$. Such behavior could be expected in advance, since acceleration appears *as an imaginary chemical potential* $\mu \to \mu \pm \frac{ia}{2}$ in Eq. (9) . In theories with an imaginary chemical potential, very similar instabilities also occur, called Roberge-Weiss phase transitions Ref. [16].

$$\rho = \frac{7\pi^2T^4}{60} + \frac{T^2a^2}{24} - \frac{17a^4}{960\pi^2} + \Big(\frac{\pi T^3a}{3} + \frac{Ta^3}{4\pi}\Big)\Big\lfloor\frac{1}{2} + \frac{a}{4\pi T}\Big\rfloor - \Big(\frac{T^2a^2}{2} + 2\pi^2T^4\Big)\Big\lfloor\frac{1}{2} + \frac{a}{4\pi T}\Big\rfloor^2 - \frac{4\pi T^3a}{3}\Big\lfloor\frac{1}{2} + \frac{a}{4\pi T}\Big\rfloor^3 + 4\pi^2T^4\Big\lfloor\frac{1}{2} + \frac{a}{4\pi T}\Big\rfloor^4 .$$

In blue: additional terms occurring in Eq. (9) at $T < T_U$ compared to Eq. (8): include integer part.

- **The polynomiality of the integral of the form of Eq. (9) is shown; it is connected with the polynomiality of the weight function in the Fermi integrals (it follows from Eq.(11)) and the appearance of symmetric combinations of integrals with $+ia$ and $-ia$.**
- **At the Unruh temperature instabilities in observables arise associated with the cross of the Fermi distribution pole $\eta = -1$ in the complex momentum plane by the integration contour.**

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