My goal for this school

Promoting a hierarchical framework

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Warning

Listen to this presentation as if you are going to the doctor.

The doctor

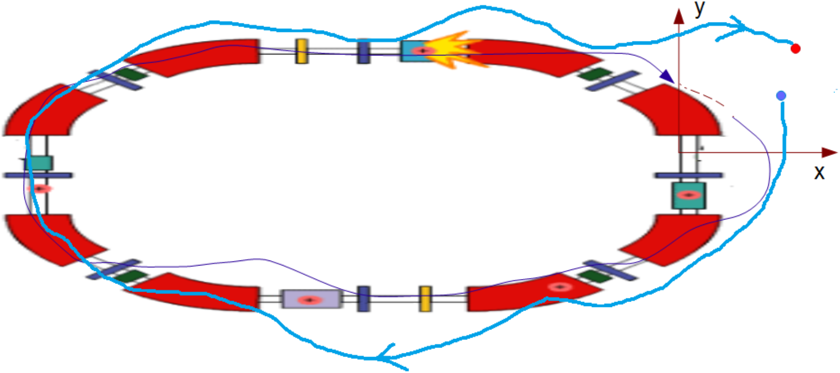
- Explains the various options
- Explains how the are tied to one another
- Explains in what order they come if needed
- Tells you the cost (the amount of work each option entails)

Then, later, in the practical section, the doctor (that is me!) can go over each option with full examples in hand.

Outline

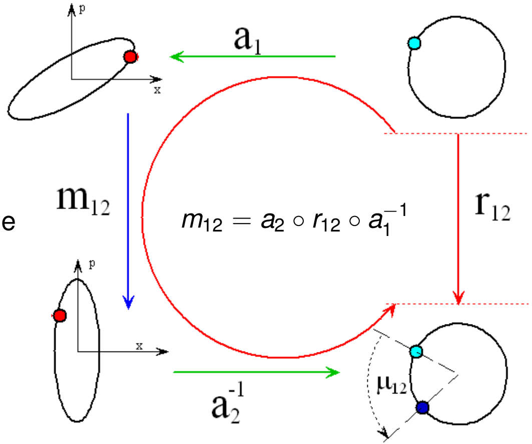
The ring and the code

A tracking code allows to compute an arbitrary trajectory around the machine: the blue curve for example



We assume the existence and the necessicity of such a code: MAD-X for example.

Generalised Courant-Snyder Theory



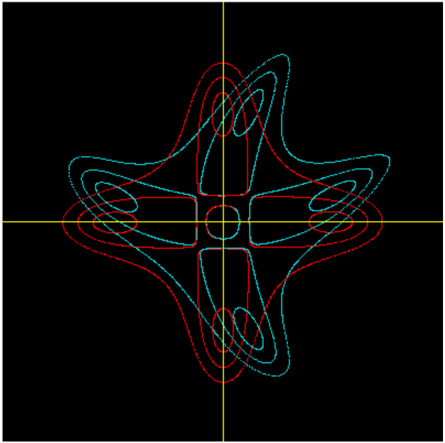
$m_1 = a_1 \circ r \circ a_1^{-1}$

$m_{12} = a_2 \circ r_{12} \circ a_1^{-1}$

m_{12} given by code

$m_2 = a_2 \circ r \circ a_2^{-1}$

One-resonance theory



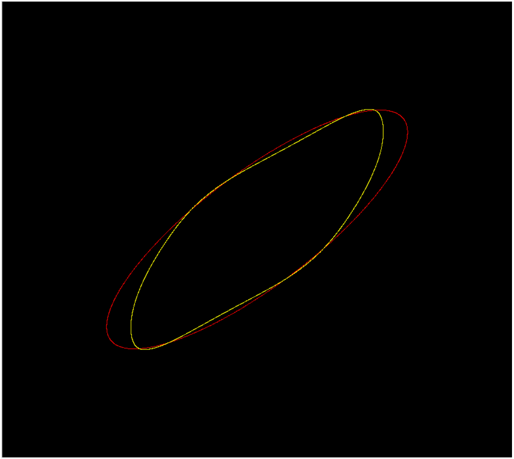
$$m = a \circ n \circ a^{-1}$$

$$n = r(\frac{-1}{4}) \circ n_\epsilon$$

$$H = log(n_\epsilon)$$

Little experiment: an octupole component in OCT

b4=500.0 in PTC units



MAD-X LATTICE

L : drift, L= 0.2;

alpha= pi/10

QF : SBEND,L= 1.0, ANGLE=ALPHA,k1=1.0;

QD : SBEND,L= 1.0, ANGLE=ALPHA,k1=-1.0;

Oct : octupole, K3= 0.0;

lattice : LINE= (QF,Oct,L,QD,Oct,L);

How can we compute the yellow curve?

1. Analyse an approximate representation of the one-turn map and propagate the analysis around the ring using the code.
2. Compute the effect of the octupole using Green's function methods
3. Concoct a smooth Hamiltonian (equation of motion) representing the entire ring and do the following: compute the effect of the octupole using Fourier transform methods on the smooth Hamiltonian.

What are the problems with each method?

How can we compute the yellow curve?

Solution: TPSA!

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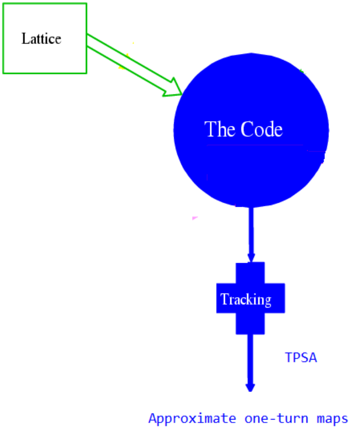
What are the problems with each method?

Example of Analytical results
Effect of Octupoles



Example of Analytical results
Effect of Octupoles

$$H = \underbrace{\frac{p^2}{2} + k_Q(s)\frac{x^2}{2}}_{H_0} + \frac{1}{4}k_o\delta(s-s_0)x^4$$

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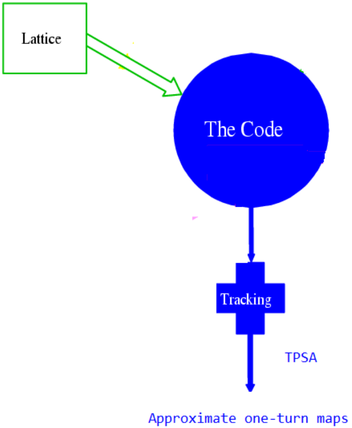

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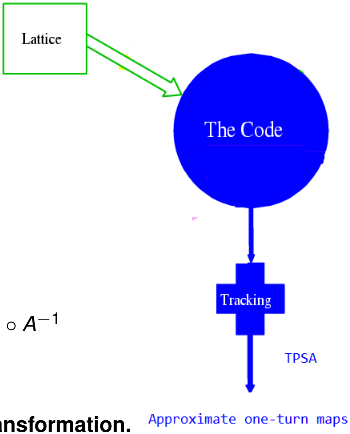

$$\Delta\varepsilon = 2i\left\{h\frac{\partial F}{\partial h} - \bar{h}\frac{\partial F}{\partial \bar{h}}\right\} \circ A^{-1}$$

where $h = x + i\,p_x$

and A^{-1} **is**
the Courant-Snyder transformation.

$$F = F_4 h^4 + F_{31} h^3 \bar{h} + C.C.$$

Example of Analytical results

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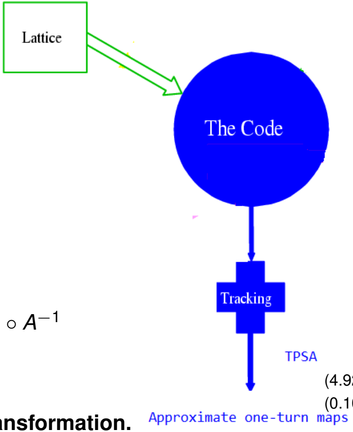

TPSA results

$$F = F_4 h^4 + F_{31} h^3 \overline{h} + C.C.$$

(4.922331235935857E-2,-4.067886707780190E-2)
(0.100630505489161,-0.213709467914908)

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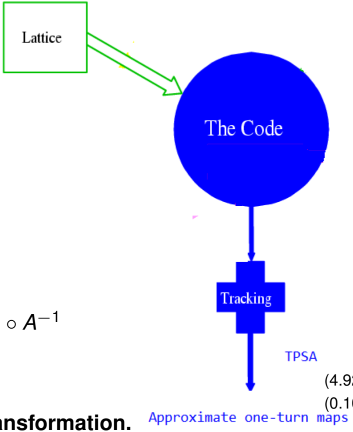

Equivalent to Green's Function analytical perturbation theory on equation of motions.

$$F = \frac{k_o\beta_0^2}{64}\left\{\frac{e^{-i4\mu_0 s_0}}{1-e^{-i4\mu}}h^4 + \frac{e^{-i2\mu_0 s_0}}{1-e^{-i2\mu}}4h^3\overline{h} + CC\right\}$$

TPSA results

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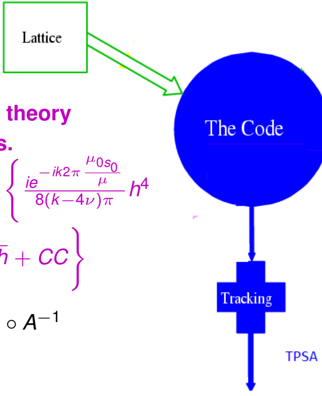

Approximate one-turn maps

Fourier Transform
analytical perturbation theory
on equation of motions.

$$F = \frac{k_o\beta_0^2}{16}\sum_{k=-\infty,+\infty}\left\{\frac{ie^{-ik2\pi\frac{\mu_0 s_0}{\mu}}}{8(k-4\nu)\pi}h^4 + \frac{ie^{-ik2\pi\frac{\mu_0 s_0}{\mu}}}{8(k-2\nu)\pi}h^3\bar{h} + CC\right\}$$

$$\Delta\varepsilon = 2i\left\{h\frac{\partial F}{\partial h} - \bar{h}\frac{\partial F}{\partial \bar{h}}\right\}\circ A^{-1}$$

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Results of the 3 calculations

```
 Fourier mode results with           10  terms
(4.782974390204826E-002,-4.059056134368263E-002)
(9.507535496091109E-002,-0.213533381429028)
 Green function results
(4.922331235935828E-002,-4.067886707780208E-002)
(0.100630505489162,-0.213709467914905)
 TPSA  results
(4.922331235935857E-002,-4.067886707780190E-002)
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```

Some details of Fourier methods

$$H = \underbrace{\frac{p^2}{2} + k_Q(s)\frac{x^2}{2}}_{H_0} + \frac{1}{4}k_o\delta(s - s_0)x^4$$

$$\Downarrow$$

$$H = \omega_s J + \beta_s^2 k_o(s)\frac{x^4}{4}$$

$$\Downarrow$$

$$K = \nu J + \frac{\nu}{\omega_s}\beta_\theta^2 k_o(\theta)\frac{x^4}{4} \quad \text{where } \theta\nu = \int_0^s \omega_\sigma d\sigma \quad \text{and} \quad J = \frac{1}{2}(x^2 + p^2)$$

$$\frac{\nu\beta_\theta^2 k_o(\theta)}{4\omega_s} = \sum_{k=-\infty,\infty} V_k e^{ik\theta} \quad \text{where} \quad V_k = \frac{1}{2\pi}\int_0^{2\pi} \frac{\nu\beta_{\theta'}^2 k_o(\theta')}{4\omega_s} e^{-ik\theta'} d\theta'$$

$$H = \nu J + \sum_{k=-\infty,\infty} V_k e^{ik\theta} x^4$$

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$$H = \underbrace{\frac{p^2}{2} + k_Q(s)\frac{x^2}{2}}_{H_0} + \frac{1}{4}k_o\delta(s-s_0)x^4$$ $\Longleftarrow$ TPSA: take the log of the step of integration

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$$H = \omega_s J + \beta_s^2 k_o(s)\frac{x^4}{4}$$ $\Longleftarrow$ TPSA: Do a linear Twiss loop to get the transformed Hamiltonian

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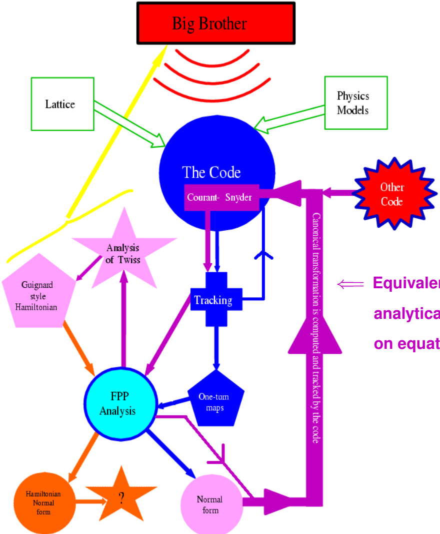
$$H = \nu J + \sum_{k=-\infty,\infty} V_k e^{ik\theta} x^4$$ $\Longleftarrow$ **Thus TPSA is essential on a realistic lattice to get to this point!**

My hierarchical view of accelerators

It respects the ubiquitous primacy of tracking codes



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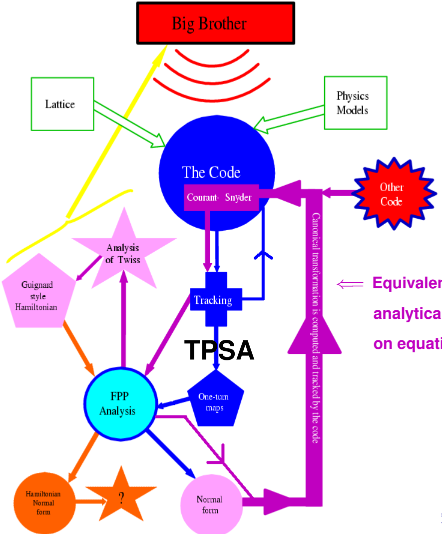

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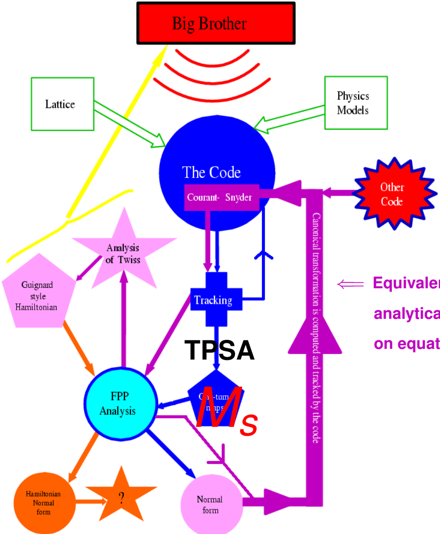

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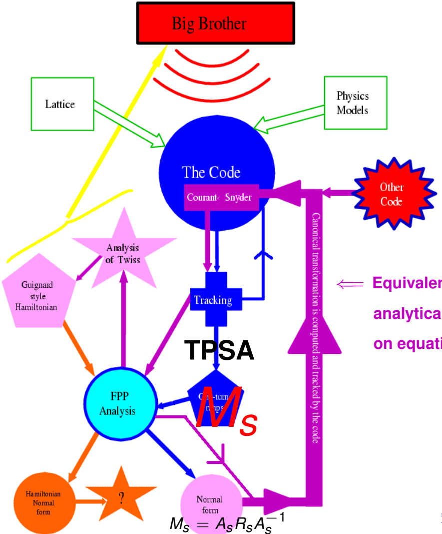

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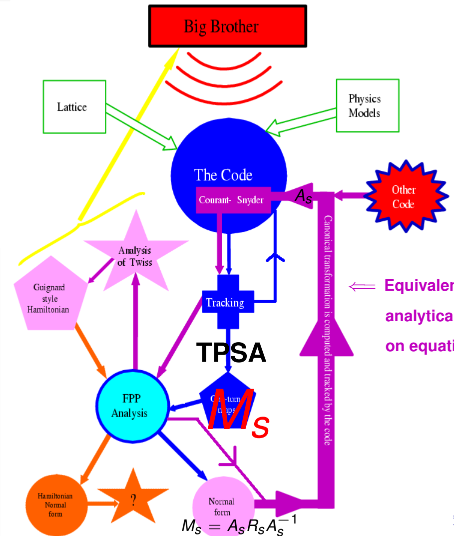

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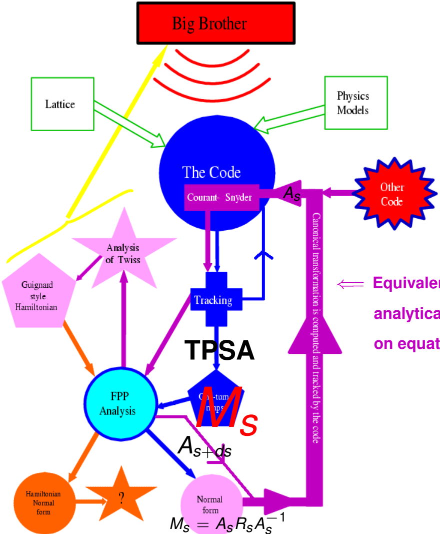

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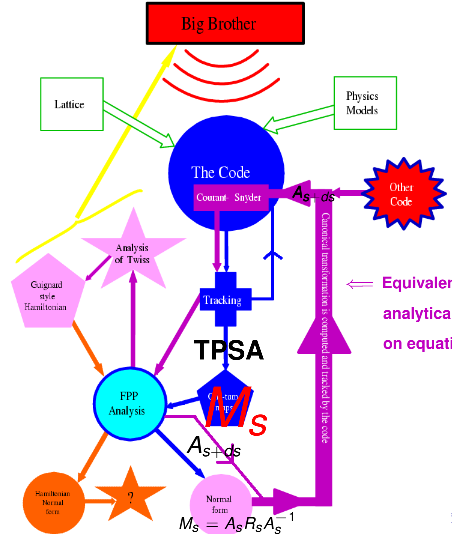

Fourier Transform $\Longrightarrow$
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What can be fed through that loop?

Consider the following list of topics:

- Courant-Snyder theory (linear phase advance, lattice functions, beam sizes)
- Coupled orbital linear theory (linear phase advance, lattice functions, beam sizes)
- Damped system with radiation (linear phase advance, lattice functions, beam sizes)
- Spin theory: (linear phase advance, lattice functions)
- Include magnet strength modulations
- Nonlinear versions of the above

Search the literature and you will likely find a series of unrelated methods

More stuff: resonances

Consider the following list of topics:

- One-dimensional phase space resonances
- Coupled phase space resonances
- Spin resonances in accelerators
- Limit cycles in electric circuits
- Track the above through the lattice

Search the literature and you will likely find a series of unrelated methods