

**CERN, Summer Theory  
Institute, Aug.2010**



# Fluctuations and sounds, The magnetic side of the sQGP

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# Outline: (3 hard pieces)

1. Initial fluctuations => Sound waves, Modified Mach cones from jets
2. magnetic plasma at RHIC => monopoles behave as Coulomb classical plasma, explain small viscosity. Confinement as monopole BEC
3. Jet quenching in magnetic plasma, flux tubes, "ridges"

(ES, CERN lectures 1982)

# The vacuum vs QGP

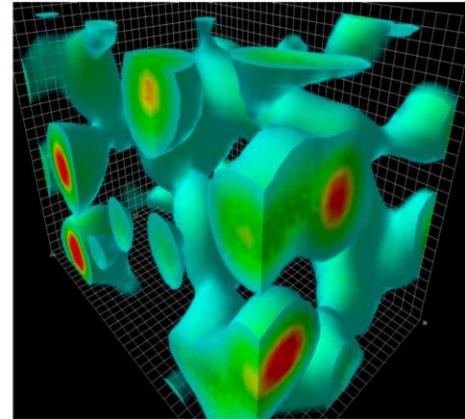
- The vacuum is very complicated, dominated by “topological objects”

**Vortices, monopoles and instantons**

- Among other changes it shifts its energy **down** compared to an “empty” vacuum, known as the Bag terms,

$$p = \frac{1}{3} \epsilon - B$$

$$\epsilon = \frac{1}{3} p + B$$



Visualization by Derek Leinweber

- The QGP, as any plasma, screens them, and thus is nearly free from them
- => when QGP is produced, **the vacuum tries to expel it**

# Magdeburg hemispheres 1656



**“We cannot pump the QCD vacuum out, but we can pump in something else, namely the Quark-Gluon Plasma and measure the pressure difference...”**

**□ Now we found a lot of topology in QGP as well, especially near  $T_c$**

# Part 1: The fluctuations and the sounds

## Fate of the initial state perturbations in heavy ion collisions

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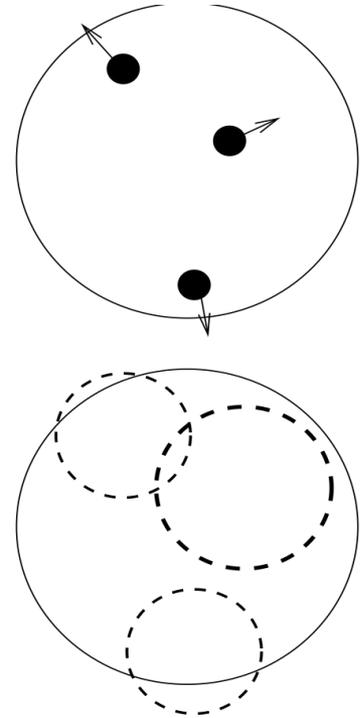
(Received 20 July 2009; revised manuscript received 14 October 2009; published 13 November 2009)

Naively, “spots” should excite a wave and get expanded to a spherical (or conical, or cylindrical) wave

Like in the case of stone thrown into the pond, nothing is left at the original position: **so how can they be observed?**

Its size => **the sound horizon** => is about twice smaller than the fireball size (at freezeout) 8 fm/c  
And thus large angular size 2 H/R about 1 rad

If one wants to get large radial flow, one has to wait the time needed for it to develop. The sound speed during this time creates large rings.



$$R_h = \int_0^{\tau_f} d\tau c_s(\tau)$$

# Two new fundamental scales, describing fluctuations **at freezeout** (V.Khachatryan,ES)

1. The sound horizon:

$$H_s = \int_0^{\text{radius}} d\tau c_s(\tau)$$

**2. The viscous horizon: the  
Width of the circle**

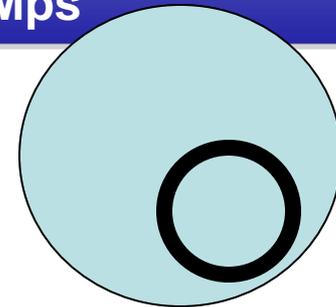
$$\delta T_{\mu\nu}(t) = \exp\left(-\frac{2}{3} \frac{\eta}{s} \frac{k^2 t}{3T}\right) \delta T_{\mu\nu}(0)$$

$$k_v = \frac{2\pi}{R_v} = \sqrt{\frac{3Ts}{2\tau_f\eta}} \sim 200 \text{ MeV}$$

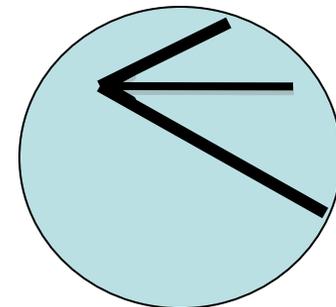
Let us finish this section by pointing out the hierarchy relation between all those four scales which we assume is true

$$R > H_s > R_v > l \quad (2.8)$$

For the Big Bang it was introduced by Sunyaev-Zeldovich about 40 years ago, was observed in CMB and galaxy correlations, it is about 150 Mps



**cylinders**



**cones**

# Visible shape of the sound

cylinder (at freezeout, boosted by radial flow)

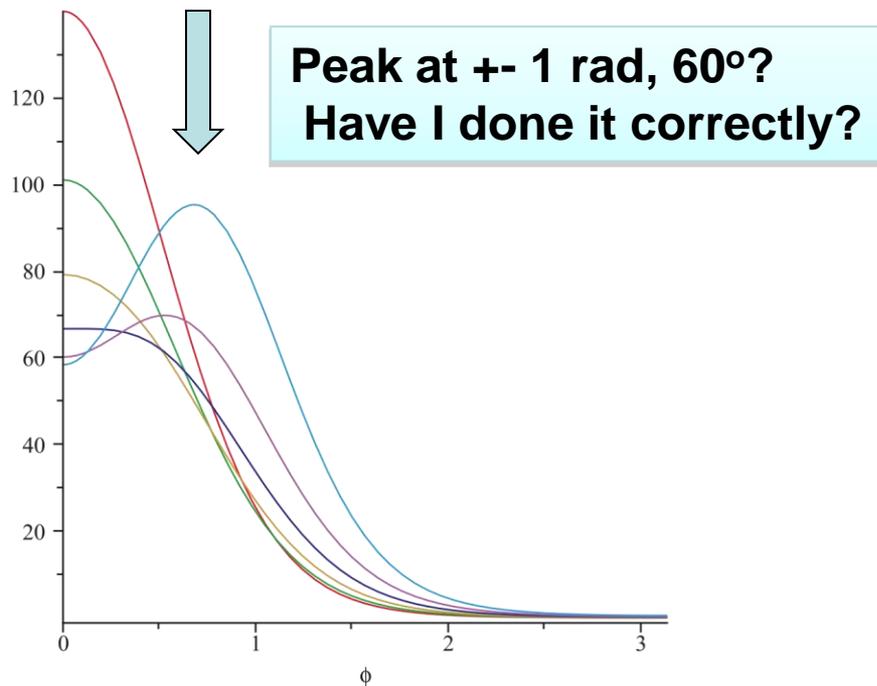


FIG. 5. (Color online) Dependence of the visible distribution in the azimuthal angle on the width of the (semi)circle at the time of freeze-out. Six curves, from the most narrow to the widest ones, correspond to the radius of the circle of 1, 2, 3, 4, 5, and 6 fm, respectively. The original spot position is selected to be at the edge of the nuclei. The distribution is calculated for a particle of  $p_t = 1$  GeV and fixed freeze-out  $T_f = 165$  MeV.

- The blue line is how asimuthal distribution would look like **for sound cylinders, double peak because part of the circle is outside of the fireball**
- comparing with data, I concluded that there no ridges at such angle
- Thus I argued it is a flux tube

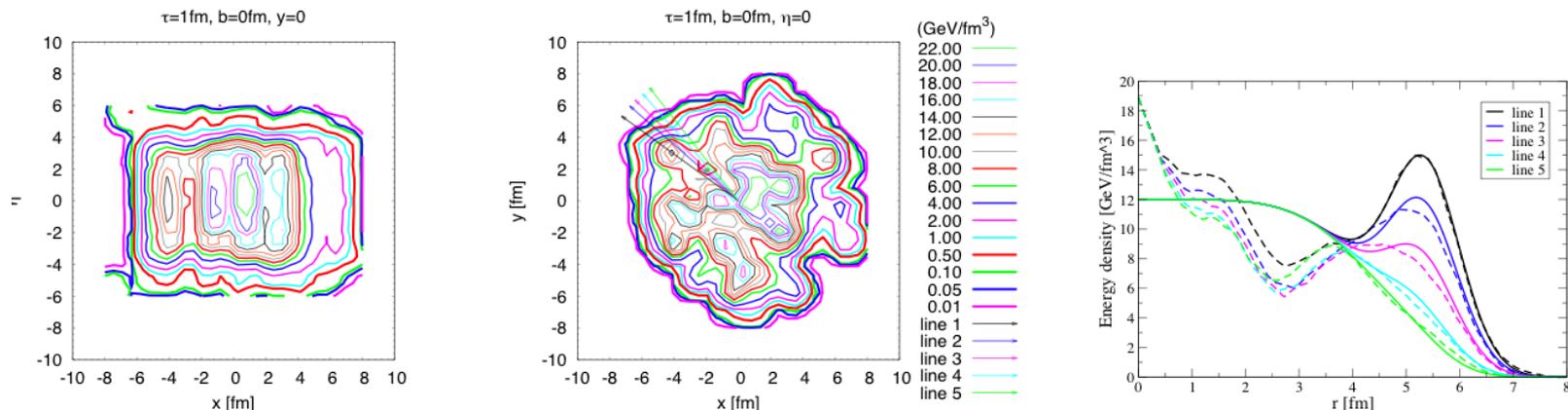
# Here is a study of F.Grassi and her student

## 2+1 hydrodynamics: one tube model (R.Andrade Ph.D. Thesis)

### WHY 3+1 HYDRO GIVES GOOD RESULTS?

→ Study transverse expansion of a slice with one tube

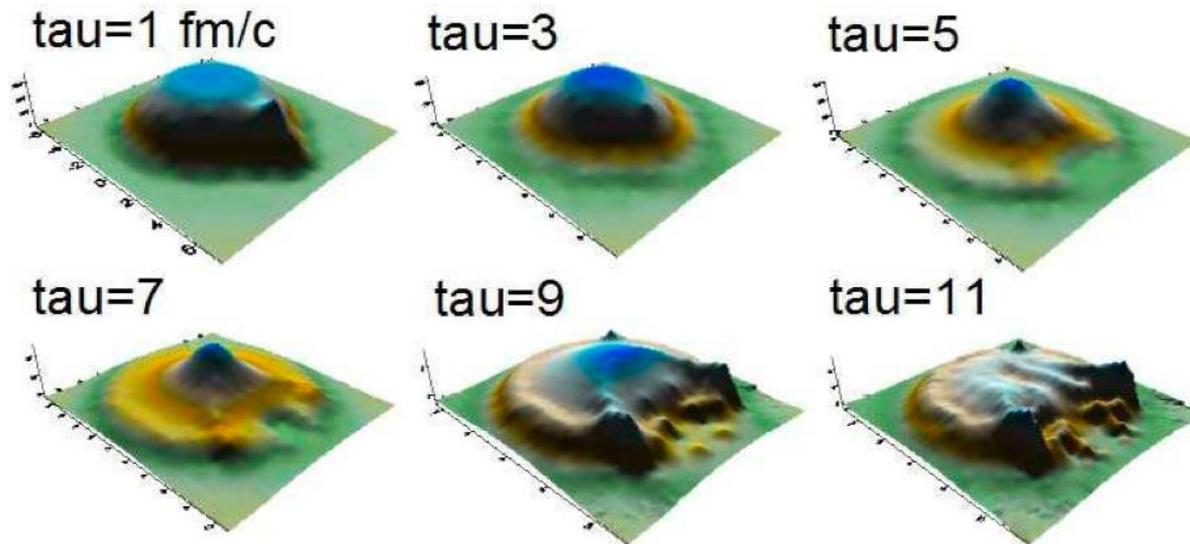
- ▶ longitudinal boost invariance assumed
- ▶ central collision
- ▶ profile inspired by NeXus initial conditions



# The sound cylinders and two peaks are also seen

Origin of the two peaks

Tube “sinks” and matter around “rises” forming a hole+two horns

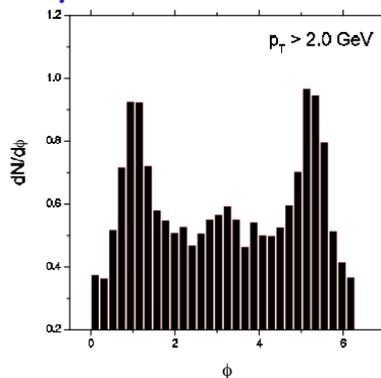


Temporal evolution of energy density for the one tube model.

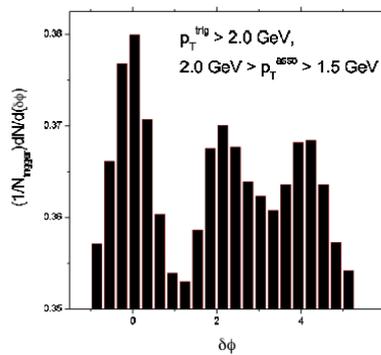
The peaks are at the same angles  
+- 1 rad (as I got) from perturbation  
but **+ -2 rad in correlations**

One tube model

**MAIN RESULT:** single particle angular distribution has TWO  
PEAKS separated by  $\Delta\phi \sim 2$



**CONSEQUENCE:** two particle angular distribution has three  
peaks



# Fluctuations from Glauber

B. Alver, G. Roland, *Phys.Rev.C*81:054905,2010. e-Print: arXiv:1003.0194 [nucl-th] =>  
triangular flow paper

$$\square \varepsilon_{mn} = \langle r^m \cos(n(\phi - \psi_{mn})) \rangle$$

- The dipole  $\varepsilon_{11}=0$  by definition but not  $\varepsilon_{31}$
- Are angles of different harmonics correlated? Odd ones are quite correlated  $\psi_{31}, \psi_{33}, \psi_{55}$  pointing in y direction, to the tips of the almond
- (Pilar Staig, ES, archive:1008.3139, aug.18<sup>th</sup>, 2010 )

# Glauber fluctuations up to 6<sup>th</sup> are all comparable

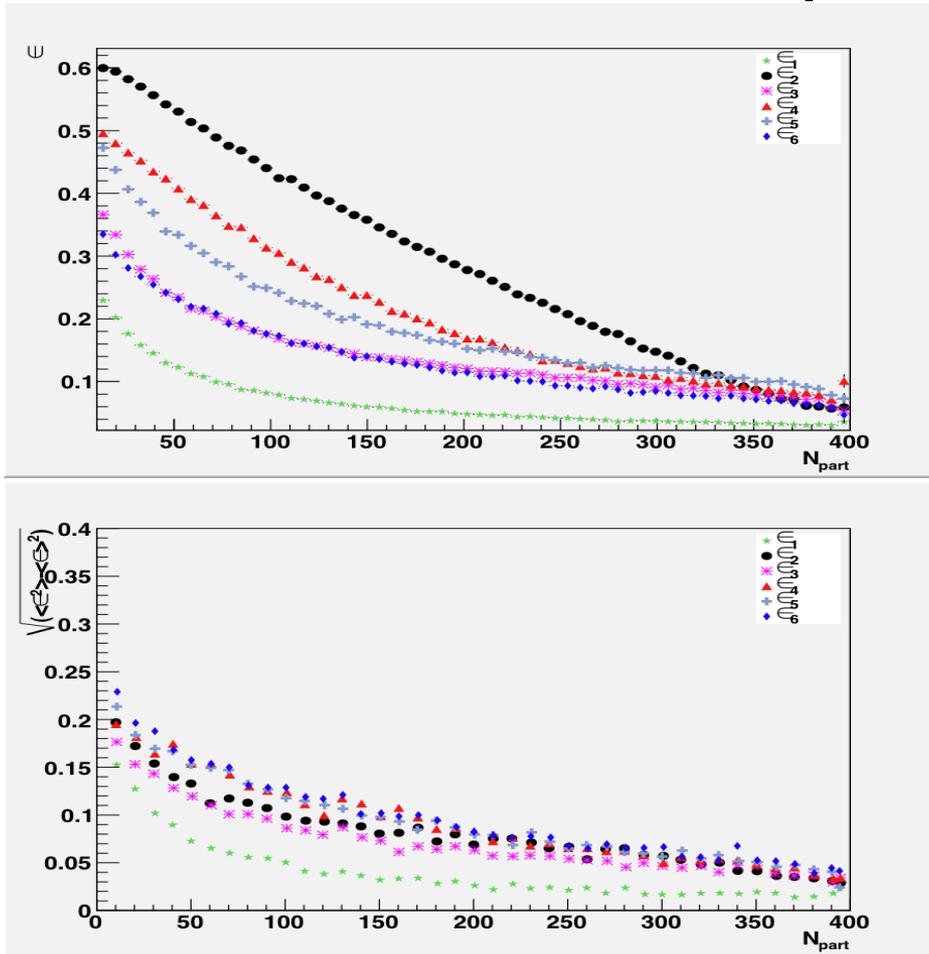


FIG. 5: Average anisotropies (upper plot) and their variations (lower), as a function of centrality expressed via the number of participants  $N_{part}$

$$\epsilon_n = \frac{\sqrt{\langle r^n \cos(n\phi) \rangle^2 + \langle r^n \sin(n\phi) \rangle^2}}{\langle r^n \rangle}$$

$$\epsilon_1 = \frac{\sqrt{\langle r^3 \cos(\phi) \rangle^2 + \langle r^3 \sin(\phi) \rangle^2}}{\langle r^3 \rangle}$$

The angles  $\psi_n$  are defined by:

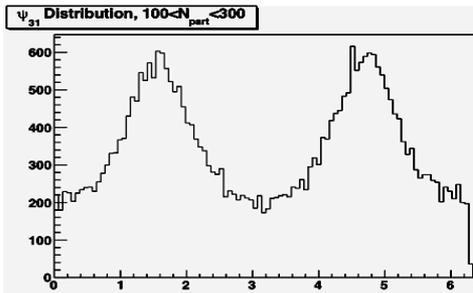
$$\tan(n\psi_n) = \frac{\langle r^n \sin(n\phi) \rangle}{\langle r^n \cos(n\phi) \rangle}$$

and to calculate  $\psi_1$  we use:

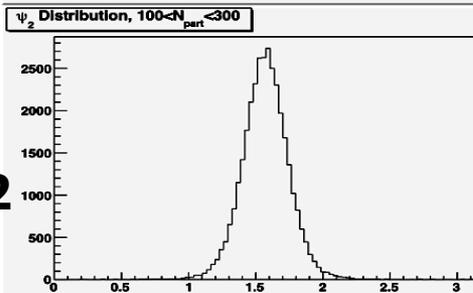
$$\tan(\psi_1) = \frac{\langle r^3 \sin(\phi) \rangle}{\langle r^3 \cos(\phi) \rangle}$$

# Distribution of the angles

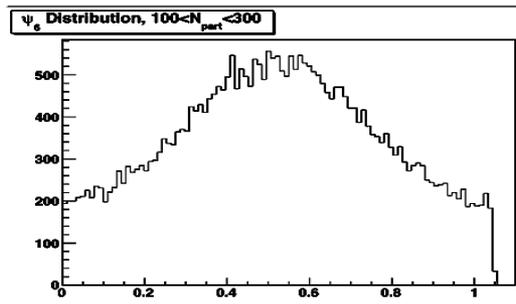
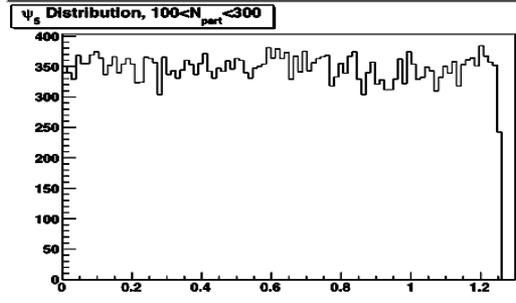
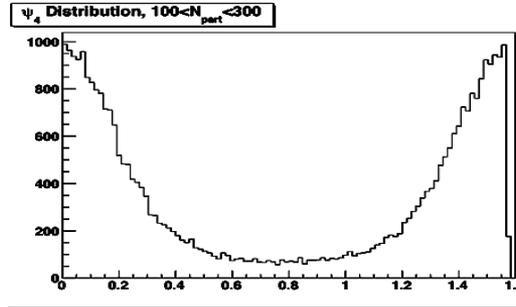
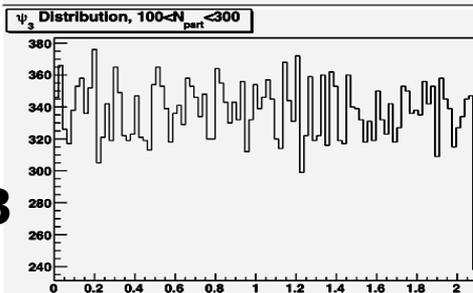
1



2

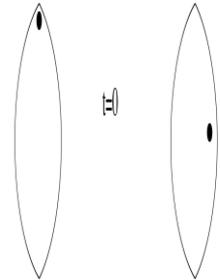


3



1<sup>st</sup> : tips and waist

4



5

3<sup>rd</sup> and 5<sup>th</sup>  
Uncorrelated?

6

- The odds are **all** correlated!
- There are “tips” and ”waist” peaks  
 geometry tells us that peripheral  
 events would be 3-peaks

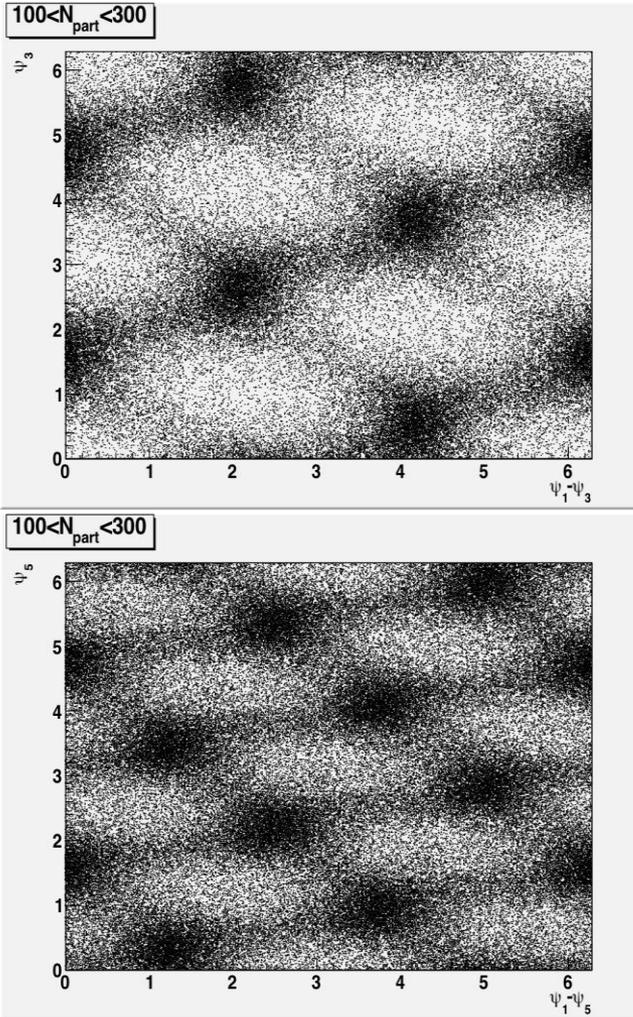


FIG. 8: Scatter plot of the  $\psi_3$  vs  $\psi_3 - \psi_1$  (above), and of the  $\psi_5$  vs  $\psi_5 - \psi_1$  (below), the same centrality

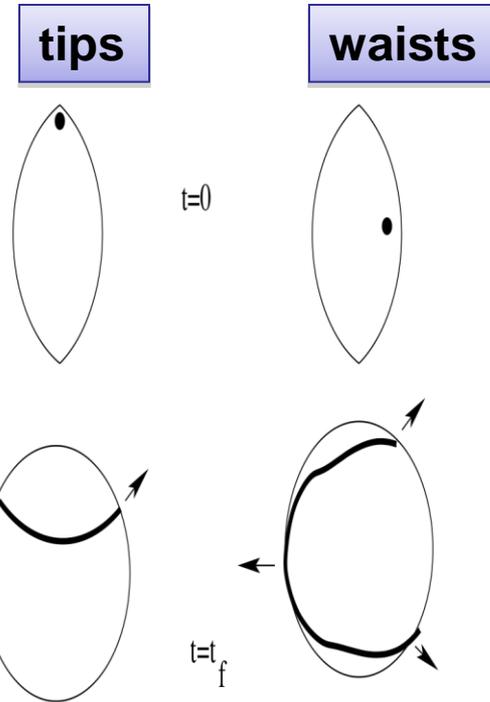


FIG. 4: Two upper picture correspond to initial time  $t = 0$ : the system has almond shape and contains perturbations (black spots). Two lower pictures show schematically location and diffuseness of the sound fronts at the freezeout time  $t_f$ . The arrows indicate the angular direction of the maxima in the angular distributions, 2 and 3 respectively.

# Both calculations agree, but they both ignored viscosity!

Thus the rms width has therefore increased as

$$\sqrt{\langle x^2(t) \rangle} = \sqrt{r_0^2 + \frac{2}{k_v^2(t)}}, \quad (3.3)$$

which is  $\sim t^{1/2}$  at large time. This is because of the diffusive nature of the viscosity. The amplitude has decreased by the factor

$$\frac{\delta T_{\mu\nu}(t, x=0)}{\delta T_{\mu\nu}(t=0, x=0)} = \left(1 + \frac{2}{r_0^2 * k_v^2(t)}\right)^{-1/2}. \quad (3.4)$$

$$k_v = \frac{2\pi}{R_v} = \sqrt{\frac{3Ts}{2\tau_f\eta}} \sim 200 \text{ MeV} \quad r_0 = .2 \text{ fm}$$

**Suppression is serious, by about factor 4**

# The observed recoil tells us the volume of the region where the sound is

- Conditional probability to get  $p_2$  provided the trigger is  $p_1$  (Borghini 0707.0436)

$$f(\vec{p}_2|p_1) = f_0(\vec{p}_2) \left( 1 - \frac{2p_1 p_{2,x}}{N \langle p_t^2 \rangle (1 + \bar{v}_2)} \right)$$

- **N is the number of particles which takes the recoil.**

where the trigger by definition is in the x direction,

## What is it?

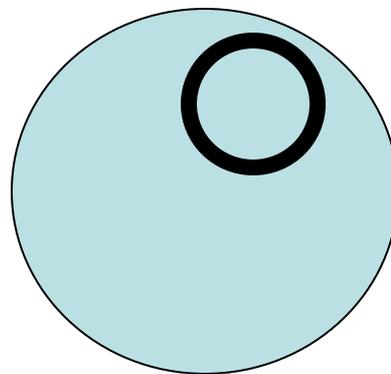
- we argue it is the black region:

Outside cannot be by causality  
Inside not because sound does not live any momentum

- $N = N_{\text{tot}} H^2/k_v/V_{\text{tot}}$  or about  $N_{\text{tot}}/10$  or 500 particles for central AuAu at RHIC

- This explains the magnitude of the first dipole harmonics in the correlation function (see e.g. 1003.0194, Alver and Roland)

$$\bar{v}_2 = \langle p_x^2 - p_y^2 \rangle / \langle p_x^2 + p_y^2 \rangle$$



# Geometric acoustics can describe modification of shapes by flow

$$\frac{d\vec{r}}{dt} = \frac{\partial\omega(\vec{k}, \vec{r})}{\partial\vec{k}},$$

$$\frac{d\vec{k}}{dt} = -\frac{\partial\omega(\vec{k}, \vec{r})}{\partial\vec{r}},$$

In this case the dispersion relation is obtained from that in the fluid at rest by a local Galilean transformation, so that

$$\omega(\vec{k}, \vec{r}) = c_s k + \vec{k}\vec{u}. \quad (4.3)$$

In the simplest case of constant flow vector  $\vec{u} = \text{const}(r)$  the first of these eqn just obtains an additive correction by flow

$$\frac{d\vec{r}}{dt} = c_s \vec{n}_{\vec{k}} + \vec{u}, \quad (4.4)$$

where  $\vec{n}_{\vec{k}} = \vec{k}/k$  is the unit vector in the direction of the momentum. The second eqn gives  $\frac{d\vec{k}}{dt} = 0$  as there is no

a (generalized) Hubble-like flow

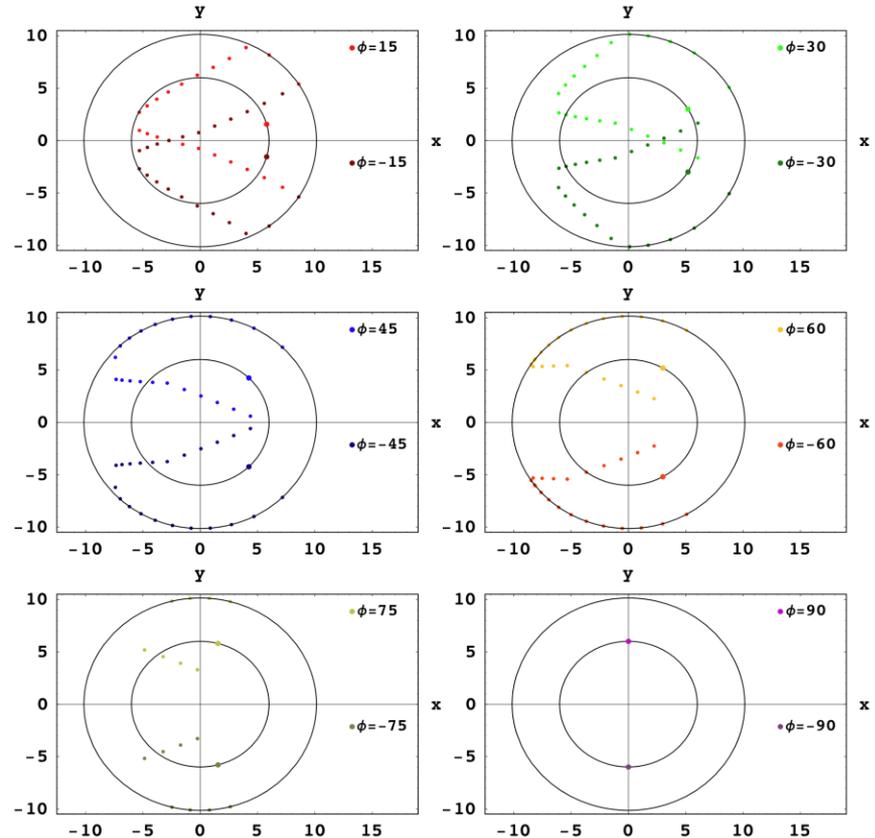
$$u_i(r) = H_{ij}r_j, \quad (4.5)$$

with some time and coordinate independent Hubble tensor. The eqn (4.2) now reads

$$\frac{dk_i}{dt} = -H_{ij}k_j, \quad (4.6)$$

$$k_i(t) = \exp(-H_i t)k_i(0). \quad \vec{r}(t) = tc_s \vec{n}_{\vec{k}} + \vec{r}(0)\exp(+Ht).$$

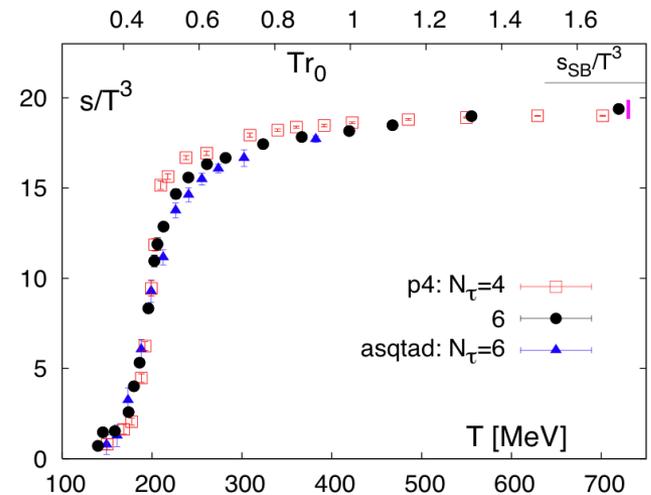
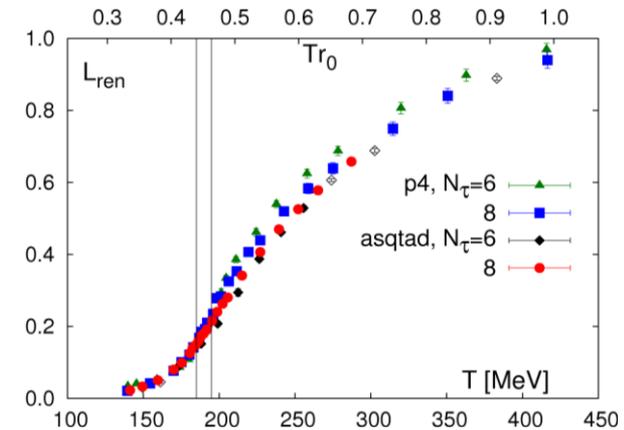
**Relativistic flow brings in Lorentz factor, easily solvable numerically: e.g.**



# Part 2: E/B duality and the magnetic plasma(?)

# The “semi-QGP puzzle” at $T \Rightarrow T_c$ (Pisarski)

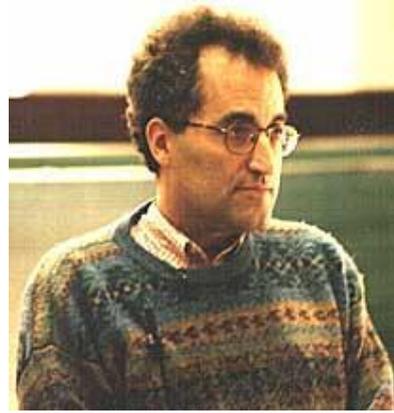
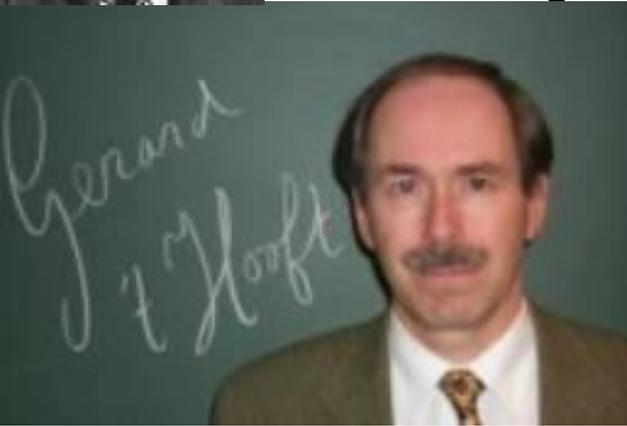
- Electric objects are suppressed by the Polyakov line
 
$$L = \frac{1}{N_c} \text{Tr} P \exp(i \int A_0 d\tau)$$
- Quarks should scale as  $\langle L \rangle$ , gluons as  $\langle L^2 \rangle$ , and they are (PNJL, Wilson-tHooft loops)
- and yet the entropy near  $T_c$  is large...



**what is the other half?**  
**Hadrons or monopoles?**  
 (no electric charge  $\Rightarrow$  no  $L$  suppression)



# Magnetic objects and their dynamics: classics



- Dirac explained how magnetic charges may coexist with quantum mechanics (1934)
- 't Hooft and Polyakov discovered **monopoles** in Non-Abelian gauge theories (1974) : VEV of adjoint scalar is needed
- 't Hooft and Mandelstamm suggested “**dual superconductor mechanism for confinement (1982), monopole BEC**”
- Seiberg and Witten shown how it works, in the **N=2 Super - Yang-Mills theory (1994)**

**“magnetic scenario”:  
(color)  
magnetic monopoles  
are important  
excitations near  $T_c$**

Four lectures on strongly coupled Quark Gluon Plasma. Edward Shuryak, (SUNY, Stony Brook) . 2009. 46pp. Published in Nucl.Phys.Proc.Suppl.195:111-156,2009.

- **Strongly coupled plasma with electric and magnetic charges.** Liao,ES, Phys.Rev.C75:054907,2007. hep-ph/0611131
- **Magnetic component of Yang-Mills plasma,**M.N.Chernodub and V.I.Zakharov, 98, 082002 (2007) [arXiv:hep-ph/0611228].
- **Electric Flux Tube in Magnetic Plasma.** Liao,ES, Phys.Rev.C77:064905,2008. arXiv:0706.4465
- Magnetic monopoles in the high temperature phase of Yang-Mills theories, A.D'Alessandro and M.D'Elia, Nucl.Phys.B 799, 241 (2008) [arXiv:0711.1266
- **Magnetic Component of Quark-Gluon Plasma is also a Liquid!** Liao,ES, Phys.Rev.Lett.101:162302,2008. e-Print: arXiv:0804.0255
- **Angular Dependence of Jet Quenching Indicates Its Strong Enhancement Near the QCD Phase Transition.** Jinfeng Liao,, Edward Shuryak Phys.Rev.Lett.102:202302,2009. e-Print: arXiv:0810.4116
- **Thermal Monopole Condensation and Confinement in**

“magnetic scenario”: Liao, ES hep-ph/0611131, Chernodub+Zakharov

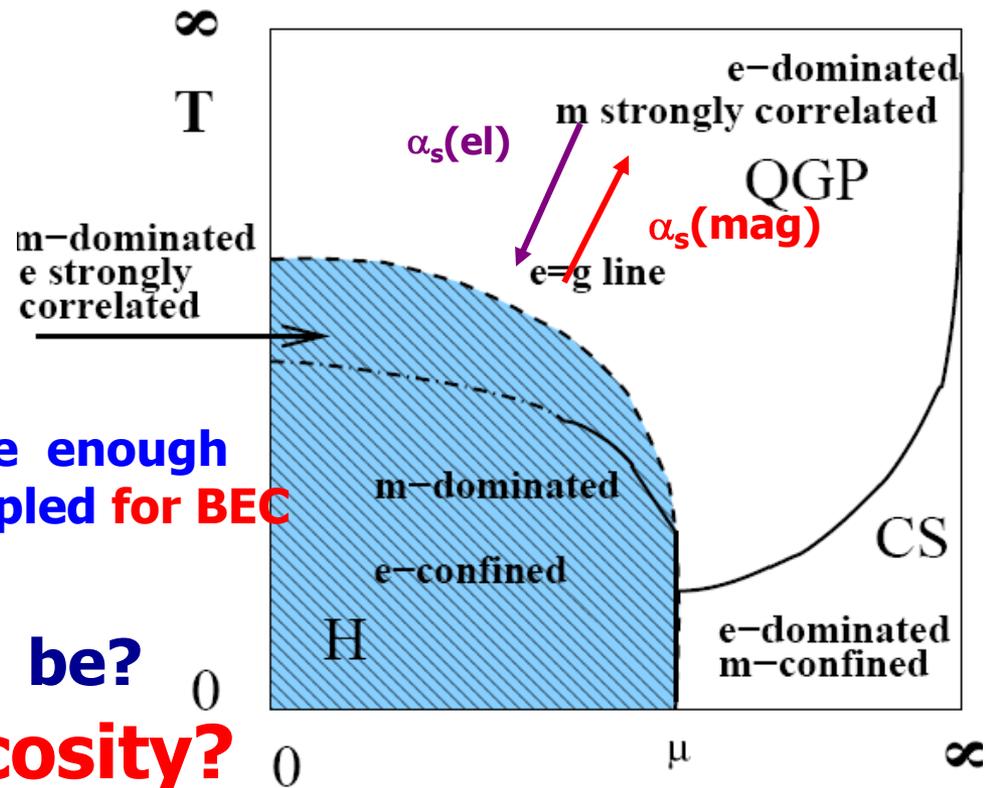
Old good Dirac condition =>

$$\alpha_s(\text{electric}) \alpha_s(\text{magnetic}) = 1$$

=> electric/magnetic couplings

must run in the **opposite directions!**

Where is the “equilibrium line”  $\alpha_s(\text{el}) = \alpha_s(\text{mag}) = 1$  ?  
Is there **the famous wall** at which density of states jumps but  $F$  does not?



monopoles should be (i) dense enough and (ii) sufficiently weakly coupled **for BEC to happen.**

how small can  $\alpha_s(\text{mag})$  be?

**Can it upset the viscosity?**

# Electric and magnetic screening masses (inverse screening lengths) from numerical simulation in lattice gauge theory

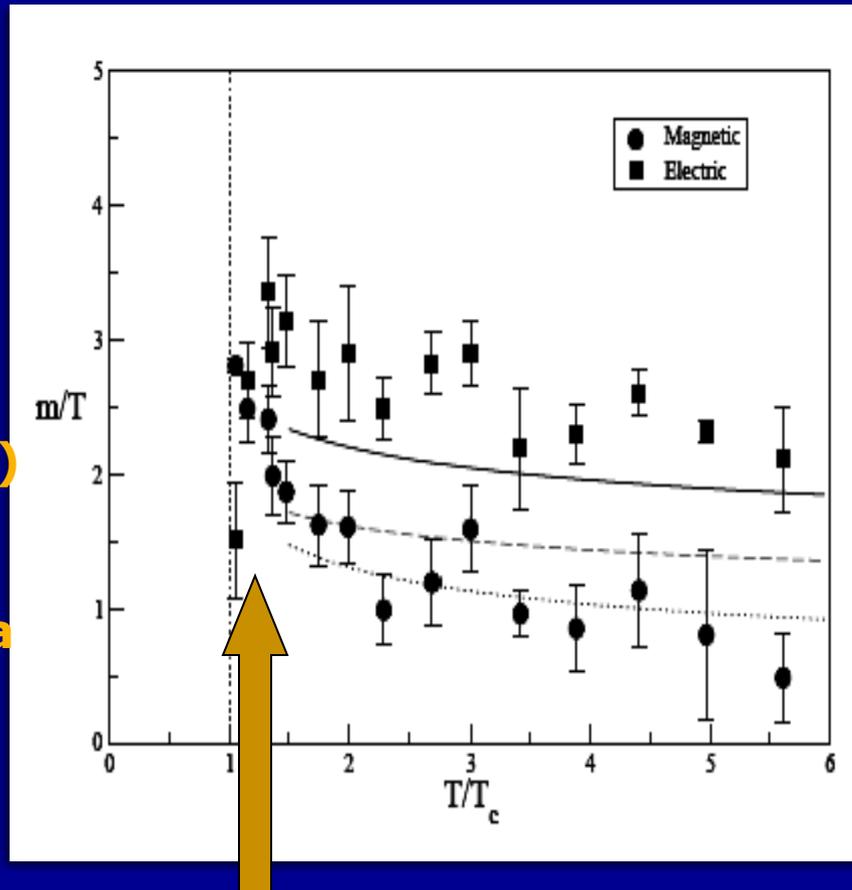
Nakamura et al, 2004

arrow shows the "equilibrium"  $E=M$  point

$m_e < m_m$   
Magnetic  
Dominated

At  $T=0$  magnetic  
Screening mass  
is about 2 GeV  
(de Forcrand et al)  
(a glueball mass)

(Other lattice data  
-Karsch et al-  
show how  $m_e$   
Vanishes at  $T_c$  in  
more detail)

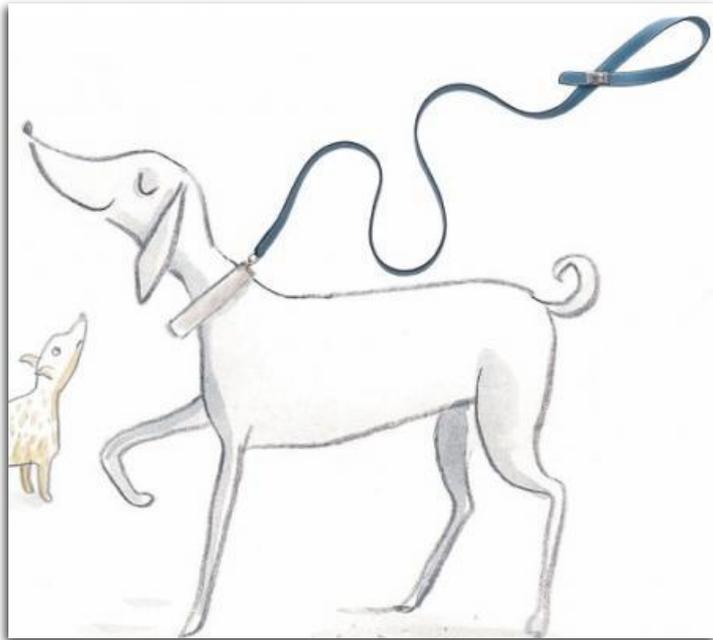


$m_e > m_m$   
Electric  
dominated

$M_E/T=0(g)$   
ES 78  
 $M_M/T=0(g^2)$   
Polyakov 79

Is QGP really getting magnetic as  $T < 1.4T_c$ ?

# magnetic objects are seen on the lattice, but we do not yet understand what they are



The leash= Dirac string is  
gauge dependent, can be directed as one  
wishes

The collar (string end) is detectable,  
It is the 3-d cube from which magnetic  
flux is going out

**The dog=magnetic object, studied  
but not understood yet.**

Many possibilities (e.g. dyons .. Fermionic zero  
modes)

$E^2$  and  $B^2$  are correlated with monopole path  
But so far statistically, not on event-by-event

The density of monopoles  
which wrap at least once  
around time is  
lattice-independent ! (D'Elia)

Experimentation with monopoles use many  
Gauges. It was shown that if one puts monopoles  
by hand "nearly all of them have collars"  
But not in some gauges

# The monopole density (vs $T/T_c$ )

## in confined and deconfined phases (Ratti, ES.08)

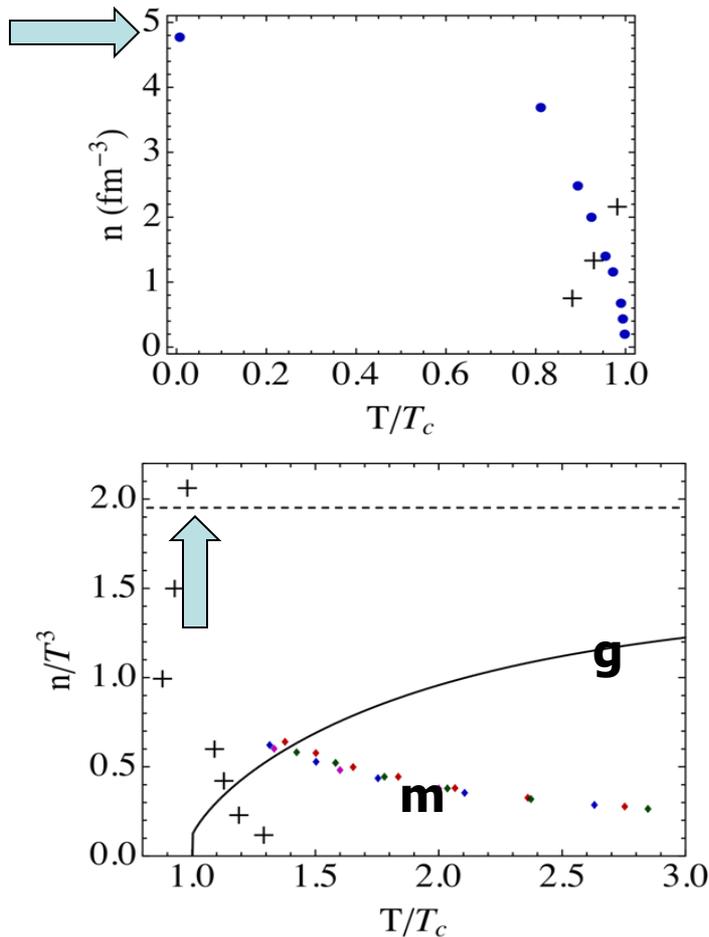


Figure 1: (a) The monopole density  $n_m$  in units of  $[\text{fm}^{-3}]$  versus  $T/T_c$  for the confined phase  $T < T_c$ . (b) The normalized density  $n/T^3$  versus  $T/T_c$  for the deconfined phase  $T > T_c$ .

- The  $T=0$  lattice point: from Bornyakov, Ilgenfritz et al
- Near- $T_c$ : condensed and uncondensed monopoles, from flux tubes (Liao ES)
- The solid line represent the density of **gluons** suppressed by  $\langle P \rangle$
- Note that the sum ( $g + \text{mono}$ ) is about  $\text{const}(T)$  except the peak at  $T_c$  (the peak is not due to dyons, as their density is flat)

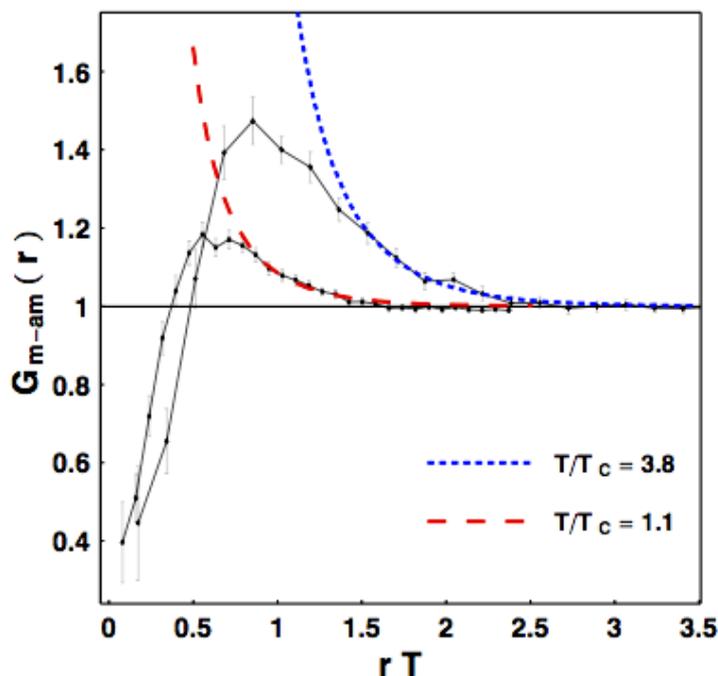
# Magnetic Component of Quark-Gluon Plasma is also a Liquid!

Jinfeng Liao and Edward Shuryak

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(April 1, 2008)

The so called magnetic scenario recently suggested in [1] emphasizes the role of monopoles in strongly coupled quark-gluon plasma (sQGP) near/above the deconfinement temperature, and specifically predicts that they help reduce its viscosity by the so called “magnetic bottle” effect. Here we present results for monopole-(anti)monopole correlation functions from the same classical molecular dynamics simulations, which are found to be in very good agreement with recent lattice results [2]. We show that the magnetic Coulomb coupling does run in the direction *opposite* to the electric one, as expected, and it is roughly inverse of the asymptotic freedom formula for the electric one. However, as  $T$  decreases to  $T_c$ , the magnetic coupling never gets weak, with the plasma parameter always large enough ( $\Gamma > 1$ ). This nicely agrees with empirical evidences from RHIC experiments, implying that magnetic objects cannot have large mean free path and should also form a good liquid



## Our MD for 50-50 MQP/EQP

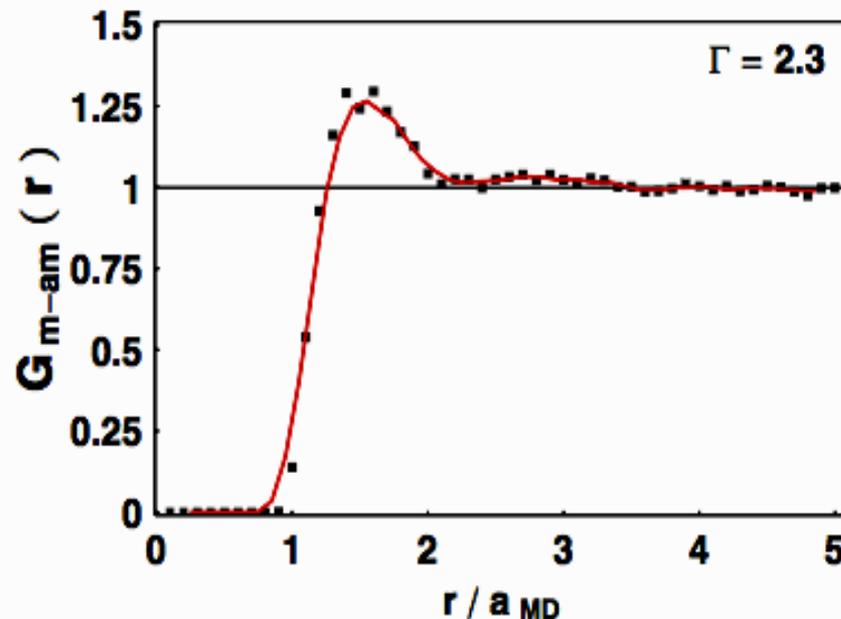


FIG. 2. (color online) Monopole-antimonopole correlators versus distance: points are lattice data [2], the dashed lines are our fits.

# $\alpha_s(\text{electric})$ and $\alpha_s(\text{magnetic})$

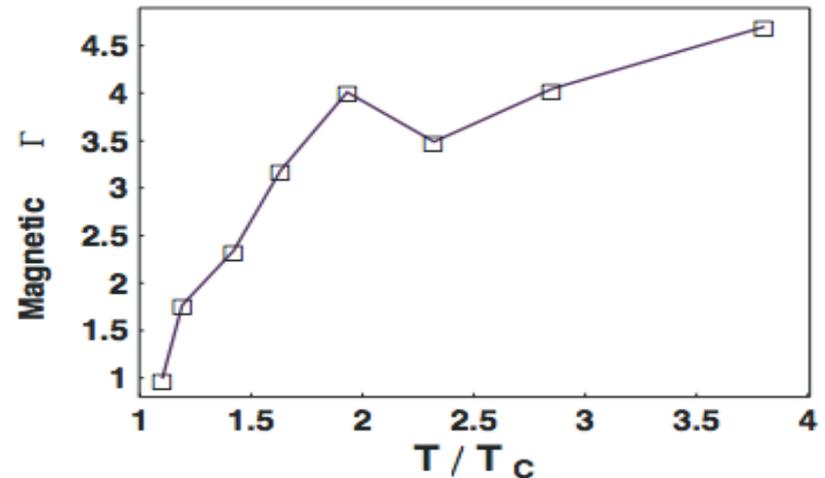
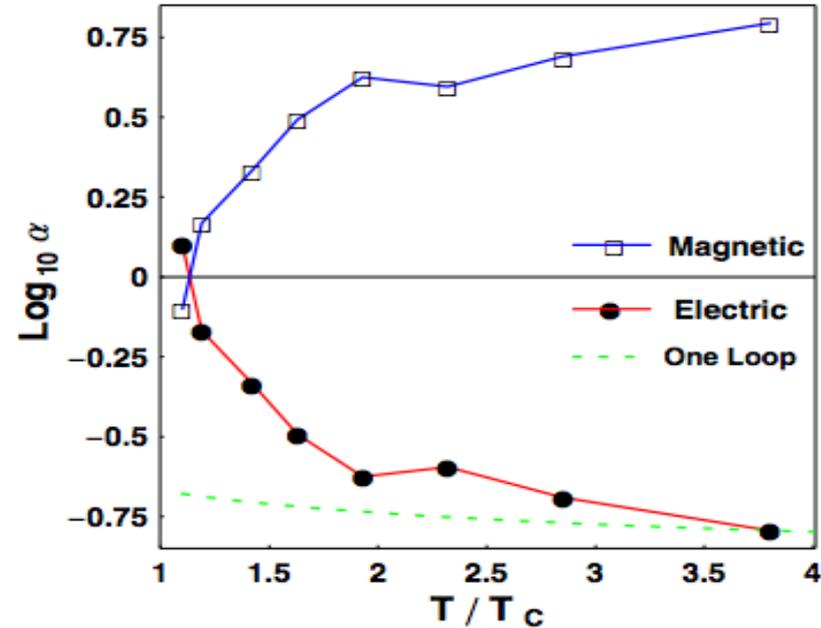
do run in opposite directions!

- Squares: fitted magnetic coupling, circles: its inverse compared to asymptotic freedom (dashed)

- Effective plasma parameter (here for magnetic)

$$\Gamma \equiv \frac{\alpha_C / \left(\frac{3}{4\pi n}\right)^{1/3}}{T}$$

- So, the monopoles are **never weakly coupled!**
- (just enough to get Bose-condensed)



# Thermal Monopole Condensation and Confinement in finite temperature Yang-Mills Theories

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(Dated: February 22, 2010)

We investigate the connection between Color Confinement and thermal Abelian monopoles populating the deconfined phase of SU(2) Yang-Mills theory, by studying how the statistical properties of the monopole ensemble change as the confinement/deconfinement temperature is approached from above. In particular we study the distribution of monopole currents with multiple wrappings in the Euclidean time direction, corresponding to two or more particle permutations, and show that multiple wrappings increase as the deconfinement temperature is approached from above, in a way compatible with a condensation of such objects happening right at the deconfining transition. We also address the question of the thermal monopole mass, showing that different definitions give consistent results only around the transition, where the monopole mass goes down and becomes of the order of the critical temperature itself.

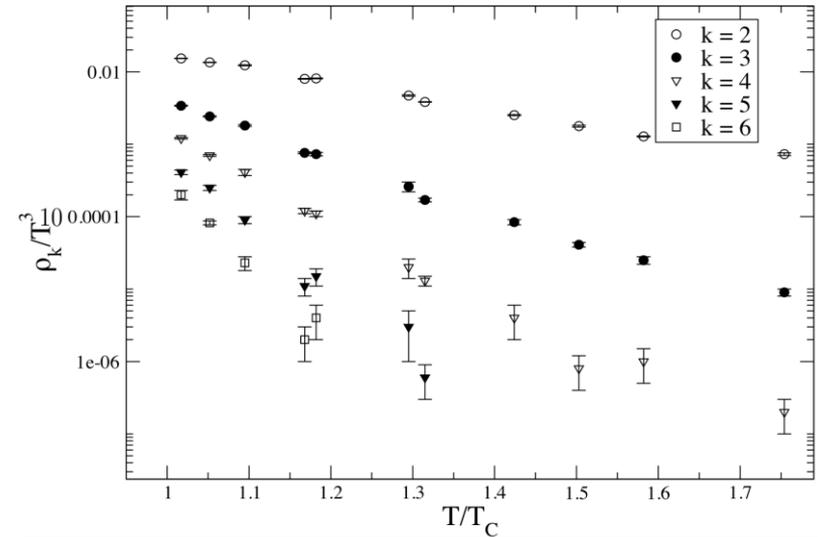
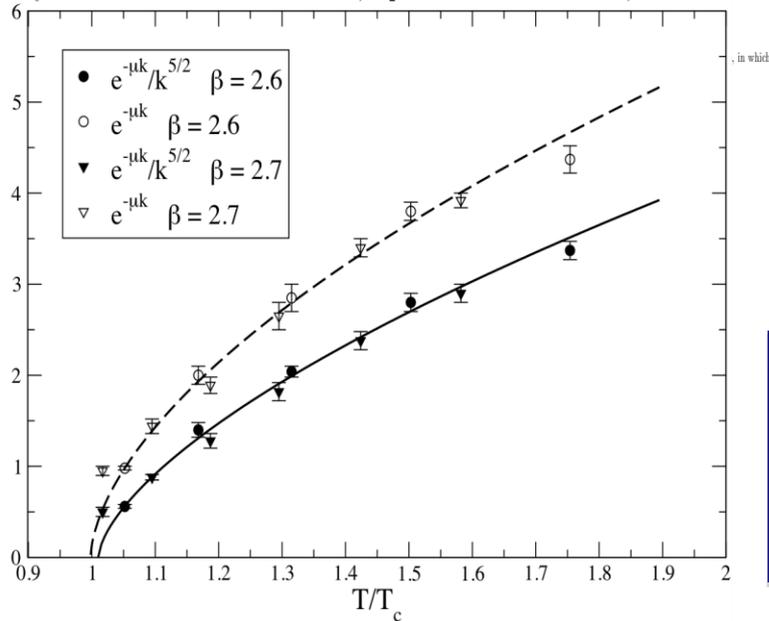
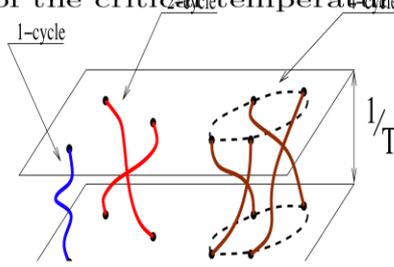


FIG. 2: Normalized densities  $\rho_k/T^3$  as a function of  $T/T_c$ .

**The lesson: monopoles at  $T_c$ ,  
behave as  $\text{He}^4 \Rightarrow$  Bose-Einstein  
condensation**

# Not surprising, large correction to transport (Ratti,ES,09)

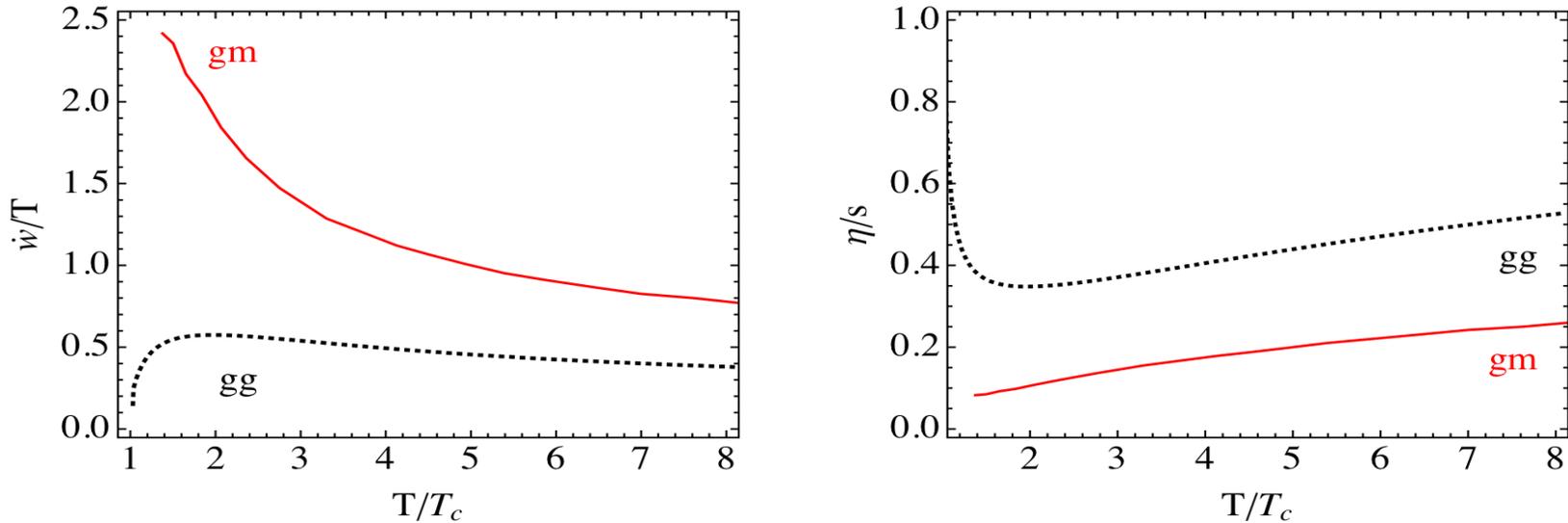
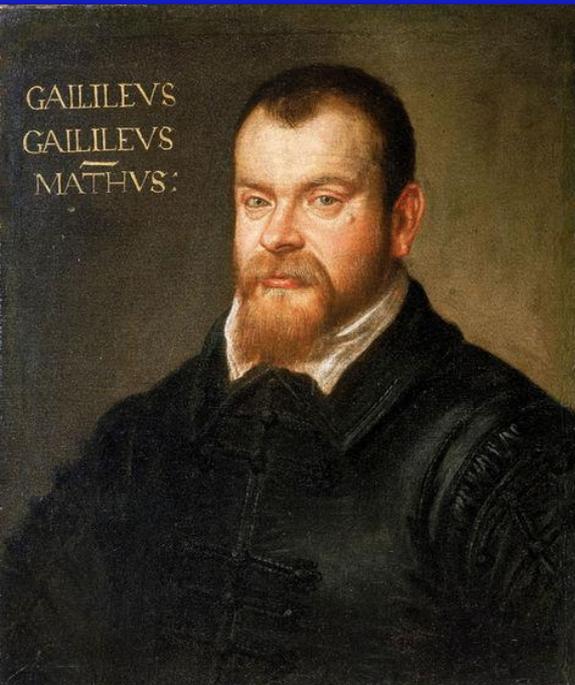


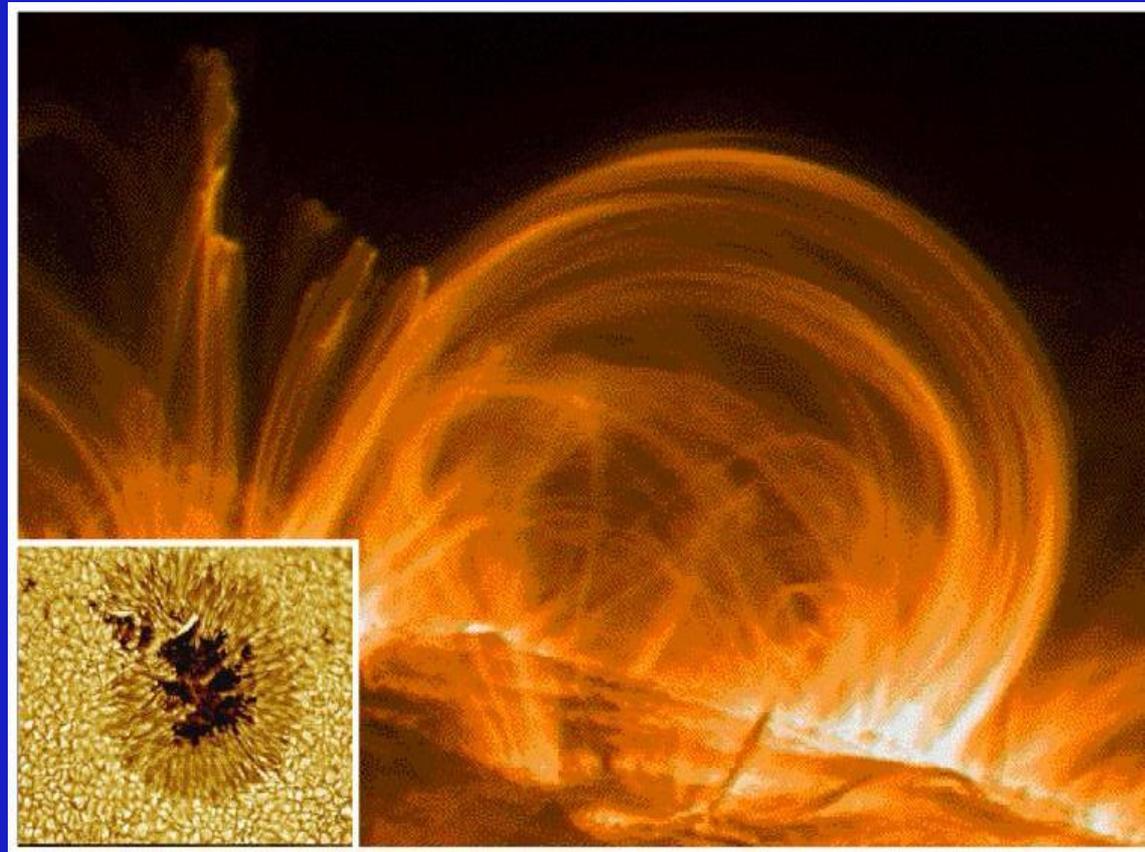
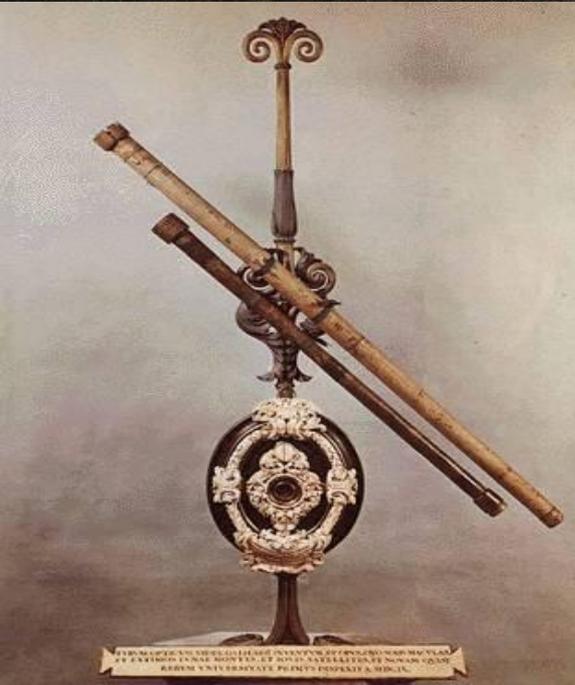
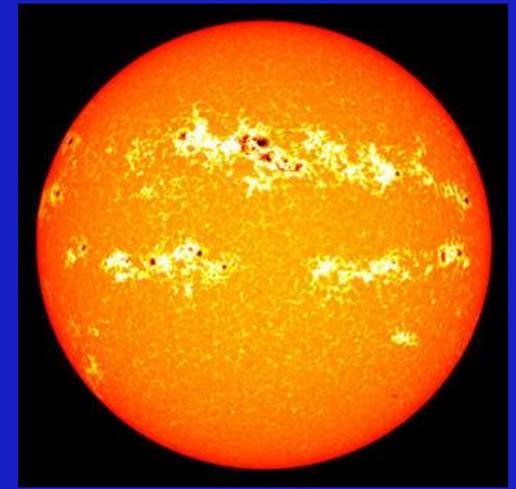
Figure 14: Left panel: gluon-monopole and gluon-gluon scattering rate. Right panel: gluon-monopole and gluon-gluon viscosity over entropy ratio,  $\eta/s$ .

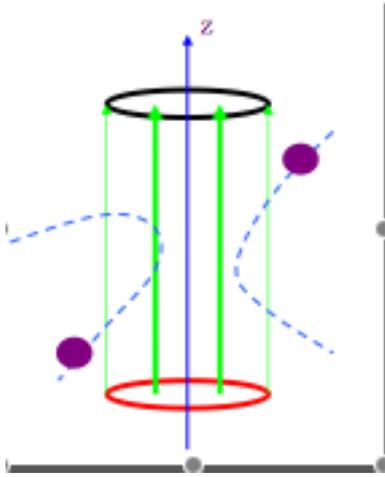
- **RHIC:  $T/T_c < 2$ , LHC  $T/T_c < 4$ :** we predict hydro will still be there, with  $\eta/s$  about .2

# Part 3: electric flux tubes



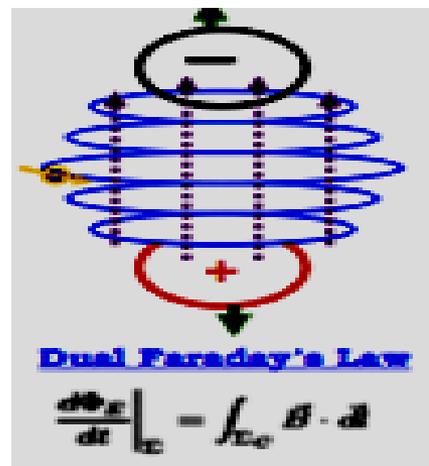
**1612:  
Galileo  
discovered what we  
now call solar  
corona**





# Moving e-charge leads to magnetic coil => e-flux tubes above $T_c$ ?

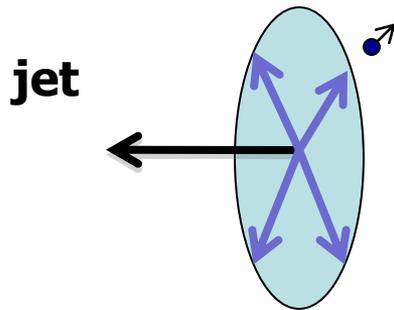
(with J.F.Liao, archive 0706.4465)



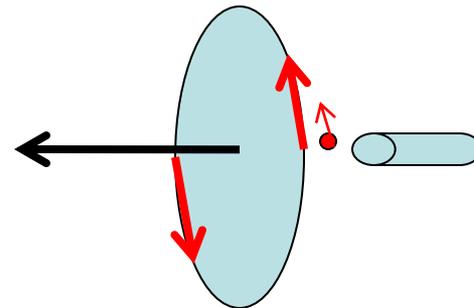
- **Dual superconductivity** at  $T < T_c$  as a confinement mechanism (t'Hooft, Mandelstam 1980's) => monopole Bose condensation => electric **flux tubes** (dual to Abrikosov-Nielsson-Olesen vortices)
- **Dual magnetohydrodynamics** at  $T > T_c$  ? Electric flux tubes in magnetic plasma (M=phase)
- monopoles are reflected from E field => pressure => metastable flux tubes

# Jet quenching is different in electric and magnetic plasmas

- Nonzero  $E_r \Rightarrow$  radial kicks to charges



**Nonzero  $B_\phi \Rightarrow$  Tangential kicks of the monopoles**



**This creates vorticity, magnetic coil and (possibly) an electric flux tube behind the jet**

# Here is my view of the “QGP corona”

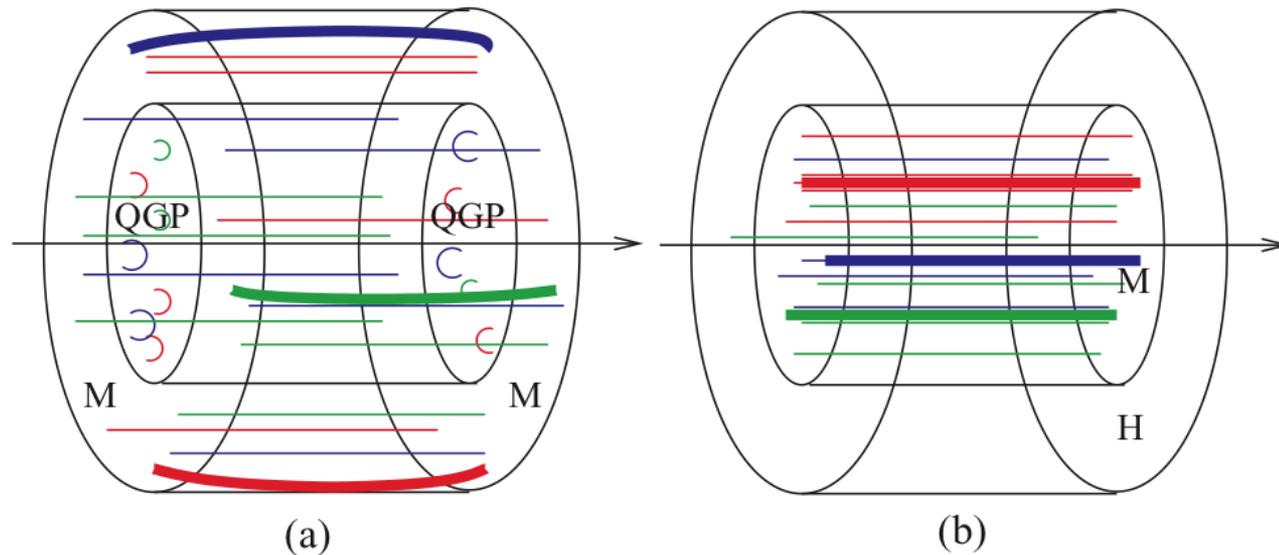
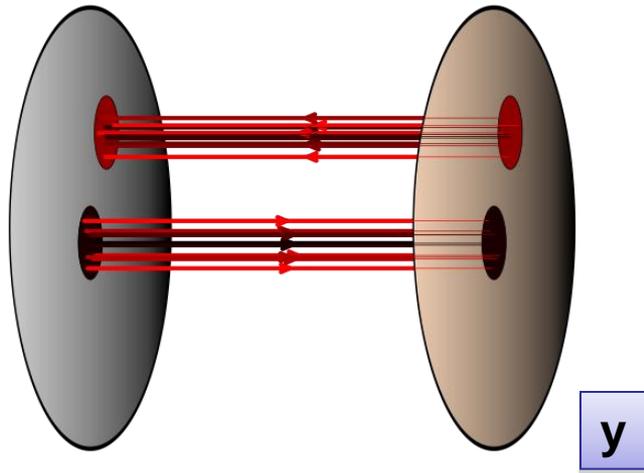


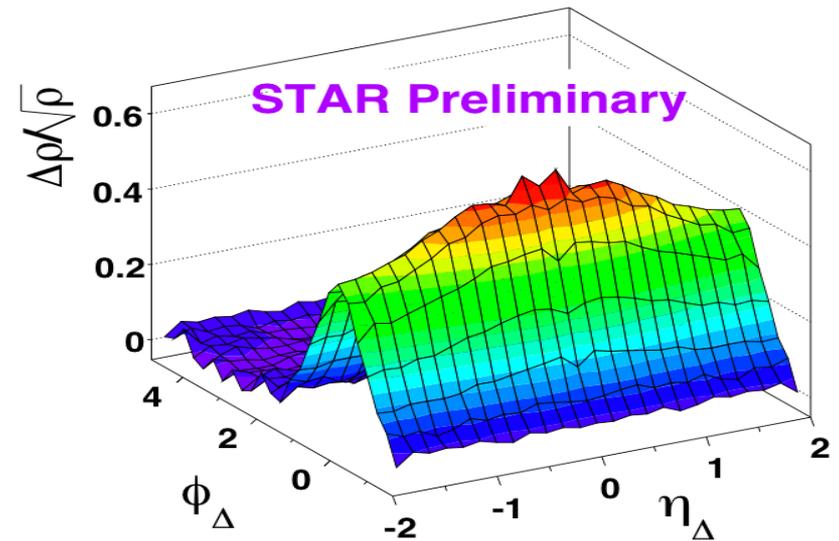
FIG. 1. (Color online) Snapshot of unscreened electric (dual-magnetic) field in the M (near- $T_c$ ) region of the fireball. (a) Full RHIC energy; (b) reduced energy (analogous to SPS).

the “soft ridge” exists even **without** any hard trigger



**McLerran, Venugopalan et al: Fluctuations of color charges at early time**

$$1/Q_s \sim .2 fm/c$$

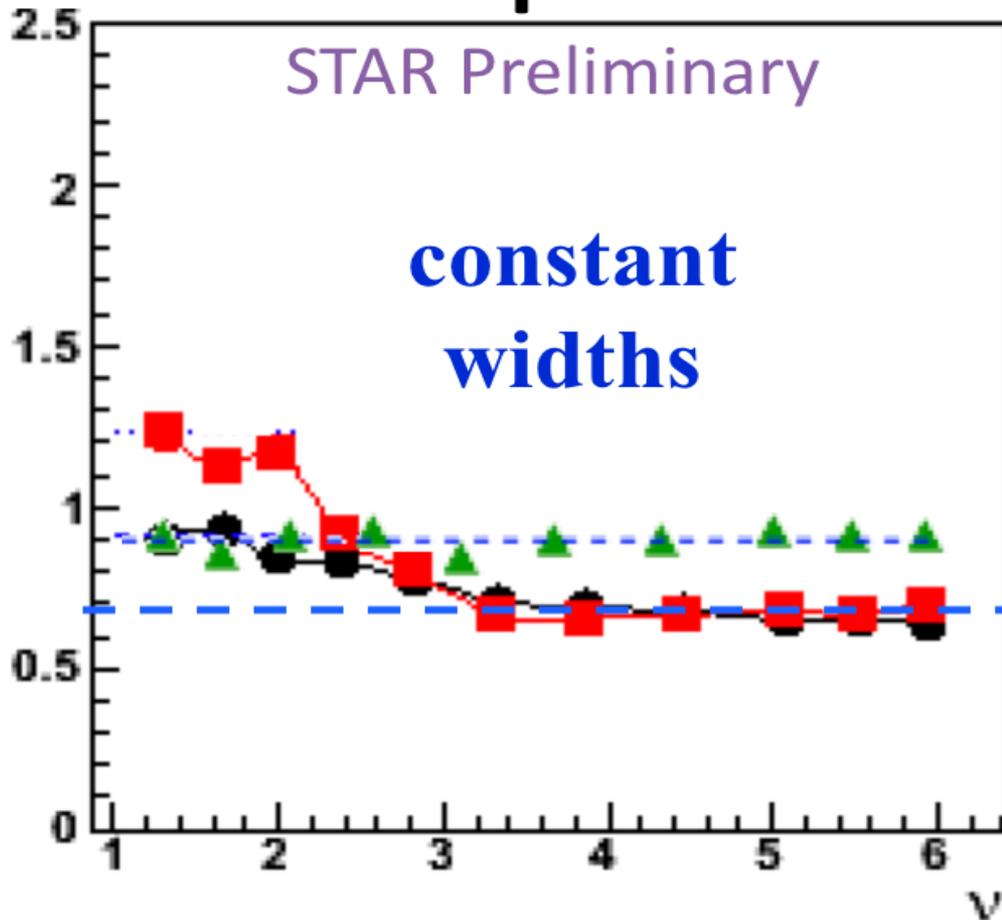


**(Phobos further observed that ridge extends at least till  $|y|=4$ )**

**What happens next, till freezeout ( $>10$  fm), is quite nontrivial**

# The peak width decreases for central collisions

## Peak $\phi$ Width



• L.Ray

The decay products of the ridge are clusters which are **larger** than in pp! they have up to 10 pions and they decay unisotropically

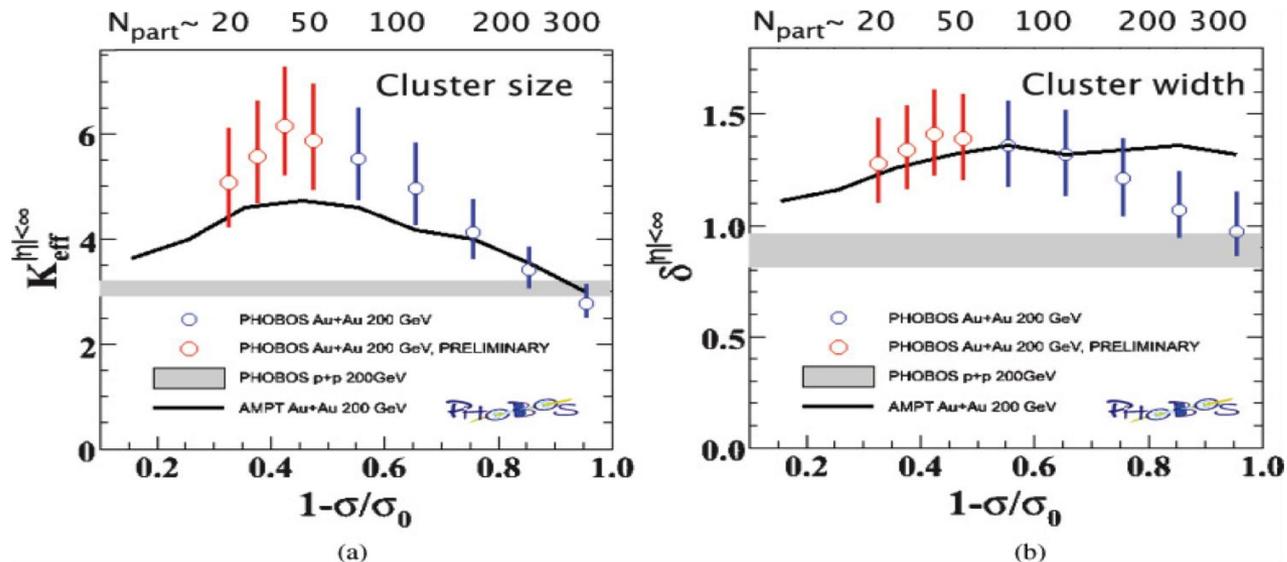
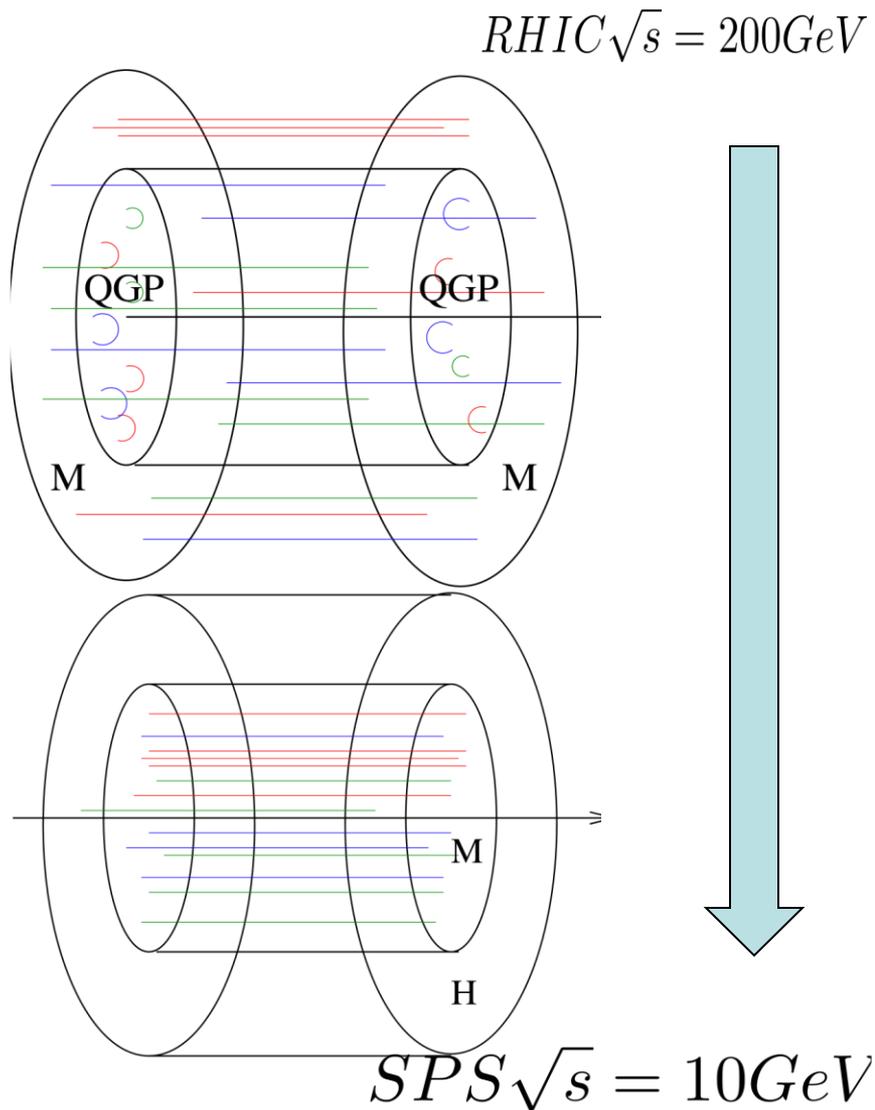
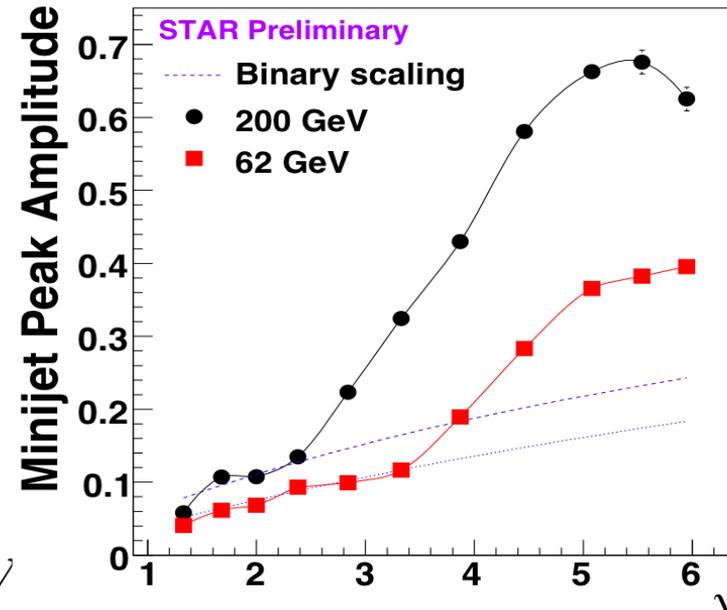


FIG. 13. (Color online) Cluster size (a) and width (b) as functions of centrality (cross section fraction) in Au-Au collisions at RHIC full energy, from PHOBOS [41]. The size is the number of charged particles associated with the cluster, and the width is in rapidity.

# Predictions for energy dependence: ridges



As energy decreases, M phase  
Goes inside the fireball =>  
Much smaller radial flow =>  
**Disappearance of the ridge  
happens at fixed density of matter!**



**STAR  
AuAu**

**L.Ray:  
Also in  
CuCu**

# “metastable flux tube” option for “cone” and “ridge”:

- there are enough monopoles to stabilize the flux tubes **mechanically** up to  $1.4T_c$ :
- They can **survive for a long time**, 5-10 fm/c due to **heavy electric quasiparticles (q,g,dyons) at  $T_c$**

$$P_{\text{breaking}} \sim \exp\left(-\frac{\pi M^2}{E}\right)$$

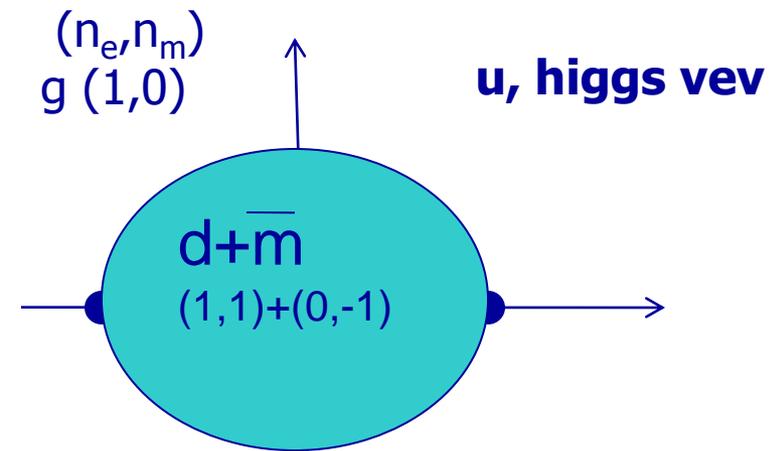
$M_q=300$  MeV at  $T=0$   
but about 600 MeV at  
 $T_c$  (lattice disp.curves)

# summary

- **Fluctuations create perturbations => sound cylinders and cones. Sound horizon and viscous spread are to be measured, by (very distorted) cylinders, "Mach peaks" and recoil**
- **Color magnetic monopoles are found on the lattice, they behave as physical objects: the Coulomb plasma, running coupling, BEC at  $T_c$**
- **What we called m-phase (from mixed) is magnetic-dominated. Different jet quenching, indications to corona made of flux tubes which are even more stable in magnetic plasma than in vacuum!**

# Recent development: the wall crossing formulae

- Black dots: SW (1994) singularities in which mono and dyon gets massless
- Blue oval is strong coupling region separated by “the wall” at which the **spectrum changes discontinuously** but not metric or charges:  
(states disappear into IR)



Dyons were promoted to dyonic black holes in N=2 supergravity and their bound states have been studied

**Recently “wall crossing formula” =>**

M. Kontsevich and Y. Soibelman, “Motivic Donaldson-Thomas invariants:summary of results,” arXiv:0910.4315

T. Dimofte, S. Gukov and Y. Soibelman, “Quantum Wall Crossing in N=2 Gauge Theories,” arXiv:0912.1346 [hep-th].