Critical dynamics of net-baryon density fluctuations

Marcus Bluhm

with N. Thouroux, G. Pihan, T. Sami and M. Nahrgang

IMT Atlantique, Nantes

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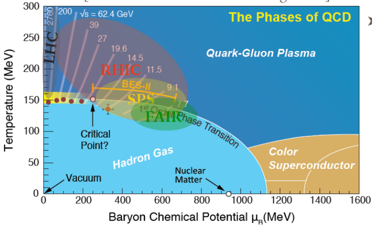
Strangeness in Quark Matter 2019, Bari, Italy – June 10-15, 2019

The search for the QCD critical point...

[B. Müller: BEST Col. Meeting 2016]



source: K. Šafařík's talk at SQM2019

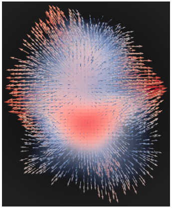
- ...is ongoing: different experiments look for non-monotonic $\sqrt{s}$-dependence of observables (conserved charge fluctuations)
- (some) expectations base on infinite-volume assumption:
  - $\rightarrow$ fluctuations of the critical mode σ diverge
  - $\rightarrow$ higher-order cumulants more sensitive to ξ:

$$\langle \Delta\sigma^2 \rangle \propto \xi^2,\ \langle \Delta\sigma^3 \rangle \propto \xi^{9/2},\ \langle \Delta\sigma^4 \rangle_c \propto \xi^7$$

M. Stephanov PRL102 (2009)

Dynamical description is very important...

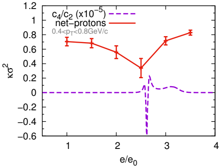
...when comparing theoretical approaches with heavy-ion collision data!



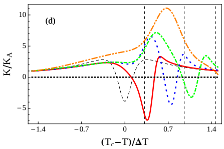
source: Madai

$\Rightarrow$ Matter created in a heavy-ion collision is small, short-lived and highly dynamical!

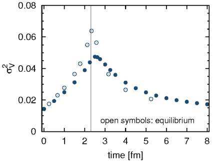
$\Rightarrow$ Finite size, non-equilibrium, ... effects are important!



C. Herold et al. PRC93 (2016)



S. Mukherjee et al. PRC92 (2015)



M. Nahrgang et al. arXiv:1804.05728

$\rightarrow$ charge conservation, critical slowing down, retardation, late stage effects...

$\Rightarrow$ In this talk: finite size and dynamical effects!

Diffusive dynamics of the net-baryon density

$$\partial_\mu N_B^\mu = 0 \quad \text{net-baryon charge conservation}$$

neglect coupling to energy and momentum density, consider space-time independent fluid
with spatially homogeneous T at fixed μ_B:

$\rightarrow$ diffusive dynamics follows the minimized free energy $\mathcal{F}$

$$\partial_t n_B(t,x) = \kappa \nabla^2 \left(\frac{\delta \mathcal{F}[n_B]}{\delta n_B} \right) + \nabla \mathbf{J}(\mathbf{t},\mathbf{x})$$

stochastic current included to study intrinsic fluctuations

$$J(t,x) = \sqrt{2T\kappa}\,\zeta(t,x)$$

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$$J(t,x) = \sqrt{2T\kappa}\,\zeta(t,x)$$

$\rightarrow$ $\zeta(t,x)$ is Gaussian and uncorrelated (white noise)

$$\langle \zeta(x,t)\zeta(0,0) \rangle = \delta(x)\delta(t)$$

$\Rightarrow$ respects the Fluctuation-Dissipation (FD) theorem!

$\rightarrow$ correlated noise possible!

Net-baryon density diffusion: Gauss+surface model

- simplest non-trivial situation:

  with $\Delta n = n_B - n_c$ we use

$$\mathcal{F}[n_B] = T\int \mathrm{d}^3 r \left(\frac{m^2}{2n_c^2}\Delta n_B^2 + \frac{K}{2n_c^2}(\nabla \Delta n_B)^2 \right)$$

  and the stochastic diffusion equation reads

$$\partial_t n_B = \frac{D}{n_c}\left(m^2 - K\nabla^2 \right)\nabla^2 n_B + \sqrt{2Dn_c}\nabla\zeta$$

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- numerical implementation:
  - $1+1$d system considered
    (noise propagated in 1 spatial direction)
  - finite size box with periodic boundary conditions
    (exact charge conservation within L)
  - solved with semi-implicit 2-stage (predictor-corrector) scheme
  - diffusion coefficient $D = \kappa T/n_c$, $n_c = 1/3\ \mathrm{fm}^{-3}$
  - equilibrium: evaluate long time behavior, $D = 1$ fm

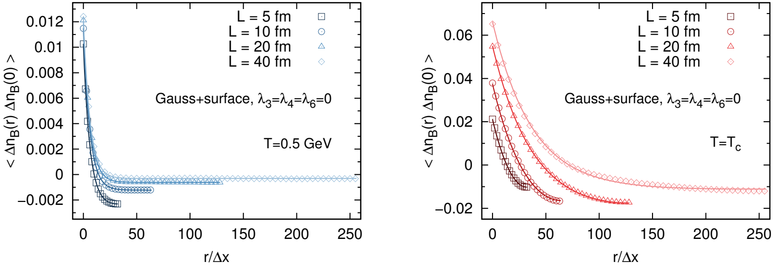
$\Rightarrow$ analytic results for continuum and discretized space-time exist!

Equal-time correlation function

- $m(T)$ from 3D Ising model; smallest at $T_c = 0.15$ GeV

$$\langle \Delta n_B(r = j\Delta x)\Delta n_B(0)\rangle = \int \frac{\mathrm{d}k}{2\pi}\,\frac{n_c^2}{m^2}\,\frac{e^{ikj\Delta x}}{1 + \frac{2K}{m^2\Delta x^2}\left(1 - \cos(k\Delta x)\right)}$$

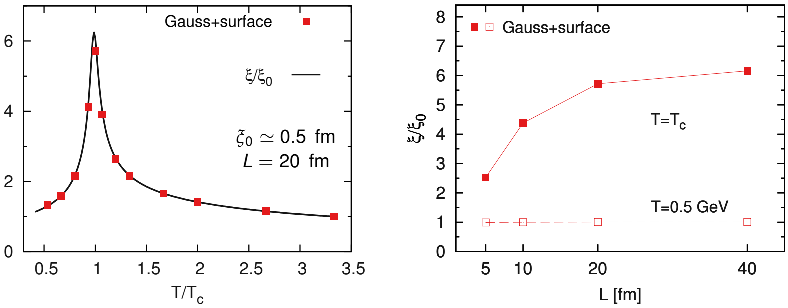
$$\to \frac{n_c^2}{2m^2}\sqrt{\frac{m^2}{K}}\,e^{-j\Delta x\sqrt{m^2/K}}$$



$\Rightarrow$ effect of finite size L visible $\leftrightarrow$ baryon number conservation!

Correlation length and finite-size effects

- numerical value for ξ from a fit of the correlation function
- expect $\xi = \sqrt{K/m^2}$ in infinite-volume limit



- already for $L = 20$ *fm* rather good realization of thermodynamic expectation!

$\Rightarrow$ For finite size L, the numerical correlation length can be strongly limited due to $\langle N_B \rangle$-conservation.

Net-baryon density diffusion: Ginzburg-Landau model

- include non-linear interactions:

  Now

$$\mathcal{F}[n_B] = T\int \mathrm{d}^3r\,\left(\frac{m^2}{2n_c^2}\Delta n_B^2 + \frac{K}{2n_c^2}(\nabla\Delta n_B)^2 + \frac{\lambda_3}{3n_c^3}\Delta n_B^3 + \frac{\lambda_4}{4n_c^4}\Delta n_B^4 + \frac{\lambda_6}{6n_c^6}\Delta n_B^6\right)$$

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  and the stochastic diffusion equation reads

$$\partial_t n_B = \frac{D}{n_c}\left(m^2 - K\nabla^2\right)\nabla^2 n_B + D\nabla^2 \left(\frac{\lambda_3}{n_c^2}\Delta n_B^2 + \frac{\lambda_4}{n_c^3}\Delta n_B^3 + \frac{\lambda_6}{n_c^5}\Delta n_B^5 \right) + \sqrt{2Dn_c}\nabla\zeta$$

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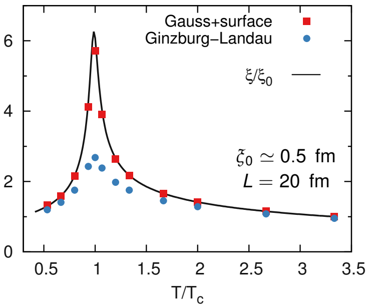
$\Rightarrow$ no analytic results at hand, but results effectively described by a Gauss+surface model with modified m^2!

Correlation length and finite-size effects

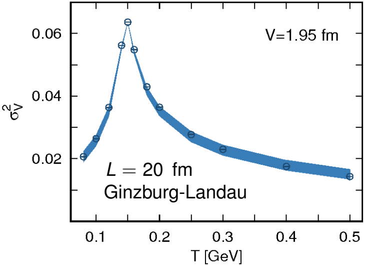


- spatial correlations significantly smaller!
- nonlinear mode couplings reduce the numerically realized correlation length!

Volume-integrated cumulants: variance

- infinite-volume expectation:
  $\sigma_V^2 = \frac{1}{V^2} \int dx \int dy \, \langle \Delta n_B(x) \Delta n_B(y) \rangle \propto \xi^2$

M. Stephanov (et al.) PRL81 (1998); PRD60 (1999); PRL102 (2009)



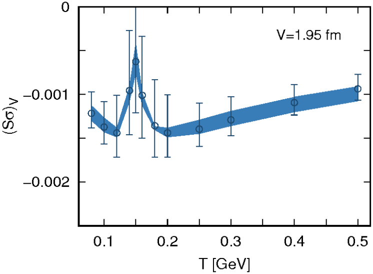
(roughly factor 4 increased signal near T_c)

- including finite-size and $\langle N_B \rangle$-conservation we find for $V \simeq 2$ fm in the Ginzburg-Landau model:

$$\sigma_V^2 \propto \xi^n \text{ with } n \simeq 1.30 \pm 0.05\,!$$

Volume-integrated cumulants: skewness

- infinite-volume expectation: $(S\sigma)_V \propto \xi^{2.5}$

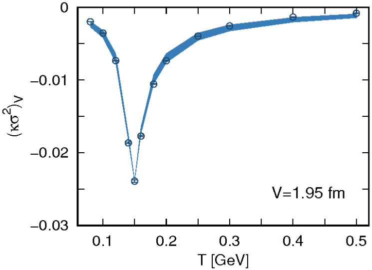


- including finite-size and $\langle N_B \rangle$-conservation we find for $V \simeq 2$ fm in the Ginzburg-Landau model:

  $(S\sigma)_V = a\xi^n - b\xi^m$ with $n \simeq 1.47 \pm 0.05$ and $m \simeq 2.40 \pm 0.05$!

Volume-integrated cumulants: kurtosis

- infinite-volume expectation: $(\kappa\sigma^2)_V \propto \xi^5$ we choose $(2\tilde{\lambda}_3^2 - \tilde{\lambda}_4) < 0$!



(roughly factor 25 increased signal near T_c)

- including finite-size and $\langle N_B \rangle$-conservation we find for $V \simeq 2$ fm in the Ginzburg-Landau model:

$$(\kappa\sigma^2)_V \propto \xi^n \text{ with } n \simeq 2.5 \pm 0.1\,!$$

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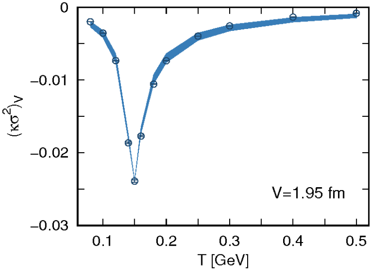

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$$(\kappa\sigma^2)_V \propto \xi^n \text{ with } n \simeq 2.5 \pm 0.1\,!$$

$\Rightarrow$ clear deviations from infinite-volume expectations!

Relaxing the infinite-volume assumption

- at leading-order $\epsilon = \sqrt{\xi^3 / L^3}$-expansion of $\mathcal{F}[\sigma]$ yields infinite-volume results

M. Stephanov PRL102 (2009); S. Mukherjee, R. Venugopalan, Y. Yin PRC92 (2015)

- in the numerics system size and observation volume not large compared to ξ!

$\Rightarrow$ impact of next-to-leading-order corrections?

Can corrections (at least) **qualitatively** explain behavior seen in the numerics?

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Can corrections (at least) **qualitatively** explain behavior seen in the numerics?

- decouple longitudinal from transverse direction (effectively 1D)!

$$\mathcal{F}[\sigma] = A \int \mathrm{d}x \left(\frac{m^2}{2}\sigma^2 + \frac{1}{2}(\nabla\sigma)^2 + \frac{\lambda_3}{3}\sigma^3 + \frac{\lambda_4}{4}\sigma^4 + \ldots \right)$$

M. Agah Nouhou et al. arXiv:1906.02647

Correlation function and integrated variance

$$
\begin{aligned}
C(x_1, x_2) &= \langle \sigma(x_1)\sigma(x_2)\rangle \\
&= \frac{1}{\mathcal{Z}} \int \mathcal{D}\sigma\, \sigma(x_1)\sigma(x_2) e^{-\mathcal{F}_0[\sigma]/T} \left(1 - \mathcal{I} + \frac{1}{2}\mathcal{I}^2 + \dots\right)
\end{aligned}
$$

with

$$
\mathcal{I} = \frac{A}{T} \int \mathrm{d}x \left(\frac{\lambda_3}{3}\sigma^3 + \frac{\lambda_4}{4}\sigma^4 + \dots\right)
$$

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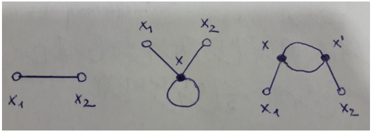

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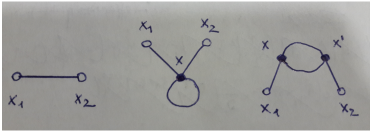


$\rightarrow$ non-linear coupling terms: $\sim \lambda_4$, $\sim \lambda_3^2$

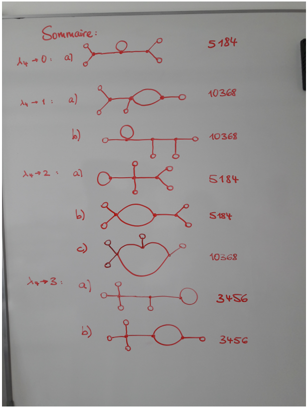
$\Rightarrow$ volume-integrated variance: $\langle \sigma_L^2 \rangle = \frac{1}{L^2} \int_0^L \mathrm{d}x_1 \mathrm{d}x_2 C(x_1, x_2)$

$$\langle \sigma_L^2 \rangle = \frac{T}{AL}\xi^2 - \frac{3}{2}\frac{T^2}{A^2 L}\xi^5 \left(\lambda_4 - \lambda_3^2 \xi^2 \right)$$

(effectively an expansion in ξ^2 / A)

Higher-order cumulants

$\rightarrow$ similarly higher orders $\langle \sigma_L^3 \rangle$ and $\langle \sigma_L^4 \rangle_c$ can be evaluated!



- third-order cumulant: $\sim \lambda_3$, $\sim \lambda_3\lambda_4$, $\sim \lambda_3^3$
- fourth-order cumulant: $\sim \lambda_3^2$, $\sim \lambda_4$, $\sim \lambda_4^2$, $\sim \lambda_3^2\lambda_4$, $\sim \lambda_3^4$

$\Rightarrow$ with alternating signs in the correction terms!

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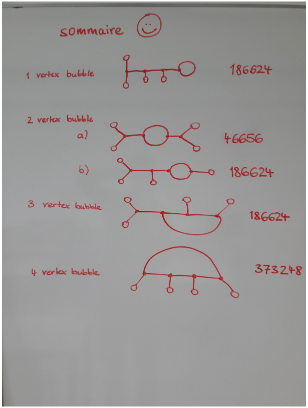

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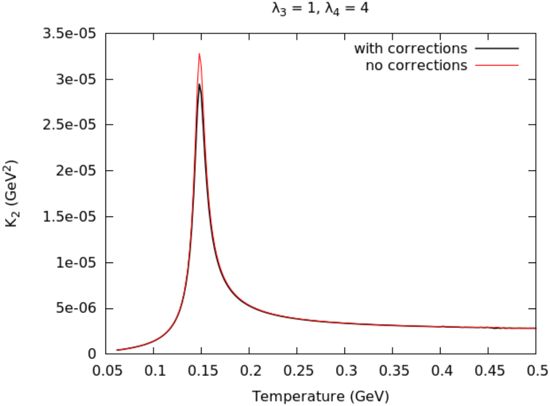
Influence of correction terms



⇒ qualitatively behavior seen in the diffusion dynamics reproduced!

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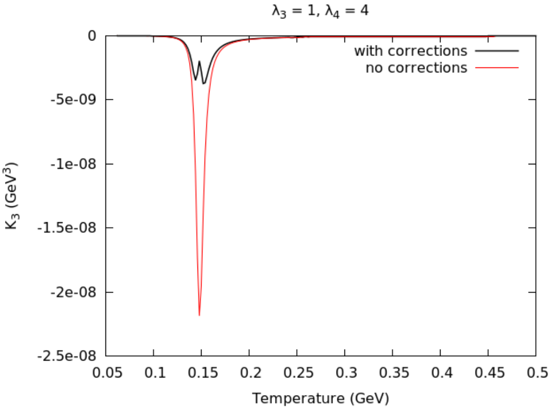

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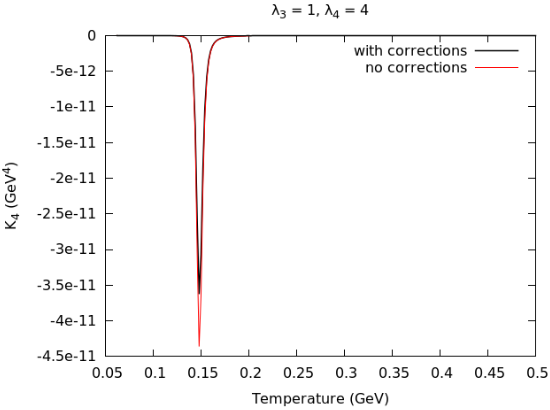

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Observation volume dependence in the numerics

- infinite-volume expectation:
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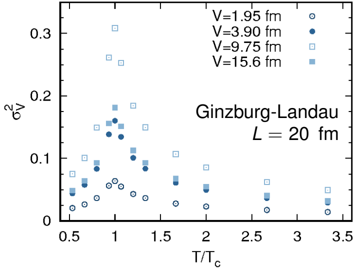
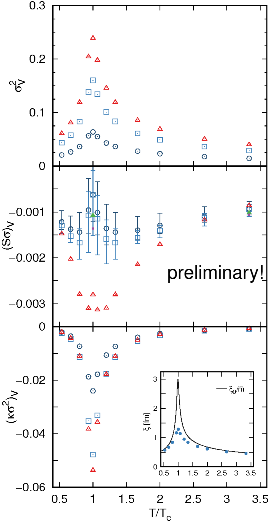


- increasing $V \to$ increase in σ_V^2 (n gets closer to 2) before impact of charge conservation visible!

Observation volume dependence in the numerics

⇒ similar non-trivial V-dependence in higher-order cumulants!



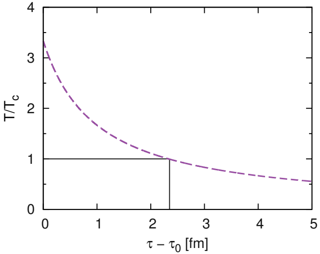
→ increase of V:

- improvement toward scaling expectations!
- scaling competition in skewness disappears!

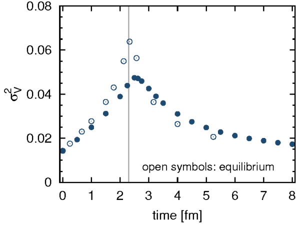
Evolution: Time-dependent temperature

no real dynamics yet but Bjorken-expansion can easily be implemented!

Hubble-like T-evolution:
$T(\tau) = T_0 \left(\frac{\tau_0}{\tau}\right)^{dc_s^2}$



- equilibrate system at $T_0 = 0.5$ GeV, $c_s^2 = 1/3$
- $D(T) = D_0 T/T_0$, $D_0 = 1$ fm
- T_c reached at $\tau - \tau_0 = 2.33$ fm



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- nonequilibrium: maximal value in dynamics smaller than in equilibrium
- expected behavior with varying D_0

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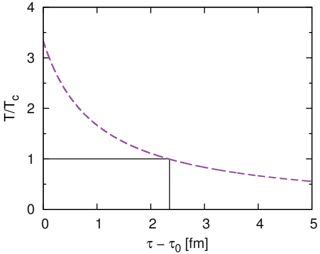

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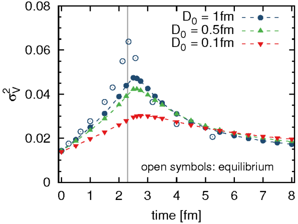

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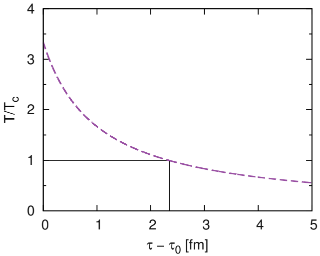

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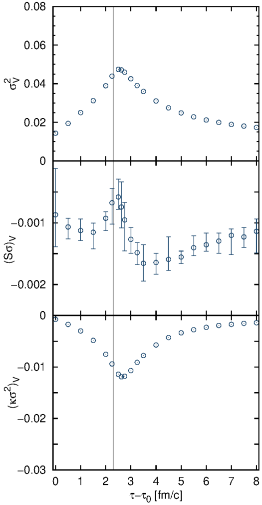


Outlook

- include Bjorken expansion of the system
- extend to $3+1$ d simulations (couple longitudinal and transverse fluctuations)
- for relativistic system - causality:

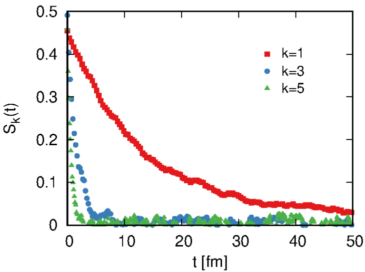
$$\partial_t n_B = -\vec{\nabla}\vec{j} + \vec{\nabla}\vec{J}$$

with two relaxation equations:

$$\partial_t \vec{J} = -\frac{1}{\tau_B}\left(\vec{J} - \sqrt{2T\kappa}\vec{\xi}\right) ,$$

$$\partial_t \vec{j} = -\frac{1}{\tau_B}\left(\vec{j} - \kappa\vec{\nabla}(\delta\mathcal{F}/\delta n_B)\right)$$

important for FD theorem!



Outlook

- include Bjorken expansion of the system
- extend to $3+1$ d simulations (couple longitudinal and transverse fluctuations)
- for relativistic system - causality:

$$\partial_t n_B = -\vec{\nabla}\vec{j} + \vec{\nabla}\vec{J}$$

with two relaxation equations:

$$\partial_t \vec{J} = -\frac{1}{\tau_B}\left(\vec{J} - \sqrt{2T\kappa}\vec{\xi}\right) \ ,$$

$$\partial_t \vec{j} = -\frac{1}{\tau_B}\left(\vec{j} - \kappa\vec{\nabla}(\delta\mathcal{F}/\delta n_B)\right)$$

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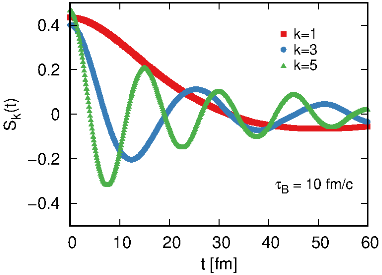

Conclusions

- ξ can be significantly affected by finite system size $L \leftrightarrow$ net-baryon charge conservation!
- nonlinear interactions reduce ξ further!
- finite size affects scaling behavior of fluctuation observables!
- qualitatively understood in terms of a systematic expansion!
- volume-integrated cumulants show non-trivial dependence on observation volume!
- dynamics: nonequilibrium and retardation effects in observables!