# Conformal viscous hydrodynamics

***and***

***finite temperature gauge/gravity duality***

Rudolf Baier

Department of Physics, University of Bielefeld

mainly based on following work - and references therein - by:

R. Baier, P. Romatschke, D. T. Son, A. O. Starinets and M. A. Stephanov

“Relativistic viscous hydrodynamics, conformal invariance, and holography”

JHEP 0804 (2008) 100 (arXiv: 0712.2451 [hep-th])

R. Loganayagam

“Entropy current in conformal hydrodynamics”

JHEP 0805 (2008) 087 (arXiv: 0801.3701 [hep-th])

P. Romatschke

”New developments in relativistic viscous hydrodynamics”

Int.J.Mod.Phys E19 (2010) 1 (arXiv: 0902.3663 [hep/ph])

# CONTENT

- motivation: viscous effects in RHIC data

viscosity $\eta$ - relaxation time $\tau_\pi$, ...

- conformal invariance,

  relativistic hydrodynamics -

  up to second-order gradients

- gauge/gravity duality -

  (AdS/CFT) correspondence

A fluid which has no shear stresses, viscosity or heat conduction is called a

# PERFECT FLUID

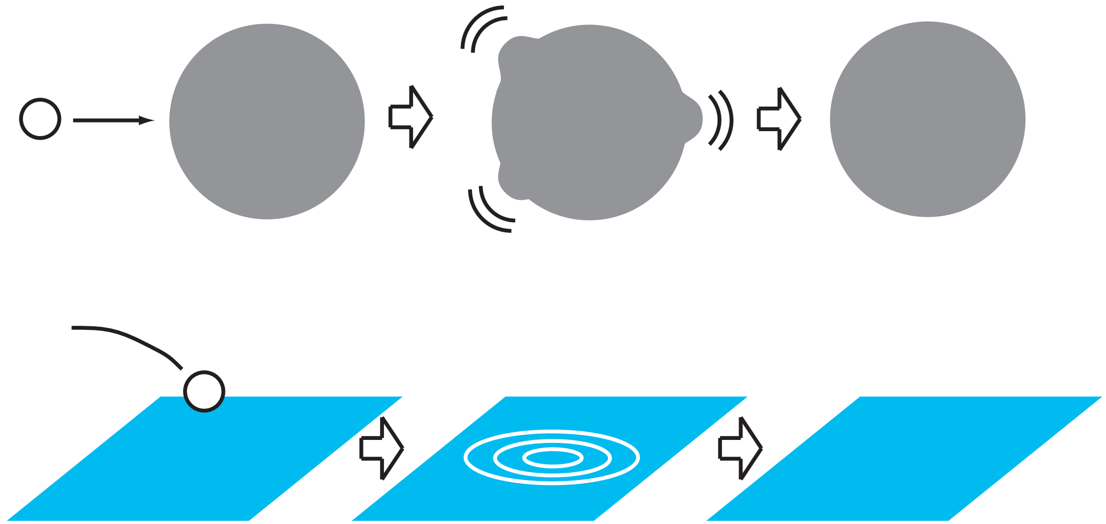
Analysis of measurements at RHIC indicate a strongly interacting quark-gluon plasma

- called sQGP -

which behaves almost like a perfect fluid, with very low viscosity  Is it correct? It requires to calculate corrections in a systematic way!

remarkable approach: **black hole theory**

is now used to explain properties of colliding nuclei dissipation: consequence of black hole absorption



dissipation: consequence of viscosity

[from M. Natsuume]

(damped) quasinormal modes - excited classical oscillations:

gravitational perturbation to a black hole ("ringing BH")

and to a hydrodynamic system ("ringing plasma")

# hydrodynamics

energy momentum tensor: $\epsilon$ energy density and $p$ pressure

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + pg^{\mu\nu} + \Pi^{\mu\nu}$$

$u^\mu$ fluid velocity, $u_\mu u^\mu = -1$, and $\Pi^{\mu\nu}$ shear stress tensor

with

$$u_\mu \Pi^{\mu\nu} = 0 \;\; (u_\mu T^{\mu\nu} = -\epsilon u^\nu),$$

momentum density is due to energy flow, and

$$\Pi^\mu_\mu = 0$$

required from conformal invariance

local conservation law (covariant derivative $\nabla_\mu$): $\nabla_\mu T^{\mu\nu} = 0$

(assume: no net charge in the system)

# a few definitions

conformal fluid in $d=4$ (curved) dimensions -

line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

metric $g_{\mu\nu}$ with signature $(-+++)$, $\mu, \nu$.. space-time indices $0, 1, 2, 3$

(geometric) covariant derivative: $\nabla_\mu$

$$\nabla_\mu \Phi(x) = \partial_\mu \Phi(x) \, ,$$

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma_{\mu\rho}{}^{\nu} V^\rho$$ etc., e.g. $\nabla_\mu g^{\nu\rho} = 0$

Christoffel symbols:

$$\Gamma_{\mu\rho}{}^{\nu} = \frac{1}{2} g^{\nu\sigma} \left( \partial_\mu g_{\rho\sigma} + \partial_\rho g_{\mu\sigma} - \partial_\sigma g_{\mu\rho} \right)$$

Riemann tensor: $[\nabla_\mu, \nabla_\nu] V^\rho = R_{\mu\nu\sigma}{}^{\rho} V^\sigma$

## approximation to $\Pi^{\mu\nu}$

only retaining shear viscosity terms - NO bulk viscosity

- Navier - Stokes $=$ first-order theory in gradients

with constitutive relation:

$$\Pi^{\mu\nu} = -2\eta < \nabla^{\mu} u^{\nu} > \equiv -2\eta\sigma^{\mu\nu}$$

with the traceless and symmetric shear strain tensor

$$\sigma^{\mu\nu} = \sigma^{\nu\mu} \equiv \frac{1}{2}\left(\Delta^{\mu\rho}\nabla_{\rho}u^{\nu} + \Delta^{\nu\rho}\nabla_{\rho}u^{\mu}\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\rho\sigma}\nabla_{\rho}u_{\sigma}$$

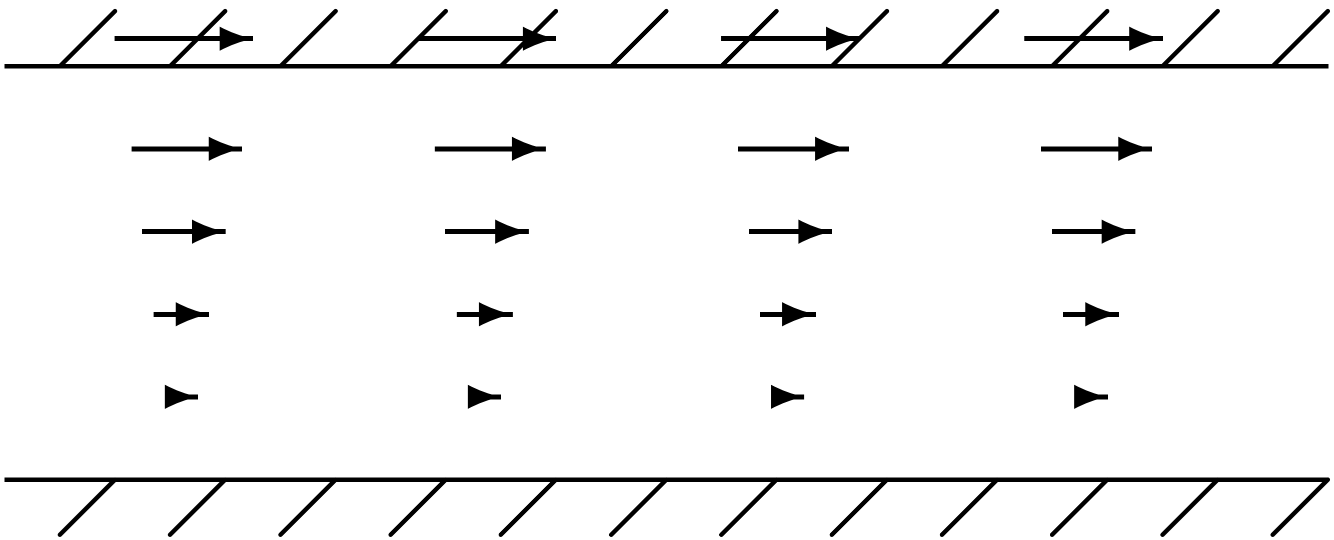
and transverse projection

$$\Delta^{\mu\nu} = g^{\mu\nu} + u^{\mu}u^{\nu}\ , \quad \Delta^{\mu\nu}u_{\nu} = 0$$

# shear viscosity $\eta$

interacting field theories:

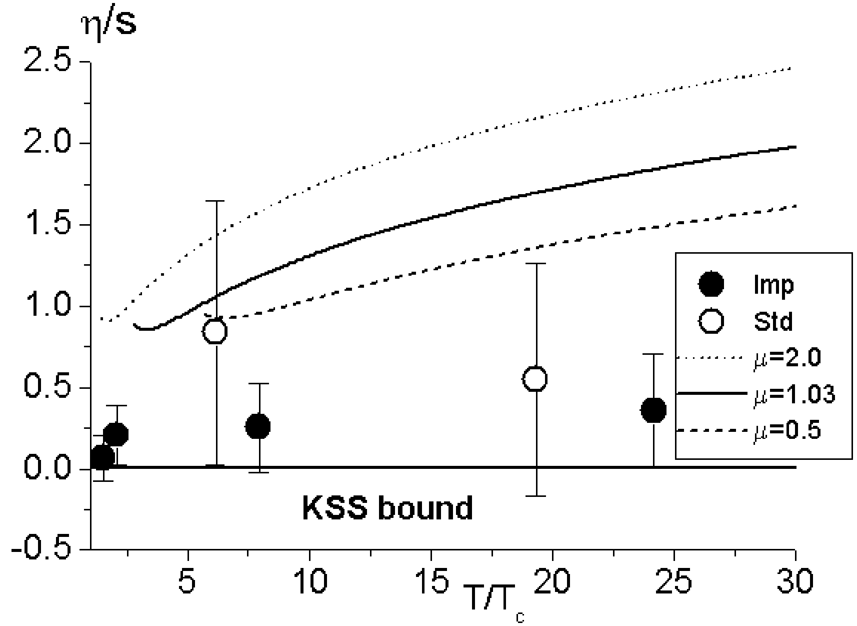
non-vanishing transport coefficients




e.g. shear flow in $x-$ direction $\rightarrow$ force per unit area/momentum transfer

$$\frac{F_x}{A} = -\Pi_{xz} = \eta \frac{\partial u_x}{\partial z}$$

# viscosity/entropy density ratio




$\eta/s$ from quenched QCD lattice simulations [Nakamura and Sakai 05],
also $\eta/s = 0.134(33)$ at $T = 1.65T_c$ [Meyer 07]
compared with the perturbative result $\sim \frac{1}{g^4 \ln(\mu/gT)} \sim \mathcal{O}(1)$

for strong coupling $\lambda = g_{YM}^2 N_c$ , $\lambda \to \infty$ [Kovtun, Son, Starinets 05]

$$\frac{\eta}{s} \geq \frac{1}{4\pi} \sim 0.08 : \quad AdS/CFT\ universal\ (?)\ lower\ bound$$

# Navier-Stokes

transverse projection of EoM $\Delta^{\mu}_{\alpha}\nabla_{\beta}T^{\alpha\beta} = 0 \;\; \longrightarrow$

relativistic Navier-Stokes equation in first-order theory

$$(\epsilon + p)u^{\alpha}\nabla_{\alpha}u^{\mu} + \Delta^{\mu\alpha}\nabla_{\alpha}p + \Delta^{\mu}_{\alpha}\nabla_{\beta}\left[-2\eta < \nabla^{\alpha}u^{\beta} >\right] = 0$$

i.e. parabolic equation:

time derivative is of first-order ($u^{\alpha}\nabla_{\alpha} \rightarrow \partial/\partial t$)

while

space derivative is of second-order ($\nabla_j\sigma^{ij} \rightarrow \partial_j\partial_i u^j$)

“relativistic first-order dissipative theory is highly pathological, and therefore should be discarded in favor of the second-order one”

[Hiscock and Lindblom 1983-1985]

# transverse mode

small linear perturbation and first-order $\longrightarrow$

diffusion equation in shear channel:

$$\delta u_\perp = \sqrt{\frac{(\epsilon + p)}{4\pi\eta t}}\,\exp\,[-\frac{(\epsilon+p)x^2}{4\eta t}]$$

i.e. propagates outside the light-cone (Gaussian)

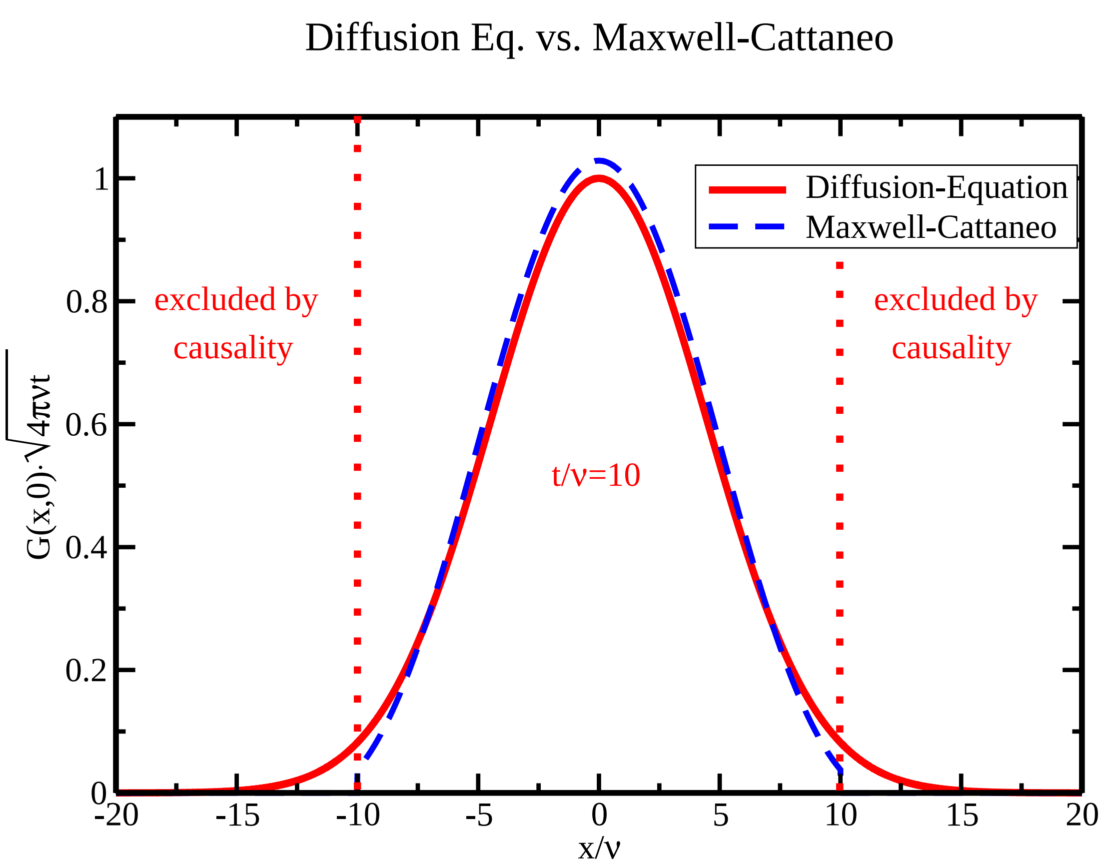
starting from $\delta u_\perp(x, t=0) = \delta(x)$ !!

minimal modification $\longrightarrow$ second-order:

relaxation time $\tau_\pi > 0$ $\longrightarrow$ hyperbolic “telegraph” equation

$$[\tau_\pi \partial_t^2 + \partial_t - \frac{\eta}{(\epsilon+p)}\partial_x^2]\;\delta u_\perp = 0$$

# diffusion vs. Cattaneo




$$\nu = \eta/(\epsilon + p) = const; \quad \tau_\pi = \nu$$

Green's function for diffusion and Maxwell-Cattaneo law

[Romatschke (2009)]

# a systematic result

*all* second-order terms classified by conformal symmetry,
by space-time dependent Weyl transformation:

$$g^{\mu\nu} = e^{-2\phi(x)}\, \tilde{g}^{\mu\nu},\;\; T^{\mu\nu} = e^{-6\phi}\, \tilde{T}^{\mu\nu},\;\; T^{\mu}_{\mu} = 0,\; \dots$$

constitutive relation (d= 4) with five new terms:

$$\begin{aligned}
\Pi^{\mu\nu} \;=\; & -2\eta\sigma^{\mu\nu} + 2\eta\tau_\pi u^\lambda \mathcal{D}_\lambda \sigma^{\mu\nu} \\
& +4\lambda_1 \sigma^{<\mu}{}_\lambda \sigma^{\nu>\lambda} + 2\lambda_2 \sigma^{<\mu}{}_\lambda \Omega^{\nu>\lambda} + \lambda_3 \Omega^{<\mu}{}_\lambda \Omega^{\nu>\lambda} \\
& +2\kappa u_\alpha C^{\alpha\mu\nu\beta} u_\beta
\end{aligned}$$

$C^{\alpha\beta\gamma\delta}$... Weyl tensor, $\Omega^{\alpha\beta}$... antisymmetric vorticity tensor, $\mathcal{D}_\lambda$... Weyl derivative

[Baier, Romatschke, Son, Starinets, Stephanov 07]
[Bhattacharyya, Hubeny, Minwalla, Rangamani 07, Loganayagam 08]

# relaxation phenomena/strong coupling limit

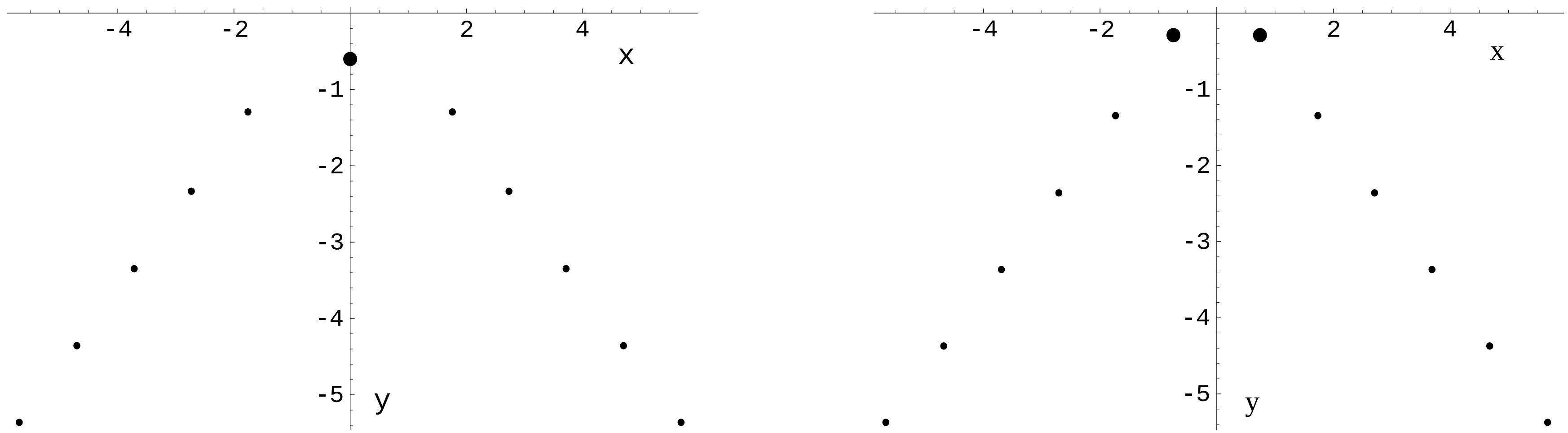
hydrodynamic transport coefficients by gauge/gravity duality:
compare with quite involved AdS/CFT-gravity calculations at Hawking temperature $T = T_H$ and for momentum $\omega, k << T_H$ i.e. long wave-length limit:

e.g. from sound channel dispersion up to $O(k^3)$, etc :

$$\frac{\eta}{s} = \frac{1}{4\pi}, \quad \tau_\pi = \frac{2 - \ln 2}{2\pi T}, \quad \kappa = \frac{\eta}{\pi T} = 2\lambda_1 \; ,$$

$$\lambda_2 = -\frac{\ln 2}{2\pi T}\eta \; , \quad \lambda_3 = 0$$

# hydrodynamic modes




quasinormal spectrum of black brane gravitational fluctuations in the shear and sound channels at fixed spatial momentum as a function of complex frequency

[Kovtun and Starinets (2005)]

hydrodynamic poles are marked by full dots, e.g. shear

$$\omega = -i\frac{\eta}{Ts}q^2 + O(q^3) = -i\frac{1}{4\pi T}q^2 + O(q^3)$$

approach the real frequency axis for $q \to 0$

[Kovtun, Son and Starinets (2005)]

# quasinormal mode (QNM)

Wikipedia:

Quasinormal modes are the modes of energy dissipation of a perturbed object or field (example: perturbation of a wine glass with a knife - the glass begins to ring, but not forever!)

Quasinormal ringing is approximated by the amplitude

$$\psi(t) \approx e^{-\omega_I t} cos(\omega_R t) \approx e^{-i\omega t}$$

complex quasinormal frequency, $Im\omega = -\omega_I < 0$

in gravity: characteristic excited classical oscillations of (AdS-) black holes (BH) and black branes responding to generic perturbations - approach to thermal equilibrium:

"ringing BH"

# transport coefficients

kinetic QCD theory (LO weak coupling: $\alpha_s < 0.7$ )

$$\frac{\eta/s}{\tau_\pi T} \simeq 0.17 - 0.2,\ \lambda_1 \simeq (5.2 - 4.1)\frac{\eta^2}{sT},\ \lambda_2 = -2\tau_\pi \eta,\ \lambda_3 = \kappa = 0$$

[York and Moore 08]

finite 't Hooft coupling $\lambda \equiv g_{YM}^2 N_c, \quad \lambda >> 1$ corrections to coefficients by gauge/gravity duality, e.g.:

$$\frac{\eta/s}{\tau_\pi T} = 0.383\ (1 - 3.52\ \lambda^{-3/2} + ...),\ \lambda_1 = \frac{2\eta^2}{sT}\ ,$$

$$\lambda_2 = -0.530\tau_\pi \eta\ ,\ \kappa = \frac{\eta}{\pi T}\ (1 - \frac{145\zeta(3)}{8}\lambda^{-3/2} + ...)$$

[Buchel and Paulos 08]

# non-conformal transport coefficients

The shear viscosity $\eta$ and the second order coefficients $\tau_\pi$ ("shear" relaxation time), $\kappa$, $\lambda_1, \lambda_2, \lambda_3$ are the only ones defined in conformal fluids, as the one of $\mathcal{N} = 4$ SYM.
All the others coefficients, i.e. the bulk viscosity $\zeta$ and the second order coefficients $\kappa^*, \tau_\pi^*, \lambda_4, \tau_\Pi$ ("bulk" relaxation time), $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6$, are only defined in non-conformal plasmas.

Defining:

$$\delta \equiv (1 - 3c_s^2)$$

transport coefficients at leading order in the deformation parameter $\delta$ :

[Bigazzi and Cotrone (2010)]

| | | | | | |
|---|---|---|---|---|---|
| $\frac{\eta}{s}$ | $\frac{1}{4\pi}$ | $T\tau_\pi$ | $\frac{2-\log 2}{2\pi} + \frac{3(16-\pi^2)}{64\pi}\delta$ | $\frac{T\kappa}{s}$ | $\frac{1}{4\pi^2}\left(1 - \frac{3}{4}\delta\right)$ |
| $\frac{T\lambda_1}{s}$ | $\frac{1}{8\pi^2}\left(1 + \frac{3}{4}\delta\right)$ | $\frac{T\lambda_2}{s}$ | $-\frac{1}{4\pi^2}\left(\log 2 + \frac{3\pi^2}{32}\delta\right)$ | $\frac{T\lambda_3}{s}$ | $0$ |
| $\frac{T\kappa^*}{s}$ | $-\frac{3}{8\pi^2}\delta$ | $T\tau_\pi^*$ | $-\frac{2-\log 2}{2\pi}\delta$ | $\frac{T\lambda_4}{s}$ | $0$ |
| $\frac{\zeta}{\eta}$ | $\frac{2}{3}\delta$ | $T\tau_\Pi$ | $\frac{2-\log 2}{2\pi}$ | $\frac{T\xi_1}{s}$ | $\frac{1}{24\pi^2}\delta$ |
| $\frac{T\xi_2}{s}$ | $\frac{2-\log 2}{36\pi^2}\delta$ | $\frac{T\xi_3}{s}$ | $0$ | $\frac{T\xi_4}{s}$ | $0$ |
| $\frac{T\xi_5}{s}$ | $\frac{1}{12\pi^2}\delta$ | $\frac{T\xi_6}{s}$ | $\frac{1}{4\pi^2}\delta$ | | |

| | | | | | |
|---|---|---|---|---|---|
| $\frac{\eta}{s}$ | $\frac{1}{4\pi}$ | $T\tau_\pi$ | $0.222$ | $\frac{T\kappa}{s}$ | $0.022$ |
| $\frac{T\lambda_1}{s}$ | $0.014$ | $\frac{T\lambda_2}{s}$ | $-0.021$ | $\frac{T\lambda_3}{s}$ | $0$ |
| $\frac{T\kappa^*}{s}$ | $-0.006$ | $T\tau_\pi^*$ | $-0.031$ | $\frac{T\lambda_4}{s}$ | $0$ |
| $\frac{\zeta}{\eta}$ | $0.101$ | $T\tau_\Pi$ | $0.208$ | $\frac{T\xi_1}{s}$ | $0.001$ |
| $\frac{T\xi_2}{s}$ | $0.001$ | $\frac{T\xi_3}{s}$ | $0$ | $\frac{T\xi_4}{s}$ | $0$ |
| $\frac{T\xi_5}{s}$ | $0.001$ | $\frac{T\xi_6}{s}$ | $0.004$ | | |

transport coefficients at $T \sim 1.5T_c$ and $c_s^2 \sim 0.283$

# conformal hydrodynamics

# Weyl transformation

scaling transformation:

$$g_{\mu\nu} = \mathrm{e}^{2\phi}\tilde{g}_{\mu\nu}; \qquad g^{\mu\nu} = \mathrm{e}^{-2\phi}\tilde{g}^{\mu\nu}\ ,\ \phi = \phi(x)$$

leads to:

$$\Gamma_{\lambda\mu}{}^{\nu} = \widetilde{\Gamma}_{\lambda\mu}{}^{\nu} + \delta^{\nu}_{\lambda}\partial_{\mu}\phi + \delta^{\nu}_{\mu}\partial_{\lambda}\phi - \tilde{g}_{\lambda\mu}\tilde{g}^{\nu\sigma}\partial_{\sigma}\phi$$

$$u^{\mu} = \mathrm{e}^{-\phi}\tilde{u}^{\mu},\ \ \Delta^{\mu\nu} = g^{\mu\nu} + u^{\mu}u^{\nu} = \mathrm{e}^{-2\phi}\widetilde{\Delta}^{\mu\nu}$$

usefull quantities (d dimensions):

$$'divergence' :\ \vartheta \equiv \nabla_{\mu}u^{\mu} = \mathrm{e}^{-\phi}\left[\tilde{\vartheta} + (d-1)\tilde{u}^{\sigma}\partial_{\sigma}\phi\right],$$

$$'acceleration' :\ a^{\nu} \equiv u^{\mu}\nabla_{\mu}u^{\nu} = \mathrm{e}^{-2\phi}\left[\tilde{a}^{\nu} + \widetilde{\Delta}^{\nu\sigma}\partial_{\sigma}\phi\right],$$

$$'gauge\ field' :\ \mathcal{A}_{\nu} \equiv a_{\nu} - \frac{\vartheta}{d-1}u_{\nu} = \tilde{\mathcal{A}}_{\nu} + \partial_{\nu}\phi$$

# Weyl derivative

cf. (abelian) gauge transform and derivative:

$$\tilde{\psi} = \mathrm{e}^{i\Lambda(x)}\psi \ , \ \tilde{A}^\mu = A^\mu - \partial^\mu \Lambda \ ,$$

$$\tilde{D}^\mu \tilde{\psi} = \mathrm{e}^{i\Lambda}(\partial^\mu + iA^\mu)\psi = \mathrm{e}^{i\Lambda} D^\mu \psi$$

elegant way: define covariant Weyl derivative [Loganayagam 07]

$$Q^{\mu\ldots}_{\nu\ldots} = e^{-w\phi}\widetilde{Q}^{\mu\ldots}_{\nu\ldots} \ , \mathcal{D}_\lambda \, Q^{\mu\ldots}_{\nu\ldots} = e^{-w\phi}\widetilde{\mathcal{D}}_\lambda \widetilde{Q}^{\mu\ldots}_{\nu\ldots} \ , \ w..integer$$

where Weyl derivative - linear in $\mathcal{A}_\lambda$ -

$$\begin{aligned}
\mathcal{D}_\lambda \, Q^{\mu\ldots}_{\nu\ldots} \equiv \nabla_\lambda \, Q^{\mu\ldots}_{\nu\ldots} + w \, \mathcal{A}_\lambda Q^{\mu\ldots}_{\nu\ldots} \\
+ \left[ g_{\lambda\alpha}\mathcal{A}^\mu - \delta^\mu_\lambda \mathcal{A}_\alpha - \delta^\mu_\alpha \mathcal{A}_\lambda \right] Q^{\alpha\ldots}_{\nu\ldots} + \ldots \\
- \left[ g_{\lambda\nu}\mathcal{A}^\alpha - \delta^\alpha_\lambda \mathcal{A}_\nu - \delta^\alpha_\nu \mathcal{A}_\lambda \right] Q^{\mu\ldots}_{\alpha\ldots} - \ldots
\end{aligned}$$

## examples

$$\mathcal{D}_\lambda g_{\mu\nu} = 0\ ,\ \mathcal{D}_\mu u^\mu = 0\ ,\ u^\mu \mathcal{D}_\mu u^\nu = 0$$

and

$$\mathcal{D}_\mu u^\nu = \nabla_\mu u^\nu + u_\mu a^\nu - \frac{\vartheta}{d-1}\Delta_\mu{}^\nu = \sigma_\mu{}^\nu + \Omega_\mu{}^\nu = \mathrm{e}^{-\phi}\widetilde{\mathcal{D}}_\mu \tilde{u}^\nu$$

$$\sigma^{\mu\nu} \equiv \frac{1}{2}\left(\Delta^{\mu\lambda}\nabla_\lambda u^\nu + \Delta^{\nu\lambda}\nabla_\lambda u^\mu\right) - \frac{1}{d-1}\vartheta\Delta^{\mu\nu} = \frac{1}{2}\left(\mathcal{D}^\mu u^\nu + \mathcal{D}^\nu u^\mu\right)$$
$$= \mathrm{e}^{-3\phi}\tilde{\sigma}^{\mu\nu}$$

$$\textit{antisymmetric vorticity tensor}:$$

$$\Omega^{\mu\nu} \equiv \frac{1}{2}\left(\Delta^{\mu\lambda}\nabla_\lambda u^\nu - \Delta^{\nu\lambda}\nabla_\lambda u^\mu\right) = \frac{1}{2}\left(\mathcal{D}^\mu u^\nu - \mathcal{D}^\nu u^\mu\right) = \mathrm{e}^{-3\phi}\widetilde{\Omega}^{\mu\nu}$$

- partition function $Z = \tilde{Z}$ is conformal invariant ($d = 4$ dimensions)

$$T^{\mu\nu} \propto \frac{1}{\sqrt{-g}} \frac{\delta \ln Z}{\delta g_{\mu\nu}} \rightarrow$$

$$T^{\mu\nu} = e^{-6\phi(x)} \tilde{T}^{\mu\nu} \;\; with \; g_{\mu\nu} = e^{2\phi} \tilde{g}_{\mu\nu}$$

- 

$$conservation : \nabla_\mu T^{\mu\nu} = 0 \rightarrow \tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = 0 \rightarrow$$

$$\tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = e^{6\phi} [\nabla_\mu T^{\mu\nu} + T^\mu_\mu \partial^\nu \phi] \rightarrow T^\mu_\mu = 0$$

- only scale is temperature $T$:

$$T^{\mu\nu}_{perfect} = (g^{\mu\nu} + 4u^\mu u^\nu) p \;,\; \epsilon = 3p$$

$$\rightarrow p = e^{-4\phi}\, \tilde{p} \;,\; p \propto T^4 \;,\; T = e^{-\phi}\, \tilde{T} \;,\; \mathcal{D}_\mu T = e^{-\phi} \widetilde{\mathcal{D}}_\mu \tilde{T}$$

# invariant Weyl tensor

$$(d-2)C_{\mu\alpha\nu\beta}u^\alpha u^\beta =$$

$$\Delta^{\mu\lambda}\Delta^{\nu\sigma}R_{\lambda\sigma} + (d-2)\Delta^{\mu\lambda}\Delta^{\nu\sigma}R_{\lambda\alpha\sigma\beta}u^\alpha u^\beta$$

$$-\frac{\Delta^{\mu\nu}}{d-1}(\Delta^{\lambda\sigma}R_{\lambda\sigma}+(d-2)\Delta^{\lambda\sigma}R_{\lambda\alpha\sigma\beta}u^\alpha u^\beta)$$

$R_{\mu\nu}\ldots$ Ricci tensor

symmetry properties (same as for the Riemann tensor):

$$C_{\mu\nu\lambda\sigma} = -C_{\nu\mu\lambda\sigma} = -C_{\mu\nu\sigma\lambda} = C_{\lambda\sigma\mu\nu}$$

and $\quad C_{\mu\alpha\lambda}{}^{\alpha} = 0$

$$C_{\alpha\mu\nu}{}^{\beta} = \tilde{C}_{\alpha\mu\nu}{}^{\beta} \implies u_\alpha C^{\alpha\mu\nu\beta}u_\beta = e^{-4\phi(x)}\,\tilde{u}_\alpha \tilde{C}^{\alpha\mu\nu\beta}\tilde{u}_\beta$$

$$u_\mu u_\alpha C^{\alpha\mu\nu\beta}u_\beta = 0$$

$\Pi^{\mu\nu}$ has in $d = 4$ dimensions 5 independent tensor structures

first-order

$$\Pi^{\mu\nu} = -2\eta\ \sigma^{\mu\nu}$$

$$\Pi^{\mu\nu} = e^{-6\phi}\tilde{\Pi}^{\mu\nu}\ ,\ \sigma^{\mu\nu} = e^{-3\phi}\tilde{\sigma}^{\mu\nu}\ \ \rightarrow$$

$$\eta = e^{-3\phi}\tilde{\eta}\ ,\ i.e.\ \eta \propto T^3$$

e.g. $\eta = (\pi T)^3\ ,\ p = (\pi T)^4$ ,

$s = \frac{\partial p}{\partial T} = 4\pi(\pi T)^3\ \rightarrow\ \eta/s = 1/4\pi$ i.e. only a number!

second-order

$$u^\lambda \mathcal{D}_\lambda \sigma^{\mu\nu} = e^{-4\phi}\tilde{u}^\lambda \widetilde{\mathcal{D}}_\lambda \tilde{\sigma}^{\mu\nu}\ ,\ \ etc.$$

$$\eta\tau_\pi = e^{-2\phi}\tilde{\eta}\tilde{\tau}_\pi\ ,\ \tau_\pi \propto \frac{1}{T}\ ,\ \kappa \propto T^2\ ,\ \lambda_{1,2,3} \propto T^2$$

## conformal $\Pi^{\mu\nu}$

*all* $5$ second-order terms classified by conformal symmetry

constitutive relation (d= 4):

$$\begin{aligned}\Pi^{\mu\nu} &= -2\eta\sigma^{\mu\nu} + 2\eta\tau_\pi u^\lambda \mathcal{D}_\lambda \sigma^{\mu\nu} \\ &\quad +4\lambda_1 \sigma^{<\mu}{}_\lambda \sigma^{\nu>\lambda} + 2\lambda_2 \sigma^{<\mu}{}_\lambda \Omega^{\nu>\lambda} + \lambda_3 \Omega^{<\mu}{}_\lambda \Omega^{\nu>\lambda} \\ &\quad +2\kappa u_\alpha C^{\alpha\mu\nu\beta} u_\beta\end{aligned}$$

$$\eta \propto T^3 \ , \ \tau_\pi \propto \frac{1}{T} \ , \ \kappa \propto \lambda_{1,2,3} \propto T^2$$

consistent evaluation of transport coefficients in strong coupling limit via

**AdS/CFT correspondence**

# **duality $\sim$ equality**

is between

Quantum Field Theory (a special one) in $d = 4$

and

Classical Gravity in $d = 5$ (for $N_c >> 1,\ g_{YM}^2 N_c >> 1$)

(at finite temperature)

# AdS/CFT [Maldacena 98, Gubser et al. 98, Witten 98]

strongly coupled quantized conformal gauge theory in $d = 4$ dimensions ($\mathcal{N} = 4$ SYM with $8N_c^2$ (1 gauge and 6 scalar) bosons and ($4N_c^2$ ) Weyl fermions)

[[ NOT QCD !]]

$$\Longleftrightarrow$$

weakly coupled classical supergravity (type IIB) in $d = 10$ dimensions (on $AdS_5 \otimes S^5$)

via holographic property: radial coordinate $r_0 \leq r < \infty$ with gauge theory on the boundary at $\infty$

# AdS/CFT cont.

crucial limit of interest:

't Hooft coupling $\lambda = g_{YM}^2 N_c$ is large, $N_c \to \infty, \;\; g_{YM}^2 << 1$

i.e. string coupling $g_s = \frac{g_{YM}^2}{4\pi} << 1$

– NO LOOPS

and

small curvature $\frac{l_s^4}{L^4} = \frac{1}{\lambda} << 1$ – RADIUS $L$ of CURVATURE is LARGE compared to the STRING SCALE $l_s = \sqrt{\alpha'}$

– CLASSICAL GRAVITY

## $AdS_5$ metric at finite $T$

generalization of the Einstein-Hilbert-Maxwell equation

- strong analogy to Reissner-Nordström black hole

near extremal black $D3-$ brane metric with horizon $r = r_0$ for $r_0 < r << L$, i.e. factorized metric for $AdS_5 \otimes S^5$ for:

$$ds^2_{(5)} = \frac{r^2}{L^2}(-f(r)dt^2 + d\vec{x}^2) + \frac{L^2}{r^2 f(r)} dr^2$$

with $f(r) = 1 - \frac{r_0^4}{r^4}$

from $AdS_5$ action - negative cosmological constant $\Lambda = -\frac{6}{L^2}$:

$$S = \frac{1}{16\pi G_{(5)}} \int d^5x \sqrt{-g} \, [R + \frac{12}{L^2}] \, ,$$

with Newton constant $G_{(5)} = \frac{\pi L^3}{2N_c^2}$

# master formula for AdS/CFT

schematically in terms of coinciding partition functions:

$$\int e^{iS_{4d}^{gauge}+i\Phi_0 O} = \int e^{iS_{5d}[\Phi]} \simeq e^{iS^{classical}[\Phi_0]}$$

$S_{5d}$ is computed with non-trivial boundary condition (holography)

$$\Phi(t,\vec{x},r) \overset{r->\infty}{=} \Phi_0(t,\vec{x})$$

$$\Longrightarrow$$

quantum correlation $=$ classical two-point function

$$G(x,y) = -i < TO(x)O(y) >= -\frac{\delta^2 S^{classical}}{\delta\Phi_0(x)\delta\Phi_0(y)}|_{r=\infty}$$

gauge: $O = T_{\mu\nu}$ .. energy-momentum tensor

gravity: $\phi = g_{\mu\nu}$ .. graviton

# correlators from gravity, $d = 4$

response of the fluid to small metric perturbations

e.g.: $g_{xy} = h_{xy}(t, z, r) = -h^{xy}(t, z, r) \neq 0,\ u^0 = 1,\ T =$ const

it leads from Christoffel symbols in the covariant derivatives to linear approximation:

$$\sigma^{xy} = \sigma_{xy} \approx \frac{1}{2}(\Gamma^y_{x0} + \Gamma^x_{y0}) \approx \frac{1}{2}\partial_t h_{xy}$$

and for the energy-momentum tensor

$$\delta T^{xy} \approx -p h_{xy} - \eta \partial_t h_{xy} + \eta \tau_\pi \partial_t^2 h_{xy} - \frac{\kappa}{2}[\partial_t^2 h_{xy} + \partial_z^2 h_{xy}]$$

Fourier transform $h(t, z) = \exp\left(-i\omega t + ikz\right)\, h(\omega, k)$, etc.

$$\delta T^{xy}(\omega, k) = -G_R^{xy,xy}(\omega, k) h_{xy}(\omega, k)$$

via linear response

# correlators, cont.

retarded Green function (in momentum space):

$$G_R^{xy,xy}(\omega, k) = p - i\eta\omega + \eta\tau_\pi\omega^2 - \frac{\kappa}{2}(\omega^2 + k^2) + \ldots .$$

i.e. Kubo formula

$$\eta = \lim \frac{1}{\omega}\; iG_R^{xy,xy}(\omega, \vec{0})|_{\omega=0}$$

$$= \lim \frac{1}{\omega} \int \, dt d^3x \;\exp\left(i\omega t\right)\,\Theta(t) < [T_{xy}(t,\vec{x}), T_{xy}(0,\vec{0})] > |_{\omega=0}$$

classical action $S^{classical}[\Phi_0]$ from the solution $\Phi_0$ for $h_{xy}(\omega, k, r)$ at the boundary $r \to \infty$ and keeping only the surface contribution:

$$\Rightarrow$$

$$G_R^{xy,xy}(\omega, k) = \frac{\pi N_c^2 T^3}{8}\left[\pi T - i\omega + \frac{1 - 2\ln 2}{2\pi T}\,\omega^2 - \frac{1}{2\pi T}\,k^2 \pm \ldots\ldots\right],$$

$$p = \frac{\pi^2 N_c^2 T^4}{8}\,,\ \eta = \frac{\pi N_c^2 T^3}{8}\,,\ \tau_\pi = \frac{2 - \ln 2}{2\pi T}\,,\ \kappa = \frac{\eta}{\pi T}$$

# sound mode

consistency requirement from sound channel:

$$\omega_{1,2} = \pm c_s k - i\Gamma\, k^2 \pm \frac{\Gamma}{c_s}(c_s^2 \tau_\pi - \frac{\Gamma}{2})\, k^3 + \mathcal{O}(k^4)$$

with $\Gamma = \frac{2}{3}\frac{\eta}{\epsilon+p}$ and

gravity perturbation $g_{tz} = h_{tz} << 1$ [i.e. poles in $G_R^{tz,tz}(\omega, k)$]

$$\omega_{1,2} = \pm\frac{1}{\sqrt{3}}\, k - i\frac{1}{6\pi T}\, k^2 \pm \frac{3 - 2\ln 2}{24\sqrt{3}\pi^2 T^2}\, k^3$$

gives sound velocity $c_s = \frac{1}{\sqrt{3}}$ and consistent values for

$$\frac{\eta}{s} = \frac{1}{4\pi}\,,\ \tau_\pi = \frac{2 - \ln 2}{2\pi T}$$

# RHIC physics

# hydro + RHIC

heavy-ion collisions require beyond hydrodynamics:

- hydrodynamics = differential equations
  initial conditions !
- initial = equilibration time
- distribution of energy density (Glauber? CGC ?)
- QCD equation of state
- hadronisation prescription (Cooper-Frye?)
- ……

# main results [Luzum and Romatschke 08]

- viscous hydrodynamics simulation give a good description of RHIC data for

$$\frac{\eta}{s} = 0.1 \pm 0.1(theory) \pm 0.08(experiment)$$

- modest estimate:

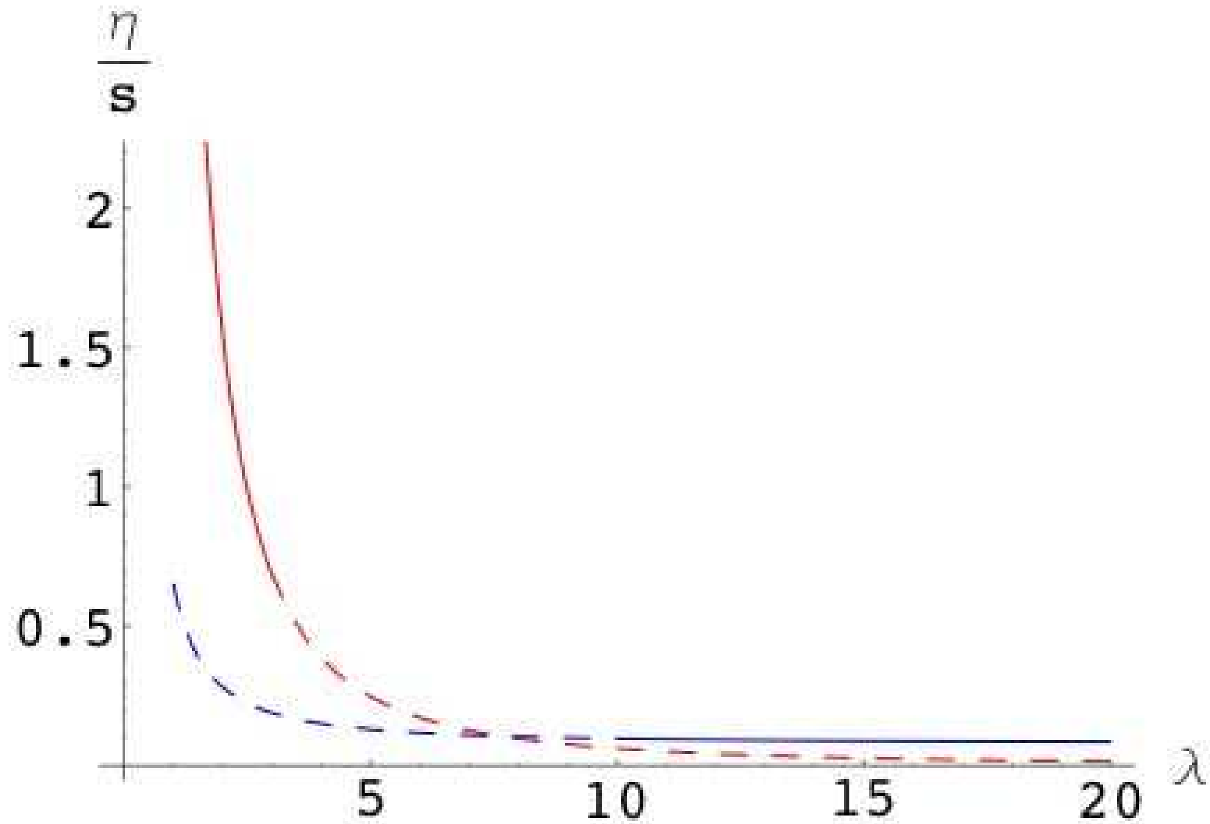
$$\frac{\eta}{s} < 0.5$$

- weak dependence on the values of the second-order parameters $\tau_\pi, \lambda_1, ..$

- early thermalisation time is questioned, but

$$\tau_0 < 2\ fm$$

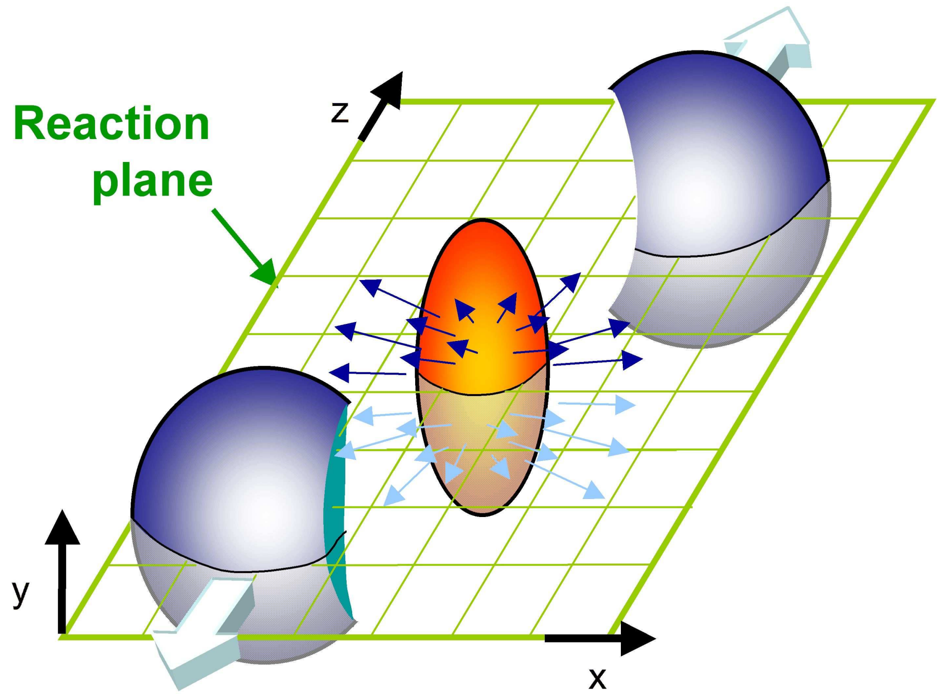
# $\eta/s$




behaviour of $\eta/s$ as a function of the 't Hooft coupling $\lambda = gYM^2N_c$

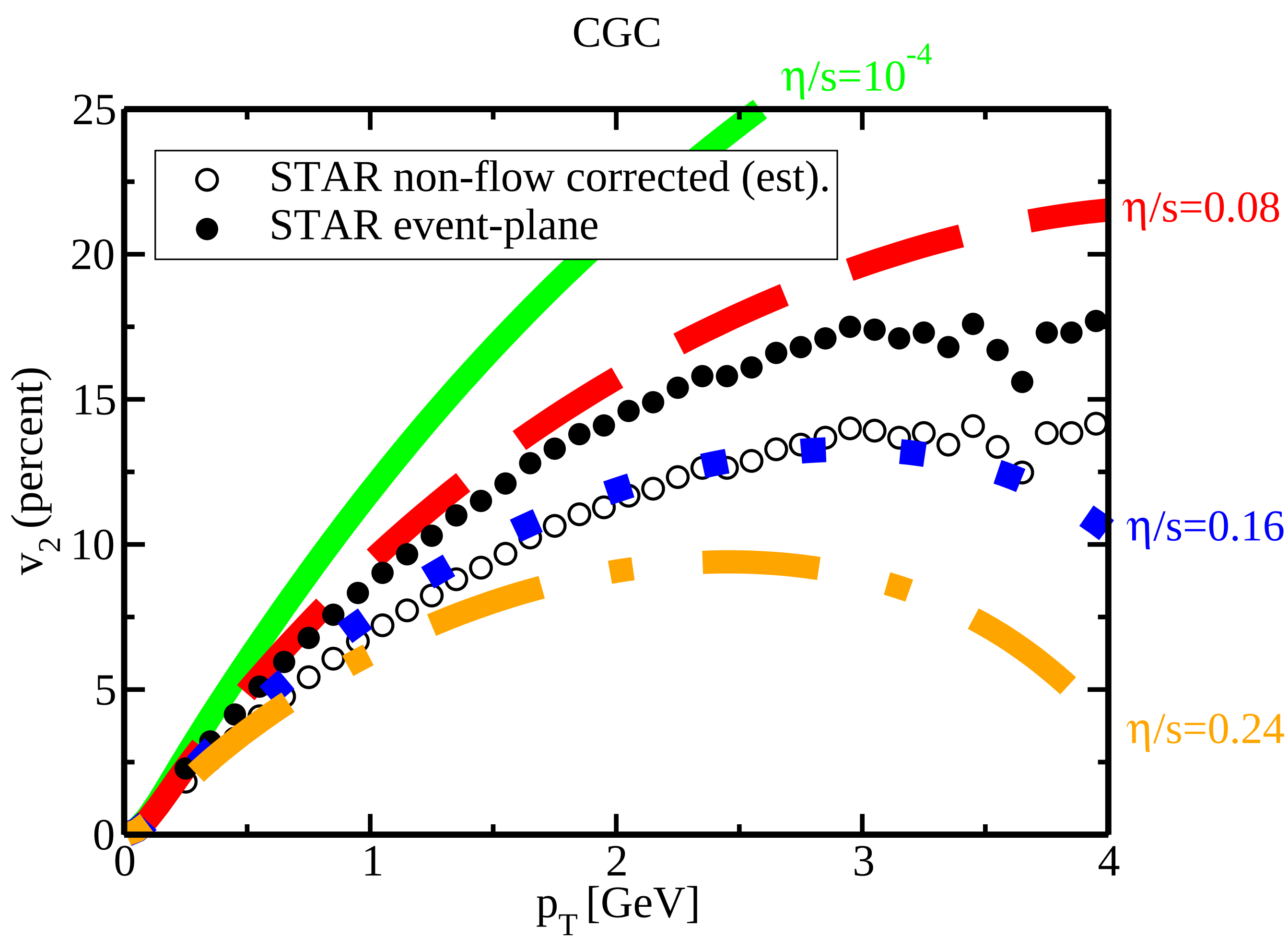
$\frac{\eta}{s} \approx 0.5:$ border-value between sQGP and pQPG

# as important example




spatial asymmetry

- $$\frac{dN}{dydp_\perp d\phi} = \langle \frac{dN}{dydp_\perp d\phi} \rangle_\phi \, (1 + 2v_2(p_\perp)\cos(2\phi) + \ldots)$$
- elliptic flow: $v_2(p_\perp)$




elliptic flow: $\eta/s > 0$ reduces $v_2$

[Luzum and Romatschke 08]

# EXTRAS

# some classical gravity

Einstein-Hilbert with nonzero cosmological constant ($d = 5$ dimensions)

- $$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = -\Lambda_{(5)} g_{\mu\nu} \ , \quad \mu, \nu = 0, 1, ...., 4$$

- ansatz:

$$ds^2 = g_{tt} dt^2 + g_{rr} dr^2 + ... \ , \quad g_{tt} = -\frac{1}{g_{rr}} = -\frac{r^2}{L^2} f(r)$$

- $$R(1 - \frac{5}{2}) = -5\Lambda_{(5)}, \quad R_{\mu\nu} = \frac{2\Lambda_{(5)}}{3} g_{\mu\nu} = -\frac{4}{L^2} g_{\mu\nu}$$

# solution

- $$R_t^t = -\frac{1}{2}\frac{[r^5 f' + 2r^4 f]'}{r^3 L^2} = -\frac{4}{L^2}$$

$$rf' + 2f = 2 + \frac{const}{r^4} \quad , \quad const = 2r_0^4$$

$$f(r) = 1 - (\frac{r_0}{r})^4$$

i.e.

$$ds^2 = -\frac{r^2}{L^2} f(r) dt^2 + \frac{L^2}{r^2 f(r)} dr^2 + \ldots$$

# Hawking temperature

black hole metric:

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + .. \, , \; g_{tt} = -\frac{1}{g_{rr}} = -[1 - (\frac{r_0}{r})^{(d-3)}]$$

expand around the horizon $r \approx r_0$:

$$g_{tt} \approx -\frac{d-3}{r_0}(r - r_0) = -\gamma_t(r - r_0) \; , \; g_{rr} \approx \frac{\gamma_r}{r - r_0}, \; \gamma_r = \frac{r_0}{d-3}$$

claim:

$$T_H = \frac{1}{4\pi}\sqrt{\frac{\gamma_t}{\gamma_r}} = \frac{d-3}{4\pi r_0}$$

and $AdS_5$: $\gamma_t = 1/\gamma_r = 4r_0/R^2$

$$T_H = \frac{r_0}{\pi R^2}$$

proof: with $\rho^2 = \frac{4r_0(r-r_0)}{d-3}$

$$ds^2 = \rho^2 (\frac{d-3}{2r_0})^2 d\tau^2 + d\rho^2 + ..., \;\; euclidean \; d\tau^2 = -dt^2$$

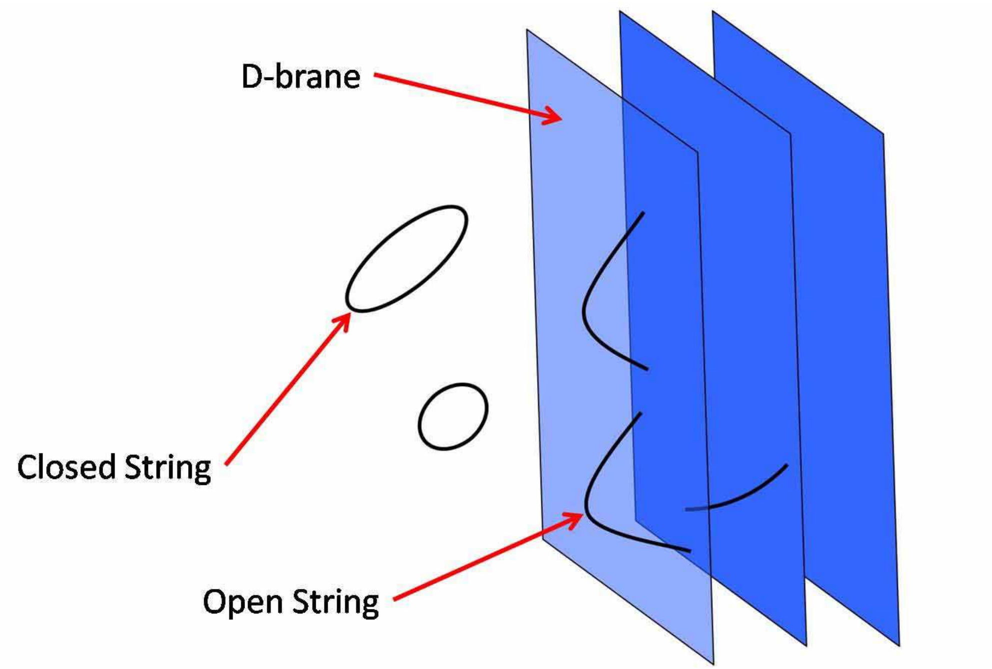
$$\longrightarrow$$

$$ds^2 = \rho^2 d\phi^2 + d\rho^2 \; , \;\; \phi = \frac{d-3}{2r_0}\tau$$

requirement: periodicity $\phi \rightarrow \phi + 2\pi$, i.e. NO conical singularity

$$2\pi = \frac{(d-3)}{2r_0}\beta \; , \;\; \beta = \frac{1}{T_H}$$

# $D-$branes




[from Myers and Vazquez 08]

dynamical walls on which strings can end:

theory of open strings living on $N_c - D3-$branes ($\mathcal{N} = 4$ SYM, $d = 4$) $\Longleftrightarrow$
gravity theory of fields living in the space curved by the branes ($AdS_5,\ d = 5$)

## symmetries $(T=0)$

duality maps the operators of QFT to the boundary values of the SUGRA fields

- QFT ($\mathcal{N}=4$, $d=4$ CFT)

$$S=-\frac{1}{g_{YM}^2}\int d^4x\,[\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}+\frac{1}{2}D_\mu\Phi^a_i D^\mu\Phi^a_i+....]$$

.... massless Weyl fermions + interactions , $i=1,..,6,a=1,...,N_c^2$

has Poincaré and conformal symmetry $SO(2,4)$:

$x^\mu\to\lambda x^\mu,\ \Phi^a_i(x)\to\lambda\Phi^a_i(\lambda x),\ F_{\mu\nu}(x)\to\lambda^2F_{\mu\nu}(\lambda x)$ etc,

$\beta(\mu)=\mathbf{0}$, and $SU(\mathcal{N}=4)$ symmetry and $SU(N_c)$

- $AdS_5\otimes S^5$:

$$SO(2,4)\otimes SO(6)=SO(2,4)\otimes SU(4)$$

besides $SU(N_c)$ due to $N_c$ parallel $D-3$ branes

# Bekenstein- Hawking entropy

thermodynamics: $dS = \frac{dE}{T}$

- Schwarzschild BH ($G = G_{(4)}, T_H = \frac{1}{4\pi r_0}$)

$$E = M = \frac{r_0}{2G} \ , \ dE = \frac{dr_0}{2G} \ , \ dS = \frac{4\pi}{2G} r_0 dr_0 \ ,$$

entropy

$$S = \frac{\pi r_0^2}{G} = \frac{A}{4G}$$

a universal relation with $A = 4\pi r_0^2$... area of the horizon

# Bekenstein-Hawking entropy, cont.

- $AdS_5$: $T_H = \frac{r_0}{\pi L^2}$ , $S = \frac{A_{(3)}}{4G_{(5)}}$

$$A_{(3)}(r = r_0) = (\frac{r_0}{L})^3 V_3 = \pi^3 T_H^3 L^3 V_3 \ , \ G_{(5)} = \frac{\pi L^3}{2N_c^2}$$

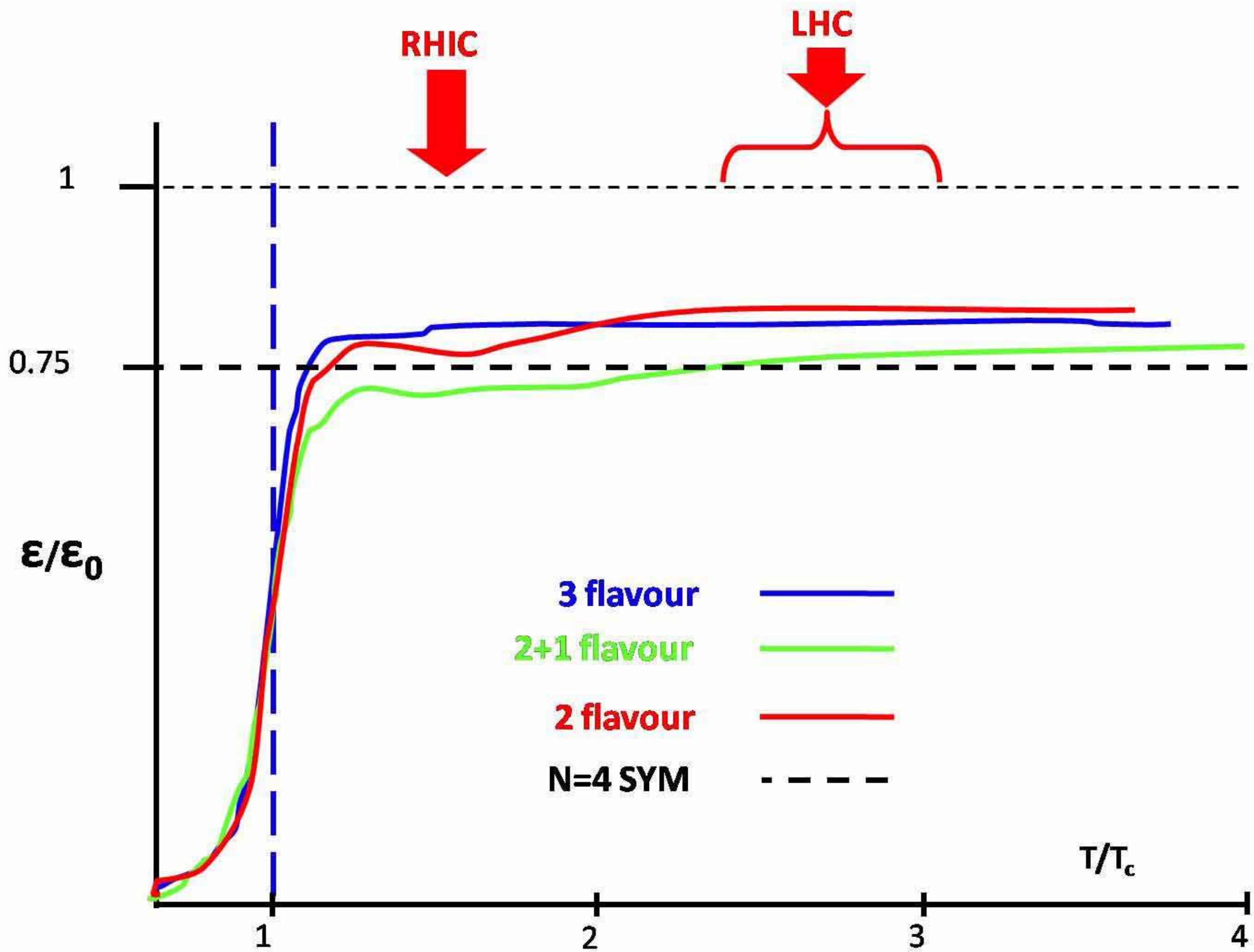
gives for the entropy density

$$s_{BH} = \frac{S}{V_3} = \frac{\pi^2 N_c^2}{2} T_H^3$$

[Gubser, Klebanov and Peet 96]

NOTE: $s_{BH} = \frac{3}{4} s_{Boltzmann}$

$$s_{Boltzmann} = \frac{4p}{T} = \frac{4\pi^2 T^3}{90}[\frac{7}{4} \cdot 4 + 2 + 6] N_c^2 = \frac{2\pi^2 N_c^2}{3} T^3$$




energy density of QCD and SYM - via BH entropy

[from Myers and Vazquez 08]

# quasinormal mode: some details

$G_R$ from gravity action with $AdS_5$ metric:

$$S_{(5d)} = \frac{N_c^2}{8\pi^2 L^3} \int d^5x \, [\sqrt{-g} R_{(5d)} + ..] \, ,$$

weak field limit: scalar mode and EoM (of Heun's type)

$$\sqrt{-g} R_{(5d)} \to -\frac{1}{2} \sqrt{-g} g^{\mu\nu} \, \partial_\mu \Phi \partial_\nu \Phi \, , \; \Phi \equiv h_{xy} \; :$$

$$\Phi(t, z, u = \tfrac{r_0^2}{r^2}) = \exp(-i\omega t + ikz) \Phi_k(u) \, , \, \Phi_k' = \tfrac{d}{du} \Phi_k \, ,$$

$$\Phi_k'' - \frac{1+u^2}{u(1-u^2)} \Phi_k' + \frac{1}{4\pi^2 T^2} \, \frac{\omega^2 - k^2(1-u^2)}{u(1-u^2)^2} \Phi_k(u) = 0$$

# quasinormal mode, cont.

solution with ansatz $\Phi_k(u) = f_k(u)\Phi_0(k)\ ,\ f_k(0) = 1$ and boundary condition: only the incoming wave, the one which moves toward the horizon at $r \to r_0 \equiv u = 1$, i.e. nothing comes out the horizon

$$f_k(u) = (1-u^2)^{-i\omega/(4\pi T)} F_k(u) \approx 1 + i\frac{\omega}{4\pi T}u^2 + \dots$$

note, near $u \sim 1$:

$$\exp\left(-i\omega t\right) f_k(u) \to \exp\left[-i\omega(t+r^*)\right]\ ,\ r^* = \frac{\ln(1-u)}{4\pi T}$$

moves from $u = 0$ at $t = 0$ to $u = 1$ at $t \to \infty$

## classical gravity action

inserting the solution into $S_{(5d)}$

$$S_{(5d)} \approx \frac{N_c^2}{8\pi^2 L^3} \int d^4x \int_0^1 du \sqrt{-g}(-\frac{1}{2}) g^{uu} \partial_u \Phi \partial_u \Phi$$

and keeping only the surface contribution at $u = 0 \quad \Rightarrow$

$$S^{classical} = -\frac{\pi^2 N_c^2 T^4}{8} \int \frac{d^4k}{(2\pi)^4} \Phi_0(-k) [\frac{(1-u^2)}{u} f_k'(u) f_k(u)]|_{u=0} \Phi_0(k)$$

and finally (including a counter term) $\quad \Rightarrow$

$$G_R^{xy,xy}(\omega, k)$$

# Bjorken flow

boost-invariant (irrotational) $1+1$ flow [Bjorken 83]
second-order equations (proper time $\tau$, $\Phi$... viscous flow):

$$\partial_\tau \epsilon = -\frac{4\epsilon}{3\tau} + \frac{\Phi}{\tau}$$

$$\tau_\pi \partial_\tau \Phi = \frac{4\eta}{3\tau} - \Phi - \frac{4\tau_\pi}{3\tau}\Phi - \frac{\lambda_1}{2\eta^2}\Phi^2$$

non-linear term NOT in MIS theory! [BRSSS 07]

compare with AdS/CFT calculation:

$$\frac{\lambda_1 T}{\eta} = \frac{1}{2\pi}\left[1 + \frac{215\,\zeta(3)}{8}\lambda^{-3/2} + \ldots\right]$$

[Janik, Peschanski, Heller 06; Buchel, Paulos 08]

# COMPARISON

| | QCD | $\mathcal{N}$=4 SYM |
|---|---|---|
| T=0 | $N_c$=3=$N_f$ , confinement, discrete spectrum, scattering, . . . . | $N_c$ large, $N_f/N_c$ small, deconfined, conformal, supersymmetric, . . . . |

**very different !!**

| | QCD | $\mathcal{N}$=4 SYM |
|---|---|---|
| $T>T_C$ | strongly-coupled plasma of gluons & fundamental matter<br>deconfined, screening, finite corr. lengths, . . . | strongly-coupled plasma of gluons & adjoint and fundamental matter<br>deconfined, screening, finite corr. lengths, . . . |

**very similar !!**

| | QCD | $\mathcal{N}$=4 SYM |
|---|---|---|
| $T>>T_C$ | runs to weak coupling | remains strongly-coupled |

**very different !!**

QCD and $\mathcal{N} = 4\ SYM$ as a function of temperature

[from Myers and Vazquez 08]

# parametric pQCD estimate

for thermalisation in an expanding gluonic medium

near equilibrium at $T(\tau)$: Knudsen number *Kn*

$$\frac{1}{Kn} = \frac{longitudinal\ expansion\ time}{mean\ free\ path} = \frac{\tau}{\lambda_f} \, (\sim\ 1/\frac{\eta}{s}) \ >> 1$$

from many gluon interactions (including saturation):

Arnold et al.: $\tau_{eq} Q_s \geq \alpha_s^{-7/3}$

'bottom-up' [Baier, Son, Mueller and Schiff 01]

$$\tau_{eq} Q_s \geq \alpha_s^{-13/5}$$

RHIC: $\tau_{eq} \geq 2 - 3$ fm