Induced surface tension EoS for nuclear and hadronic matter and quantum virial coefficients

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Outline of lecture II

1. Basics on the hard-core repulsion

2. Heuristic derivation of the IST EoS for Boltzmann statistics

3. ALICE data fit with IST EoS

4. Quantum IST EoS for Nuclear Matter

5. Virial coefficient for Quantum VdWaals and Quantum IST EoS

6. Summary

Origin of Hard-core Repulsion

Hard-core repulsion is observed at short distances among ALL composite particles which consist from fermions: atoms, nuclei, hadrons etc

For noble gases the potential behaves as

$$U(r) \simeq \frac{1}{(r - R_{core})^k}\,, \quad \text{with} \quad k \in [28; 32]$$

Hence hard-core repulsion is a very good approximation!

Its origin is due to Pauli blocking among the identical fermions interior of composite particles!

For nuclei (or hadrons) it is hard to measure the power, but physics is similar and, hence, we can use such an approximation!

Virial Expansion for Classical Hard Spheres Interaction

Virial expansion for one-component Boltzmann gas

change of free energy F due to interaction (β=1/T): $e^{\beta(F_{id}-F)} = V^{-N} \int d^3r_1 \dots d^3r_N \exp\left(-\beta \sum_{i<j} u_{ij}\right)$

$\exp\left(-\beta \sum_{i<j} u_{ij}\right) = \prod_{i<j}(1-f_{ij})$ Mayer function: $f_{ij} = 1 - e^{-\beta u(r_{ij})} \underset{\text{HSI}}{\rightarrow} \Theta(2R - r_{ij})$

(no energy scale in HSI)

⇒ virial expansion of EoS (H.K. Onnes, 1901)

compressibility function: $Z \equiv \frac{P}{P_{id}} = 1 + \sum_{i=1}^{\infty} B_{i+1}(T) n^i \qquad (P_{id} = nT)$

virial coefficients:

HSI: $B_i(T) = \text{const} \propto v^{i-1} \rightarrow Z = Z(\eta),\ \eta = nv$

$B_2 = \frac{1}{2V} \int d^3r_1 d^3r_2 f_{12} \underset{\text{HSI}}{\rightarrow} b = 4v$ - contribution of binary interactions $(v = 4\pi R^3/3)$

$B_3 = \frac{1}{3V} \int d^3r_1 d^3r_2 d^3r_3 f_{12} f_{23} f_{31} \underset{\text{HSI}}{\rightarrow} 10v^2$ - contribution of three particle interactions

Monte Carlo calculation for HSI (van Rensburg, 1993):

$Z = 1 + 4\eta + 10\eta^2 + 18.36\eta^3 + 28.23\eta^4 + 39.74\eta^5 + 53.5\eta^6 + 70.8\eta^7 + \dots$

4 10 18 28 40 54 70

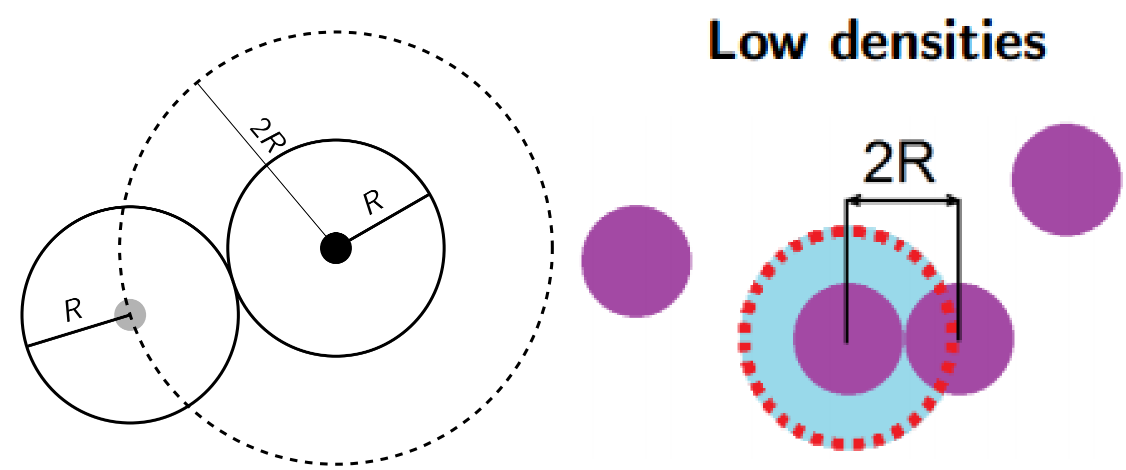
only first two terms are correctly reproduced in the EVM: Z=1/(1-4η)

red numbers - coefficients in the Carnahan-Starling approximation (CSA): J. Chem. Phys. 51 (1969) 635, decomposed in powers of η

this expansion works well at $\eta \lesssim 0.5$

Van der Waals EoS with Hard-core repulsion

Excluded Volume



$$V_{excluded} = \frac{2\pi}{3}(2R)^3$$

b is excluded volume per particle,

V_0 is eigenvolume (proper volume)

VdWaals pressure correctly reproduces 2-nd virial coefficient only:

$$p = \frac{TN}{V - bN} = Tn\left[1 + 4V_0 n + (4V_0 n)^2 + (4V_0 n)^3 + \ldots\right]$$

$$= Tn\left[1 + 4V_0 n + 16(V_0 n)^2 + 64(V_0 n)^3 + \ldots\right]$$

$n = \frac{N}{V}$ **is particle number density**

V >> bN => ideal gas pressure p =T N/V

V => bN => pressure diverges, i.e. there is dense packing

VdWaals is applicable at low densities only, at high densities it is too stiff!

Maximum Term Method

Let's find VdWaals EOS with repulsion in GC Ensemble:

$$Z_{\rm can}(T,V,N) = \frac{\phi^N}{N!}\,(V-bN)^N \quad \Rightarrow \quad Z_{\rm gce}(\mu,T,V) = \sum_{n=0}\frac{\phi^N}{N!}\,(V-bN)^N\, e^{\frac{\mu N}{T}}$$

ϕ (T) is thermal density

Statement:

For $V \to \infty$ the GCE pressure is determined by a single term $N = N^*$ in GCE partition

Proof: for finite V the number of terms in GCE partition is finite and all are nonnegative

Evidently, there is maximal term $N = N^*$: $\frac{\partial}{\partial N}\ln\left[\frac{\phi^N}{N!}\,(V-bN)^N\, e^{\frac{\mu N}{T}}\right] = 0 \quad \Rightarrow$

$$\boxed{\xi \;=\; \phi\, e^{\frac{\mu}{T}-b\xi}} \quad \text{with} \quad \xi \equiv \frac{N^*}{V-bN^*} \quad \Rightarrow \quad Z_{can}(T,V,N^*) = e^{N^*(1+b\xi)} \quad \Rightarrow$$

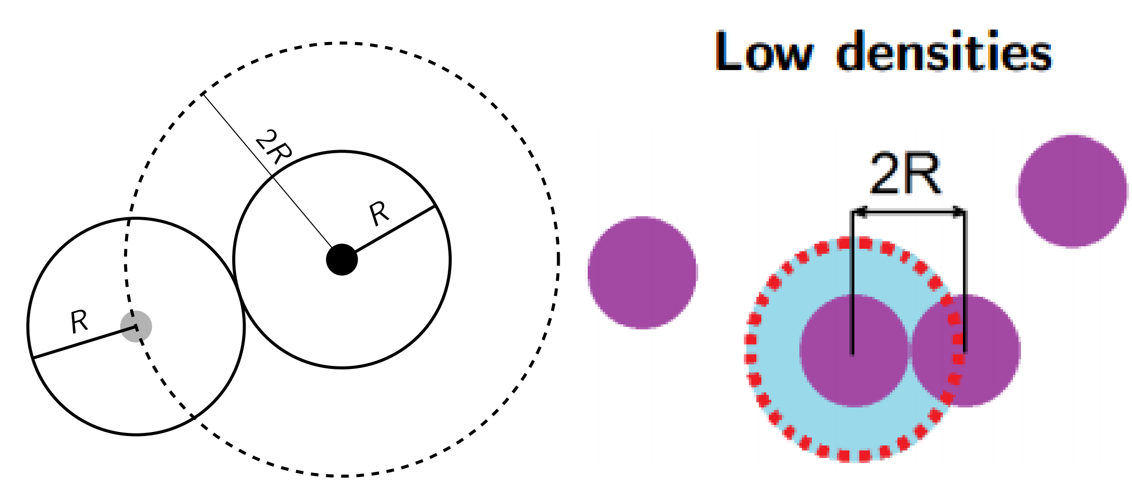
$$p_{can} \;=\; \lim_{V\to\infty}\frac{T\,N^*}{V}(1+b\xi) = \xi\,T \;=\; \frac{T\,N^*}{V-bN^*}$$

b is excluded volume per particle,

V_0 is eigenvolume (proper volume)

$$b \;=\; \frac{1}{2}\cdot\frac{4}{3}\pi(2R)^3 = 4\frac{4}{3}\pi(R)^3 = 4\,V_0$$

Excluded Volume



$V_{excluded} = \frac{2\pi}{3}(2R)^3$

Homework: please derive p_can

VdWaals EOS in GCEnsemble

Evidently: $Z_{can}(T, V, N^*) \quad < \quad Z_{\rm gce}(\mu, T, V) \quad < \quad N^* \; Z_{can}(T, V, N^*)$

Take ln, get pressure and find limit $V \to \infty$ and $\xi = const$

$$p_{can}(T, V, N^*) < p_{gce}(T, \mu) < T \frac{\ln N^*}{V} + p_{can}(T, V, N^*) = p_{can}(T, V, N^*)$$

In fact, we showed that both ensembles are EQUIVALENT!

$$p(T, \mu) = T\, \phi(T)\, e^{\frac{\mu - bp}{T}} = e^{\frac{\mu - bp}{T}} T \int \frac{d^3 k}{(2\,\pi)^3} e^{-\frac{\sqrt{k^2 + m^2}}{T}}$$

This is the VdWaals gas pressure for Boltzmann particles in GCE

«Derivation» of Van der Waals EoS from Virial Expansion

Van der Waals EoS cannot be derived! It is a postulate.

Let's derive it in three steps:

Consider first the virial (cluster) expansion in GCE:

ideal gas pressure

$$p = T\,\phi\,e^{\frac{\mu - b\,p}{T}} \simeq T\,\phi\,e^{\frac{\mu}{T}}\left(1 - \frac{b\,p}{T} + ...\right) \simeq T\,\phi\,e^{\frac{\mu}{T}}\left(1 - b\,\phi\,e^{\frac{\mu}{T}} + ...\right)$$

expand exponential — substitute ideal gas pressure

Step No 1: start from this expansion and replace density by p/T

$$p = T\,\phi\,e^{\frac{\mu}{T}}\left(1 - b\,\phi\,e^{\frac{\mu}{T}} + ...\right) \simeq T\,\phi\,e^{\frac{\mu}{T}}\left(1 - \frac{b\,p}{T}\right)$$

G. Zeeb, K. A. Bugaev, P. T. Reuter and H. Stoecker, Ukr. J. Phys. 53 (2008) 279

Step No 2: move b p/T into exponential

$$p \simeq T\,\phi\,e^{\frac{\mu}{T}}\left(1 - \frac{b\,p}{T}\right) = T\,\phi\,e^{\frac{\mu - b\,p}{T}}$$

Step No 3: extrapolate this EoS to all densities

Source of Surface Tension

Pressure of N-sorts particles with hard core radii R_k up to 2-nd virial coefficient

$$p(T,\mu) \approx T \sum_{k=1}^{N} \phi_k e^{\frac{\mu_k}{T}} \left(1 - \sum_{n=1}^{N} a_{kn}\, \phi_n\, e^{\frac{\mu_n}{T}}\right), \quad \phi_n(T) \quad \text{is thermal particle density}$$

density

a_{kn} is the 2-nd virial coefficient between hard core radii R_k and R_n

$$a_{kn} = \frac{2}{3\pi}(R_k + R_n)^3 = \frac{2}{3\pi}(R_k^3 + 3R_k^2 R_n + 3R_k R_n^2 + R_n^3)$$

Usual VdWaals approximation: the pressure is extrapolated to high density as

$$p = \sum_{k=1}^{N} p_k \approx T \sum_{k=1}^{N} \phi_k e^{\frac{\mu_k}{T}} \left(1 - \sum_{n=1}^{N} a_{kn} \frac{p_n}{T}\right) \approx T \sum_{k=1}^{N} \phi_k \exp\left[\frac{\mu_k}{T} - \sum_{n=1}^{N} a_{kn} \frac{p_n}{T}\right]$$ GCensemble

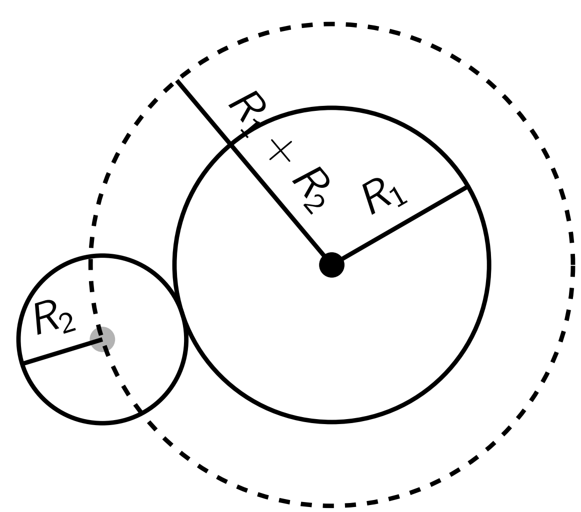
~pressure/T

p_n is partial pressure

Multi-component

V. V. Sagun, A. I. Ivanytskyi, K. A. Bugaev and I. N. Mishustin, Nucl. Phys. A 2014, 924, 24

But this procedure is not unique!



Multiple Boltzmann particle species

$\frac{p}{T} = n + a_2 n^2 + a_3 n^3 + \ldots$

second virial coefficient:

$a_2^{ij} = \frac{2\pi}{3}(R_i + R_j)^3$

$$V_{ij}^{exc} = \frac{2\pi}{3}(R_i + R_j)^3$$

Source of Surface Tension

Pressure of N-sorts particles with hard core radii R_k up to 2-nd virial coefficient

$$p(T,\mu) \approx T \sum_{k=1}^{N} \phi_k e^{\frac{\mu_k}{T}} \left(1 - \sum_{n=1}^{N} a_{kn}\, \underbrace{\phi_n\, e^{\frac{\mu_n}{T}}}_{\text{density}}\right), \quad \phi_n(T) \quad \text{is thermal particle density}$$

a_{kn} is the 2-nd virial coefficient between hard core radii R_k and R_n

$$a_{kn} = \frac{2}{3\pi}(R_k + R_n)^3 = \frac{2}{3\pi}(R_k^3 + 3R_k^2 R_n + 3R_k R_n^2 + R_n^3)$$

V. V. Sagun, A. I. Ivanytskyi, K. A. Bugaev and I. N. Mishustin, Nucl. Phys. A 2014, 924, 24

But it is not unique procedure! Substituting a_{nk} and regrouping terms we have

$$p \approx T \sum_{k=1}^{N} \phi_k e^{\frac{\mu_k}{T}} \left[1 - \frac{4}{3}\pi R_k^3 \cdot \underbrace{\sum_{n=1}^{N} \phi_n e^{\frac{\mu_n}{T}}}_{\sim \text{pressure}/T} - 4\pi R_k^2 \cdot \sum_{n=1}^{N} R_n \phi_n e^{\frac{\mu_n}{T}}\right]$$

$$= T \sum_{k=1}^{N} \phi_k e^{\frac{\mu_k}{T}} \left[1 - \frac{4}{3}\pi R_k^3 \cdot \frac{p}{T} - 4\pi R_k^2 \cdot \frac{\Sigma}{T}\right] \simeq T \sum_{k=1}^{N} \phi_k e^{\frac{\mu_k}{T}} \exp\left[\underbrace{-\frac{4}{3}\pi R_k^3 \cdot \frac{p}{T}}_{volume\ part} \underbrace{- 4\pi R_k^2 \cdot \frac{\Sigma}{T}}_{surface\ part}\right]$$

with $$\Sigma(T,\mu) = T \sum_{k=1}^{N} R_k \phi_k e^{\frac{\mu_k}{T}} \exp\left[\underbrace{-\frac{4}{3}\pi R_k^3 \cdot \frac{p}{T}}_{volume\ part} \underbrace{- \alpha 4\pi R_k^2 \cdot \frac{\Sigma}{T}}_{surface\ part}\right]$$

α is important!

Induced Surface Tension EOS (2017)

pressure

$$\frac{p}{T} = \sum_i \phi_i \exp\left(\frac{\mu_i - pV_i - \Sigma S_i}{T}\right)$$

induced surface tension

$$\frac{\Sigma}{T} = \sum_i R_i \phi_i \exp\left(\frac{\mu_i - pV_i - \Sigma S_i}{T}\right) \cdot \underbrace{\exp\left(\frac{(1-\alpha) S_i \Sigma}{T}\right)}_{\text{new term}}$$

new term

V_k and S_k are eigenvolume and eigensurface of hadron of sort k

- One component case with $\alpha > 1$

$$\Sigma = pR \exp\left(\frac{(1-\alpha)S\Sigma}{T}\right)$$
$$p = T\phi \exp\left(\frac{\mu - pV_eff}{T}\right)$$
$$V_{eff} = V_o\left[1 + 3\exp\left(\frac{(1-\alpha)S_i\Sigma}{T}\right)\right]$$

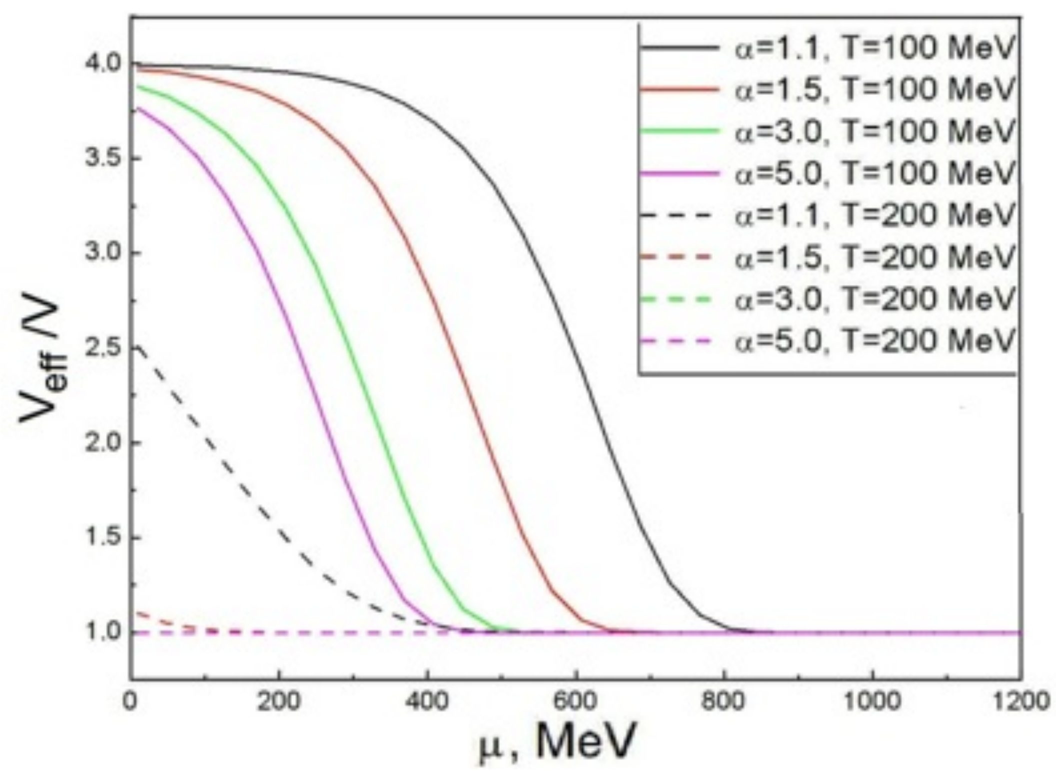
$\Rightarrow$

α switches excluded and eigen volume regimes

high order virial coefficients?

low densities $(\Sigma \to 0)$: $V_{eff} = 4V_o$

high densities $(\Sigma \to \infty)$: $V_{eff} = V_o$



Advantages

1. Allows to go beyond the Van der Waals approximation

2. Number of equations is 2 and it does not depend on the number different hard-core radii!

Higher Virial Coefficients of IST EOS

- Virial expansion of one component EoS with induced surface tension

$$p = nT\Big[1 + \overbrace{4V}^{a_2} n + \overbrace{\Big(16 - 18(\alpha-1)\Big)}^{a_3} V^2 n^2 + \underbrace{\Big(64 - 216(\alpha-1) + \frac{243}{2}(\alpha-1)^2\Big)}_{a_4} V^3 n^3\Big] + \mathcal{O}(n^5)$$

- Second virial coefficient of hard spheres $a_2 = 4V$ is reproduced always
- Fourth virial coefficient of hard spheres

$a_4 \simeq 18.365 V^3 \;\Rightarrow\; \alpha \simeq 2.537,\; a_3 \simeq -11.666 V^2$ - not reproduced

$\alpha \simeq 1.245,\; a_3 \simeq 11.59 V^2$ - reproduced with 16 % accuracy

One parameter reproduces two (3rd and 4th) virial coefficients and allows generalization for multicomponent case

=> IST EoS is valid for packing fractions $\eta < 0.22$

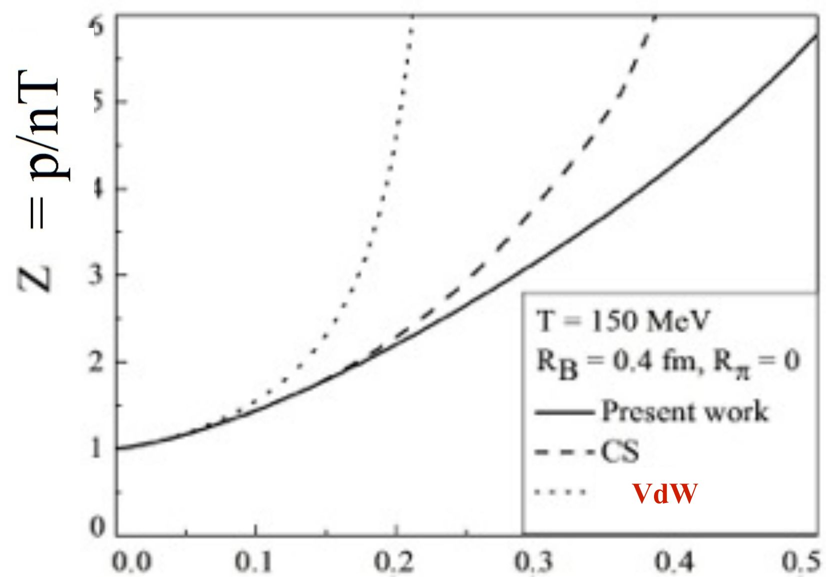
V.V. Sagun, K.A.Bugaev, A.I. Ivanytskyi, D.R. Oliinychenko, EPJ Web Conf 137 (2017);

K.A.Bugaev, V.V. Sagun, A.I. Ivanytskyi, E. G. Nikonov, G.M. Zinovjev et. al., Nucl. Phys. A 970 (2018) 133-155

Comparison with Carnahan-Straling EOS

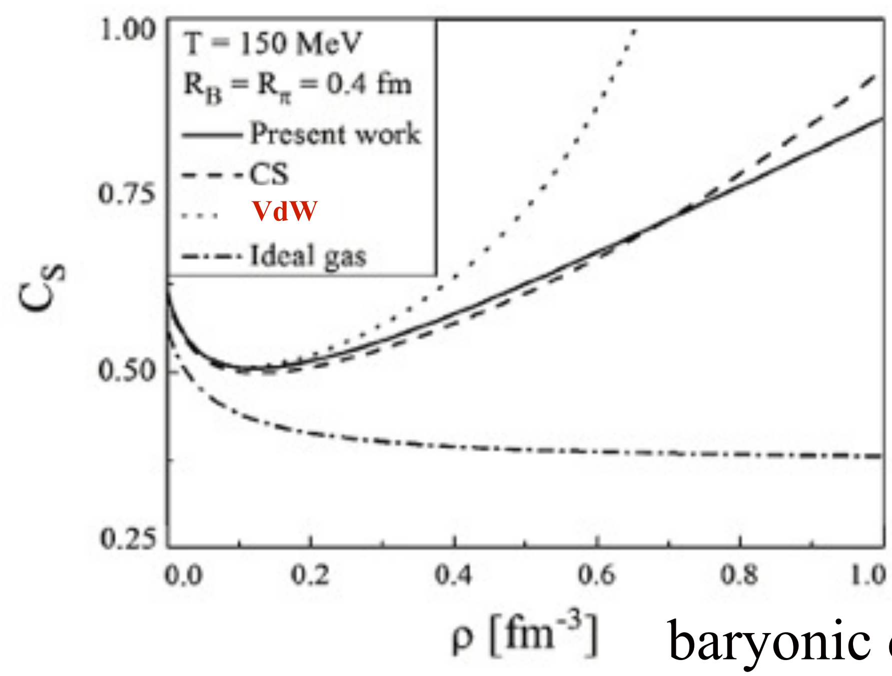
IST EOS with $\alpha = 1.25$ vs **Two component CS EOS: point-like pions and nucleons and Δ-isobar with finite hard-core radius**

Compressibility for 2-component EOS



packing fractions $\eta = \sum_{all\ \ hadrons} \rho_h V_h$

Speed of sound for 1-component EOS



baryonic density

$$c_s^2 = \left(\frac{\partial p}{\partial \epsilon}\right)_{s/n}$$

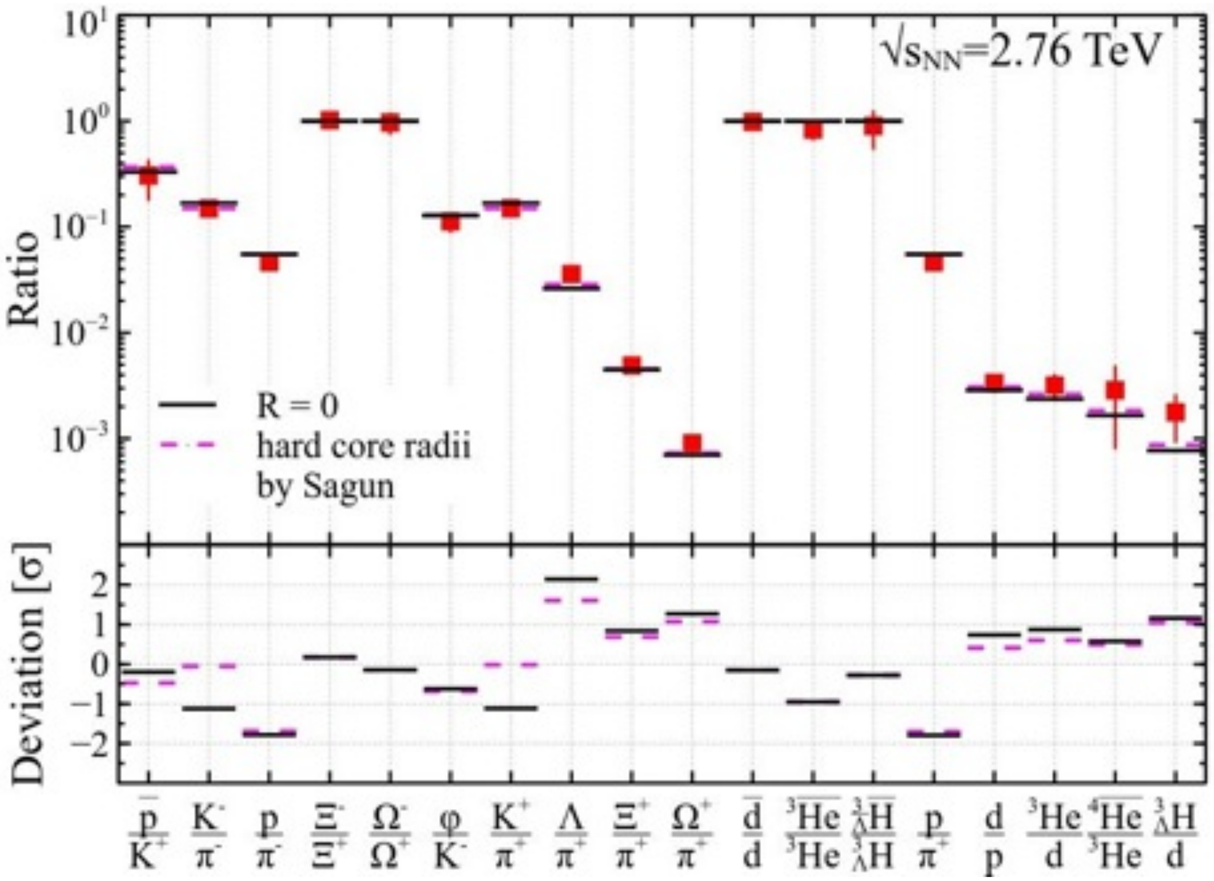
Carnahan-Starling EoS (reproduces 7 virial coefficients): $Z = \frac{1+\eta+\eta^2-\eta^3}{(1-\eta)^3}$

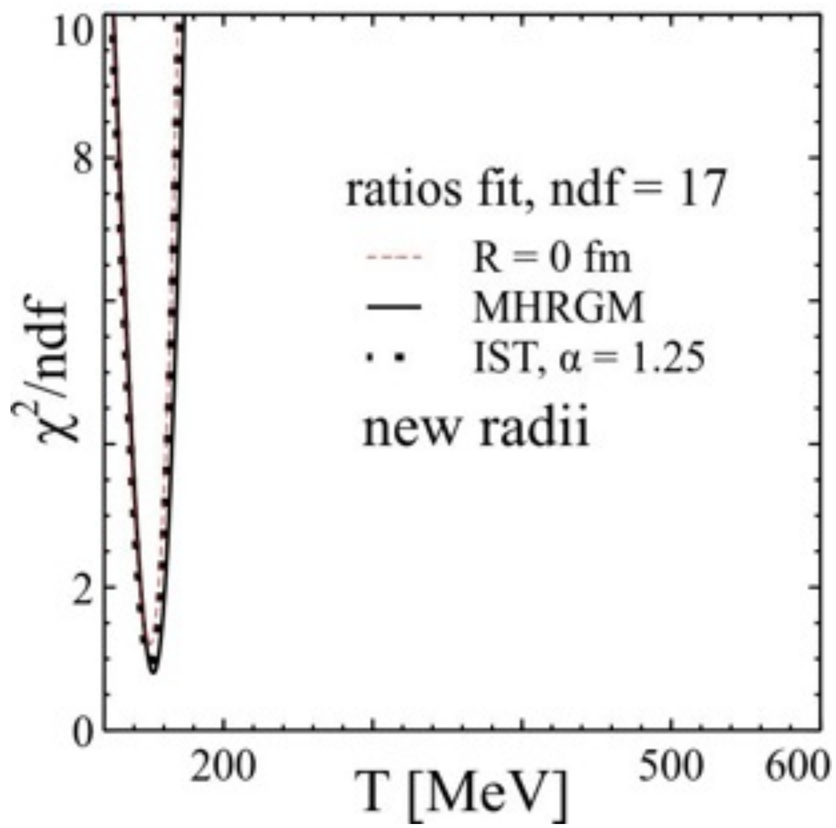
- **reproduced up to** $\eta \simeq 0.22$ **, but IST EOS is softer at higher packing fractions** =>

IST EOS is causal at very high densities (up to 7 normal nuclear densities) at which the Quark-Gluon Plasma is expected

HRGM Results for LHC energy

Light (anti)nuclei are included into fit





Ideal gas

$T_{CFO} \simeq 151 \pm 7$ MeV

$\chi^2/dof \simeq 17/17 \simeq 1$

Conventional HRGM:

V. V. Sagun, Ukr. J. Phys. **59**, 755 (2014)

R_π=**0.10 fm**, R_K=**0.395 fm**, R_Λ=**0.11 fm**, R_b=**0.355 fm**, R_m=**0.40 fm**

$T_{CFO} \simeq 153 \pm 7$ MeV

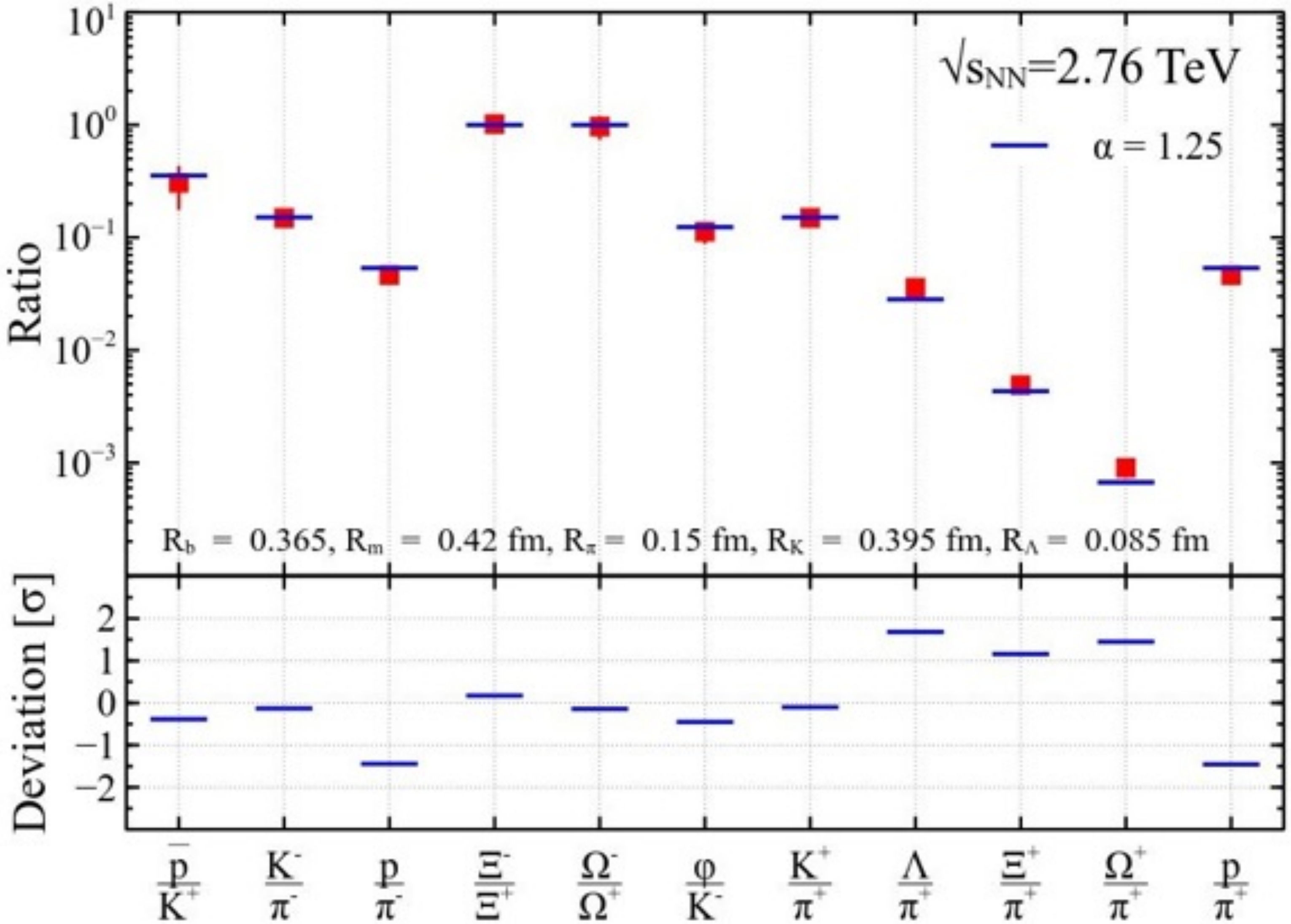
χ^2/dof = 13.58/17 =0.8 !

Similar to J. Stachel, A. Andronic, P. Braun-Munzinger and K. Redlich, J. Phys. Conf. Ser. 509, 012019 (2014) the (anti)nuclei have the same hard-core radius as baryons!

Compare J. Stachel et al. fit quality for Tcfo = 156 MeV χ^2/dof = 2.4 with our one!

IST EOS Results for LHC energy

Light (anti)nuclei are NOT included into fit



χ^2/dof = 9.1/10 =0.91 !

In all our fits (anti)protons and (anti)Ξ-s do not show any anomaly compared to J. Stachel et.al. fit, since we have right physics!

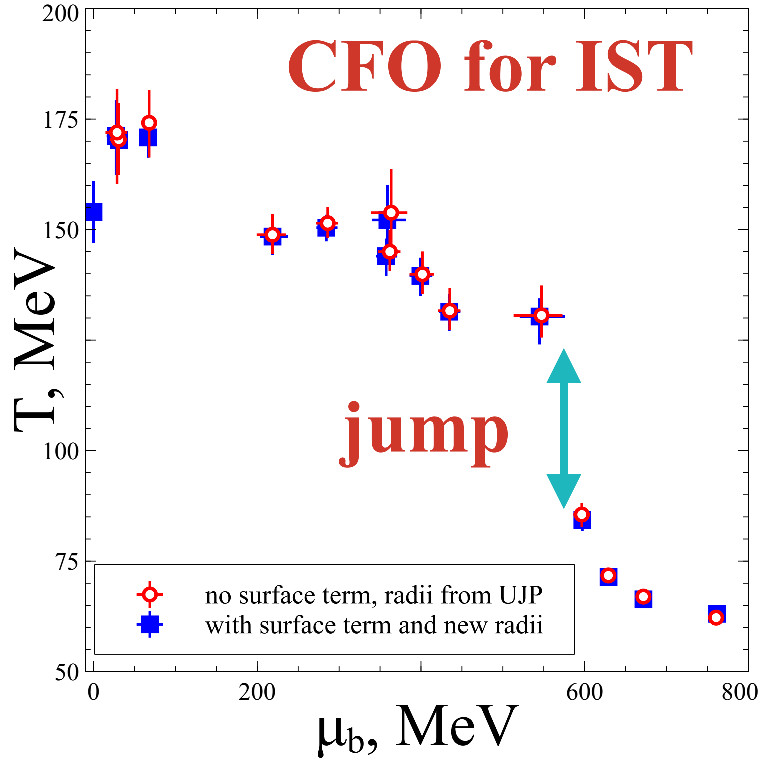
In contrast to J. Stachel, A. Andronic, P. Braun-Munzinger and K. Redlich, J. Phys. Conf. Ser. 509, 012019 (2014) (anti)nuclei are NOT included into the fit!

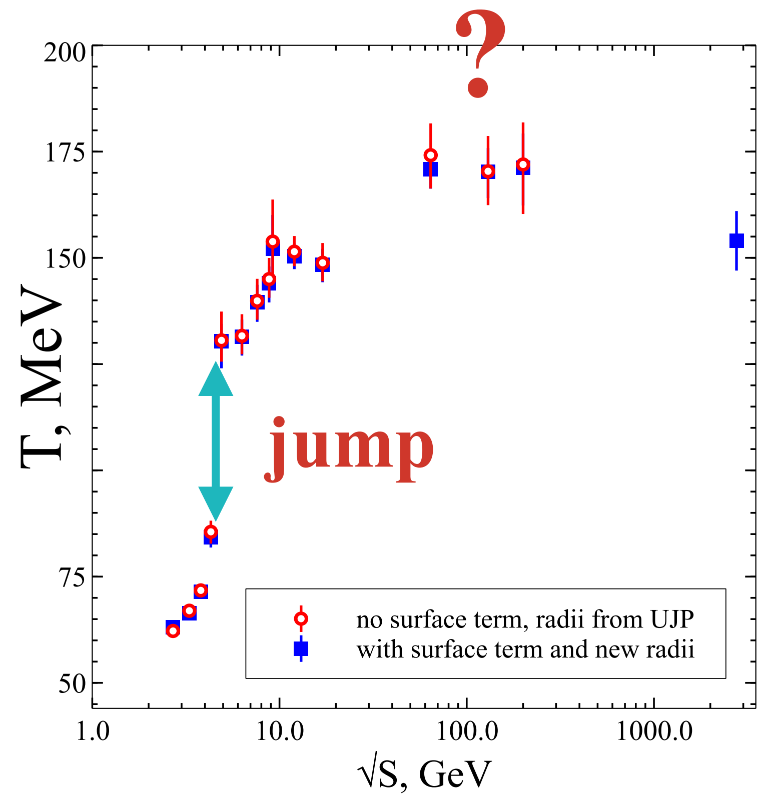
Radii are taken from the fit of AGS, SPS and RHIC data

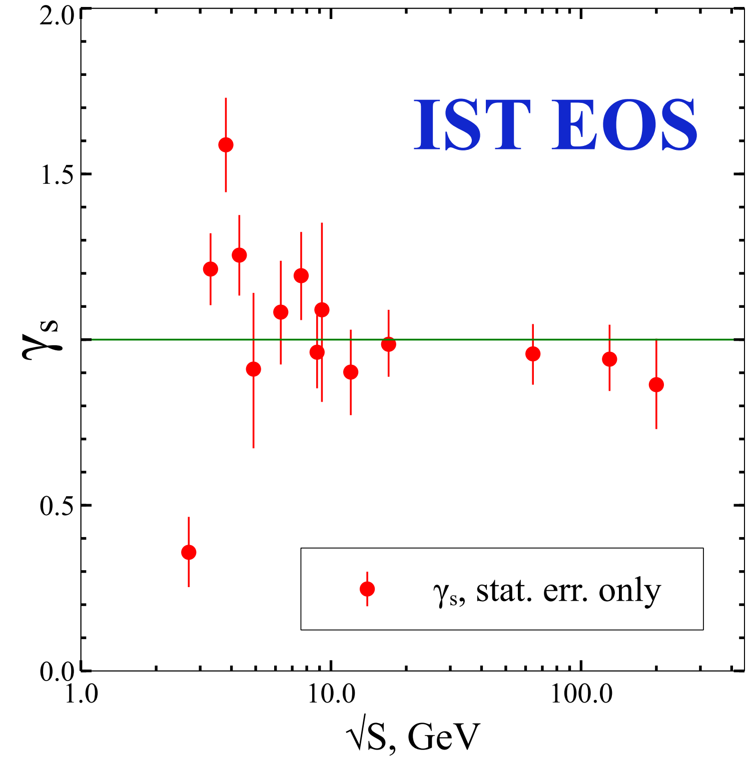
Combined fit of AGS, SPS, RHIC and LHC data $\chi^2_{tot}/dof \simeq 64.8/60 \simeq 1.08$

BUT the puzzle of light (anti)nuclei remains unresolved!

Main Results for AGS, SPS and RHIC energies







IST EOS (without ALICE):

$$R_\pi = 0.15 \text{ fm}, \quad R_K = 0.395 \text{ fm}, \quad R_\Lambda = 0.085 \text{ fm}, \quad R_b = 0.365 \text{ fm}, \quad R_m = 0.42 \text{ fm}$$

Only pion and Λ hyperon radii are changed, but no effect on T and μ_B

1. **We confirm that there is a jump of T_{CFO} between √s = 4.3 GeV and √s = 4.9 GeV**
2. **We confirm that there is a strangeness enhancement peak at √s = 3.8 GeV**

V.V. Sagun et al., NPA (2018) and arXiv:1703.00009 [hep-ph]

Quantum IST EOS for Nuclear Matter

A. I. Ivanytskyi, K. A. Bugaev, V. V. Sagun, L.V. Bravina and E. E. Zabrodin, PRC (2018)

$$p = p_{id}(T,\nu_p) - p_{int}\left(n_{id}(T,\nu_p)\right),$$ total pressure

$$\Sigma = R\, p_{id}(T,\nu_\Sigma),$$ IST coefficient

$$p_{id}(T,\nu) = Tg\int\frac{d^3p}{(2\pi)^3}\ln\left[1+\exp\left(\frac{\nu-\sqrt{p^2+m^2}}{T}\right)\right]$$ ideal gas pressure of nucleons

$$\nu_p = \mu - pV_0 - \Sigma S_0 + U\left(n_{id}(T,\nu_p)\right),$$ eff. chemical potentials

$$\nu_\Sigma = \mu - pV_0 - \alpha\Sigma S_0 + U_0,$$ in non-relativ. case U_0 shifts the mass

Vo, So are, respectively, eigenvolume and eigensurface of nucleon => R

$$n_{id}(T,\nu) = \frac{\partial p_{id}}{\partial\nu} = g\int\frac{d^3p}{(2\pi)^3}\frac{1}{\exp\left(\frac{\sqrt{p^2+m^2}-\nu}{T}\right)+1}.$$ ideal gas density of nucleons

$$U(n) = C_d^2 n^\kappa \quad\Rightarrow\quad p_{int}(n) = \frac{\kappa}{\kappa+1}C_d^2 n^{\kappa+1},$$ interaction pressure of nucleons

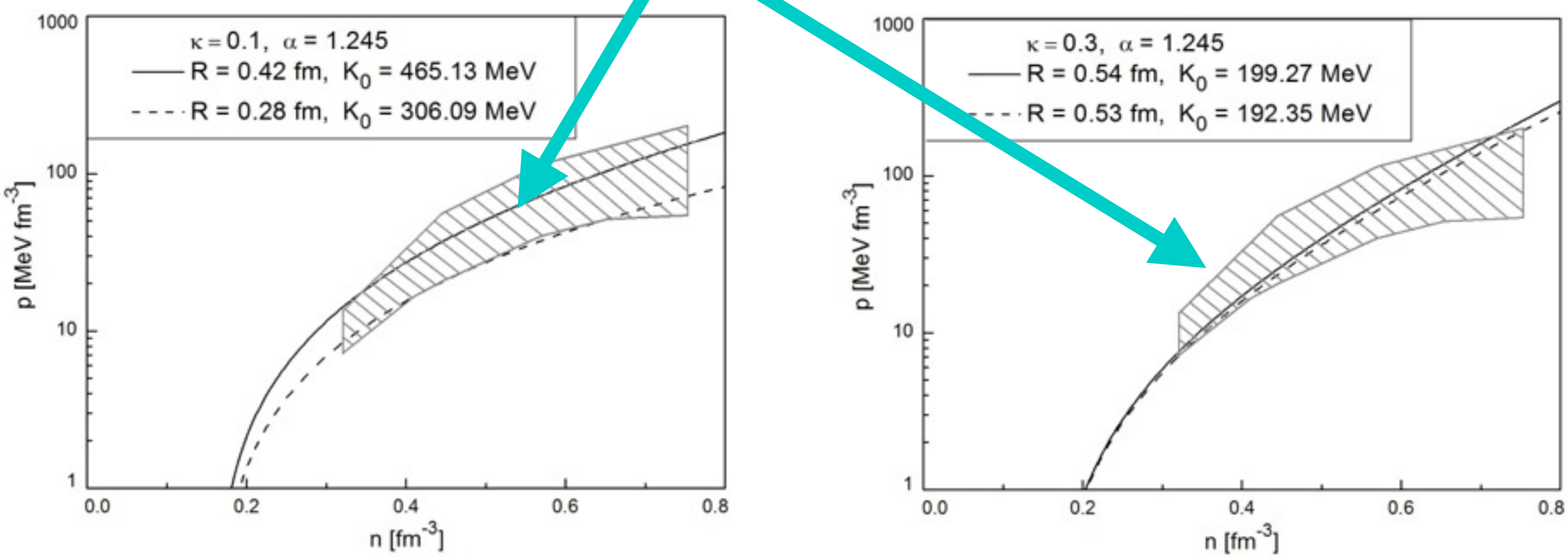
density dependent mean-field potential

α=1.245 => 3 parameters of the model: nucl. hard-core radius R, κ and Cd

Results for Dense Nuclear Matter at T=0

This simple model with 3 parameters reproduces 3 properties of normal nuclear matter and proton flow constraint (8 independent conditions)!

P. Danielewicz, R. Lacey and W. G. Lynch, Science 298, 1593 (2002)



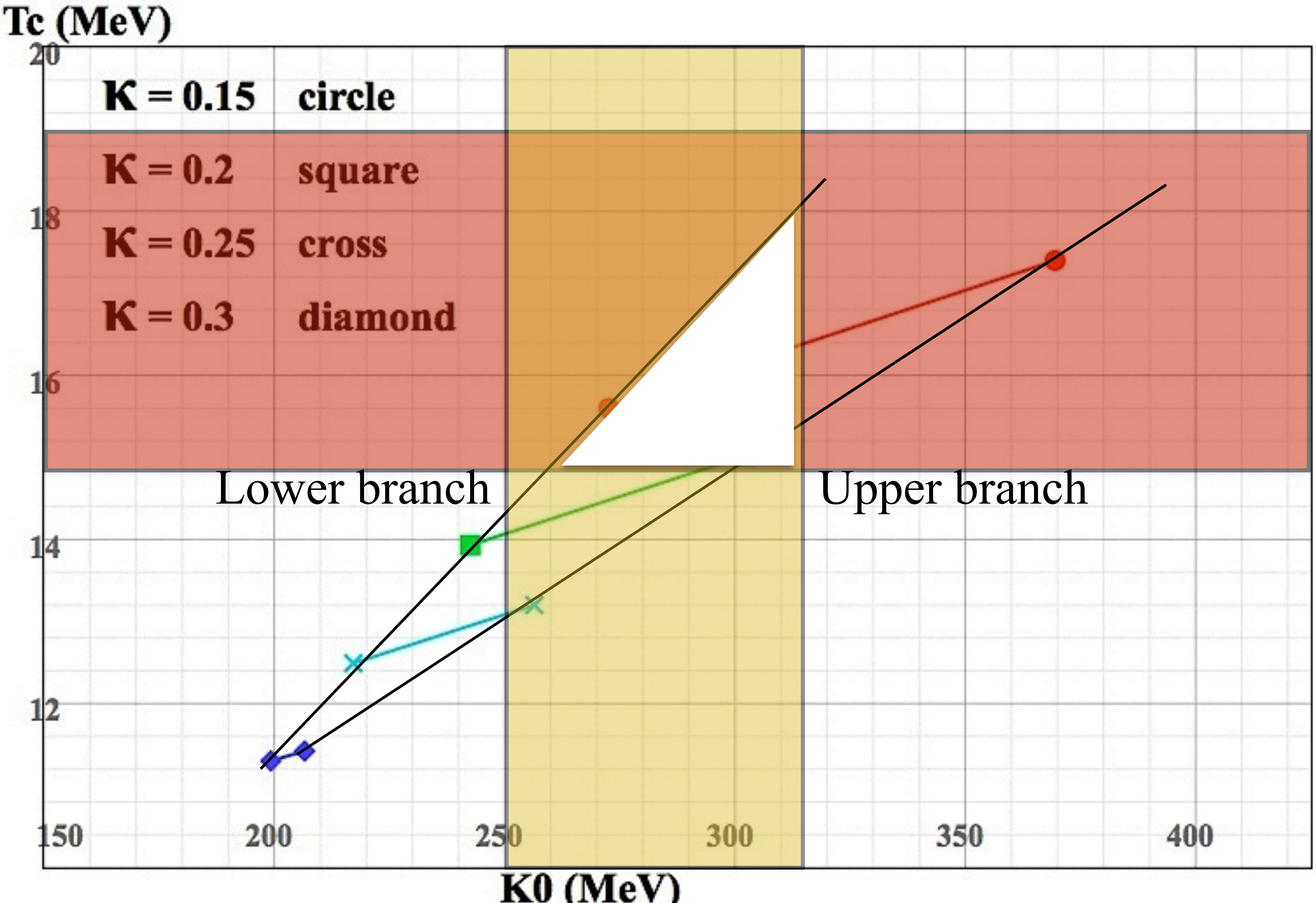
	$\kappa = 0.1$		$\kappa = 0.15$		$\kappa = 0.2$		$\kappa = 0.25$		$\kappa = 0.3$	
$R\ [fm]$	0.28	0.42	0.35	0.48	0.41	0.50	0.47	0.52	0.53	0.54
$C_d^2\ [MeV \cdot fm^{3\kappa}]$	284.98	325.06	206.05	229.57	168.15	179.67	146.97	152.00	133.79	134.60
$U_0\ [MeV]$	567.32	501.65	343.93	312.83	231.42	217.76	162.03	157.41	114.32	113.84
$K_0\ [MeV]$	306.09	465.13	272.55	405.97	242.56	322.80	217.16	256.44	192.35	199.27
$\mu_c\ [MeV]$	890.94	881.01	900.08	895.08	906.44	904.49	911.11	910.53	914.74	914.70
$T_c\ [MeV]$	17.62	20.60	15.60	17.97	13.93	15.36	12.49	13.20	11.16	11.30
$n_c\ [fm^{-3}]$	0.009	0.010	0.013	0.014	0.016	0.017	0.018	0.020	0.022	0.022
$p_c\ [MeV \cdot fm^{-3}]$	0.0186	0.028	0.031	0.045	0.043	0.055	0.053	0.061	0.060	0.062
Z_c	0.1173	0.1359	0.1529	0.1789	0.1929	0.2106	0.2357	0.2311	0.2444	0.2494

Values of κ > 0.3

do not obey the proton

flow constraint!

Finding Compatible Experimental Data



Remarkable Features of IST EOS of Dense Nuclear Matter at T=0

In contrast to Quantum VdWaals EOS the interaction pressure

cannot be expanded in Taylor series at zero density!

In contrast to Quantum VdWaals EOS

IST EOS has a wide range of critical compressibility

constant values!

$$Z_c = \frac{p_c}{T_c\, n_c}$$ = crit. pressure/(crit. T x crit. density)

For Classical or Quantum VdWaals EOS => Zc = 0.375

For all advanced mean-field model EOS of nuclear matter => Zc = 0.28-0.35

For IST EOS => Zc = [0.11-0.294] <=> For ordinary liquids => Zc = [0.11-0.4]

What is the universality class of IST EOS?

Virial Coefficients for Quantum VdWaals and Quantum IST EOS

Quantum mean-field VdWaals EOS

$$p(T,\mu,n_{id}) = p_{id}(T,\nu(\mu,n_{id})) - P_{int}(T,n_{id})$$

$$p_{id}(T,\nu) = d_p \int \frac{dk}{(2\pi^3)} \frac{k^2}{3\,E(k)} \frac{1}{e^{\left(\frac{E(k)-\nu}{T}\right)}+\zeta}\,,$$

$$\nu(\mu,n_{id}) = \mu - b\,p + U(T,n_{id})\,,$$

Quantum mean-field IST EOS

$$p = p_{id}(T,\nu_1) - P_{int\,1}(T,n_{id\,1})\,,$$

$$\Sigma = R_p\left[p_{id}(T,\nu_2) - P_{int\,2}(T,n_{id\,2})\right]\,,$$

$$\nu_1 = \mu - V_0\,p - S_0\,\Sigma + U_1(T,n_{id\,1})\,,$$

$$\nu_2 = \mu - V_0\,p - \alpha S_0\,\Sigma + U_2(T,n_{id\,2})\,,$$

K. A. Bugaev et al., arXiv:1704.06846 [nucl-th] (published in UJP)

Virial expansion of quantum ideal gas

$$p_{id}(T,\nu) = T\sum_{l=1}^{\infty} a_l^{(0)}\left[n_{id}(T,\nu)\right]^l\,, \quad \text{where}$$

$$a_1^{(0)} = 1\,,$$

$$a_2^{(0)} = -b_2^{(0)}\,,$$

$$a_3^{(0)} = 4\left[b_2^{(0)}\right]^2 - 2\,b_3^{(0)}\,,$$

$$a_4^{(0)} = -20\left[b_2^{(0)}\right]^3 + 18\,b_2^{(0)}\,b_3^{(0)} - 3\,b_4^{(0)}\,,$$

$$\ldots\ldots$$

Virial coeff. of quantum ideal gas are expressed in terms of cluster integral coefficients

Cluster integrals of quantum ideal gas

$$b_l^{(0)} = \frac{(\mp 1)^{l+1}}{l} n_{id}^{(0)}(T/l,\nu)\left[n_{id}^{(0)}(T,\nu)\right]^{-l}\,,$$

$$b_l^{(0)}\Big|_{nonrel} \simeq \frac{(\mp 1)^{l+1}}{l^{\frac{5}{2}}}\left(\frac{1}{d_p}\left[\frac{2\,\pi}{T\,m_p}\right]^{\frac{3}{2}}\right)^{l-1}$$

$$b_l^{(0)}\Big|_{urel} \simeq \frac{(\mp 1)^{l+1}}{l^4}\left[\frac{\pi^2}{d_p\,T^3}\right]^{l-1}$$

(-) for fermions, (+) for bosons

d_p is particle degeneracy factor

m_p is particle mass

Virial Coefficients for Quantum VdWaals and Quantum IST EOS II

Quantum mean-field VdWaals EOS

$$p(T,\mu,n_{id}) = p_{id}(T,\nu(\mu,n_{id})) - P_{int}(T,n_{id})$$
$$p_{id}(T,\nu) = d_p \int \frac{dk}{(2\pi^3)} \frac{k^2}{3\,E(k)} \frac{1}{e^{\left(\frac{E(k)-\nu}{T}\right)} + \zeta}\,,$$
$$\nu(\mu,n_{id}) = \mu - b\,p + U(T,n_{id})\,,$$

Quantum mean-field IST EOS

$$p = p_{id}(T,\nu_1) - P_{int\,1}(T,n_{id\,1})\,,$$
$$\Sigma = R_p\left[p_{id}(T,\nu_2) - P_{int\,2}(T,n_{id\,2})\right]\,,$$
$$\nu_1 = \mu - V_0\,p - S_0\,\Sigma + U_1(T,n_{id\,1})\,,$$
$$\nu_2 = \mu - V_0\,p - \alpha S_0\,\Sigma + U_2(T,n_{id\,2})\,,$$

Virial expansion of quantum ideal gas

$$p_{id}(T,\nu) = T\sum_{l=1}^{\infty} a_l^{(0)}\left[n_{id}(T,\nu)\right]^l\,, \quad \text{where}$$
$$a_1^{(0)} = 1\,,$$
$$a_2^{(0)} = -b_2^{(0)}\,,$$
$$a_3^{(0)} = 4\left[b_2^{(0)}\right]^2 - 2\,b_3^{(0)}\,,$$
$$a_4^{(0)} = -20\left[b_2^{(0)}\right]^3 + 18\,b_2^{(0)}\,b_3^{(0)} - 3\,b_4^{(0)}\,,$$
$$\ldots\ldots$$

Virial coeff. of quantum ideal gas are expressed in terms of cluster integral coefficients

Substituting the VdWaals relation

$$n_{id} = \frac{n}{1-b\,n} \quad \Rightarrow$$

$$\frac{p_{id}(T,\nu)}{T\,n} = \frac{1}{1-b\,n} + \sum_{l=2}^{\infty} a_l^{(0)}\frac{[n]^{l-1}}{[1-b\,n]^l}\,.$$

Expanding all denominators, one gets the true virial expansion for quantum VdWaals EoS

Virial Coefficients for Quantum VdWaals and Quantum IST EOS III

Quantum mean-field VdWaals EOS

$$p_{id}(T,\nu) = T\left[n + \sum_{k=2}^{\infty} a_k^Q n^k\right], \quad \text{where}$$

$$a_2^Q = b + a_2^{(0)},$$
$$a_3^Q = b^2 + 2\,b\,a_2^{(0)} + a_3^{(0)},$$
$$a_4^Q = b^3 + 3\,b^2\,a_2^{(0)} + 3\,b^1\,a_3^{(0)} + a_4^{(0)},$$
$$a_k^Q = b^{k-1} + \sum_{l=2}^{k} \frac{(k-1)!}{(l-1)!(k-l)!} b^{k-l} a_l^{(0)}.$$

b = 4Vo is classical virial coefficient

Vo is particle eigenvolume

Quantum mean-field IST EOS

$$a_2^{Q,tot} = V_0 + a_2^{(0)} + 3V_0 B_1 = 4V_0 + a_2^{(0)},$$

$$a_{k\geq 3}^{Q,tot} \simeq \sum_{l=1}^{k} C_l^{(k)} + 3V_0 \sum_{l=1}^{k-1} C_l^{(k-1)} l + 3V_0^2 \sum_{l=1}^{k-2} C_l^{(k-2)}\left[\frac{3}{2} l(l+1) + (7-6\alpha)\right]$$

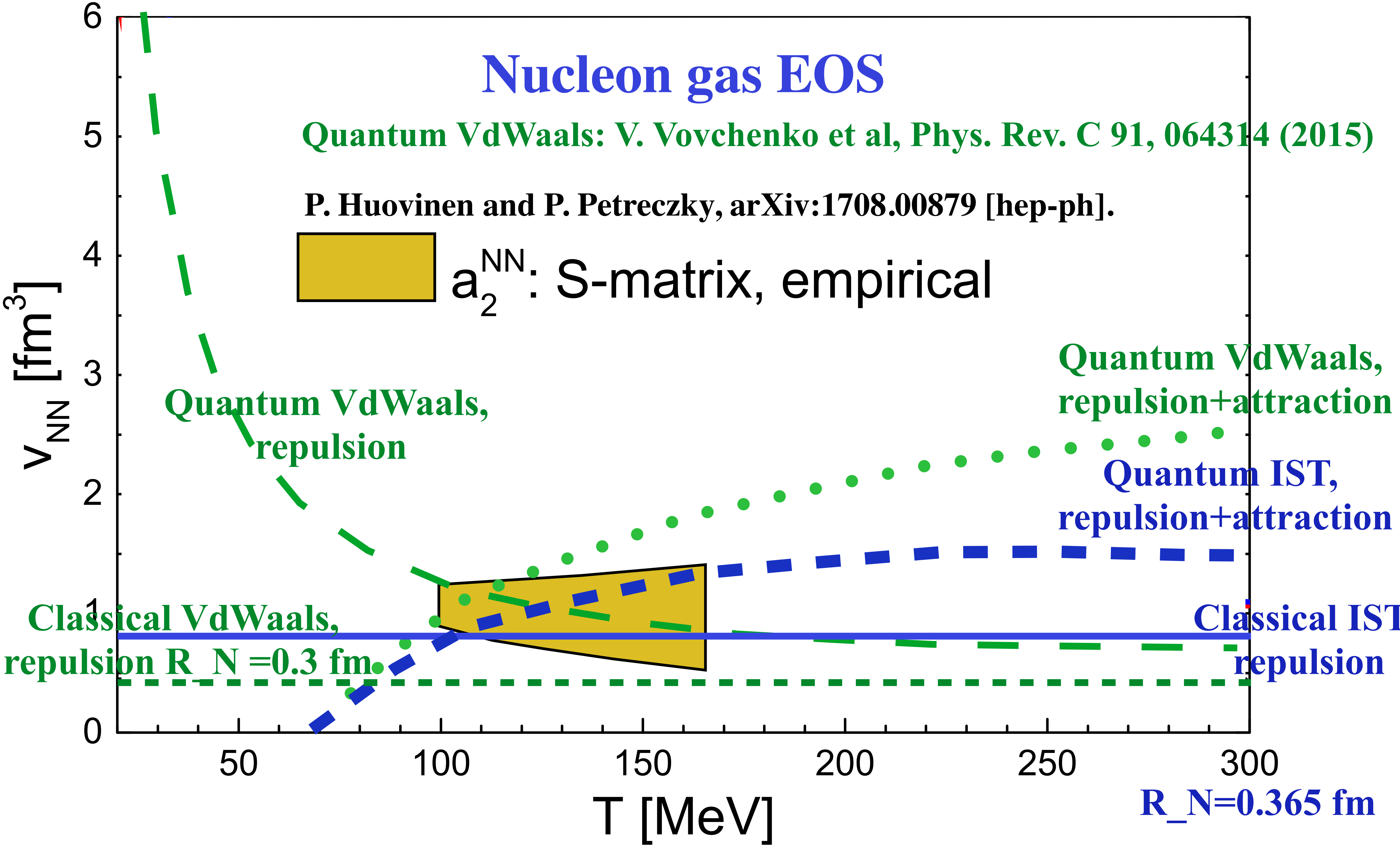
where

$$C_l^{(k)} = \frac{(k-1)!}{(l-1)!(k-l)!} V_0^{k-l} a_l^{(0)}$$

For IST EoS it is bit more complicated

$$a_2^{Q,tot} = b + a_2^{(0)} - \frac{a_{attr}}{T} \simeq b + \frac{1}{2^{\frac{5}{2}} d_p}\left[\frac{2\,\pi}{T\,m_p}\right]^{\frac{3}{2}} - \frac{a_{attr}}{T}$$

for nucleons it is good approximation



Classical and Quantum VdWaals EOS fail to reproduce S-matrix approach results!

Classical and Quantum IST EOS reproduce S-matrix approach results very well!

Role of 3-rd Virial Coefficient for Quantum IST EoS

Explicit formulae K. A. Bugaev et al., arXiv:1810.00486

$$a_2^{IST} = 4V_0 + a_2^{(0)}\,, \qquad a_3^{IST} \simeq [16 - 18(\alpha - 1)]V_0^2 + 5V_0 a_2^{(0)} + a_3^{(0)}\,,$$

$$a_2^{(0)} \simeq 2^{-\frac{5}{2}}\omega_N \simeq 0.177\omega_N\,,\ a_3^{(0)} \simeq 2\left[2^{-4} - 3^{-\frac{5}{2}}\right]\omega_N^2 \simeq -3.4\cdot 10^{-3}\omega_N^2\,,$$

where $\omega_N = \frac{1}{g_N}\left[\frac{2\pi\hbar^2}{Tm_N}\right]^{\frac{3}{2}}$

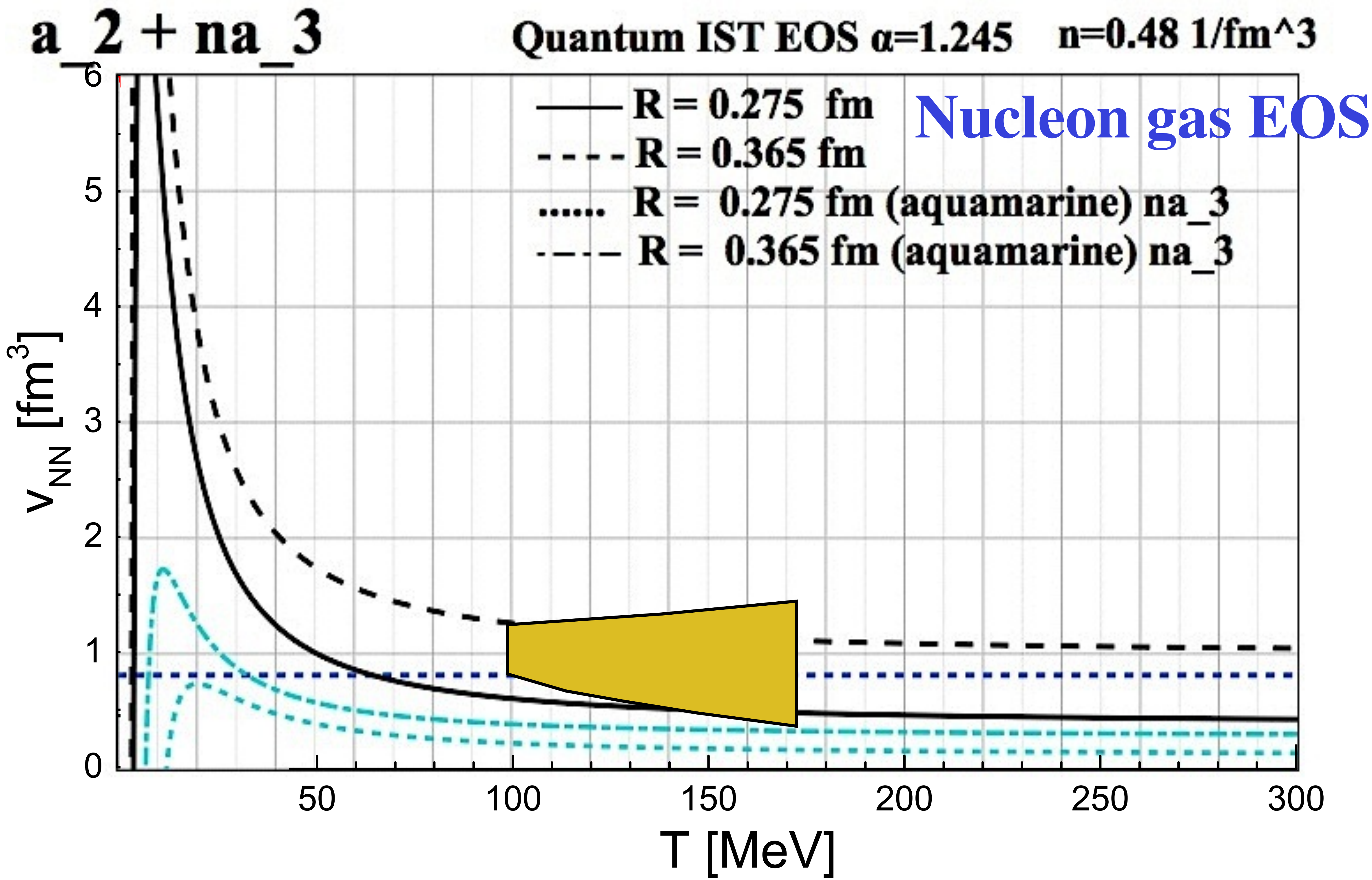
very small for T> 20 MeV!

Introducing effective virial coefficient

$$a_2^{eff} \;=\; a_2^{IST} + n a_3^{IST}$$

one can study whether it is compatible with S-matrix approach

Role of 3-rd Virial Coefficient

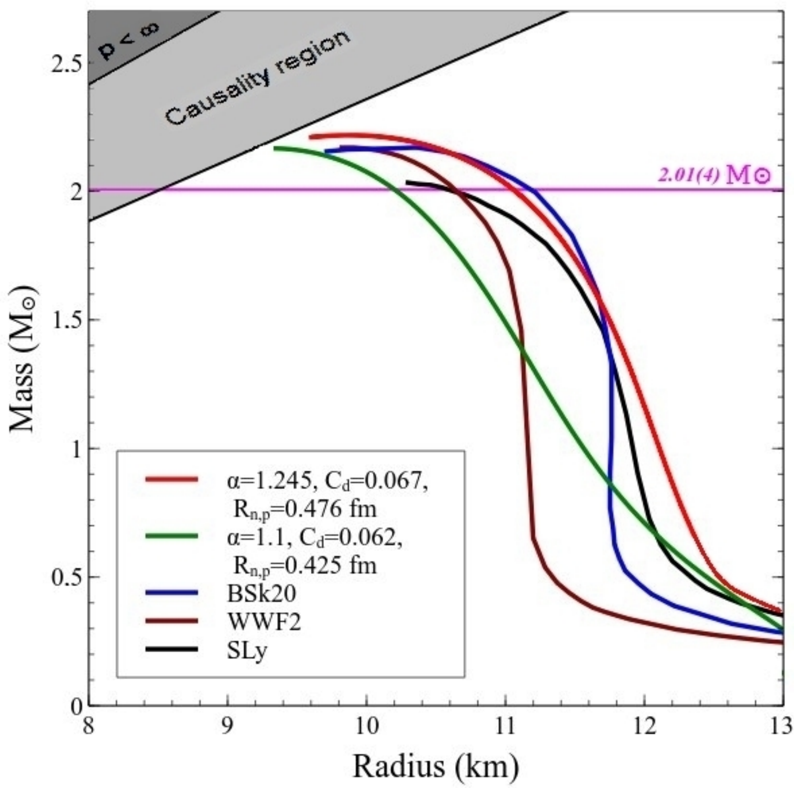


For T > 50 MeV the 3-rd virial coefficient does not play ANY role in the nucleon gas! Also there is range of R_N values [0.3; 0.36] fm

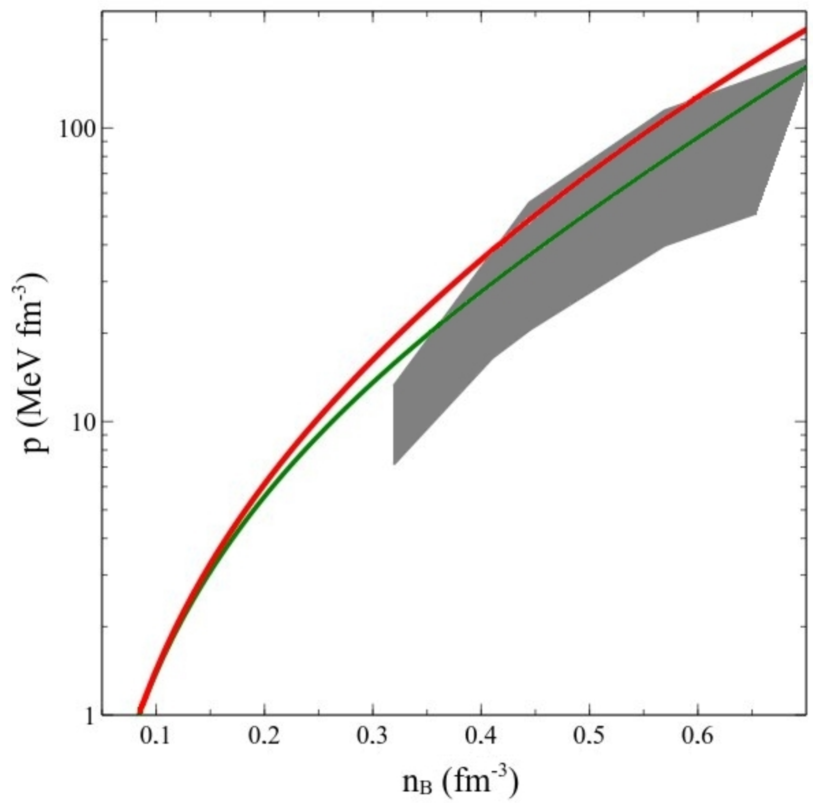
Further Applications of IST EoS

1. Quantum IST EoS was successfully applied to the neutron star properties description

V. Sagun and I. Lopes, Astrophys. J 2017, 850, 75



Mass-radius relation for neutron stars



This EoS obeys proton flow constraint

$$U(n_{id}) = C_d^2 n_{id}^{\frac{1}{3}}$$

	baryon radius (fm)	α	C	max $M_{NS}/M_{\odot}$
red curve on Fig. (1)	0.476	1.245	0.067	2.217
green curve on Fig. (1)	0.425	1.06	0.062	2.166
A+A collisions	0.355	1.245	0.067	1.544

2. N. Yakovenko and A. Oshlyansky successfully apply IST EoS to hard disks and now they develop more sophisticated EoS which includes induced curvature

Problem With Generalized Quantum VdWaals EoS

In some papers it was suggested to improve QVdWaals by choosing the T-dependent interaction pressure to replace the VdWaals virial coefficients by the virial coefficients of hard-spheres

$$P_{int}(T, n(n_{id})) = Tn\left[(b^2 - B_3)n^2 + (b^3 - B_4)n^3 + (b^4 - B_5)n^4 + ...\right]$$

D. Anchishkin and V. Vovchenko, arXiv:1411.1444 [nucl-th];

V. Vovchenko, D. V. Anchishkin and M. I. Gorenstein, Phys. Rev. C 91, (2015) 064314

In this case one has to use the self-consistency condition:

$$n_{id}\frac{\partial U(T, n_{id})}{\partial n_{id}} = \frac{\partial P_{int}(T, n_{id})}{\partial n_{id}} \quad \Rightarrow$$

$$P_{int}(T, n_{id}) = n_{id}\, U(T, n_{id}) - \int_0^{n_{id}} dn\, U(T, n)$$

The problem appears, when one calculates the entropy density at T=0!

Problem With Generalized Quantum VdWaals EoS II

Assume $U(T, n(n_{id})) = g(T) f(n(n_{id}))$

and find the entropy densities $s = \frac{\partial p(T,\mu)}{\partial T}$ and $s_{id} = \frac{\partial p_{id}(T,\nu)}{\partial T}$

$$
\begin{aligned}
s(T,\mu) &= \left[s_{id} + \left[n_{id} \frac{\partial U}{\partial T} - \frac{\partial P_{int}}{\partial T} \right] \right] [1 + b\, n_{id}]^{-1} \\
&= \left[s_{id} + \frac{d g(T)}{d\, T} \int_0^{n_{id}} d\tilde{n}\, f(n(\tilde{n})) \right] [1 + b\, n_{id}]^{-1}
\end{aligned}
$$

For $g(T) = T \Rightarrow \frac{d g(T)}{d\, T} = 1$ and $P_{int}(T, n(n_{id})) \sim T$

one finds For $T \rightarrow 0 \Rightarrow s_{id} \rightarrow 0$

but $s \rightarrow [1 + b\, n_{id}]^{-1} \cdot \frac{d g(T)}{d\, T} \int_0^{n_{id}} d\tilde{n}\, f(n(\tilde{n})) \neq 0$

I.e. for U~T this EoS breaks down the Third Law of Thermodynamics!

Problem With Generalized Quantum VdWaals EoS III

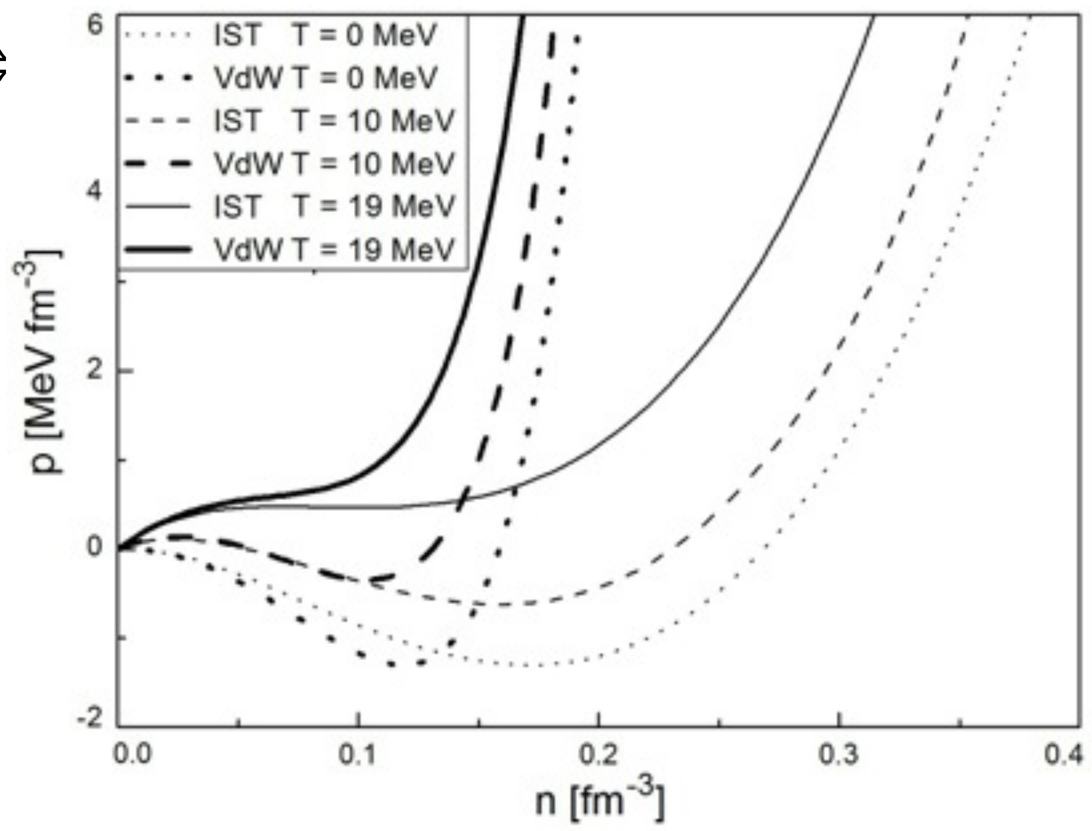
Example: QVdWaals and QIST far same parameters a and b

$$p^{QVdW}(T,n_{id}) = T n_{id} - P^{VdW}_{int}(T,n_{id}) \quad \text{with} \quad g(T) \equiv \frac{T^2}{T+T_{SW}}$$

T_sw = 1 MeV

$$P^{VdW}_{int}(T,n_{id}) = a\left[\frac{n_{id}}{1+b\,n_{id}}\right]^2 + T n_{id} - \frac{g(T)\,n_{id}}{1+b\,n_{id}} - \frac{g(T) b\,n_{id}^2}{[1+b\,n_{id}]^2} - \frac{g(T)\,B_3\,n_{id}^3}{[1+b\,n_{id}]^3} - \frac{g(T)\,B_4\,n_{id}^4}{[1+b\,n_{id}]^4} \quad \Rightarrow$$

$$p^{QVdW}(T,n_{id}) = g(T)\left[\frac{n_{id}}{1+b\,n_{id}} + \frac{b\,n_{id}^2}{[1+b\,n_{id}]^2} - \frac{B_3\,n_{id}^3}{[1+b\,n_{id}]^3} - \frac{B_4\,n_{id}^4}{[1+b\,n_{id}]^4}\right] - a\left[\frac{n_{id}}{1+b\,n_{id}}\right]^2$$



This QVdWaals the Third Law of Thermodynamics and coincides with QIST at T > 120 MeV

Thus, if U~T then QVdWaals breaks down the Third Law of Thermodynamics, if U~T^2 then then this EoS cannot go beyond the usual VdWaals approximation at low T!

Summary

1. **We discussed the basic properties of VdWaals EoS**

2. **A heuristic derivation of the IST EoS is presented**

3. **The Quantum IST EoS of normal nuclear matter is developed. It obeys 11 conditions using 4 parameters!**

4. **We discussed the quantum virial coefficients of VdWaals and IST EoS and the problems of generalized QVdWaals**