

Standard Model or Standard Theory?

The many ways Beyond the SM

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CMS School, Pisa, January 2019

Some general introductory remarks

The potential of precision at LHC

More than one (motivated) scalar (if time permits)

The SM Lagrangian (since 1973 in its full content)

$$\mathcal{L}_{\sim SM} = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + i\bar{\psi} \not{D}\psi \quad (\sim 1975-2000)$$

$$+ |D_\mu h|^2 - V(h) \quad (\sim 1990-2012\text{-now})$$

$$+ \psi_i \lambda_{ij} \psi_j h + h.c. \quad (\sim 2000\text{-now})$$

In () the approximate dates of the experimental confirmation of the various lines (at different levels)

The synthetic nature of PP exhibited

All of Particle Physics in 1 page

1. Symmetry group $L \times \mathcal{G}$

L = Lorentz (space-time)

$\mathcal{G} = SU(3) \times SU(2) \times U(1)$ (local)

2. Particle content (rep.s of $L \times \mathcal{G}$)

	h	Q	L	u	d	e
Lorentz	0	$1/2_L$	$1/2_L$	$1/2_R$	$1/2_R$	$1/2_R$
$SU(3)$	1	3	1	3	3	1
$SU(2)$	2	2	2	1	1	1
$U(1)$	$-1/2$	$1/6$	$-1/2$	$2/3$	$-1/3$	-1

3. All "operators" (products of $\Phi, \partial_\mu \Phi$) in \mathcal{L} of dimension ≤ 4

Problems of (questions for) the SM

0. Which rationale for matter quantum numbers?

$$|Q_p + Q_e| < 10^{-21} e$$

1. Phenomena unaccounted for

neutrino masses
Dark matter

matter-antimatter asymmetry
inflation?

2. Why $\theta \lesssim 10^{-10}$?

$$\theta G_{\mu\nu} \tilde{G}^{\mu\nu}$$

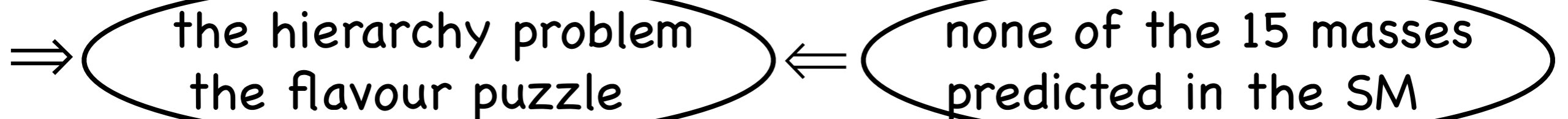
Axions

3. $\mathcal{O}_i : d(\mathcal{O}_i) \leq 4$ only?

neutrino masses
Gravity

Are the protons forever?

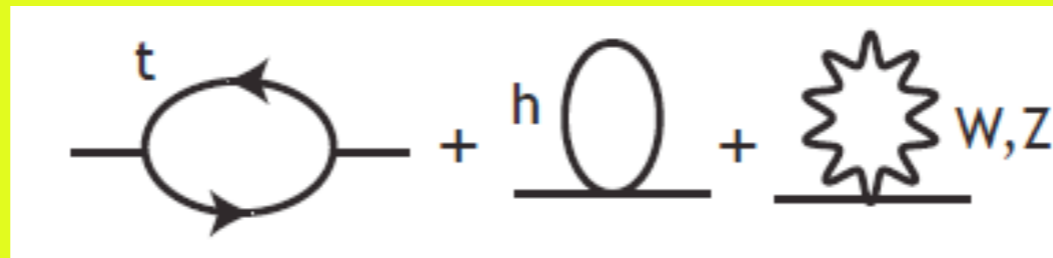
4. Lack of calculability (a euphemism)



The hierarchy problem, once again

Can we compute the Higgs mass/vev in terms of some fundamental dynamics?

NOT in the SM



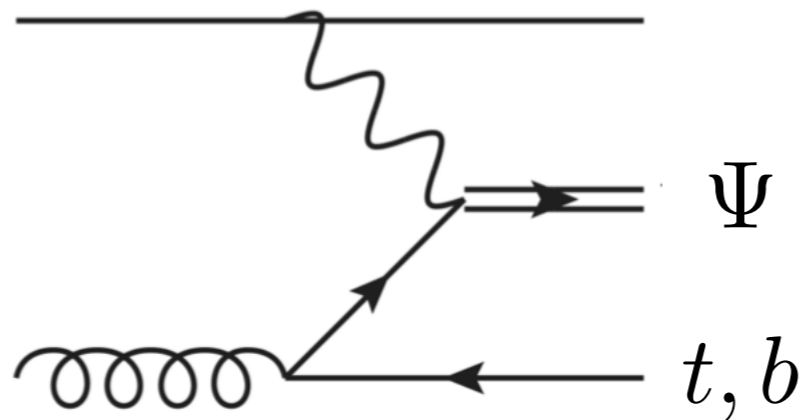
$$\delta m_h^2 = \frac{3y_t^2}{4\pi^2} \Lambda_t^2 - \frac{9g^2}{32\pi^2} \Lambda_g^2 - \frac{3g'^2}{32\pi^2} \Lambda_{g'}^2$$

The standard reaction

Look for top "partners", $J=0$ or $1/2$,
coloured or uncoloured,
with a mass not far from a TeV,
capable to cutoff the Λ^2 divergence

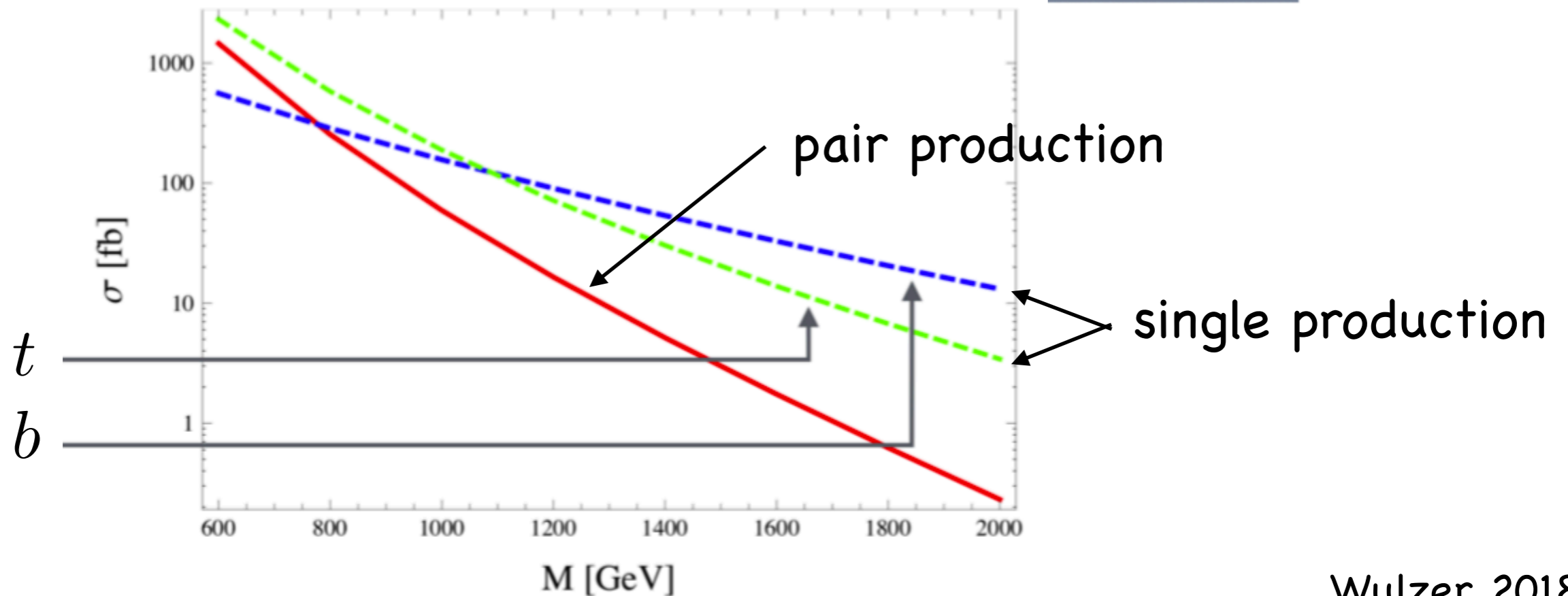
Single production deserves attention

(although NOT generically present)

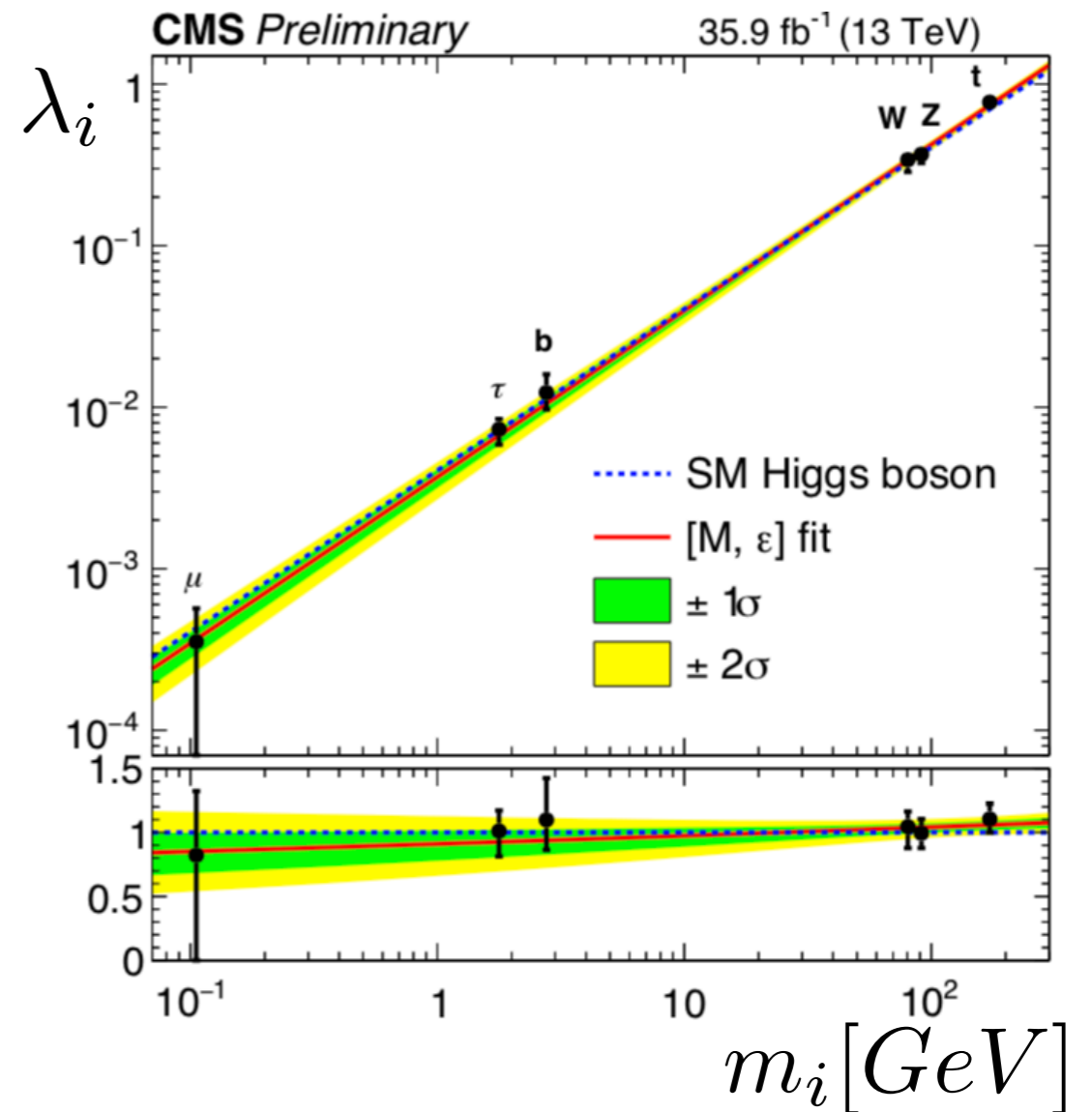
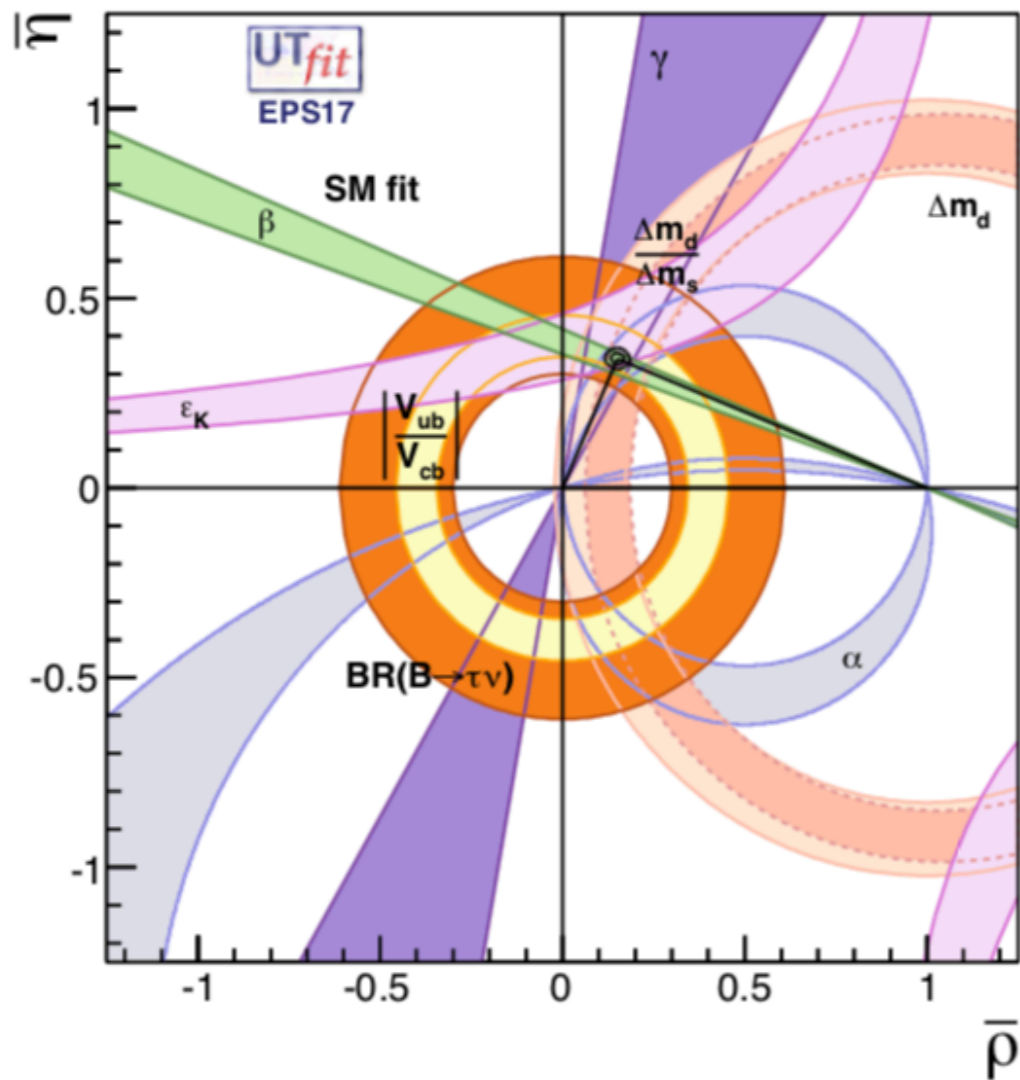


Branching Ratios

\tilde{T}	Wb	Zt	ht
$X_{5/3}$	Wt		
$\tilde{X}_{2/3}$	Zt	ht	
T	Zt	ht	
B	Wt		



The flavour paradox $\lambda_{ij} \Psi_i \Psi_j$



As opposed to the hard time in trying to explain the spectrum and the mixing of quarks and leptons

Not easy to improve without observing deviations from the SM
(See below)

The many different directions in BSM

(for an audience of philosophers, sic)

1. Explore the space of theories

- Address a specific problem, theoretical or experimental
E.g.: Supersymmetry, DM axions, Baryogenesis, ...
- Expand the set of consistent and potentially "true" theories
E.g.: Supersymmetry, conformal field theory, string theory, ...

2. Explore the space of observables

- Test a "true" theory
E.g.: Precision tests of the SM
- Extend the explorable territory
E.g.: Where can one look for "DM"? Are there new light particles?

The emphasis on the specific direction is time dependent
To concentrate now on a single direction is very dangerous

The potential of precision at LHC

- Higgs couplings

$$\mathcal{L} = -\lambda k_\lambda H^4 + g_f k_f H \bar{f} f + g_V k_V V_\mu H^\dagger \partial_\mu H$$

- ElectroWeak observables

Pole observables: $m_W, \sin\theta_{eff}^l$

Drell-Yan $l^+ l^-, l\nu$ at high m_{ll}, m_{ll}^T

DiBosons Wh, Zh, WZ, WW

- Flavour observables

Testing the FCNC loops

Lepton Flavour Violation

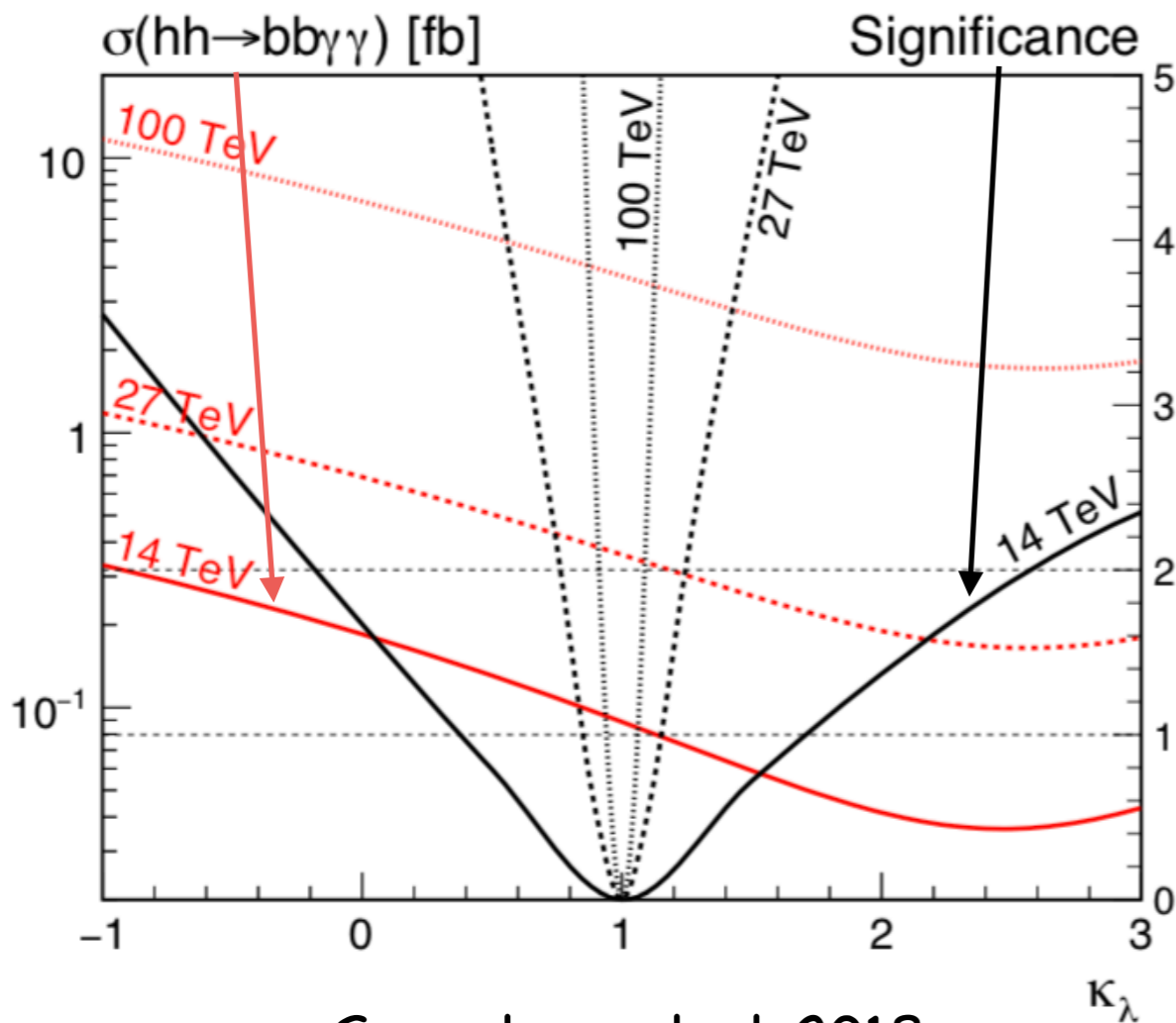
The role of flavour in BSM

$$V = \mu^2 H^2 + \lambda H^4$$

$$\lambda = \frac{G_\mu}{\sqrt{2}} m_h^2 + \text{rad. corr.} = 0.12$$

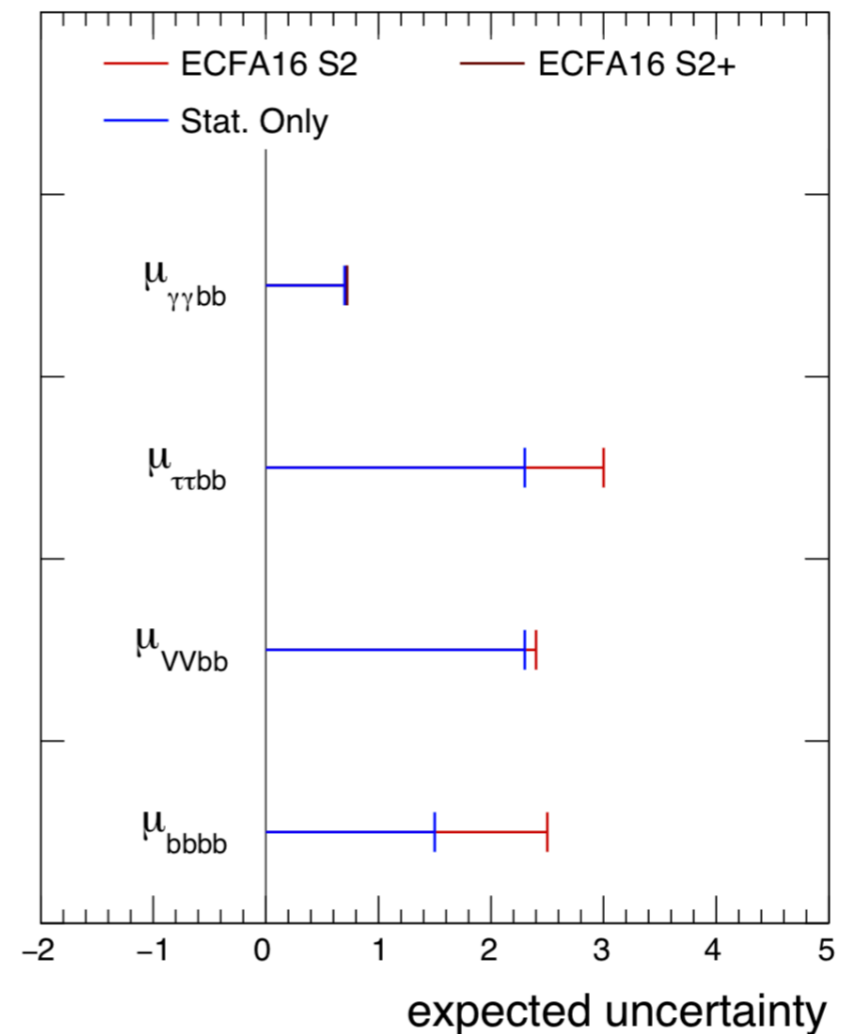
Can one measure it directly?

CMS-PAS-FTR-16-002



Goncalves et al 2018

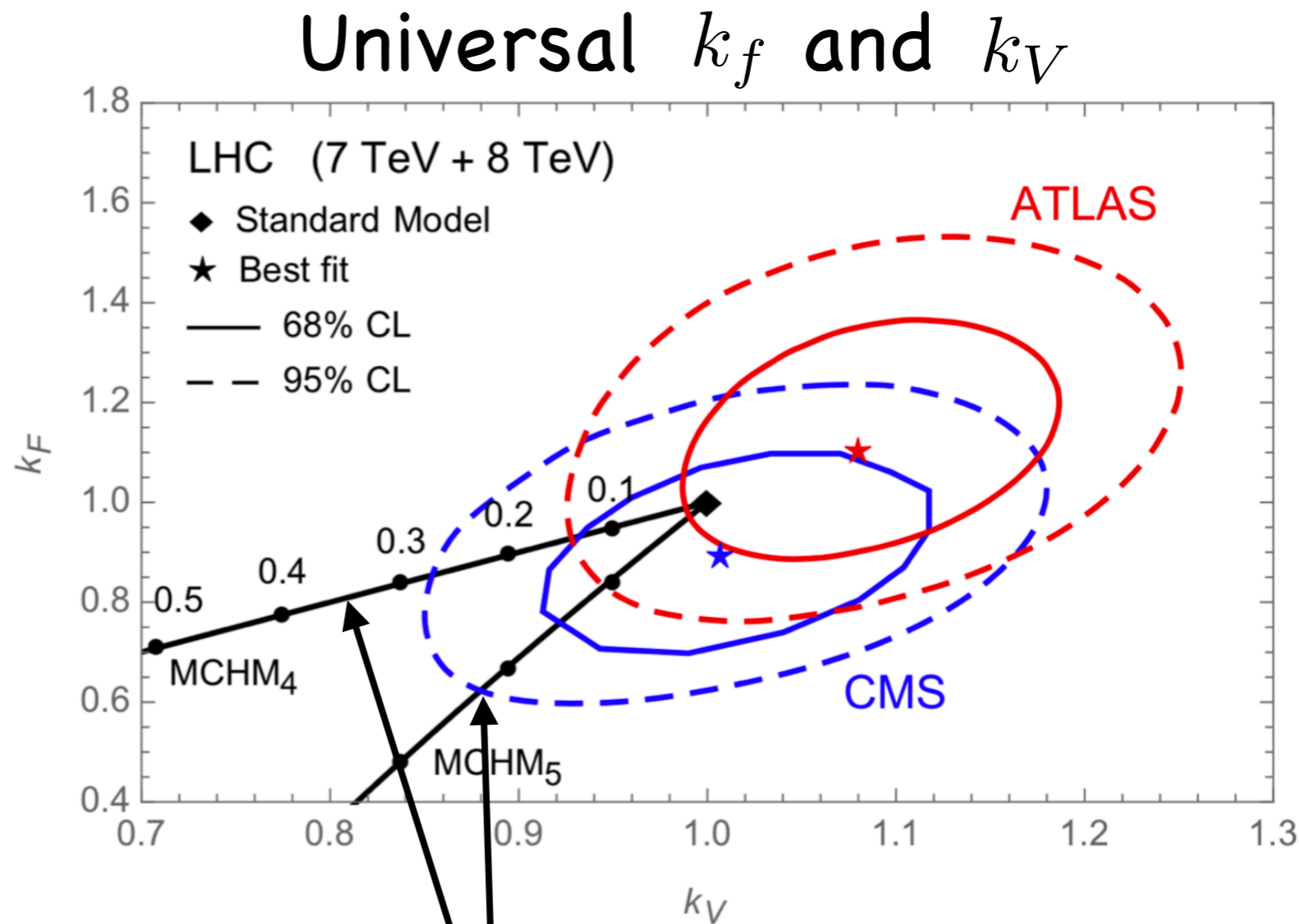
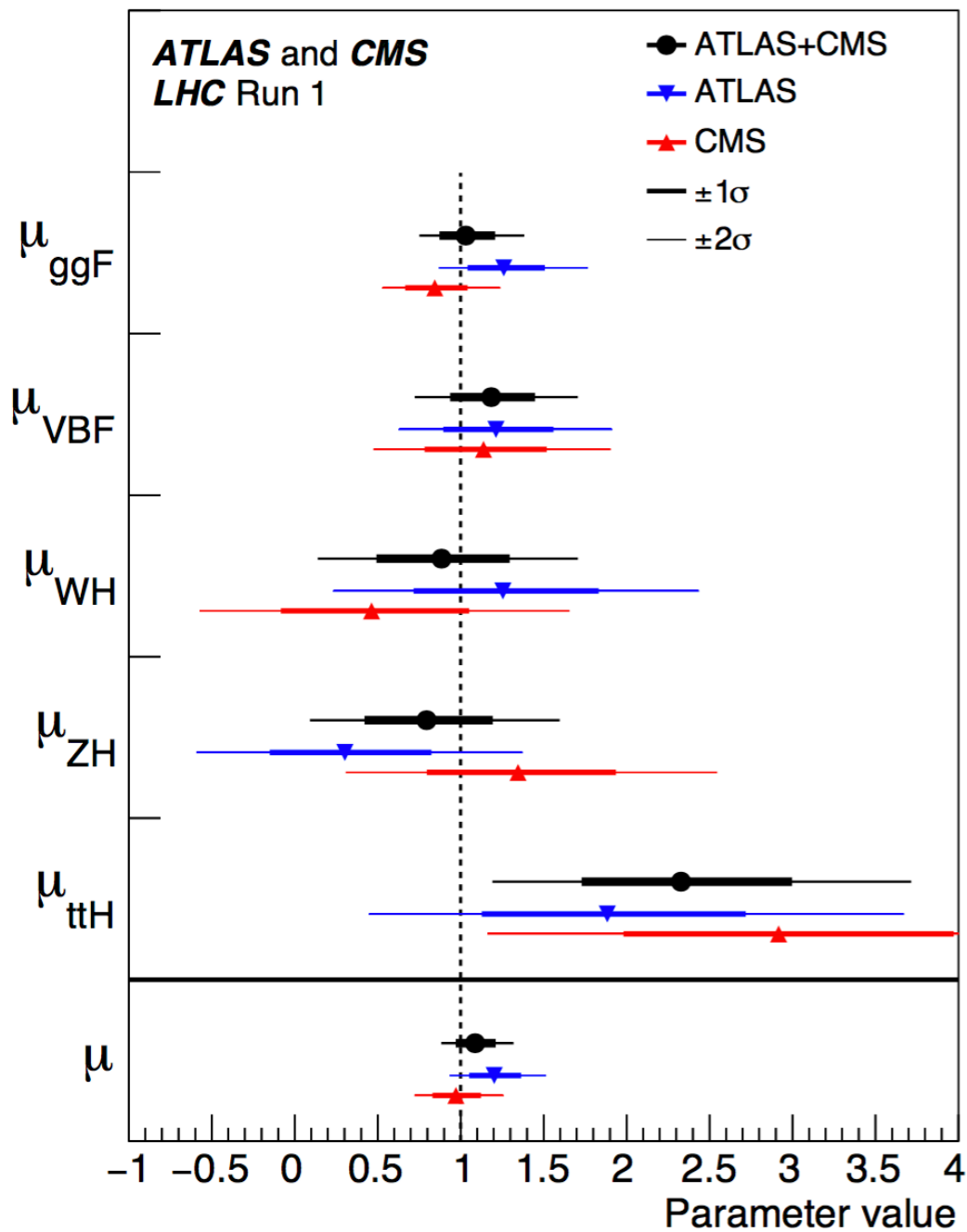
CMS Projection $\sqrt{s} = 13$ TeV SM $gg \rightarrow HH$



As difficult as important
large deviations conceivable in BSM

Higgs couplings

$$\mathcal{L} = -\lambda k_\lambda H^4 + g_f k_f H \bar{f} f + g_V k_V V_\mu H^+ \partial_\mu H$$



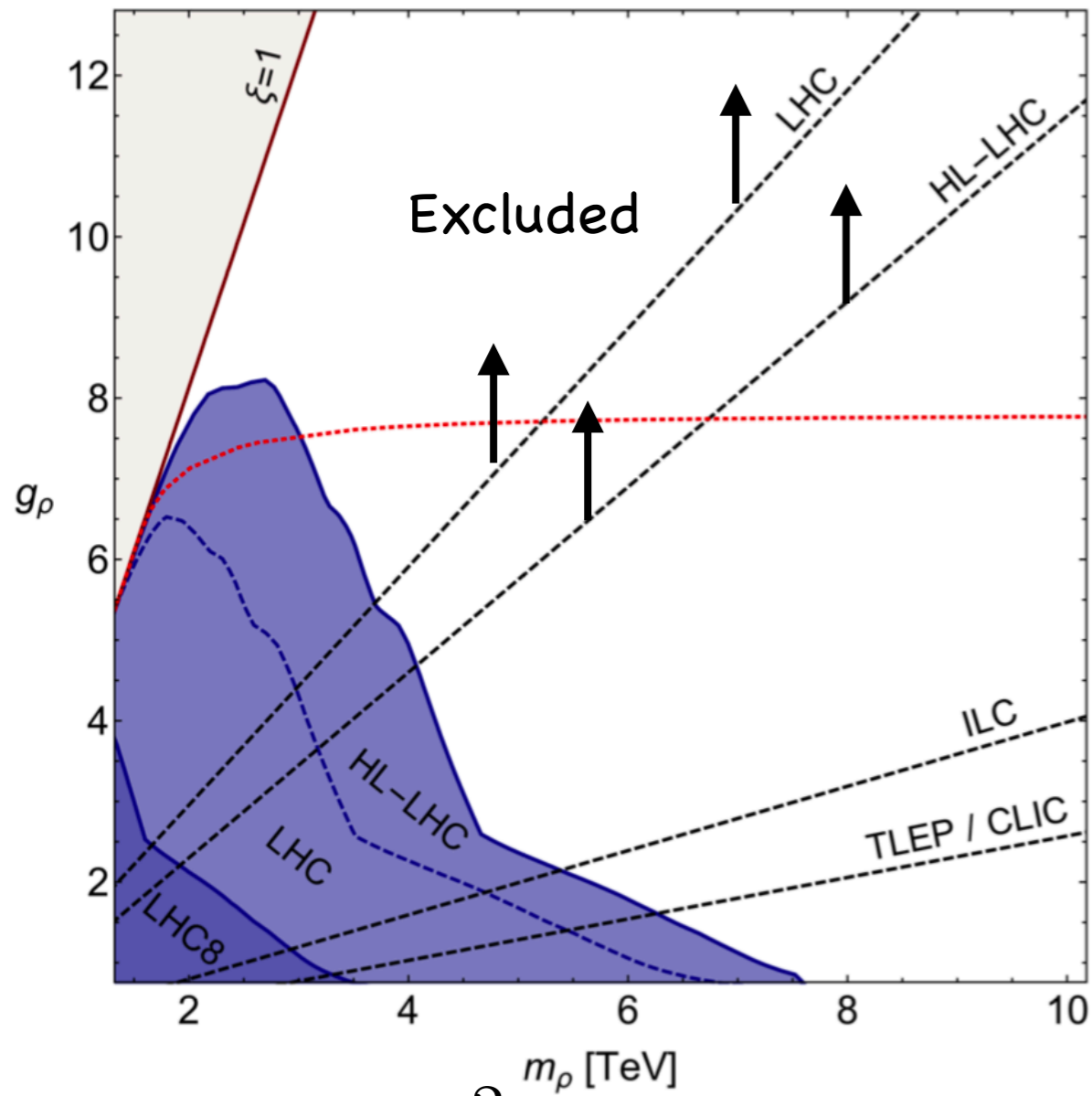
the scale f of Higgs compositeness $\xi = \frac{v^2}{f^2}$ $k_V = \sqrt{1 - \xi}$

Now $f \gtrsim 600 \text{ GeV}$ only!

Direct versus indirect searches

$$pp \rightarrow \rho \rightarrow WZ$$

$$\xi = \frac{v^2}{f^2} = g_\rho^2 \frac{v^2}{m_\rho^2}$$



$$g_f = \frac{g^2}{g_\rho}$$

Collider	Energy	Luminosity	ξ [1σ]
LHC	14 TeV	300 fb^{-1}	$6.6 - 11.4 \times 10^{-2}$
LHC	14 TeV	3 ab^{-1}	$4 - 10 \times 10^{-2}$
ILC	250 GeV + 500 GeV	250 fb^{-1} 500 fb^{-1}	$4.8-7.8 \times 10^{-3}$
CLIC	350 GeV + 1.4 TeV + 3.0 TeV	500 fb^{-1} 1.5 ab^{-1} 2 ab^{-1}	2.2×10^{-3}
TLEP	240 GeV + 350 GeV	10 ab^{-1} 2.6 ab^{-1}	2×10^{-3}

The potential of precision at LHC

- Higgs couplings

$$\mathcal{L} = -\lambda k_\lambda H^4 + g_f k_f H \bar{f} f + g_V k_V V_\mu H^\dagger \partial_\mu H$$

- ElectroWeak observables

Pole observables: $m_W, \sin\theta_{eff}^l$

Drell-Yan $l^+ l^-, l\nu$ at high m_{ll}, m_{ll}^T

DiBosons Wh, Zh, WZ, WW

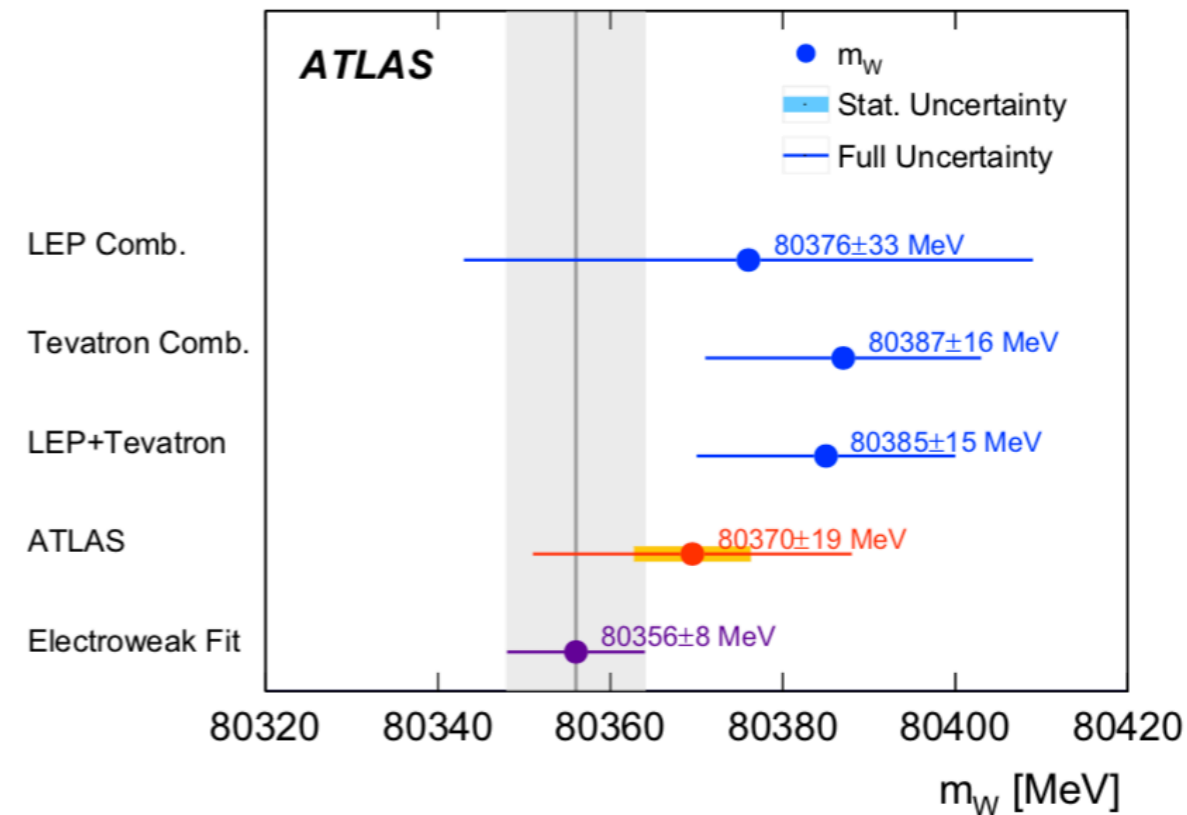
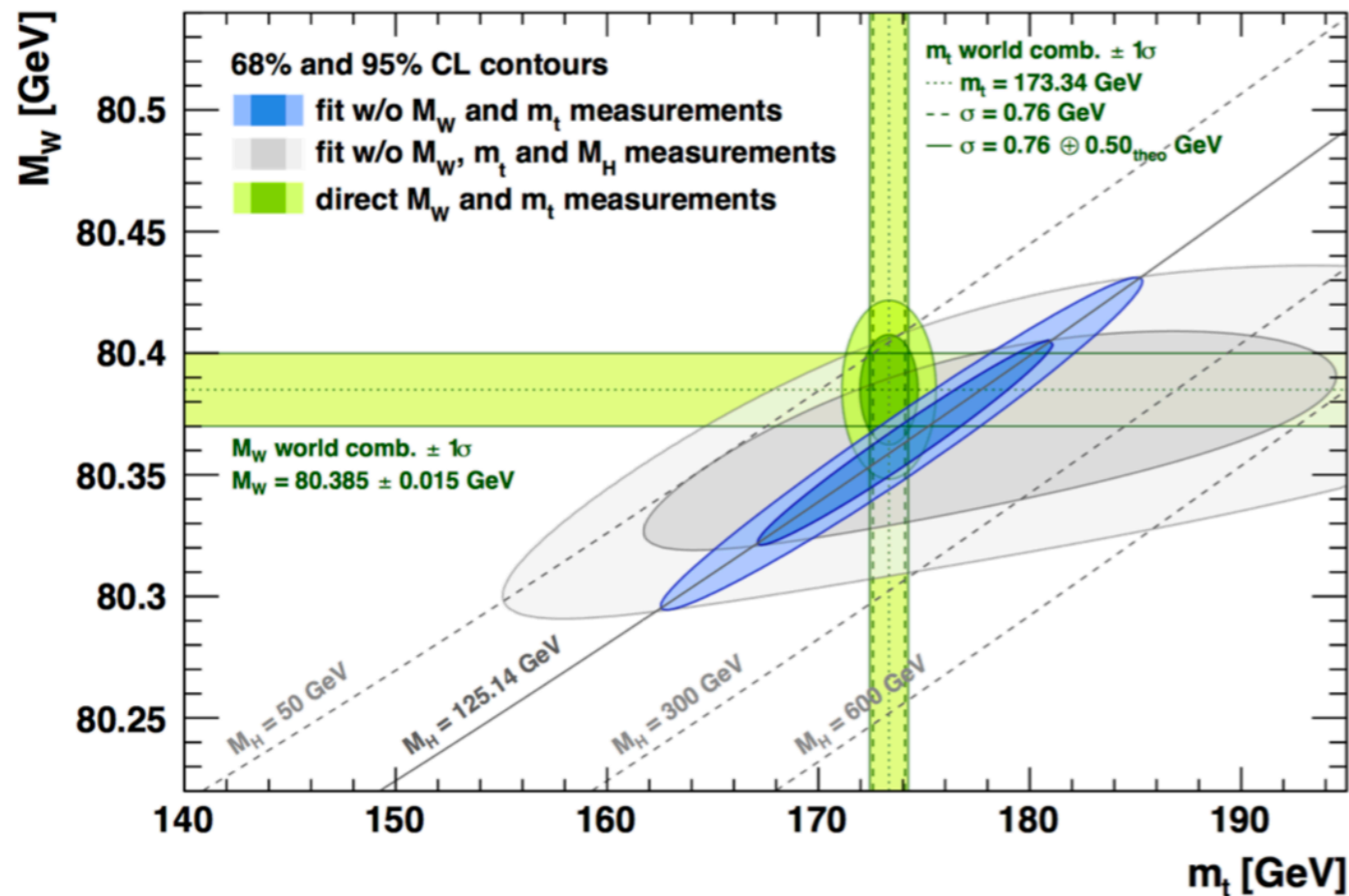
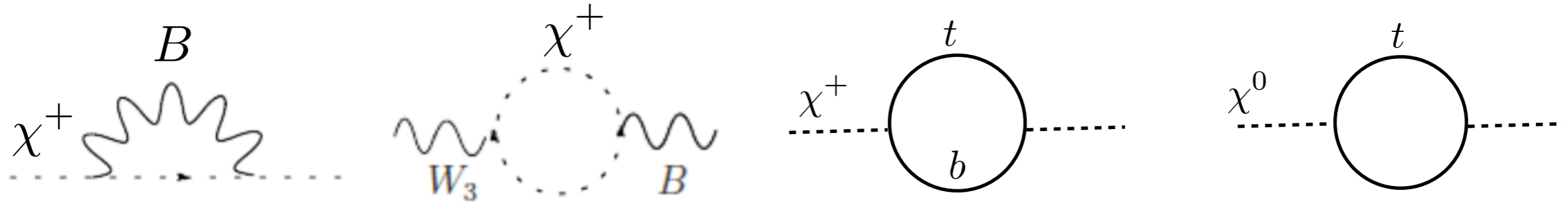
- Flavour observables

Testing the FCNC loops

Lepton Flavour Violation

The role of flavour in BSM

Comparing direct measurements with virtual effects

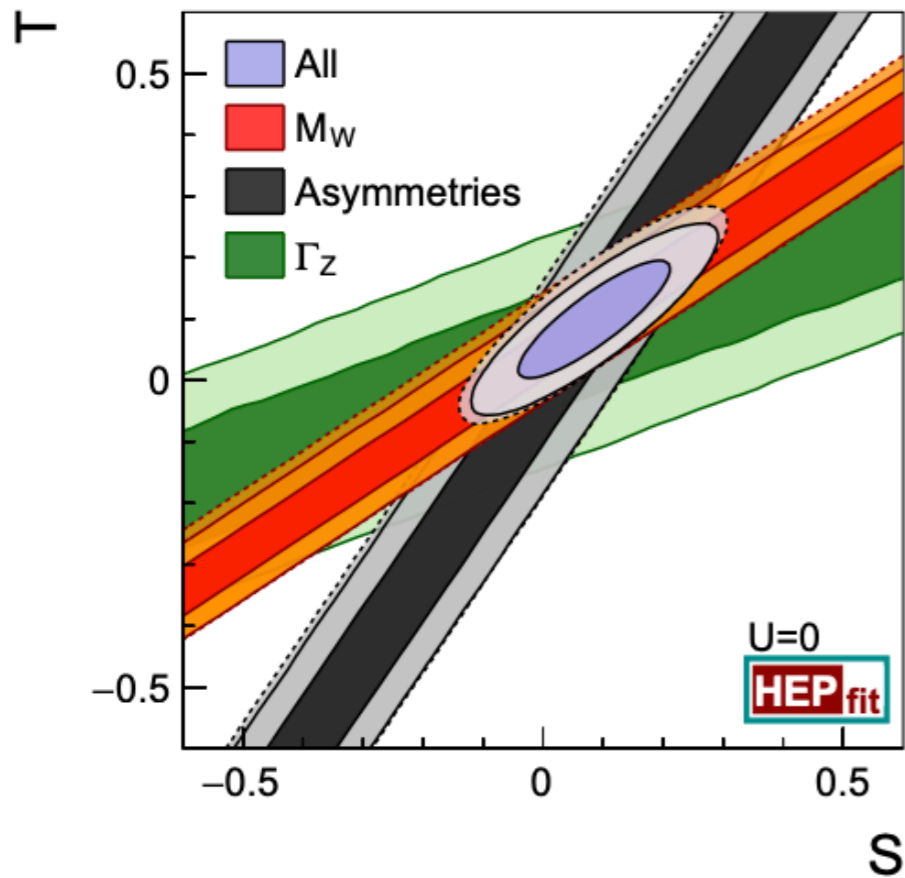


Blue = prediction of m_t, M_W by fitting "pole observables" in the SM, with crucial inclusion of loop effects

Green = direct measurements of m_t, M_W

Constraints from pole observables

Standard parameters: \hat{S}, \hat{T} or ϵ_3, ϵ_1



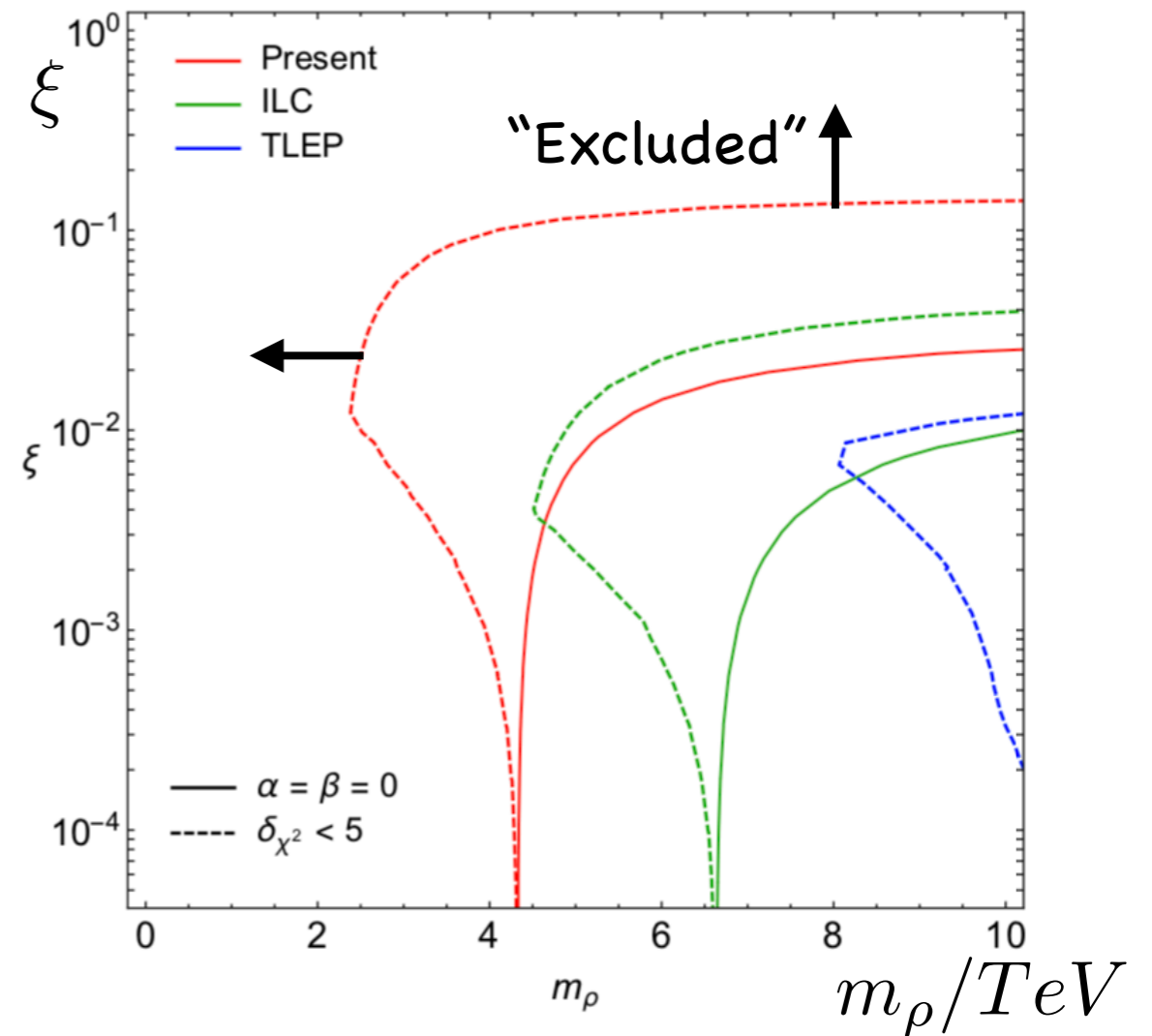
In a composite Higgs picture:

$$\Delta \hat{S} = \frac{g^2}{96\pi^2} \xi \log \left(\frac{\Lambda}{m_h} \right) + \frac{m_W^2}{m_\rho^2} + \alpha \frac{g^2}{16\pi^2} \xi,$$

$$\Delta \hat{T} = -\frac{3g'^2}{32\pi^2} \xi \log \left(\frac{\Lambda}{m_h} \right) + \beta \frac{3y_t^2}{16\pi^2} \xi,$$

$$k_V = \sqrt{1 - \xi}$$

Thamm, Torre, Wulzer 2015

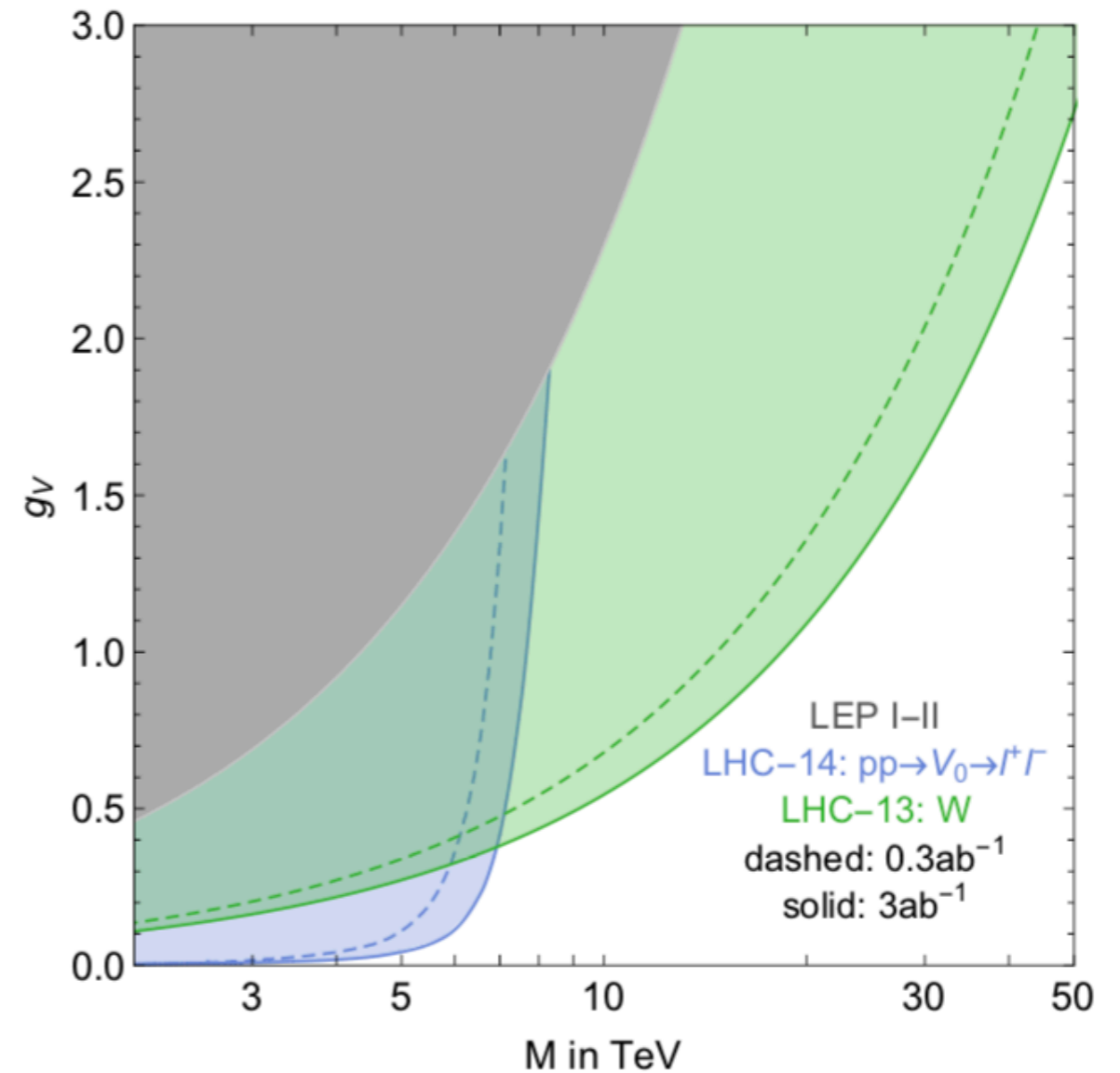
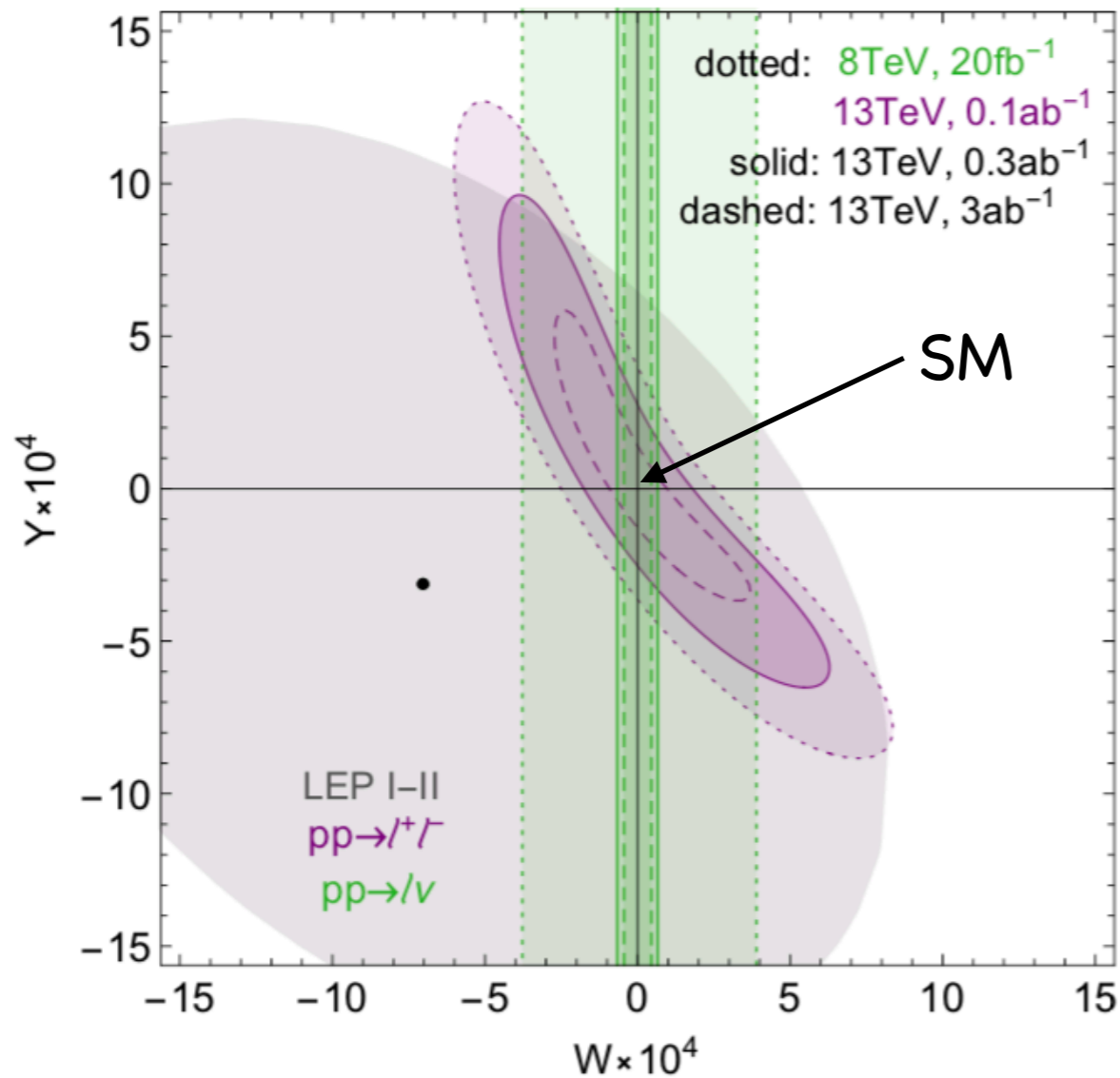


Nominally the limit on ξ , or on f better than from Higgs couplings, but the fudge factors $\alpha, \beta \dots$

Drell-Yan l^+l^- , $l\nu$ at high m_{ll}, m_{ll}^T

$$pp \rightarrow l^+l^-, l\nu$$

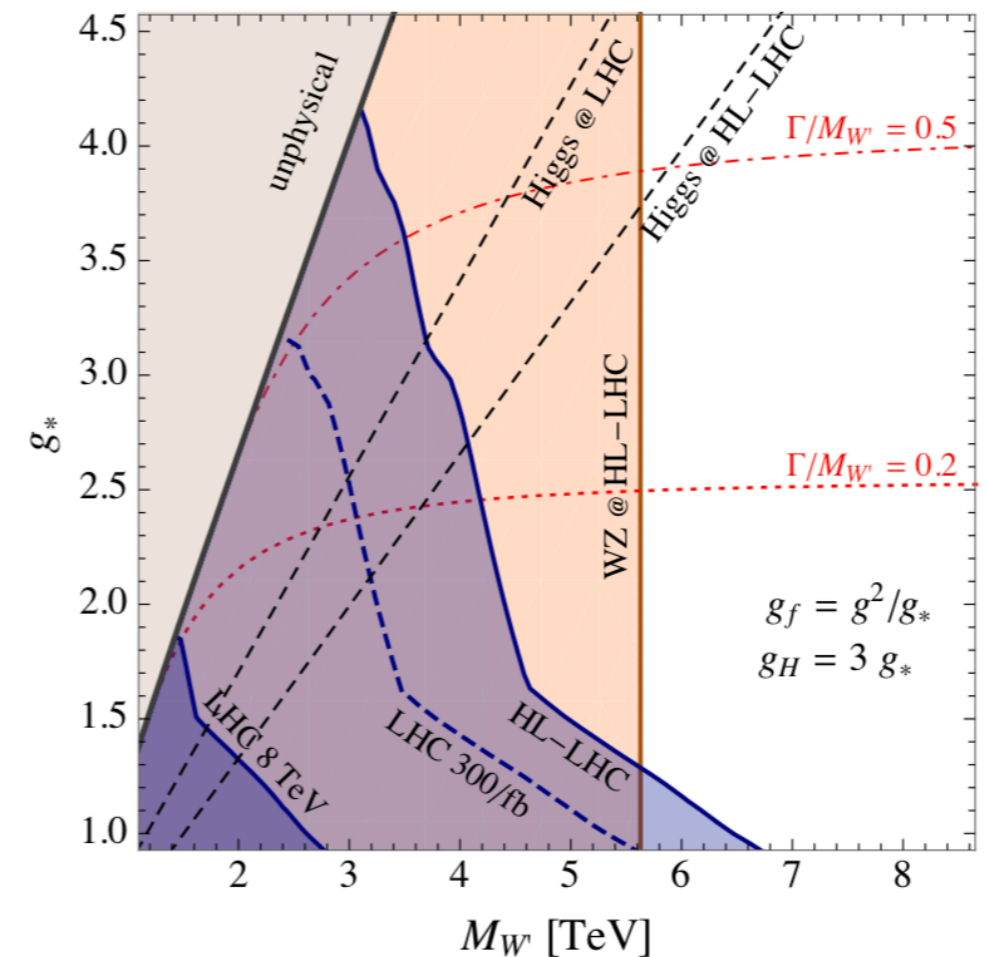
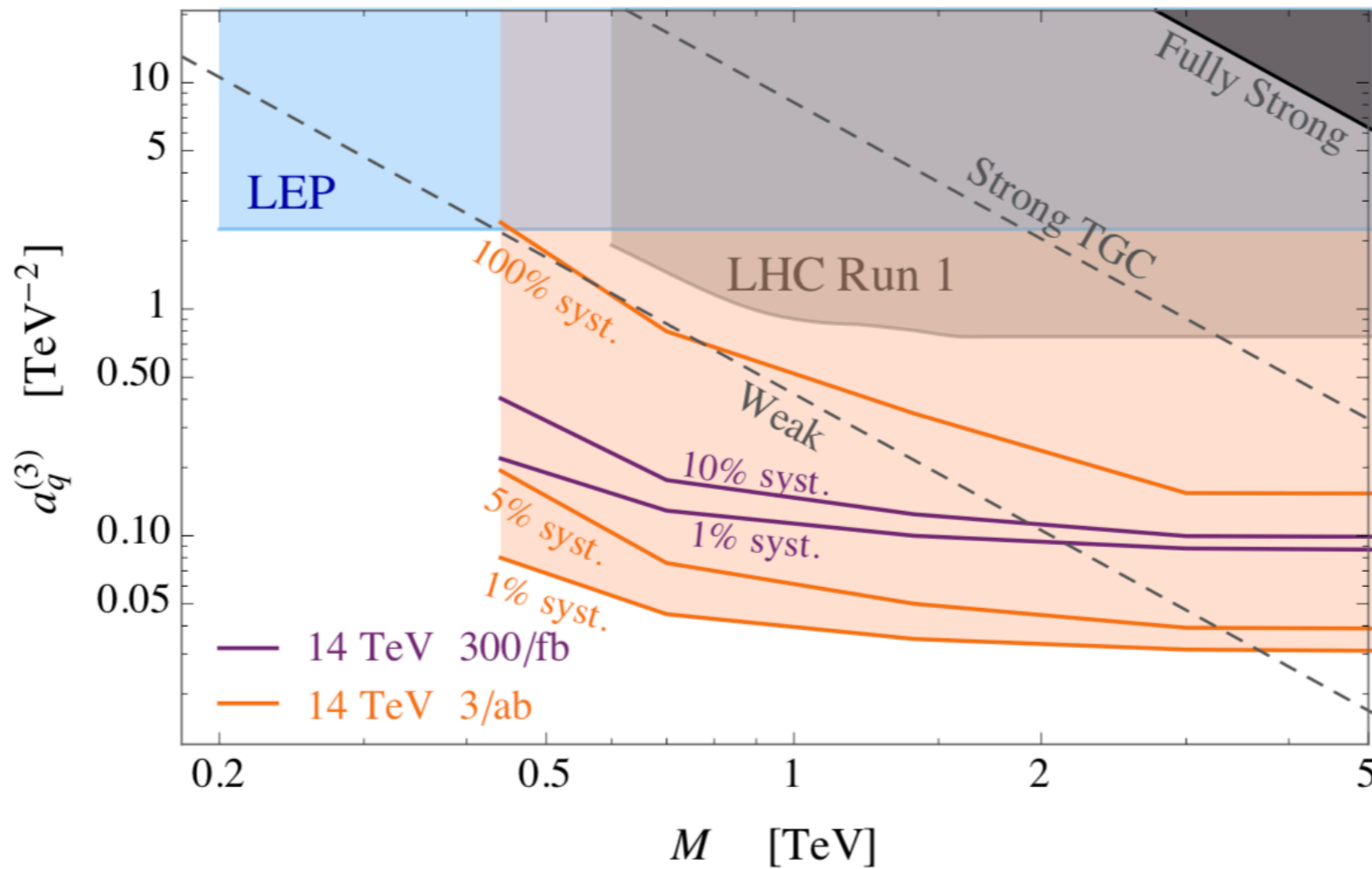
Farina et al 2016



$$\mathcal{L} = g_V V_\mu^a (f \tau^a \gamma_\mu f + i H^\dagger D_\mu H)$$

On some observables (W,Y) LEP beaten by LHC
 (if suitable precision pursued)

DiBoson differential cross section with suitable angular analyses



$$\delta A(\bar{q}q' \rightarrow WZ) \approx a_q^{(3)} E^2$$

$$a_q^{(3)} = \frac{g^2}{M^2} \div \frac{16\pi^2}{M^2} \quad \mathcal{L} = V_\mu^a (g_f \bar{f} \tau^a \gamma_\mu f + g_H i H^\dagger D_\mu H)$$

The potential of precision at LHC

- Higgs couplings

$$\mathcal{L} = -\lambda k_\lambda H^4 + g_f k_f H \bar{f} f + g_V k_V V_\mu H^\dagger \partial_\mu H$$

- ElectroWeak observables

Pole observables: $m_W, \sin\theta_{eff}^l$

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DiBosons Wh, Zh, WZ, WW

- Flavour observables

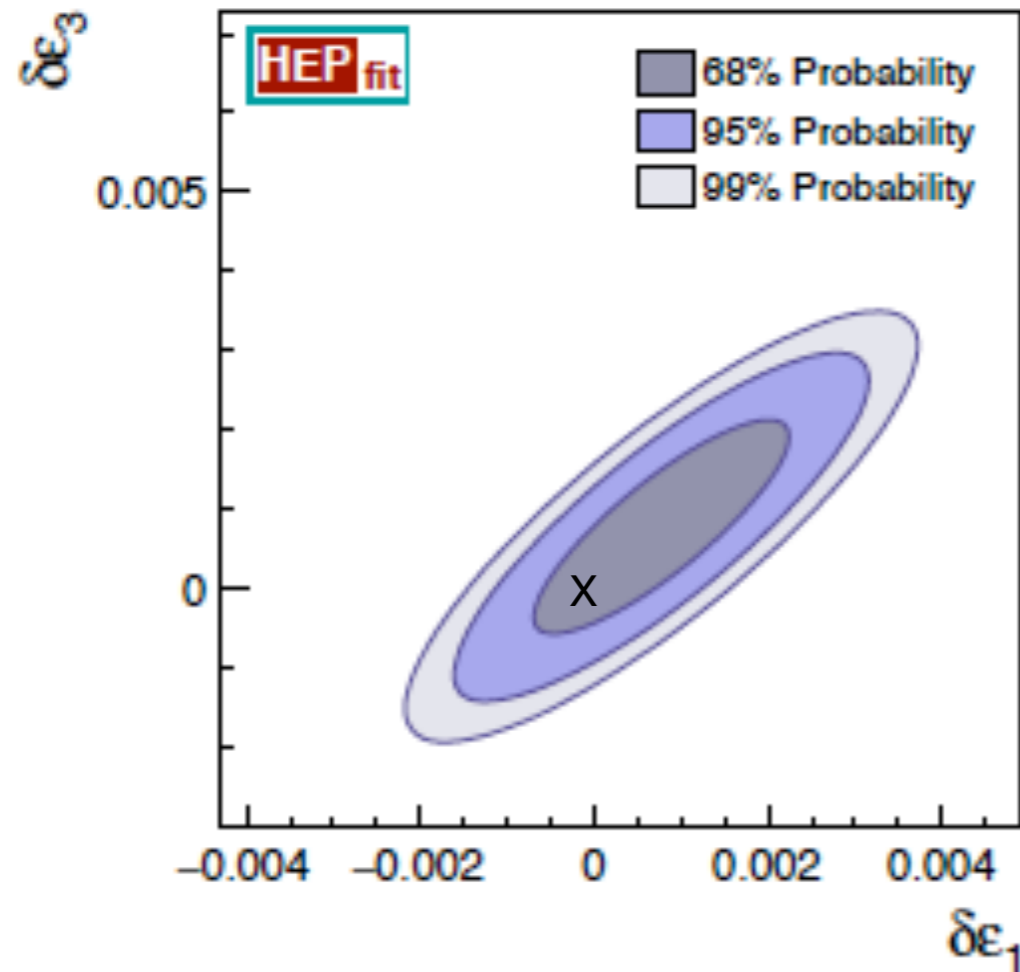
Testing the FCNC loops

Lepton Flavour Violation

The role of flavour in BSM

FCNC versus EWPT: a significant comparison

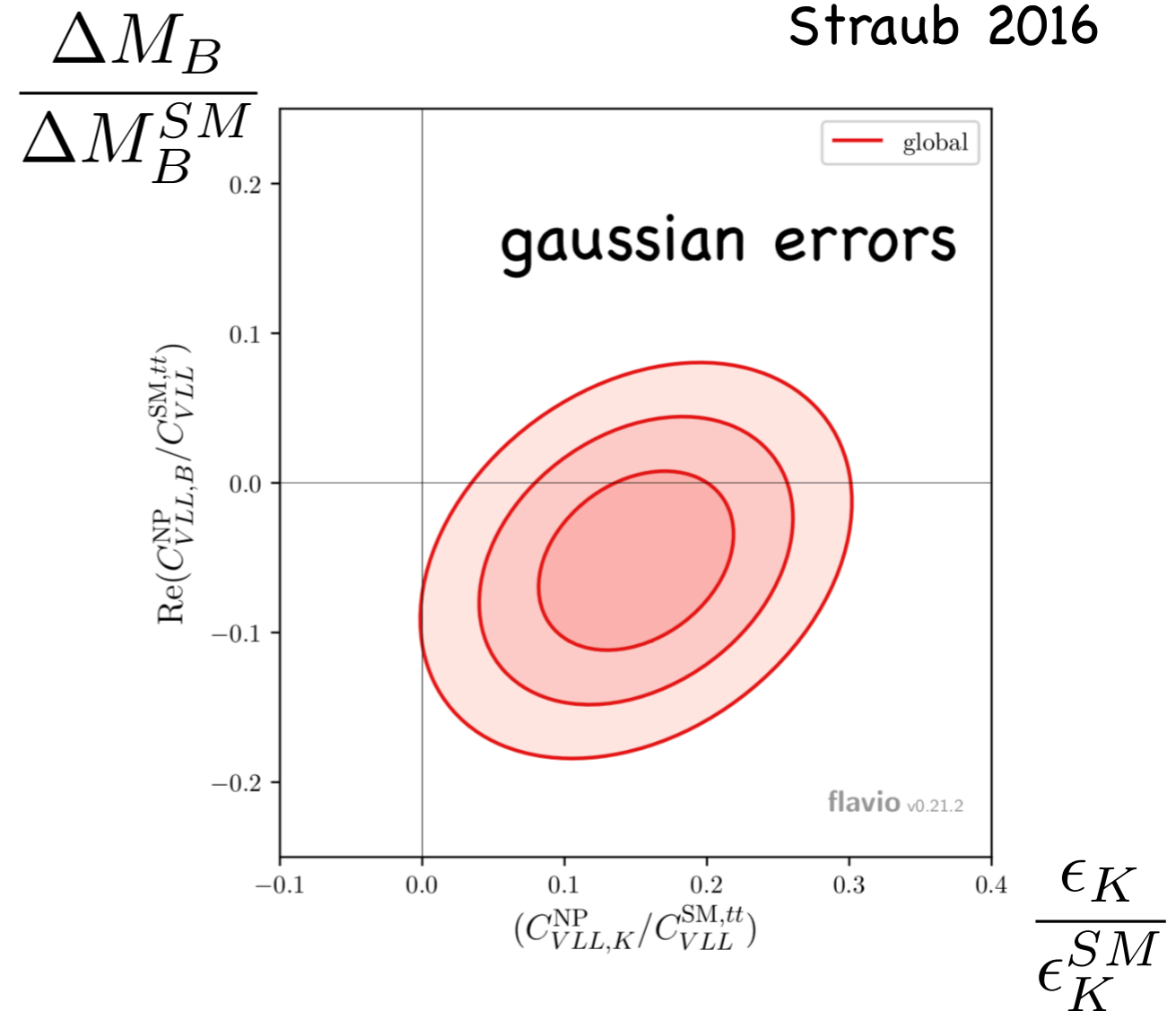
$$\epsilon_1^{SM} = 5.21 \cdot 10^{-3}, \quad \epsilon_3^{SM} = 5.28 \cdot 10^{-3}$$



measures EW loops
at about 20% level

A future facility (FCCee, ...) could go to 2% level

Straub 2016

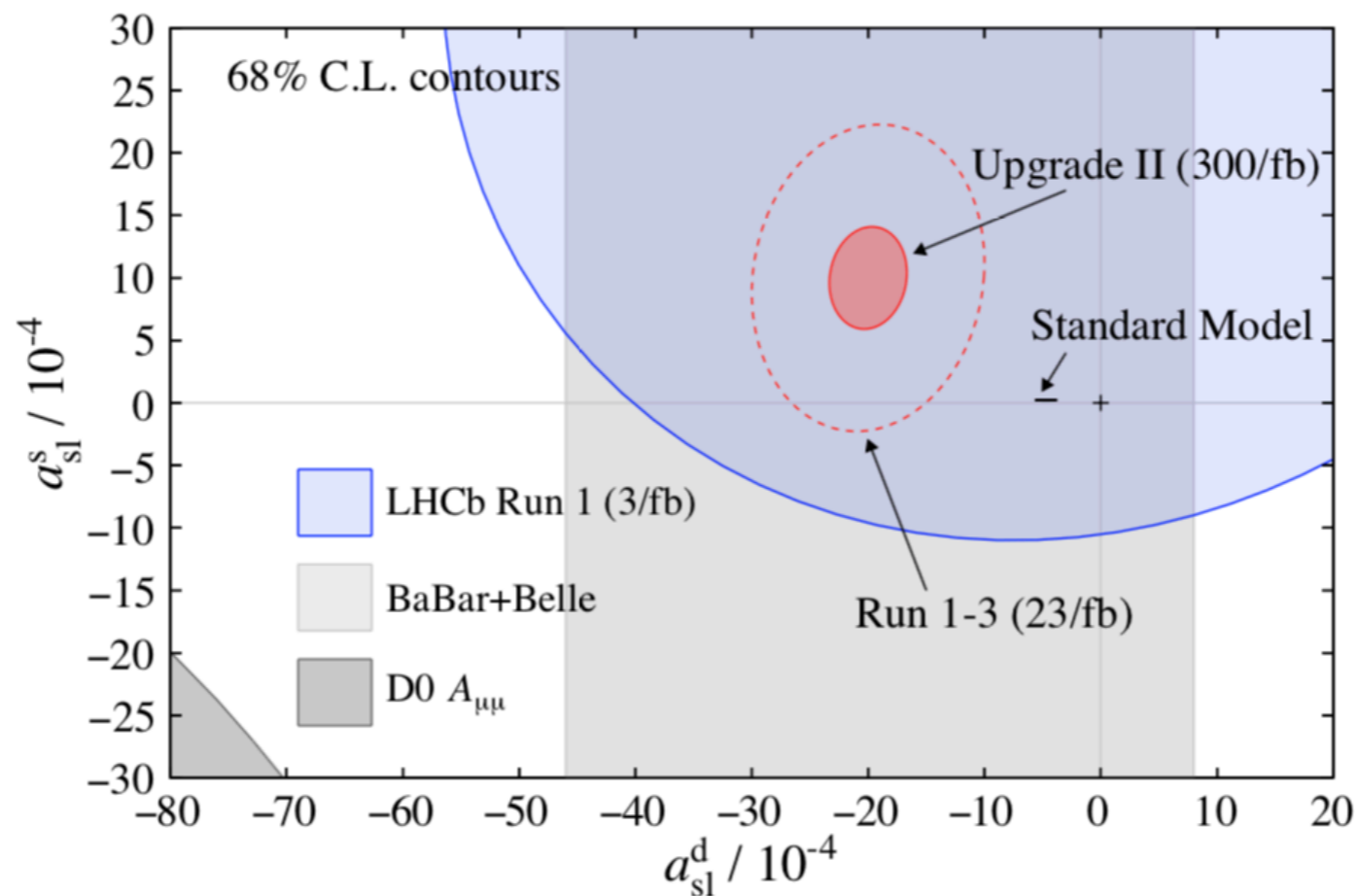


measures FCNC loops
at about 20% level

An "aggressive" flavour program could go to 2% level

Several totally clean observables

$$a_{\text{sl}}^q = \frac{\Gamma(\bar{B}_q^0 \rightarrow f) - \Gamma(B_q^0 \rightarrow \bar{f})}{\Gamma(\bar{B}_q^0 \rightarrow f) + \Gamma(B_q^0 \rightarrow \bar{f})} \approx \frac{\Delta\Gamma_q}{\Delta M_q} \tan \phi_{12}^q$$

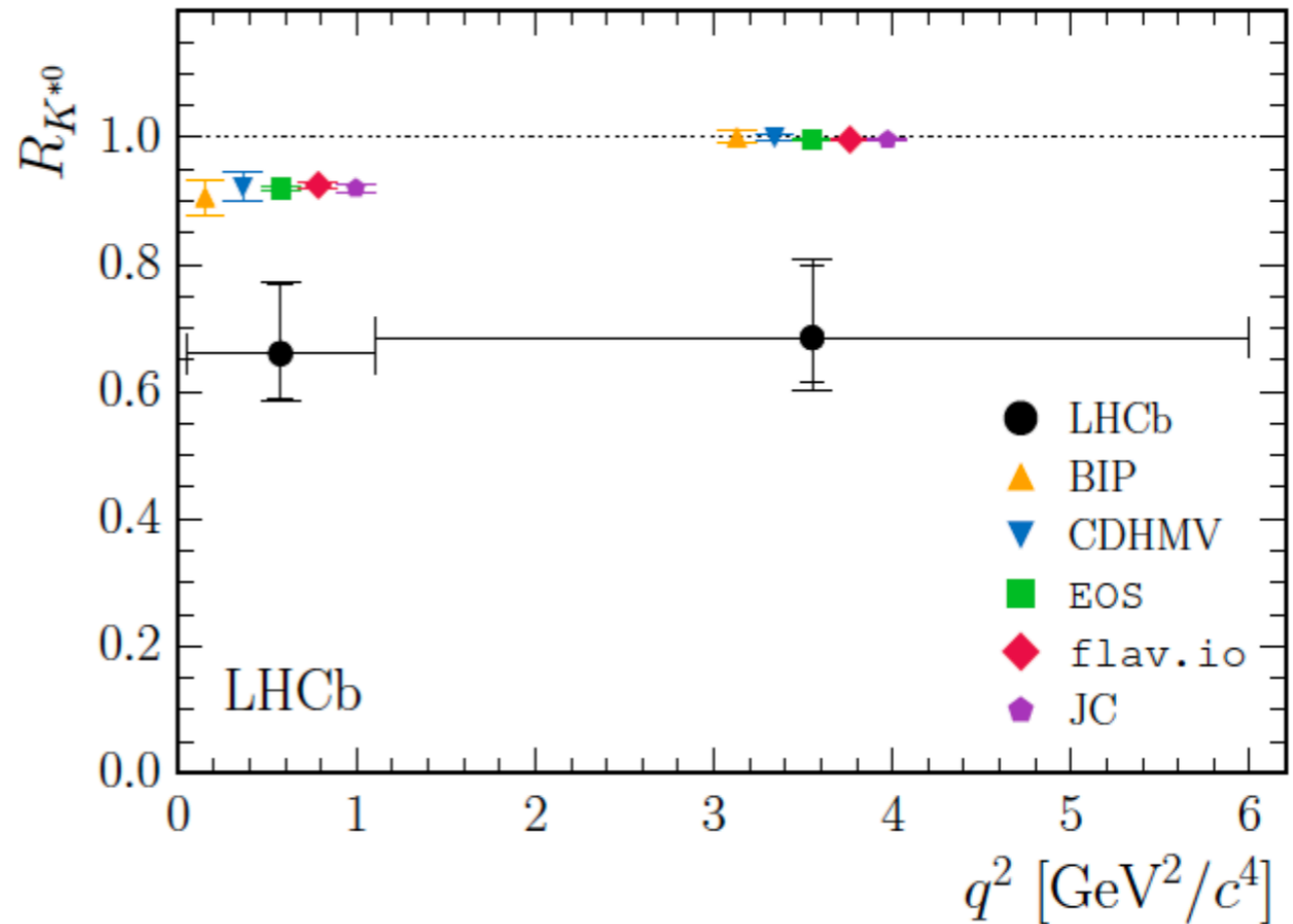
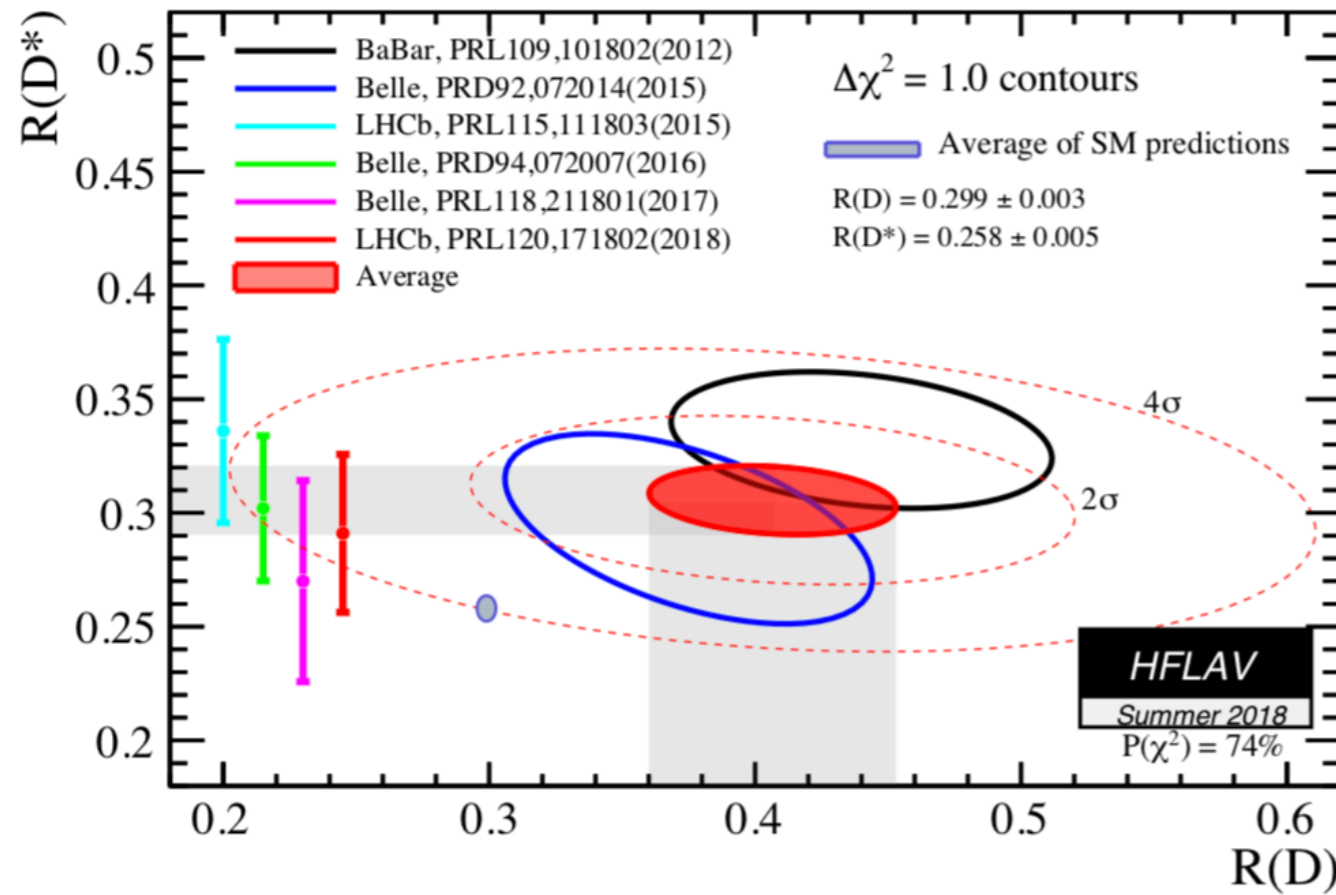


and many others controllable by multiple channel measurements
(especially in the charm case)

Lepton Flavour Violation

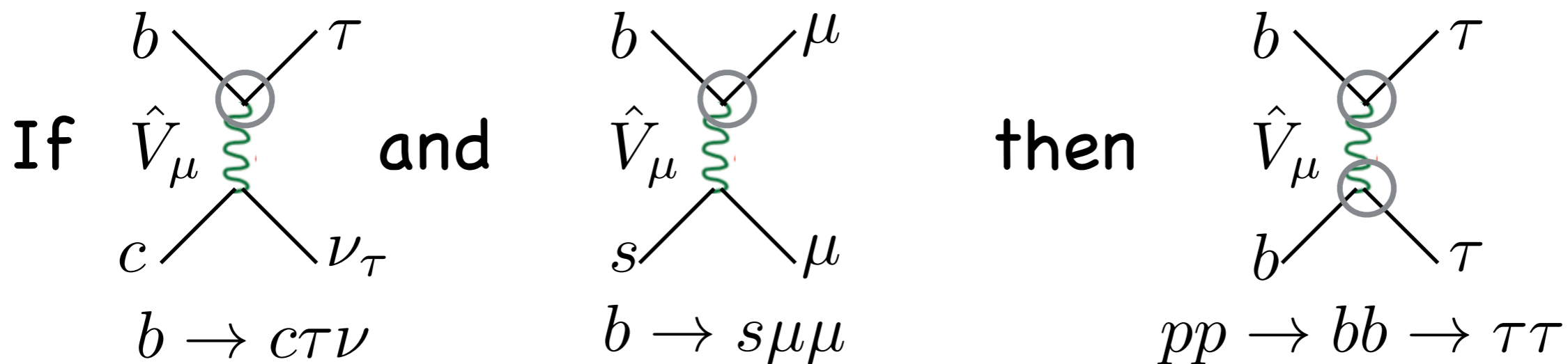
$$R_{D^{(*)}} = \frac{BR(B \rightarrow D^{(*)}\tau\nu)}{BR(B \rightarrow D^{(*)}l\nu, l = \mu, e)}$$

$$R_{K^{(*)}} = \frac{BR(B \rightarrow K^{(*)}\mu\mu)}{BR(B \rightarrow K^{(*)}ee)}$$



Observable	Current LHCb	LHCb 2025	Upgrade II
EW Penguins			
$R_K (1 < q^2 < 6 \text{ GeV}^2 c^4)$	0.1 [4]	0.025	0.007
$R_{K^{*}} (1 < q^2 < 6 \text{ GeV}^2 c^4)$	0.1 [5]	0.031	0.008
$b \rightarrow c\ell^{-}\bar{\nu}_l$ LUV studies			
$R(D^{*})$	0.026 [15, 16]	0.0072	0.002
$R(J/\psi)$	0.24 [17]	0.071	0.02

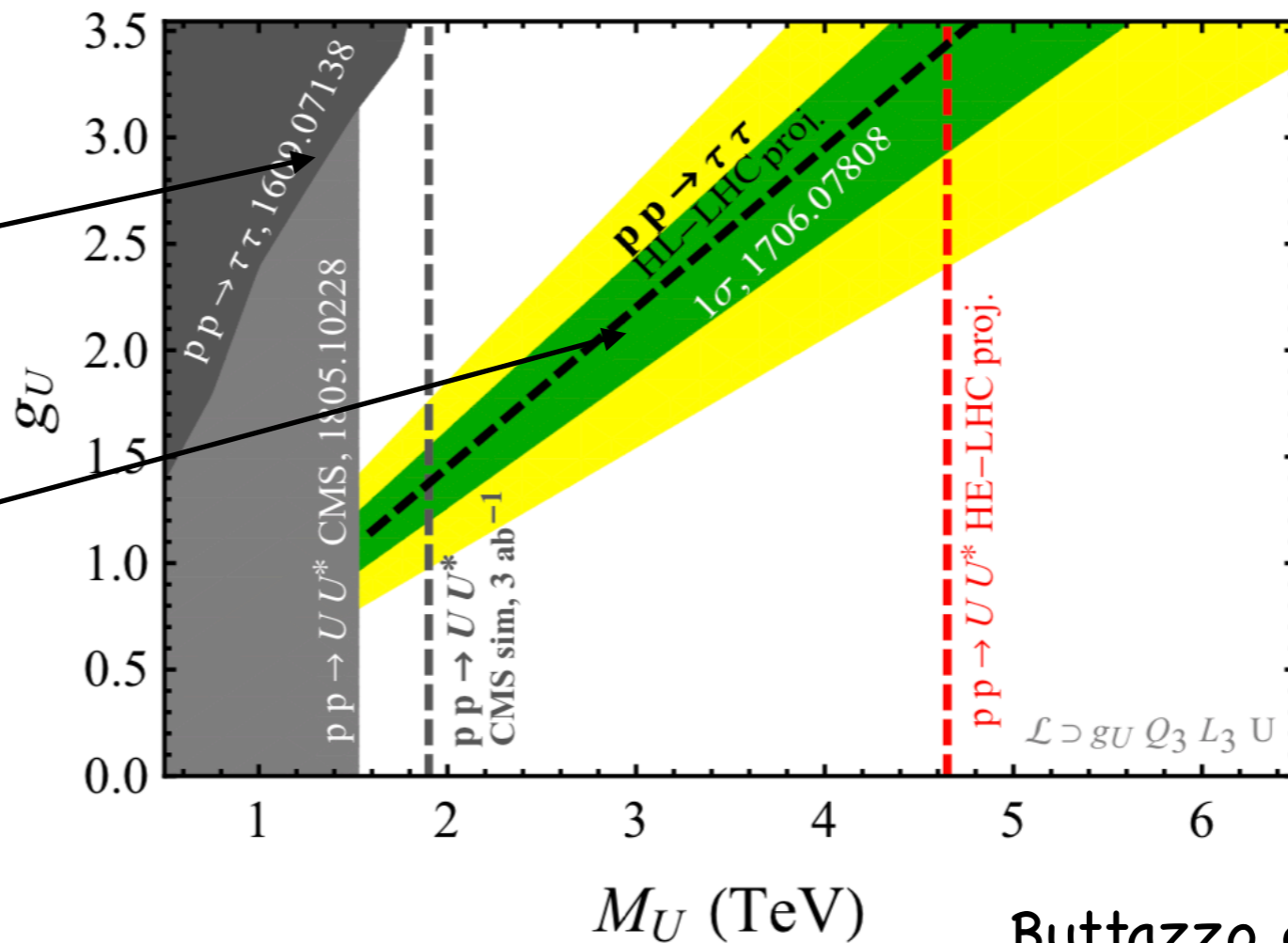
A perfect example of complementarity



The only unknown is

$$\frac{g_V^{c\nu}}{g_V^{b\tau}} \approx V_{cb}$$

$$\frac{g_V^{c\nu}}{g_V^{b\tau}} \approx 5V_{cb}$$

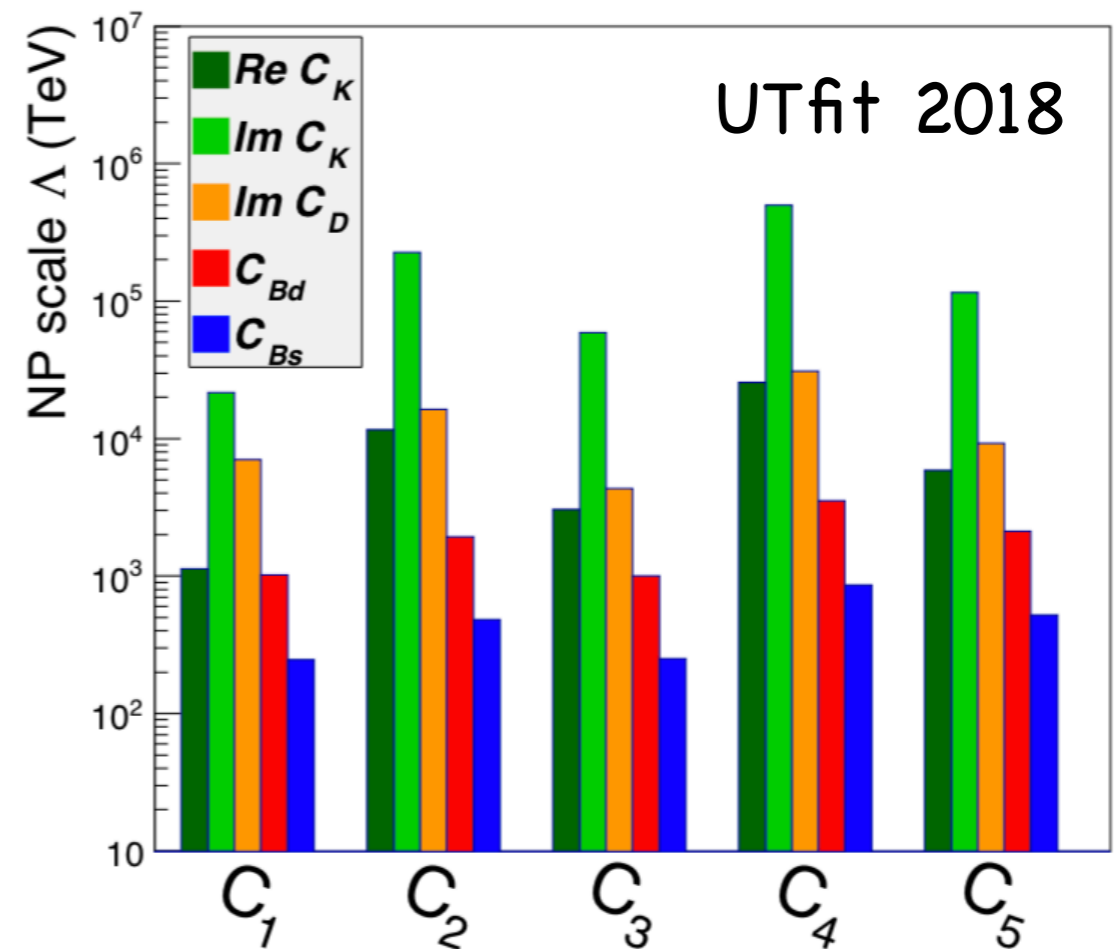


Which attitude towards flavour in BSM?

1. Flavour physics confined to high energy
(the prevailing lore)

$$\mathcal{L} = \mathcal{L}_{SM} + \sum_i^\alpha \frac{C_i^\alpha}{\Lambda_i^\alpha} (\bar{f} f \bar{f} f)_i^\alpha$$

$i = 1, \dots, 5$ = different Lorentz structures



2. New physics at the TeV scale hidden by a suitable (approximate) flavour symmetry

If so, a special role played by the third generation, special because of its masses and (in the quarks) its small mixing with the first two generations $10^{-(2 \div 3)}$

An “Extreme Flavour” experiment?

Vagnoni - SNS, 7-10 Dec 2014

- Currently planned experiments at the HL-LHC will only exploit a small fraction of the huge rate of heavy-flavoured hadrons produced
 - ATLAS/CMS: full LHC integrated luminosity of 3000 fb^{-1} , but limited efficiency due to lepton high p_T requirements
 - LHCb: high efficiency, also on charm events and hadronic final states, but limited in luminosity, 50 fb^{-1} vs 3000 fb^{-1}

• Would an experiment capable of exploiting the full HL-LHC luminosity for flavour physics be conceivable?

- Aiming at collecting $O(100)$ times the LHCb upgrade luminosity
→ 10^{14} b and 10^{15} c hadrons in acceptance at $L=10^{35} \text{ cm}^{-2}\text{s}^{-1}$

Motivation: test CKM (FCNC loops)
from $\approx 20\%$ to $\approx 1\%$

More than one (motivated) scalar

(MSSM, NMSSM, etc)

- “Inert” doublet Dark Matter: H_1, H_2

$$H_2 : \quad \langle H_2 \rangle = 0, \quad H_2 \bar{f} f \text{ forbidden}$$

The lightest member of H_2 , if neutral, is a DM candidate

- “Singlet-Catalysed” EW phase transition: H, S

$$\Delta V = \lambda_1 M (H^+ H) S + \lambda_2 (H^+ H) S^2$$

Can induce a first order phase transition, crucial to Baryogenesis

- “Twin” Higgs: H, H'

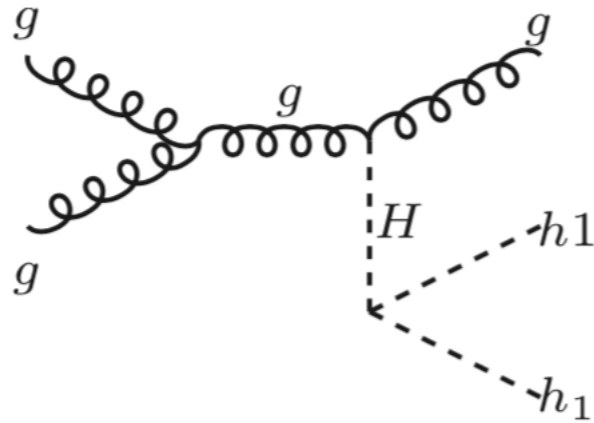
H' = doublet of a “twin” SU(2)

$$V(H, H') \rightarrow V(\mathcal{H}), \quad |\mathcal{H}|^2 = |H|^2 + |H'|^2$$

h is a pseudo-Goldstone

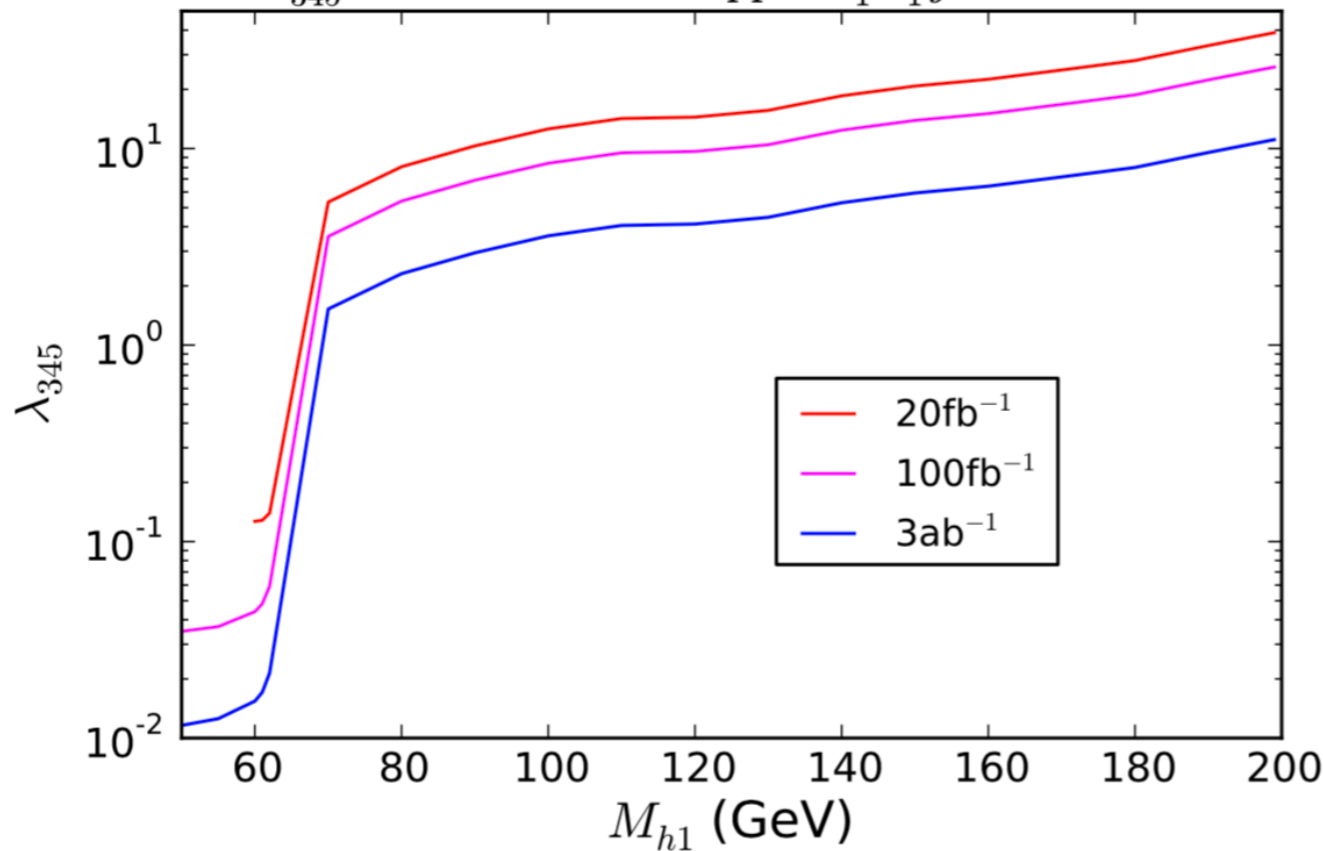
- "Inert" doublet Dark Matter: H_1, H_2

$h_1 = \text{Dark Matter}$

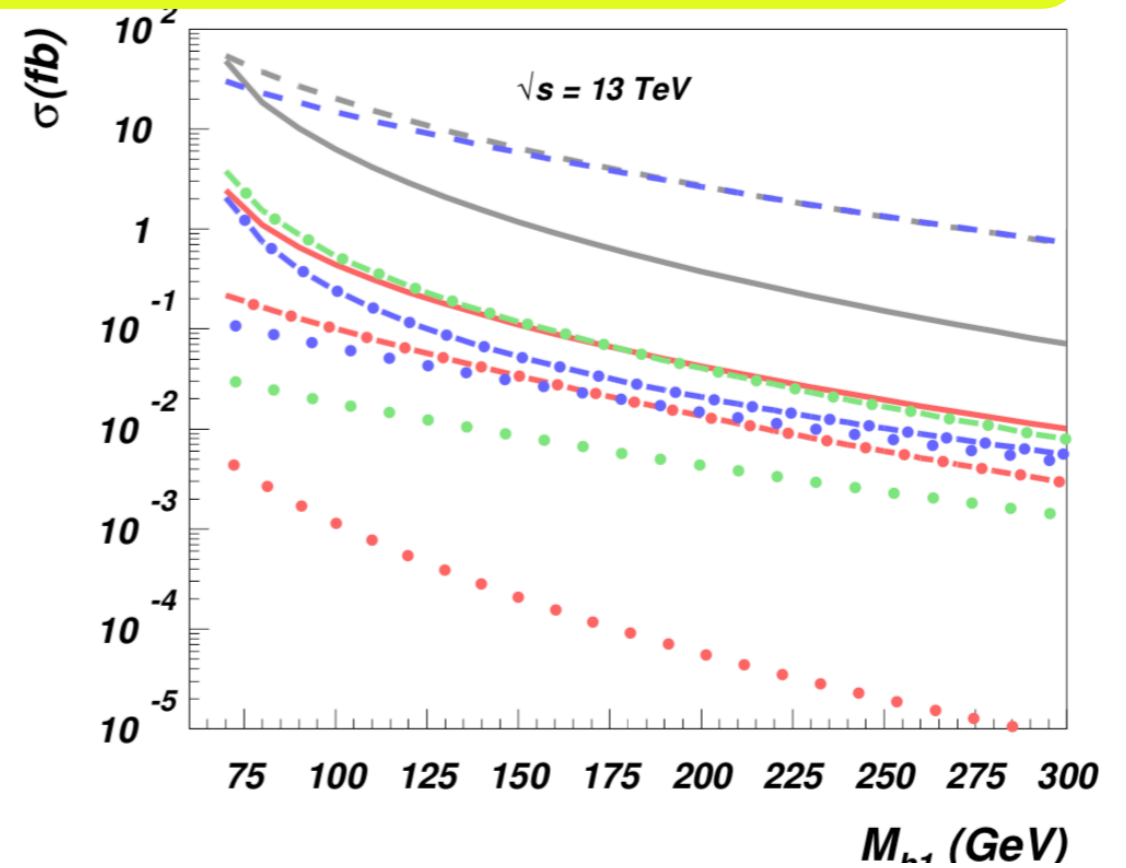


monojets

λ_{345} exclusion from $pp \rightarrow h_1 h_1 j$ at 13 TeV

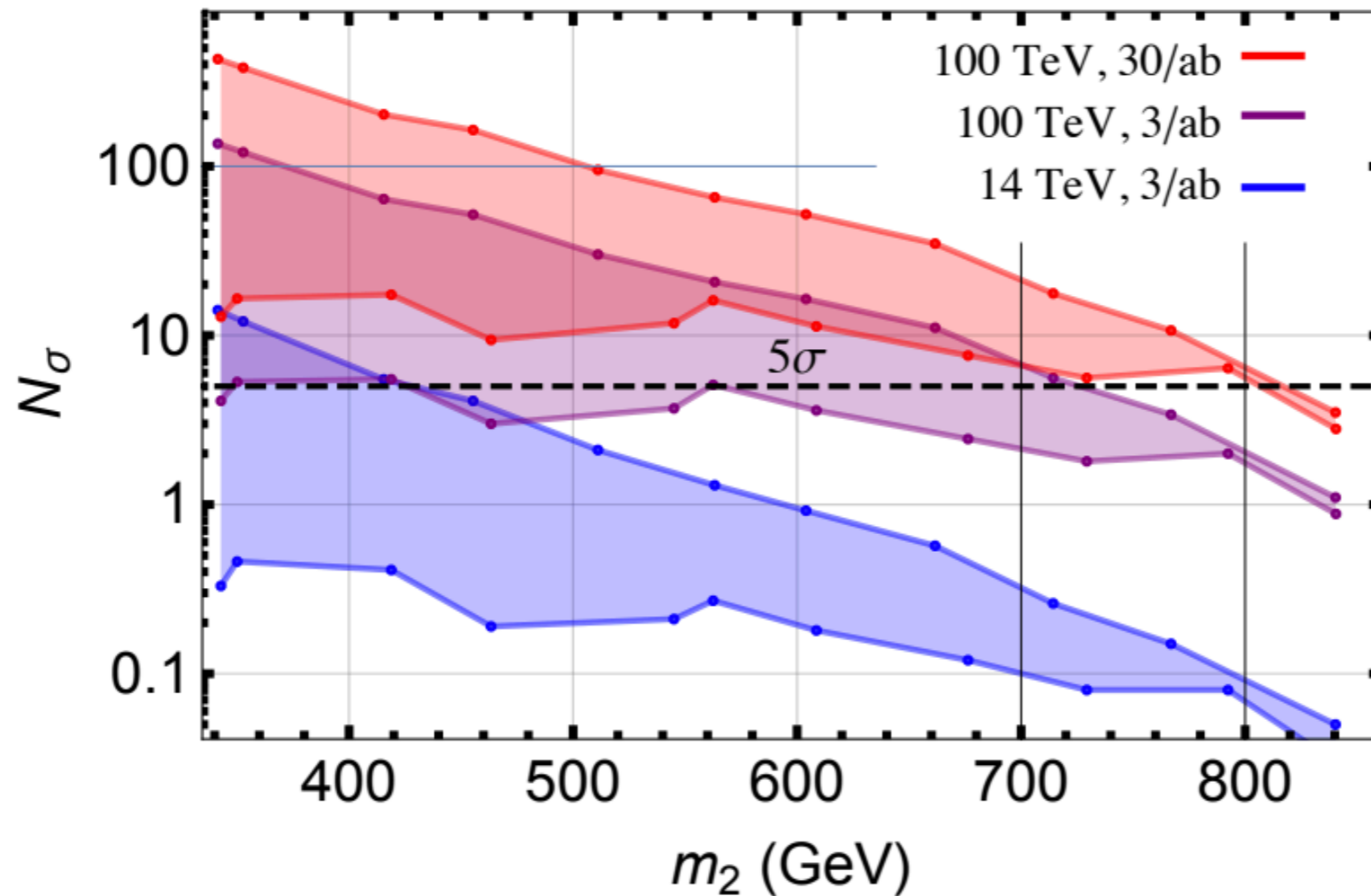


Cacciapaglia et al 2016



- $pp \rightarrow jh_1 h_1$ ($\lambda_{345}=1$)
- - $pp \rightarrow jh_1 h_2$ ($\Delta M_{h_1 h_2}=100 \text{ GeV}$)
- $gg \rightarrow Hh_1 h_1$ ($\lambda_{345}=1$)
- $gg \rightarrow Hh_1 h_2$ ($\lambda_{345}=1, \Delta M_{h_1 h_2}=100 \text{ GeV}$)
- $q\bar{q} \rightarrow Hh_1 h_2$ ($\lambda_{345}=0.002, \Delta M_{h_1 h_2}=100 \text{ GeV}$)
- $q\bar{q} \rightarrow Zh_1 h_1$ ($\lambda_{345}=1$)
- $q\bar{q} \rightarrow Zh_1 h_1$ ($\lambda_{345}=0.02$)
- - $q\bar{q} \rightarrow Zh_1 h_1$ ($\Delta M_{h_1 h_2}=100 \text{ GeV}$)
- VBF**
- $q\bar{q} \rightarrow jjh_1 h_1$ ($\lambda_{345}=1$)
- $q\bar{q} \rightarrow jjh_1 h_1$ ($\lambda_{345}=0.02$)

- "Singlet-Catalysed" EW phase transition: H, S



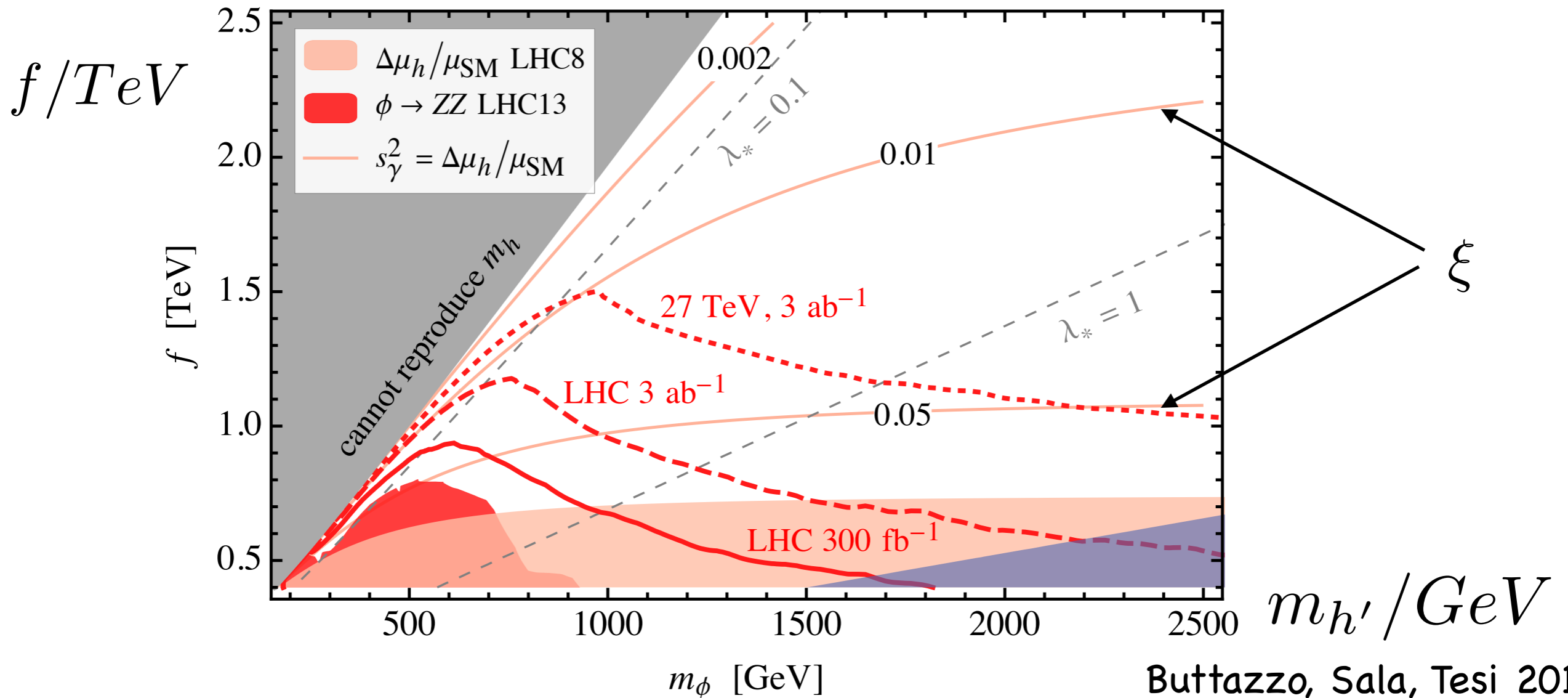
$$pp \rightarrow h_2 \rightarrow h_1 h_1 \rightarrow b\bar{b}\gamma\gamma, \tau\bar{\tau}\tau\bar{\tau}$$

- "Twin" Higgs: H, H'

$\sigma(pp \rightarrow h') \approx \xi \sigma(pp \rightarrow h_{SM}(m = m_{h'}))$ via a top loop

Neglecting phase space $\frac{\Gamma_L}{\Gamma_L + \Gamma_T} \rightarrow 1$

f	ZZ	WW	hh	$W'W'$	$Z'Z'$
$\Gamma(\tilde{h}' \rightarrow f)$	1	2	1	2	1



Summary

1. To turn the SM into a ST still premature
2. BSM more relevant than ever, though in more diversified directions than 10 years ago, rightly so
3. A significant discovery potential in precision at LHC
 - Higgs couplings
 - Extended EW precision tests
 - Flavour observableshighly complementary between themselves and with direct searches
4. A pending question: why a single scalar?

Backup on B-anomalies

general caveats

$$R_{D^{(*)}} = \frac{BR(B \rightarrow D^{(*)}\tau\nu)}{BR(B \rightarrow D^{(*)}l\nu, l = \mu, e)} \quad R_{K^{(*)}} = \frac{BR(B \rightarrow K^{(*)}\mu\mu)}{BR(B \rightarrow K^{(*)}ee)}$$

Difficult and/or statistically limited experiments

Lepton Flavour Violation never seen before

in charged leptons $BR(K_L \rightarrow \mu e) < 4.7 \cdot 10^{-12}$

No “mediator” seen in LHC searches

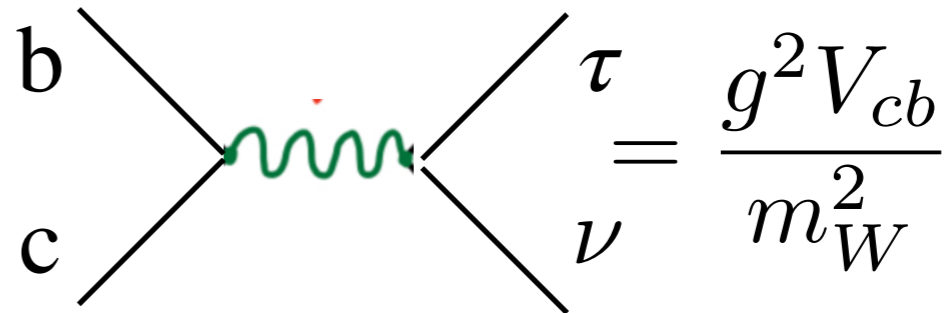
In case one wants to see them correlated:

$b \rightarrow c l \nu$ tree level, $b \rightarrow s ll$ loop level

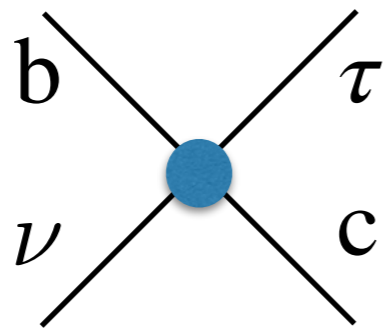
The need of a "mediator"

$\frac{R_{D^{(*)}}}{R_{D^{(*)}}^{SM}} = 1.237 \pm 0.053$ is a deviation from the SM
at about 20% level in $b \rightarrow c\tau\nu$

Need to interfere with



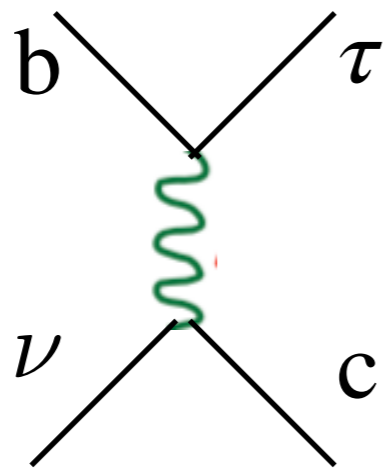
From



need

$$\Lambda \approx 500 \text{ GeV} \left(\frac{V_{cb} V_{\tau\nu}}{V_{cb}} \right)^{1/2}$$

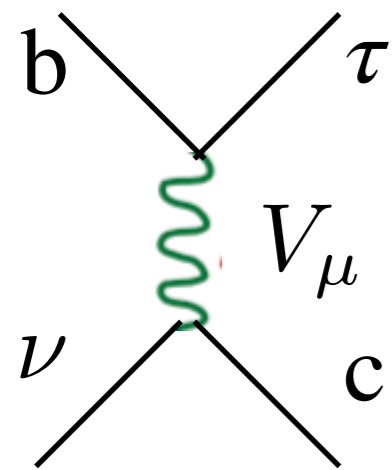
From



need

$$\frac{g}{m} \approx \frac{2}{\text{TeV}} \left(\frac{V_{cb}}{V_{cb} V_{\tau\nu}} \right)^{1/2}$$

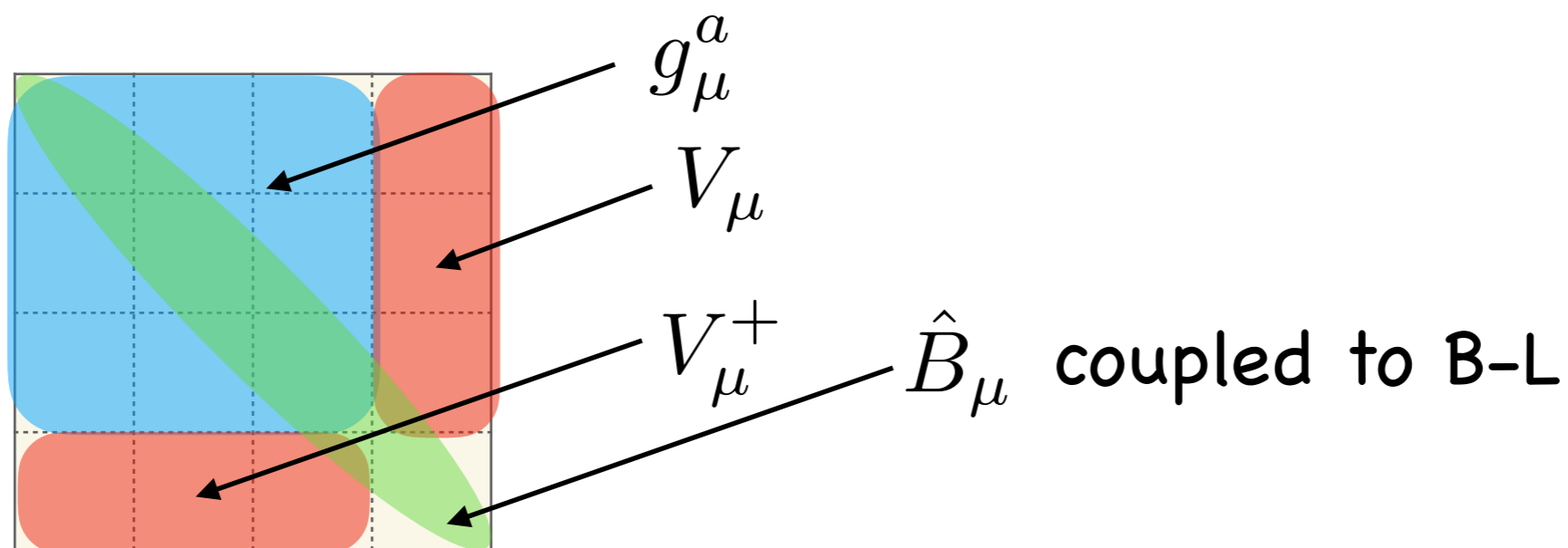
Can one make sense of a vector leptoquark?



$$V_\mu = (3, 1)_{2/3}$$

$$V_\mu^a (\bar{q}_L^a \gamma_\mu l_L) = V_\mu^a (\bar{u}_L^a \gamma_\mu \nu_L + \bar{d}_L^a \gamma_\mu e_L)$$

Pati-Salam SU(4): L as a fourth colour

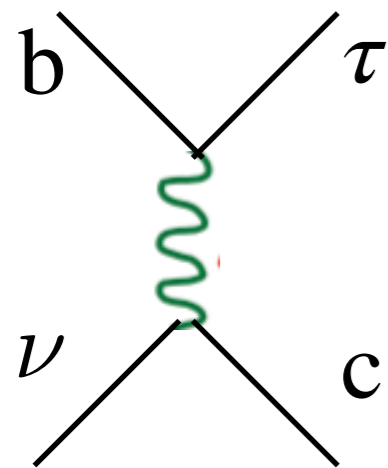


B, Isidori, Pattori, Senia 2015

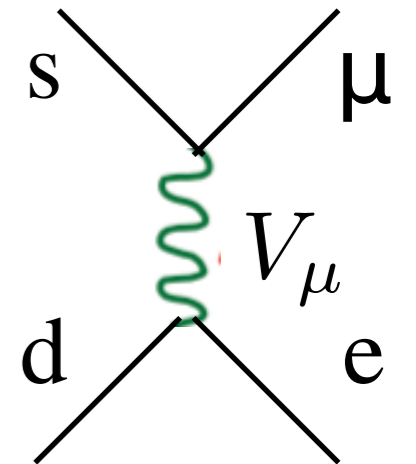
B, Murphy, Senia 2016

B, Tesi 2017

Back to $K_L \rightarrow \mu e$



$$V_{\mu}^a (\bar{q}_L^a \gamma_{\mu} l_L) = V_{\mu}^a (\bar{u}_L^a \gamma_{\mu} \nu_L + \bar{d}_L^a \gamma_{\mu} e_L)$$



$$BR(K_L \rightarrow \mu e) < 4.7 \cdot 10^{-12}$$

(s^a, μ) and (d^a, e) cannot live in the same SU(4) quartet

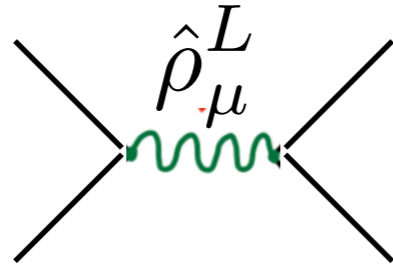
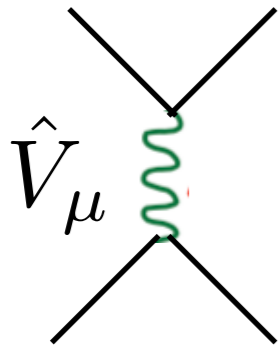
Way out:

Consider heavy $(Q^a, L)_{Dirac}$ with $V_{\mu}^a (\bar{Q}_L^a \gamma_{\mu} L_L)$
and mix them appropriately with standard q_L, l_L

(not trivial if SU(4) is a standard gauge group)

Observed anomalies

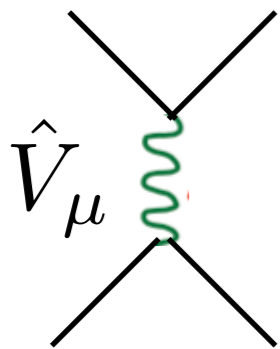
$b \rightarrow cl\nu$



$$\frac{R_{D^{(*)}}}{R_{D^{(*)}}^{SM}} = 1.237 \pm 0.053$$

$$\left(\frac{\hat{g}_G^2}{m_G^2} + \frac{\hat{g}_\rho^2}{m_\rho^2} \right) s_{l3}^2 s_{q3}^2 \approx 5/TeV^2$$

$b \rightarrow s\mu\mu$



$$\frac{R_{K^{(*)}}}{R_{K^{(*)}}^{SM}} = 0.70 \pm 0.10$$

$$\frac{s_{q2} s_{l2}}{s_{q3} s_{l3}} \frac{E_{\mu 3}}{V_{ts}} \sim 5 \cdot 10^{-3}$$

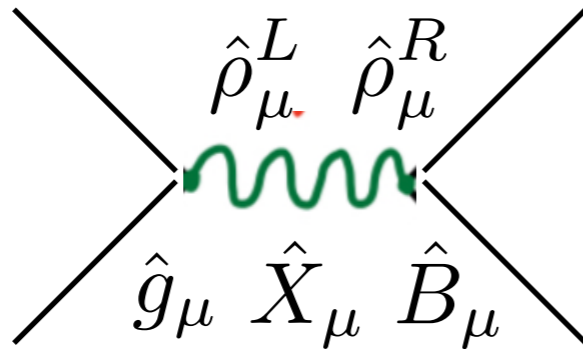
Low energy observables

$$\frac{s_{q2}s_{l2}}{s_{q3}s_{l3}} \frac{E_{\mu 3}}{V_{ts}} \sim 5 \cdot 10^{-3}$$

$b \rightarrow c(u) l \nu$

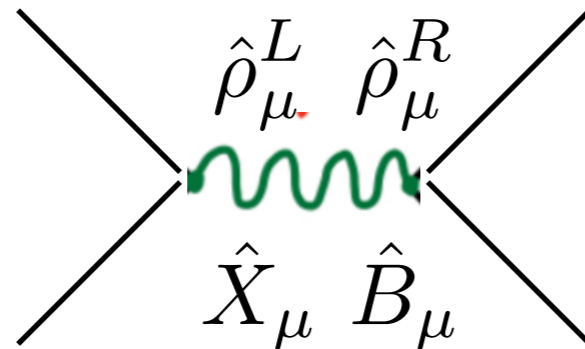
$$\begin{aligned} \text{BR}(B \rightarrow D^* \tau \nu) / \text{BR}_{\text{SM}} &= \text{BR}(B \rightarrow D \tau \nu) / \text{BR}_{\text{SM}} = \text{BR}(\Lambda_b \rightarrow \Lambda_c \tau \nu) / \text{BR}_{\text{SM}} \\ &= \text{BR}(B \rightarrow \pi \tau \nu) / \text{BR}_{\text{SM}} = \text{BR}(\Lambda_b \rightarrow p \tau \nu) / \text{BR}_{\text{SM}} = \text{BR}(B_u \rightarrow \tau \nu) / \text{BR}_{\text{SM}} \end{aligned}$$

$$\Delta C = 2$$



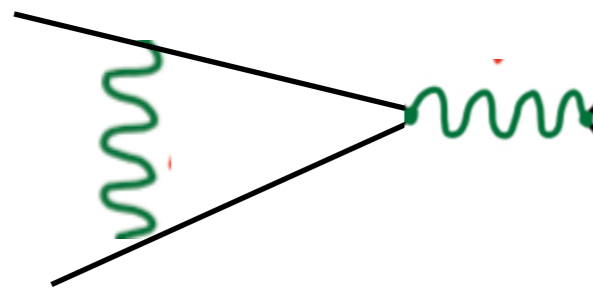
$$\frac{s_{q2}^2}{s_{q3}s_{l3}} \lesssim 10^{-3}$$

$$\tau \rightarrow 3\mu$$



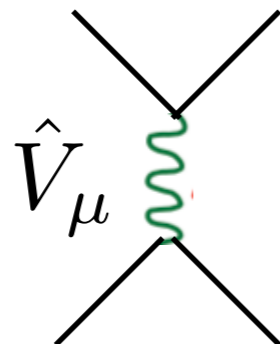
$$E_{\mu 3} \left(\frac{s_{l2}^2}{s_{l3}^2} + |E_{\mu 3}|^2 \right) \lesssim 3 \cdot 10^{-3}$$

$$\tau \rightarrow \mu \gamma$$



$$\left(A_G + \left(\frac{s_{l3}}{s_{q3}} \right)^2 A_\rho \right) E_{\mu 3} \lesssim 0.1$$

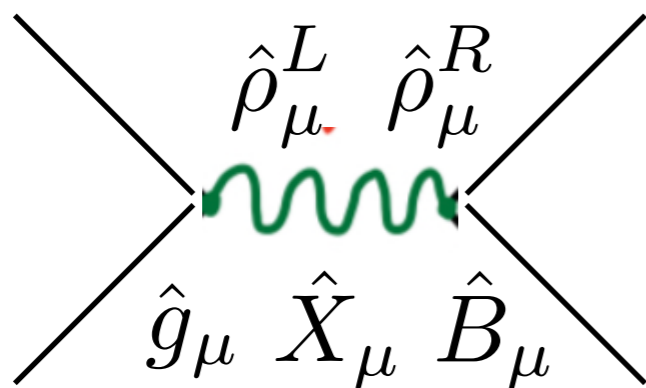
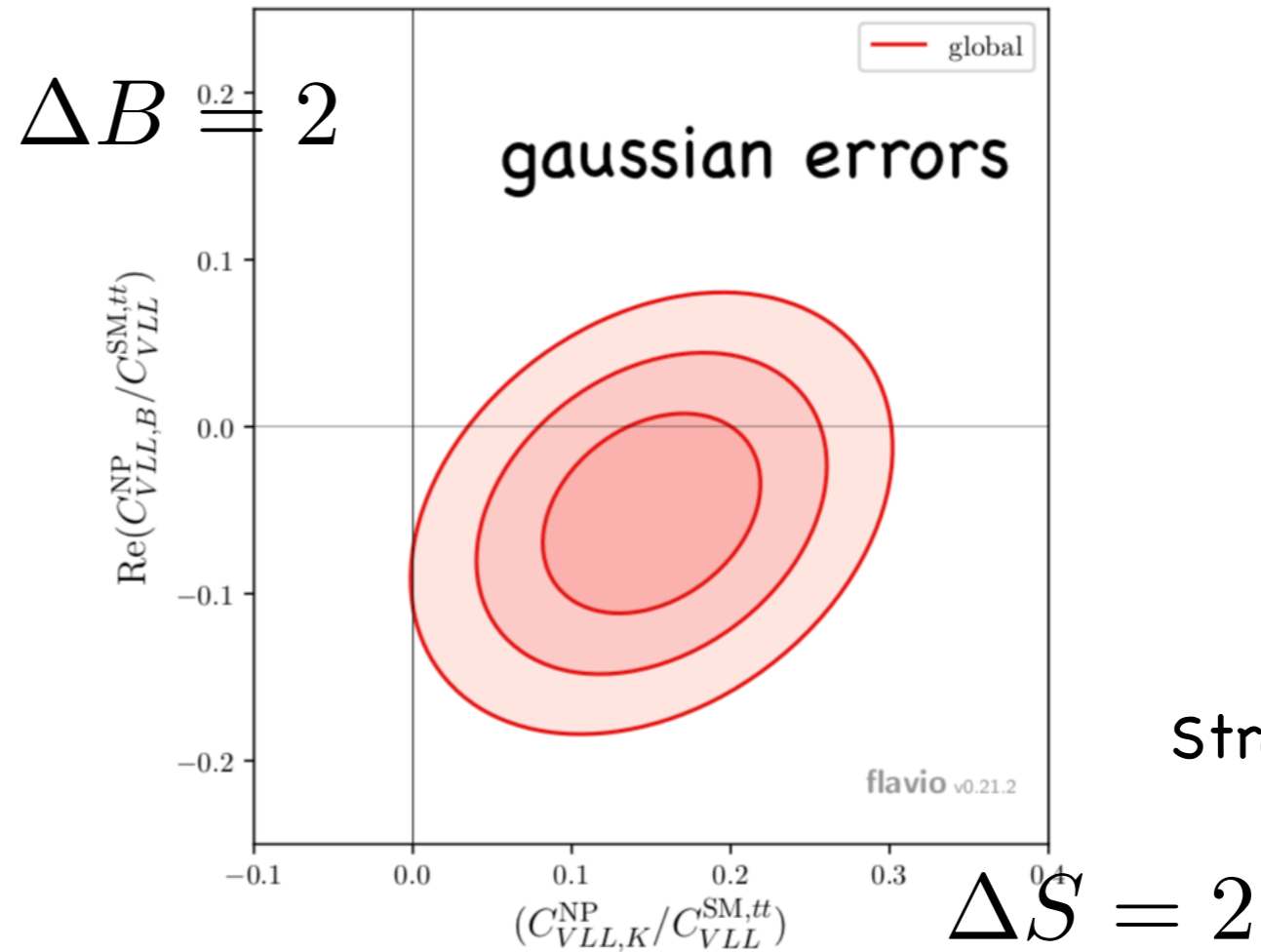
$$B^+ \rightarrow K^+ \mu^+ \tau^-$$



$$\frac{s_{q2}s_{l2}}{s_{q3}s_{l3}} \lesssim 10^{-2}$$

$$\Delta B = 2$$

Current status



$$\frac{s_{q3}}{s_{l3}} D_{s3} \lesssim 2 \cdot 10^{-3}$$

(against $V_{ts} \approx U_{t2} + D_{s3} = 4 \cdot 10^{-2}$)

Direct searches of the heavy vectors

Leptoquarks \hat{V}_μ pair produced: $gg \rightarrow \hat{V}_\mu^+ \hat{V}_\mu^-$
 $\hat{V}_\mu \rightarrow t\nu, b\tau$

\hat{V}_μ exchanged in the t-channel: $b\bar{b} \rightarrow \tau\bar{\tau}$

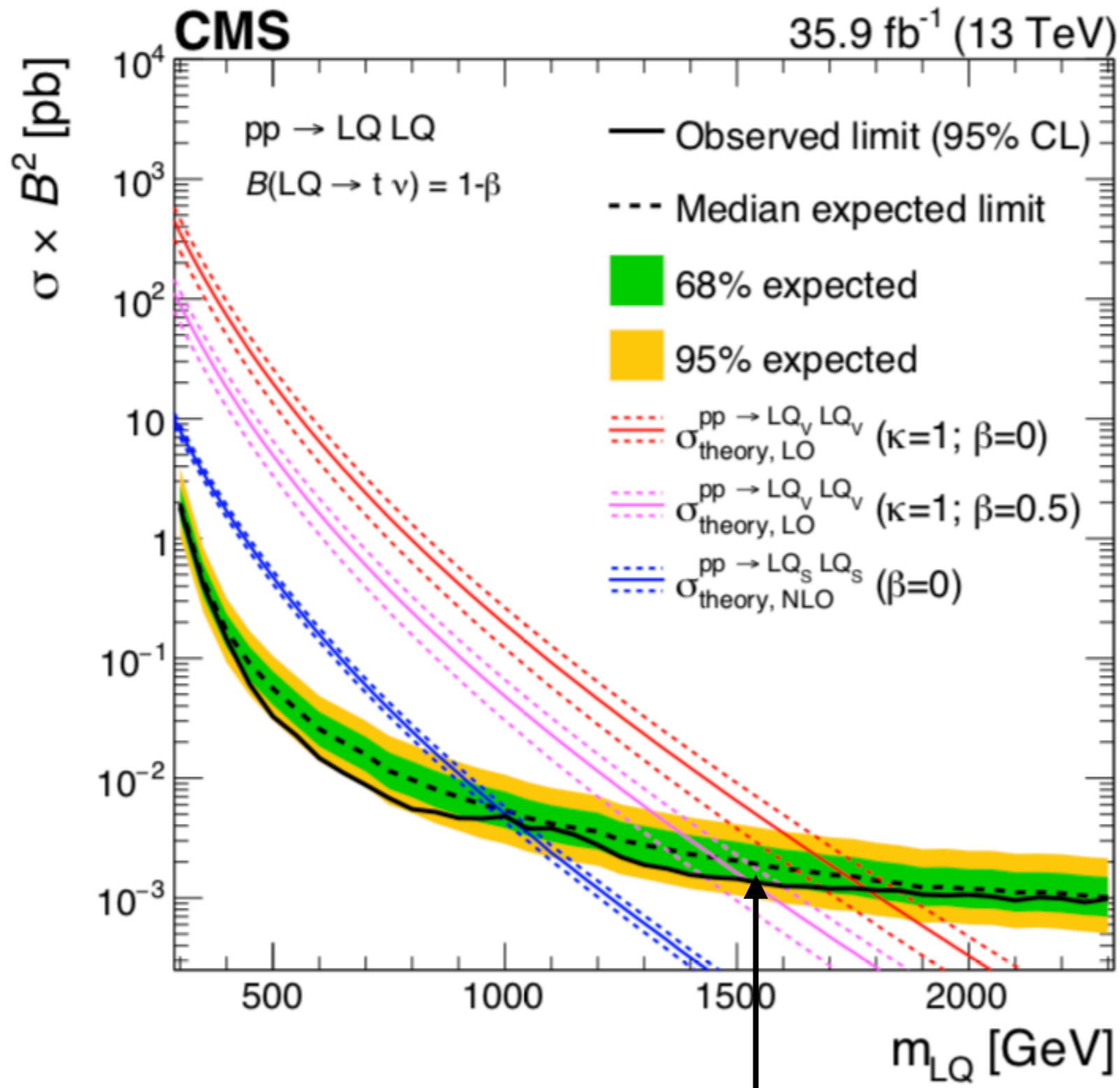
Single \hat{V}_μ production $gb \rightarrow \hat{V}_\mu \tau$

All other vectors but $\hat{\rho}_\mu^{R\pm}$: $\hat{G}_\mu^a, \hat{B}_\mu, \hat{\rho}_\mu^{La}, \hat{\rho}_\mu^{R3}, \hat{X}_\mu$
 couple to the light fermions by $F - f$ mixing (mostly f_3)
 and, flavour universally, by vector mixing

$$\hat{G}_\mu^a = \frac{g_G \mathcal{G}_\mu^a - g_3 G_\mu^a}{\sqrt{g_G^2 + g_3^2}} \implies \frac{\Gamma_{\hat{G} \rightarrow t\bar{t}}}{m_G} \approx \frac{\Gamma_{\hat{G} \rightarrow b\bar{b}}}{m_G} \approx \frac{\hat{g}_G^2 s_{q3}^4}{48\pi}$$

$$\frac{\Gamma_{\hat{G} \rightarrow u\bar{u}}}{m_G} \approx \frac{\Gamma_{\hat{G} \rightarrow d\bar{d}}}{m_G} \approx \frac{g_3^4}{24\pi g_G^2}$$

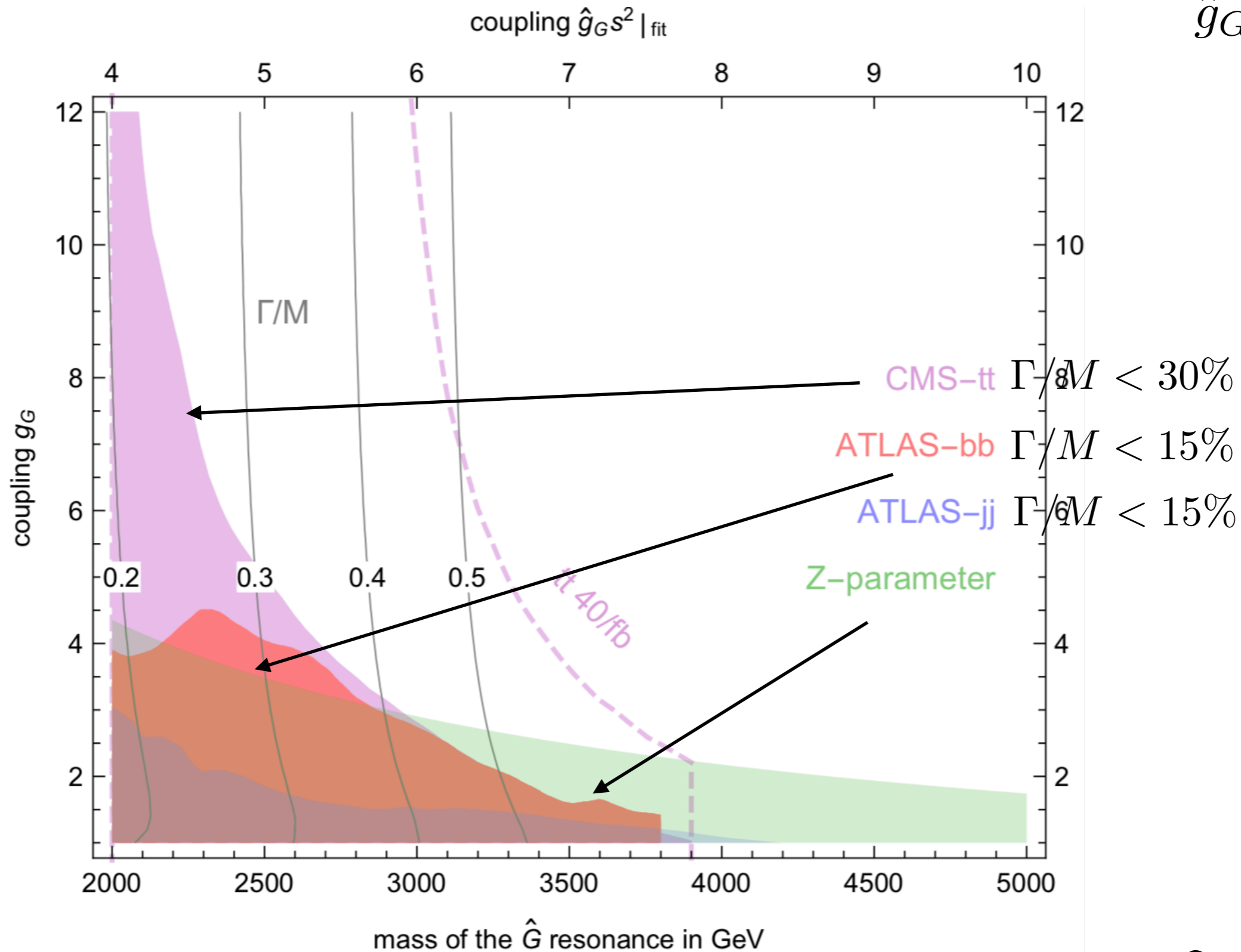
$$gg \rightarrow \hat{V}_\mu^+ \hat{V}_\mu^- \rightarrow (t\bar{\nu}_\tau)(\bar{t}\nu_\tau)$$



$$m_{\hat{V}} > 1.5 \text{ TeV}$$

$$u\bar{u}, d\bar{d}, b\bar{b} \rightarrow \hat{G} \rightarrow t\bar{t}, b\bar{b}, jj$$

$$\hat{g}_G s_{q3} s_{l3} = 2 \frac{m_G}{TeV}$$



$$m_{\hat{G}} > 2 \div 2.5 TeV$$