Lectures on Partial Differential Equations

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Introduction

- General solution
- Particular solution
- Singular solution
- Eigensolutions
- Exact solution
- Fundamental solution

- You can't solve differential equations, because if you did, they would name them after yourself; Euler, Laplace, Cauchy, Dirichlet, Bessel, Bernoulli, Poisson, Lagrange, Schroedinger, Hill, anybody?
- We look them up in a book or throw a (FD/FEM) mesh on them; separation of variables, variation of variables, integral transforms, FD, FEM (Galerkin), Runge-Kutta, perturbation theory.
- We then match what is found in books to the given boundary value problems on trivial domains and make sure that the required mathematical structures are compatible with the physical problem at hand.

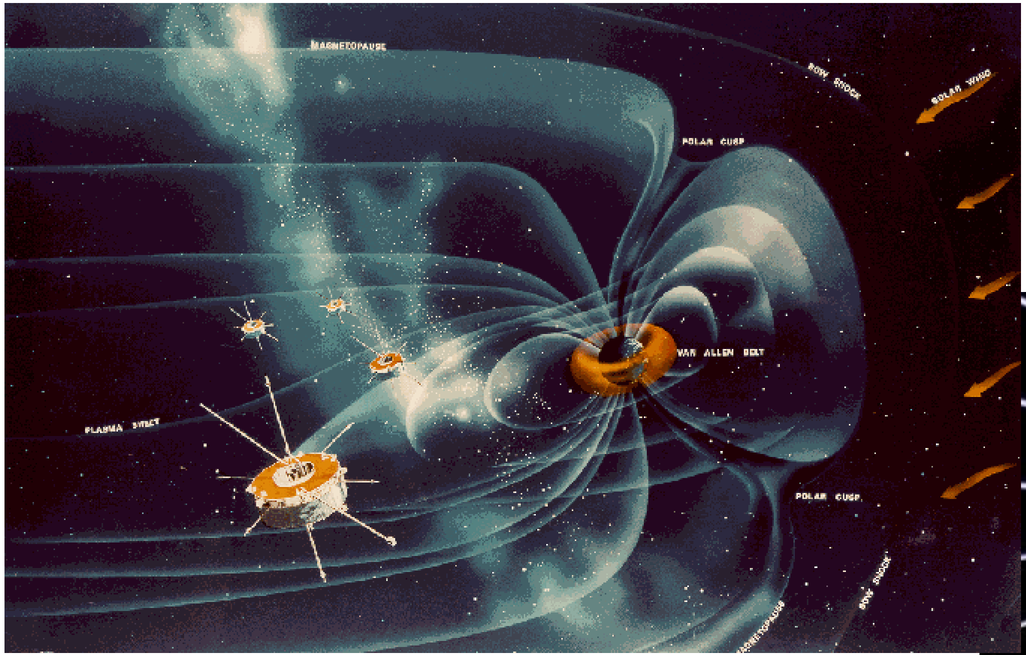
Episodes

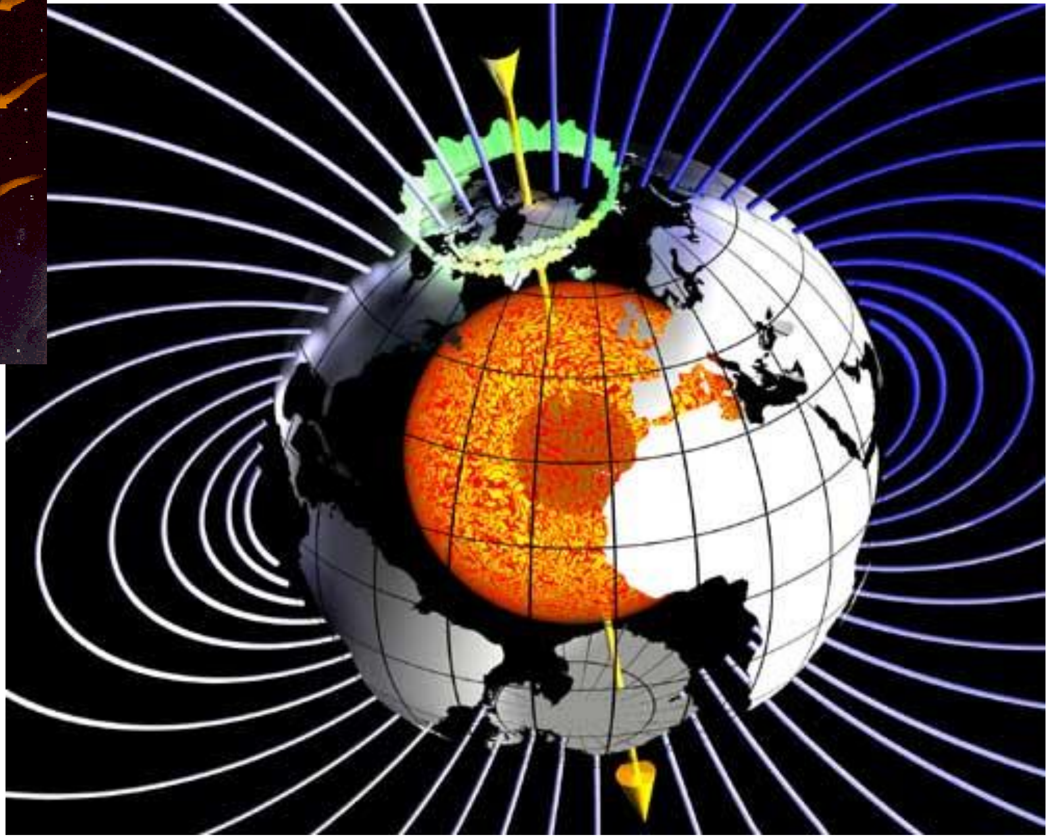
- Affine Spaces and Vector Fields
- Fourier Series
- The Taut String (applications to stretched-wire field measurements)
- Foundations of Vector Analysis
- Maxwell’s Equations in Different Avatars
- Harmonic Fields
- Field Singularities – The Green’s Functions
- Finite-Element Shape Functions
- Numerical Methods for the curl-curl Equation

Vector Fields and their Associated Affine Spaces

Flux Tubes of Mother Earth (or What IS a Magnetic Field)

Erdmagnetfeld





Different Renderings of the Same Vector Field (ROXIE)

In which way can we declare an algebra or an analysis on this

Vector and Scalar Fields

$$\operatorname{curl}\mathbf{H} = \mathbf{J} + \frac{\partial}{\partial t}\mathbf{D},$$

$$\operatorname{curl}\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B},$$

$$\operatorname{div}\ \mathbf{B} = 0,$$

$$\operatorname{div}\ \mathbf{D} = \rho.$$

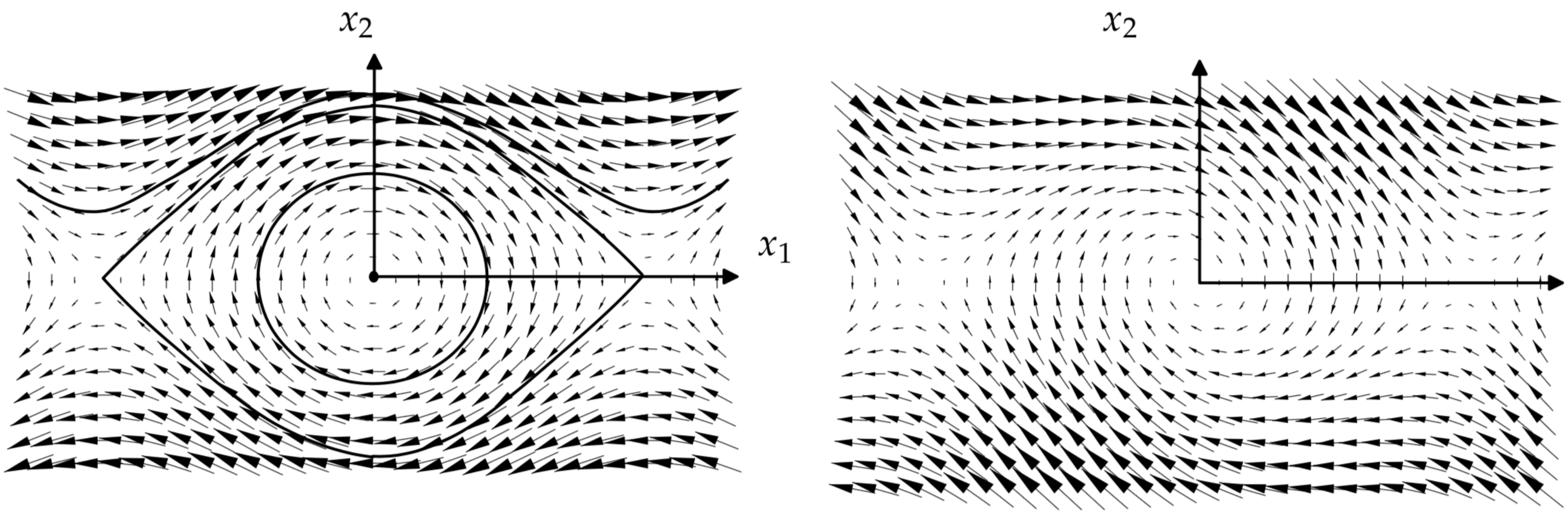
2-Dimensional Trace Space

$$\mathbf{v} : P_2 \to \bigcup_{\mathbf{x}\in P_2} T_{\mathscr{P}} P_2 : \mathbf{x}(t) \mapsto \mathbf{v}(\mathbf{x}(t))$$

$$\mathbf{x}(t) := (x_1(t), x_2(t))^T = \left(\varphi(t), \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t}\right)^T$$

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{v}(t) = \left(\frac{\mathrm{d}x_1(t)}{\mathrm{d}t}, \frac{\mathrm{d}x_2(t)}{\mathrm{d}t}\right)^T = \left(\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t}, \frac{\mathrm{d}^2\varphi(t)}{\mathrm{d}t^2}\right)^T$$

$$= \left(x_2(t), -\alpha x_2(t) - \frac{g}{l}\sin x_1(t)\right)^T .$$



Find flow maps:

$$\phi(t,\mathbf{x}) : P_2 \to P_2 : \mathbf{x}(0) \mapsto \mathbf{x}(t)$$

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} + \alpha\frac{\mathrm{d}\varphi}{\mathrm{d}t} + \frac{g}{l}\sin\varphi = 0$$

Mathematical Structure of Vector Fields

Affine Space (physical)

1. $\mathscr{P} + \mathbf{x} \in A$ if $\mathscr{P} \in A$ and $\mathbf{x} \in V$.
2. $(\mathscr{P} + \mathbf{x}) + \mathbf{y} = \mathscr{P} + (\mathbf{x} + \mathbf{y})$ for $\mathscr{P} \in A$ and $\mathbf{x}, \mathbf{y} \in V$.
3. There is a unique $\mathbf{x} \in V$ such that $\mathscr{P}_1 = \mathscr{P}_2 + \mathbf{x}$ for $\mathscr{P}_1, \mathscr{P}_2 \in A$.

Vector (Linear) Space (algebraic)

1. For any vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V : (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$.
2. There is a zero vector $\mathbf{0}$ for which $\mathbf{a} + \mathbf{0} = \mathbf{a}$ for any vector $\mathbf{a}$.
3. For each vector $\mathbf{a} \in V$ there is a vector $-\mathbf{a}$ in V for which $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$.
4. For any vectors $\mathbf{a}, \mathbf{b} \in V : \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.
5. For any scalar $\lambda \in \mathbb{F}$ and any vectors $\mathbf{a}, \mathbf{b} \in V$: $\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$.
6. For any scalars $\lambda, \mu \in \mathbb{F}$ and any vector $\mathbf{a} \in V$: $(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$.
7. For any scalars $\lambda, \mu \in \mathbb{F}$ and any vector $\mathbf{a} \in V$: $(\lambda\mu)\mathbf{a} = \lambda(\mu\mathbf{a})$.
8. For the unit scalar $1 \in \mathbb{F}$ and any vector $\mathbf{a} \in V$: $1\mathbf{a} = \mathbf{a}$.

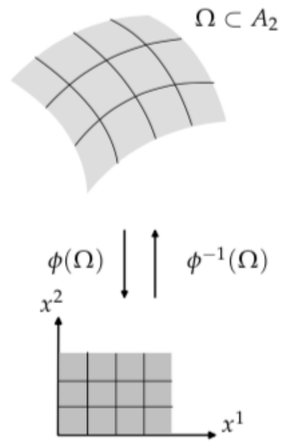
Inner product space (metric)

1. $\langle \mathbf{a} + \mathbf{b}, \mathbf{c} \rangle = \langle \mathbf{a}, \mathbf{c} \rangle + \langle \mathbf{b}, \mathbf{c} \rangle$ and $\langle \mathbf{a}, \lambda\mathbf{b} + \mu\mathbf{c} \rangle = \lambda\langle \mathbf{a}, \mathbf{b} \rangle + \mu\langle \mathbf{a}, \mathbf{c} \rangle$.
2. $\langle \mathbf{a}, \mathbf{b} \rangle = \langle \mathbf{b}, \mathbf{a} \rangle$.
3. $\langle \mathbf{a}, \mathbf{a} \rangle > 0$ and $\langle \mathbf{a}, \mathbf{a} \rangle = 0$ if and only if $\mathbf{a} = \mathbf{0}$.

$$\mathbf{a} \cdot \mathbf{b} := a^1 b^1 + a^2 b^2 + a^3 b^3$$

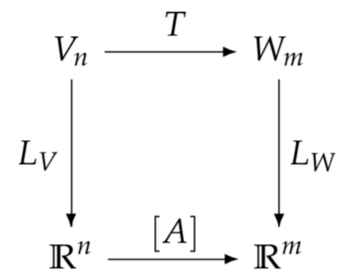
$$\langle f, g \rangle := \int_{-1}^{1} f(x) g(x) \mathrm{d}x.$$

Coordinate map



Isomorphism

$$\mathscr{P} \in A_n \xrightarrow{\text{Origin}} \mathbf{r} \in V_n \xrightarrow{\text{Basis}} (x^1, \ldots, x^n) \in \mathbb{R}^n .$$



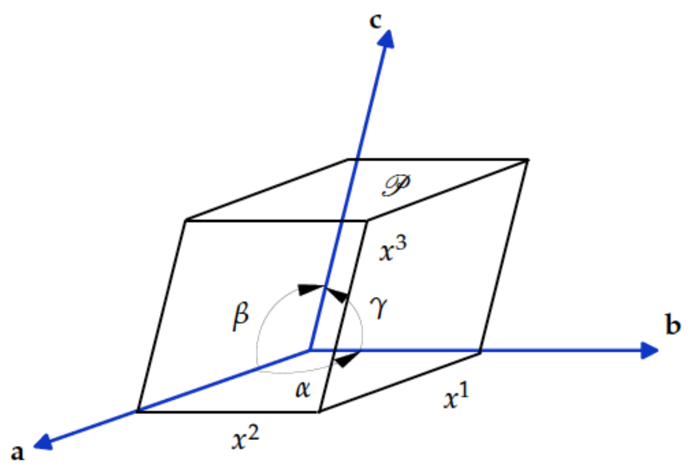
Norm and Distance

Length (Norm induced by the scalar product) $\| \mathbf{a} \| = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle},$

Cauchy Schwarz inequality $| \langle \mathbf{a}, \mathbf{b} \rangle | \leq \| \mathbf{a} \| \, \| \mathbf{b} \|,$

If a basis is present: $\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^{3} \sum_{j=1}^{3} a^i b^j \langle \mathbf{g}_i, \mathbf{g}_j \rangle \equiv a^i b^j \langle \mathbf{g}_i, \mathbf{g}_j \rangle =: a^i b^j g_{ij},$

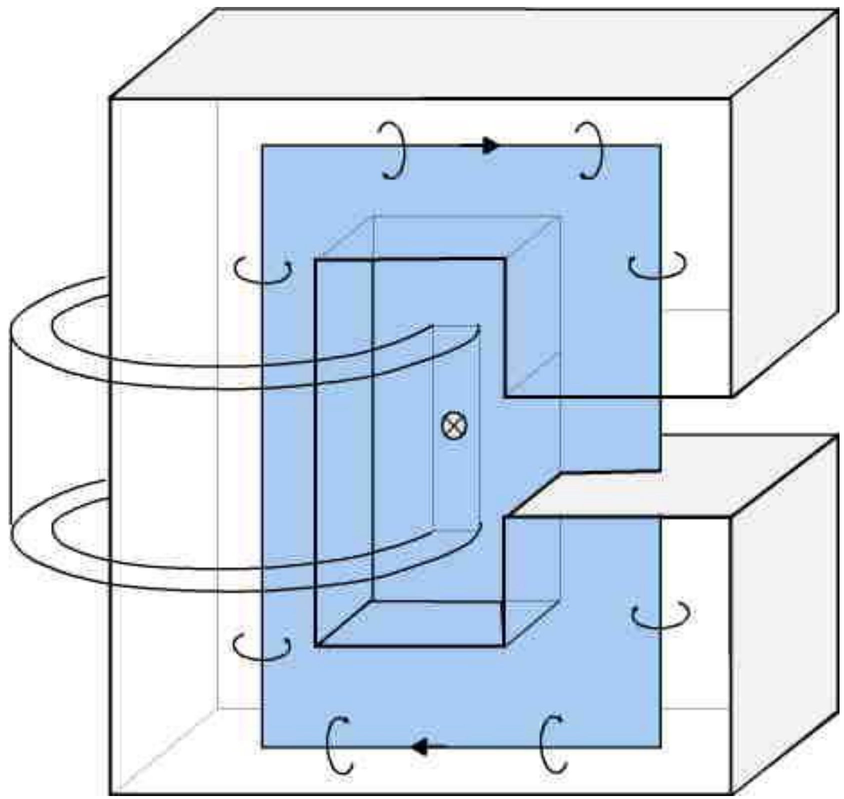
(Generalized Pythagoras)



$$[G] = \begin{pmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{pmatrix}$$

Applications: Calibration of Helmholtz coils, Calibration of 3-axis displacement stages and robots

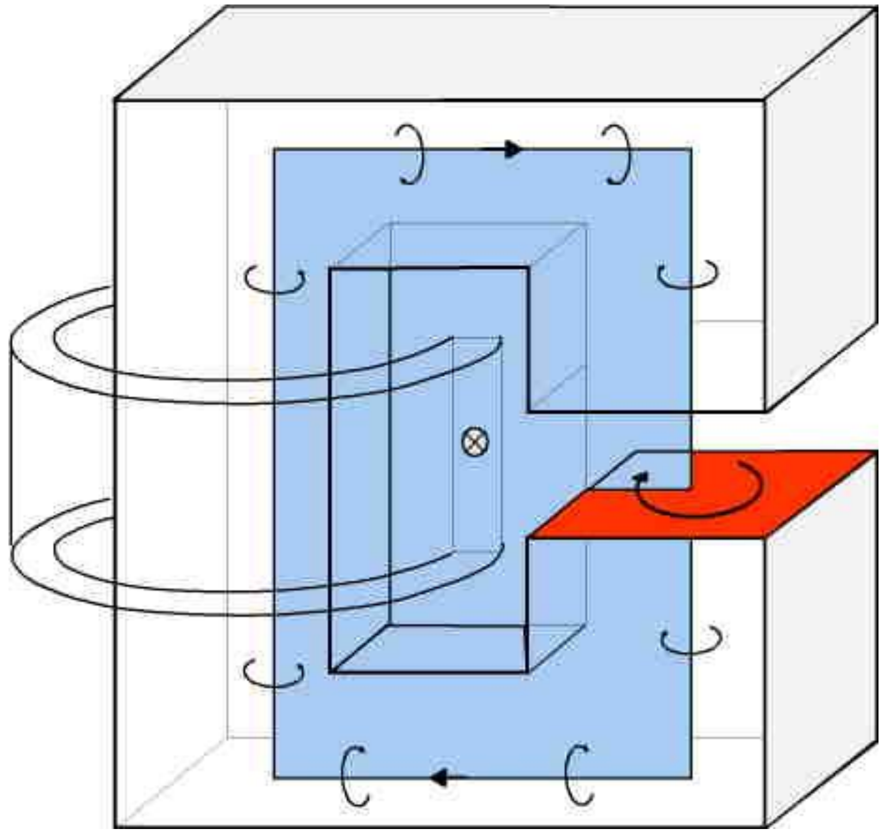
Inner and Outer Oriented Surfaces



Outer oriented
by the current

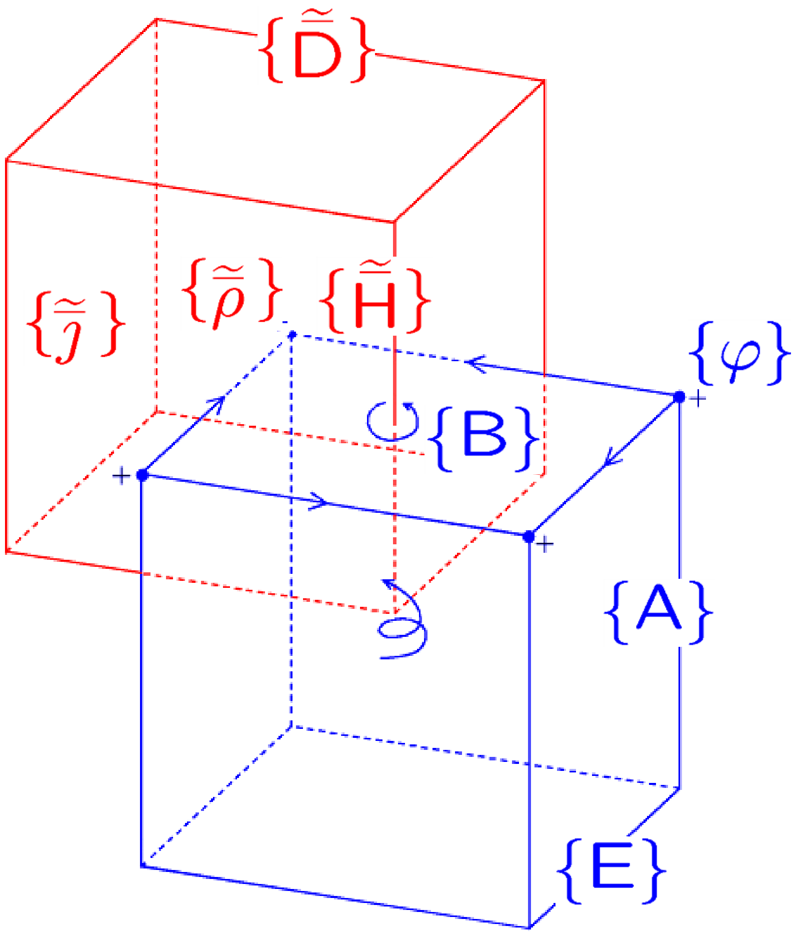
Inner and Outer Oriented Surfaces

$$\int_{\partial a} \mathbf{H} \cdot \mathrm{d}\mathbf{s} = \int_a \mathbf{J} \cdot \mathrm{d}\mathbf{a}$$



Inner oriented because flux is a measure for the voltage that can be generated on the rim

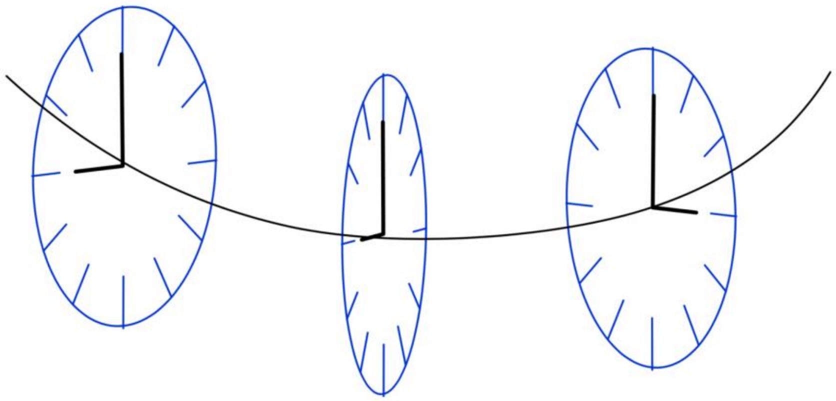
Faraday Fields are discretized on the primal grid



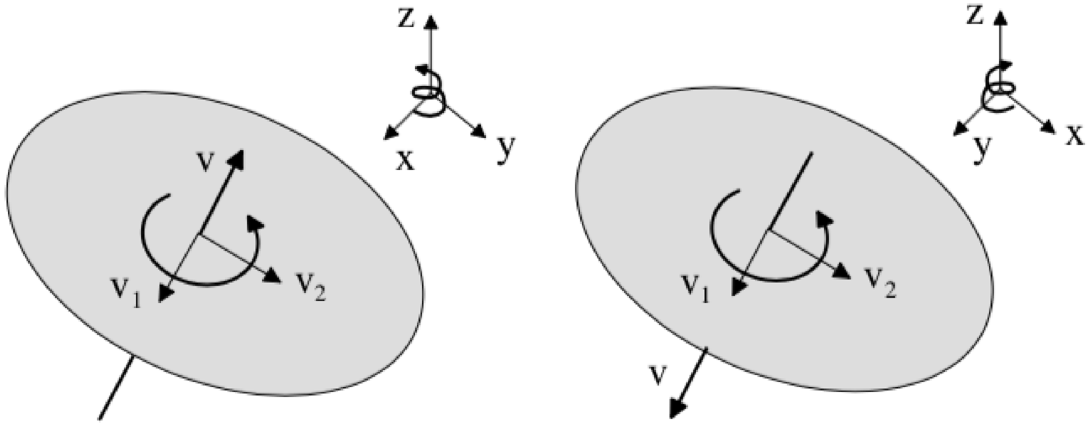
Ampère-Maxwell Fields are discretized on the dual grid

Consistent Inner and Outer Orientation

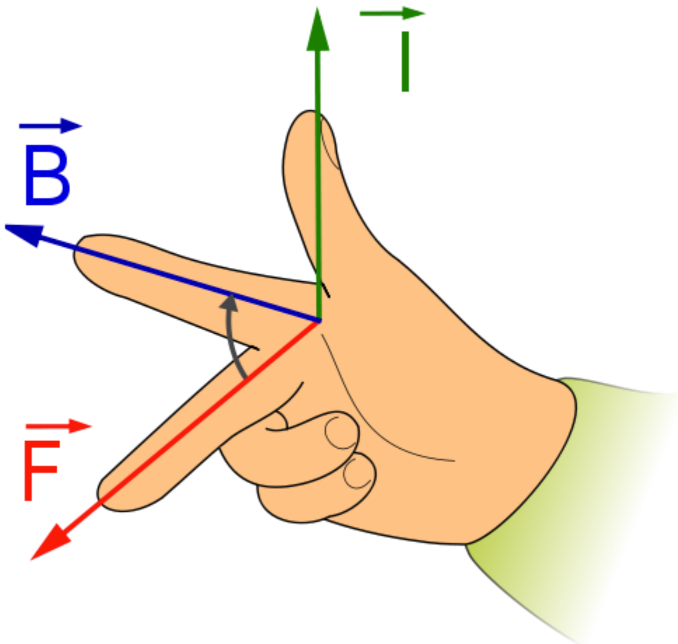
Consistent inner and outer orientation of a manifold embedded in an encompassing oriented space (requires an origin and a coordinate frame)



09:00 or 15:00 ?



The Right-Hand Rule or "Magnetic Discussion"



Bruno Touschek (1921-1978)

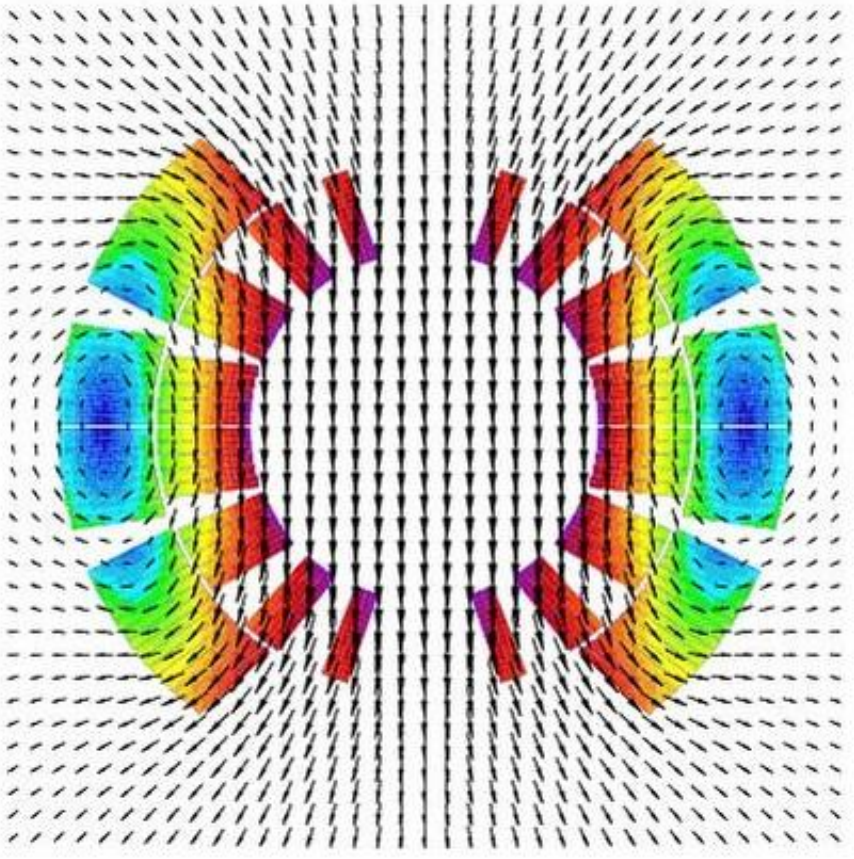
Framework of our Vectorfields E_3 (Euclidean Affine Space)

- ➔ E_3 has the structure of the affine point space
- ➔ It carries the vector (linear) space structure of its associated vector space
- ➔ It is equipped with a metric that gives rise to distance and angles

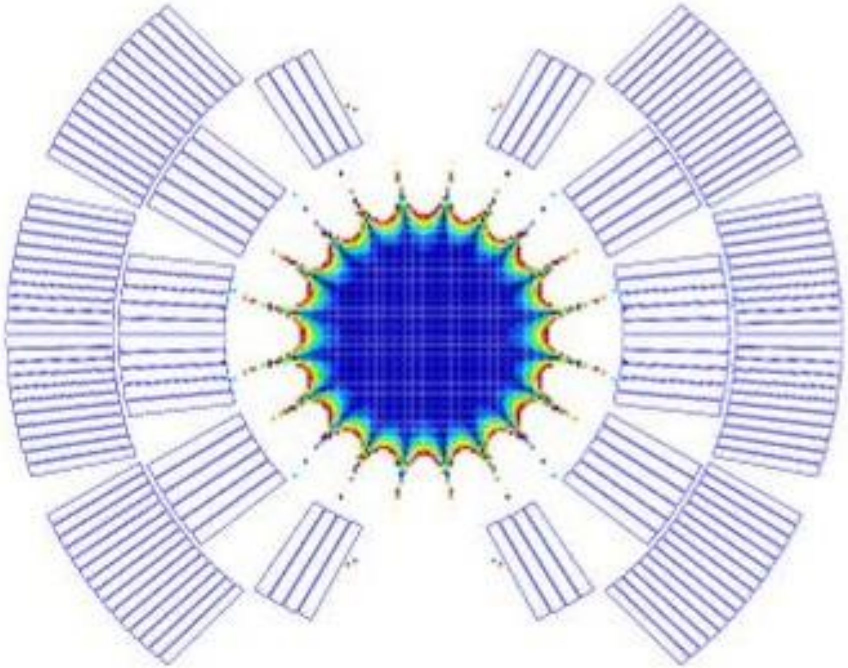
- ➔ If an origin and basis is selected, the projection of the position vector on the basis yields the coordinates (in $\mathbb{R}^3$)
- ➔ The canonical basis can be made to a basis field by translation
- ➔ The components of the field at some point are then the projection on this basis field

Fourier Series

Field Quality



Field map



Good field region

Fourier Series (an Infinite Dimensional Vector Space)

$$B_r(r_0,\varphi) = \sum_{n=1}^{\infty} \left(B_n(r_0) \sin n\varphi + A_n(r_0) \cos n\varphi \right),$$

$$A_n(r_0) = \frac{1}{\pi} \int_0^{2\pi} B_r(r_0,\varphi) \cos n\varphi \, d\varphi, \qquad n = 1,2,3,\ldots,$$

$$B_n(r_0) = \frac{1}{\pi} \int_0^{2\pi} B_r(r_0,\varphi) \sin n\varphi \, d\varphi, \qquad n = 1,2,3,\ldots.$$

And on the computer: Discrete setting (don't bother with the FFT)

$$A_n(r_0) \approx \frac{2}{N} \sum_{k=0}^{N-1} B_r(r_0,\varphi_k) \cos n\varphi_k, \qquad \varphi_k = \frac{2\pi k}{N}$$

$$B_n(r_0) \approx \frac{2}{N} \sum_{k=0}^{N-1} B_r(r_0,\varphi_k) \sin n\varphi_k. \qquad k = 0,1,2,\ldots,N-1\,.$$

E_3 and L^2

Structure	Euclidean E_3	Hilbert $L^2(\Omega)$
Vector	$\mathbf{x}, \mathbf{y}$	$f(t), g(t)$
Basis	$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$	$\{g_n(t)\}$
Scalar product	$\sum_{n=1}^{3} x_n y_n$	$\langle f, g\rangle = \int_\Omega f(t) g(t) \mathrm{d}t$
Norm	$\parallel \mathbf{x} \parallel = \sqrt{\sum_{n=1}^{3} x_n^2}$	$\parallel f \parallel = \sqrt{\int_\Omega \lvert f(t)\rvert^2 \mathrm{d}t}$
Orthonormality	$\mathbf{e}_n \cdot \mathbf{e}_k = \delta_{nk}$	$\langle g_n, g_k\rangle = \delta_{nk}$
Expansion	$\mathbf{x} = \sum_{n=1}^{3} x_n \mathbf{e}_n$	$f(t) = \sum_{n=1}^{\infty} x_n g_n(t)$
Coefficients	$x_n = \mathbf{x} \cdot \mathbf{e}_n$	$x_n = \langle g_n, f\rangle$

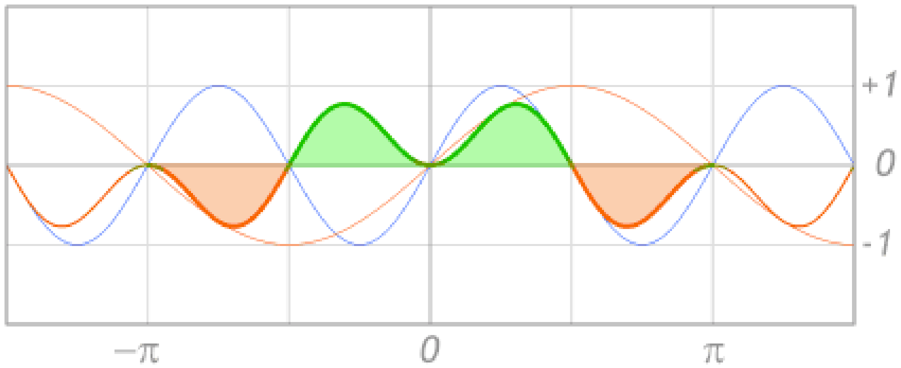
Hilbert spaces are those in which notation and concepts of ordinary Euclidean geometry hold without any restrictions on the dimension.

The Road Map to Convergence of Fourier Series

- → The trigonometric functions are orthogonal
- → The Fourier polynomial of grade n is the best approximation of f in V_n
- → The projections onto the trigonometric functions (scalar product) induces a norm (the RMS error)
- → Riemann Lebesque Lemma: Within this norm, the coefficients converge to zero.

- → 3 Convergence theorems
  - For a C^1 function Pn converges uniformly to f(x) in any x
  - For “clean jumps” Pn converges pointwise to 0.5 $(f_+(x) + f_-(x))$
  - The Fourier polynomial converges for every square integrable function in the RMS sense (allows jump discontinuities, e.g., at material boundaries)

Trigonometric Functions as Orthogonal Function Set (from Wikipedia)



$$\int_{-\pi}^{+\pi} \sin(2x)\sin(1x)\; dx = 0$$

$$\int_{-\pi}^{\pi} \cos(mx)\, \cos(nx)\, dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos((n-m)x) + \cos((n+m)x)\, dx = \pi\delta_{mn},$$

$$\int_{-\pi}^{\pi} \sin(mx)\, \sin(nx)\, dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos((n-m)x) - \cos((n+m)x)\, dx = \pi\delta_{mn},$$

$$\int_{-\pi}^{\pi} \cos(mx)\, \sin(nx)\, dx$$

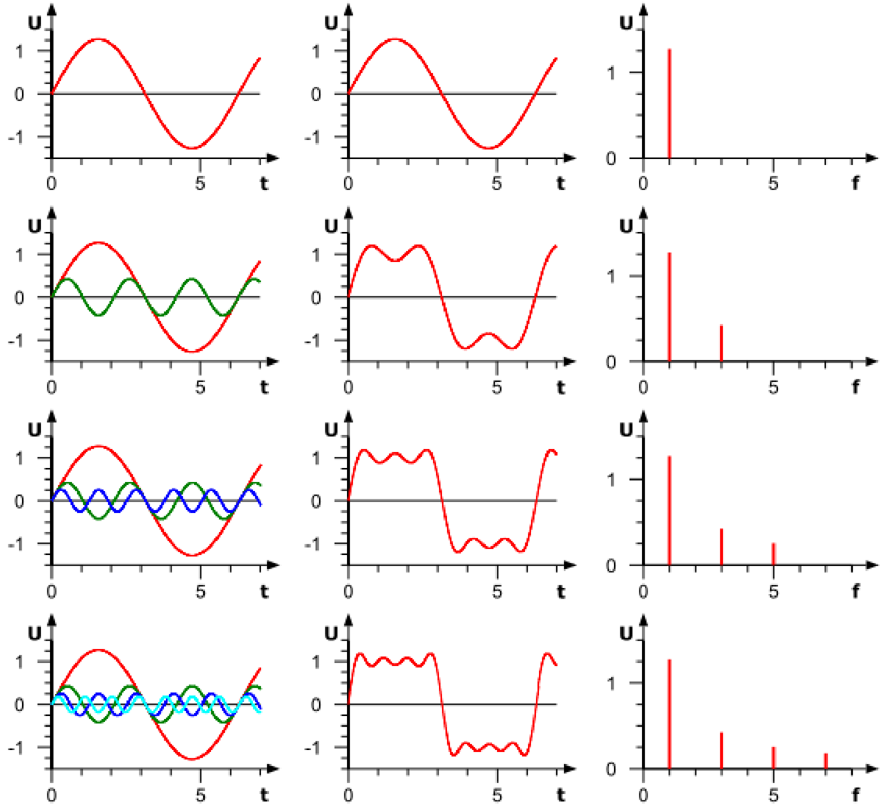
$$= \frac{1}{2}\int_{-\pi}^{\pi} \sin((n+m)x) + \sin((n-m)x)\, dx = 0\,.$$

The Fourier Polynomial is the best Approximation of f within Polynomial Approximations of order m

$$\| f - \sum_{n=0}^{m} \gamma_n g_n \|^2 = \left\langle f - \sum_{n=0}^{m} \gamma_n g_n \, , \, f - \sum_{n=0}^{m} \gamma_n g_n \right\rangle$$

$$= \langle f, f \rangle - 2 \sum_{n=0}^{m} \gamma_n \langle f, g_n \rangle + \sum_{n=0}^{m} \gamma_n^2$$

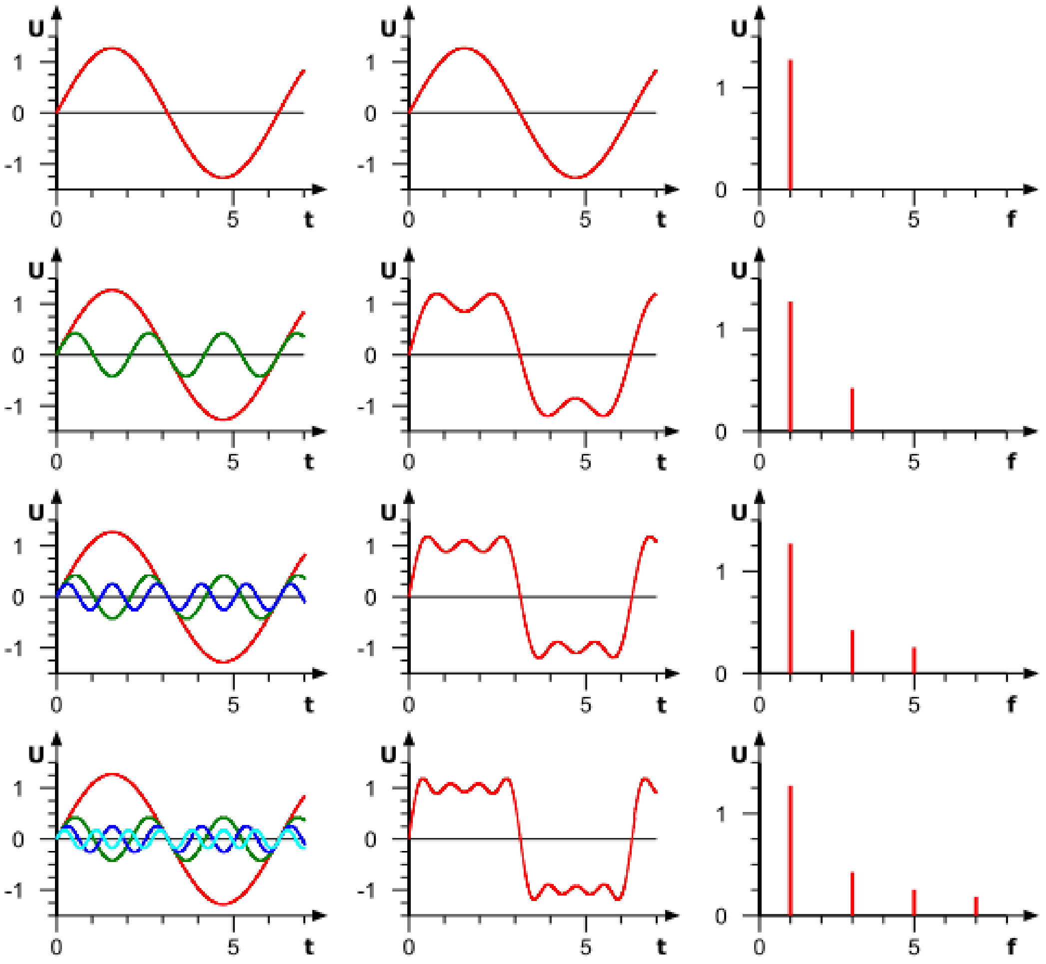
$$= \| f \|^2 - \sum_{n=0}^{m} |\langle f, g_n \rangle|^2 + \sum_{n=0}^{m} |\langle f, g_n \rangle - \gamma_n|^2$$



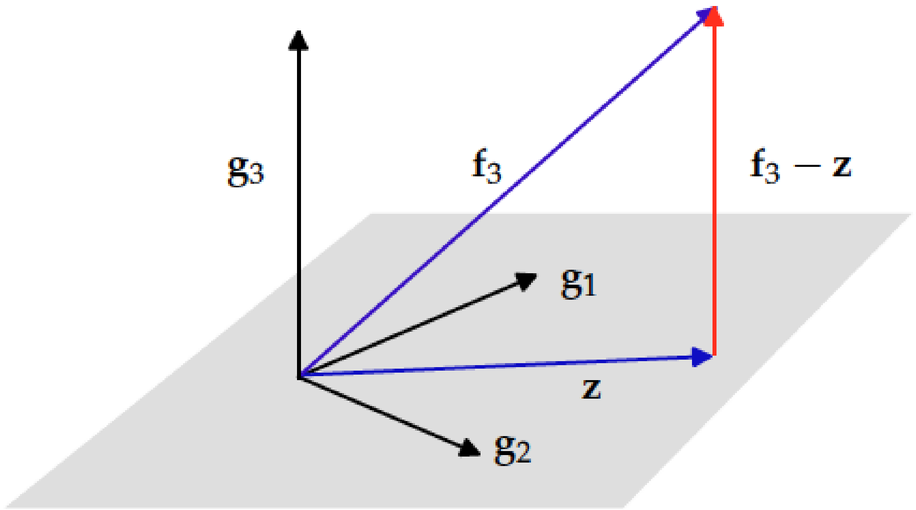
Minimum for $\gamma_n = c_n = \langle f, g_n \rangle$

Projection of the square wave onto the “shape” of the trigonometric functions

The Fourier Polynomial Pn is the best Approximation in Vn



Projection of the square wave onto the "shape" of the trigonometric functions



Solving Boundary Value Problems

Take any 2π periodic function and develop according to

$$\frac{C_0}{2} + \sum_{n=1}^{\infty} (C_n(r_0) \sin n\varphi + D_n(r_0) \cos n\varphi) \,.$$

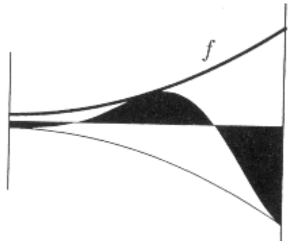
	B_r	B_φ	B_x	B_y	A_z	ϕ_{m}
$B_n =$	C_n	D_n	C_{n-1}	D_{n-1}	$\frac{-nD_n}{r_0}$	$\frac{-n\mu_0 C_n}{r_0}$
$A_n =$	D_n	$-C_n$	D_{n-1}	$-C_{n-1}$	$\frac{nC_n}{r_0}$	$\frac{-n\mu_0 D_n}{r_0}$

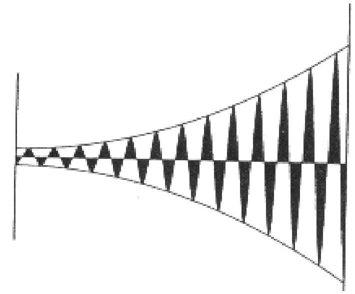
We can use fields, potentials, fluxes, or wire-oscillation amplitudes as “raw data”. The linear differential operators grad and rot transform into simple algebra in the L2 space of Fourier coefficients.

Method of Superposition

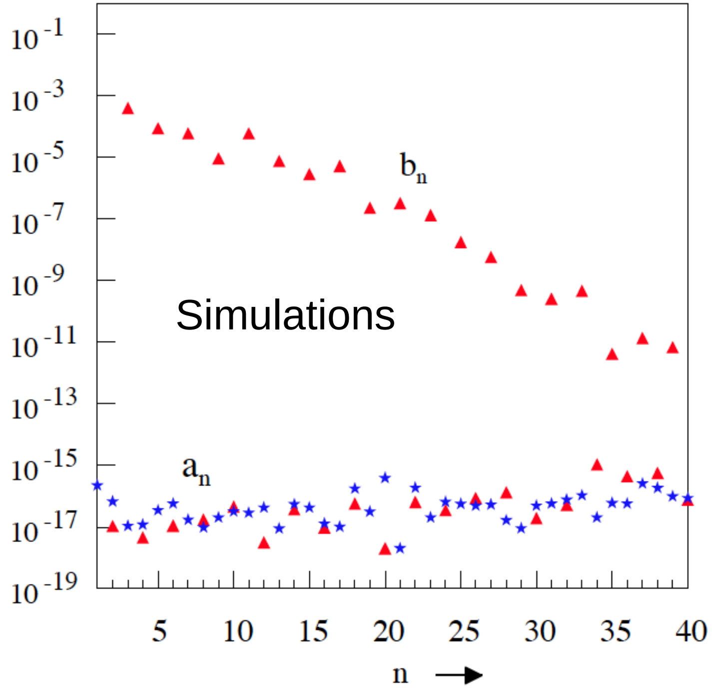
The Fourier coefficients tend to zero as n goes to infinity

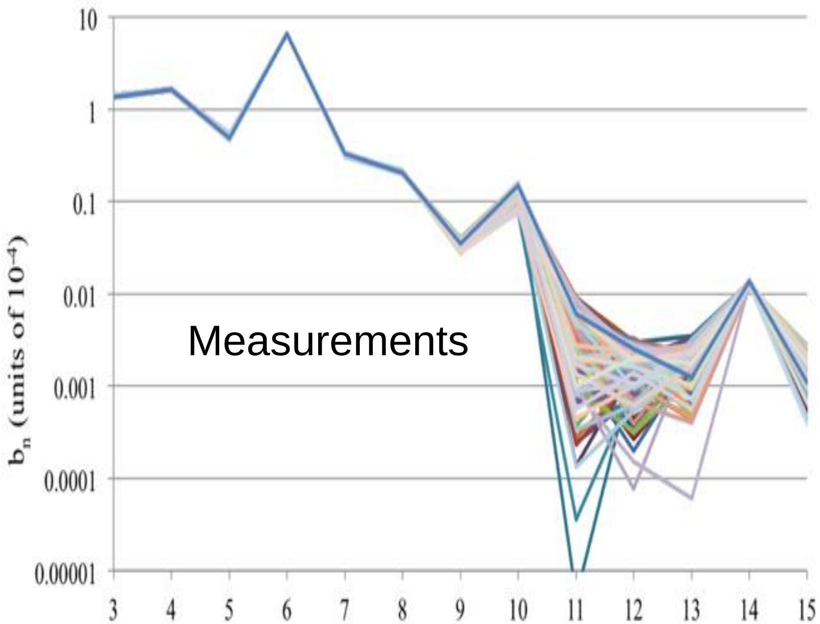
$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2$$





Limits: 10^{-6} T, 10^{-8} Vs

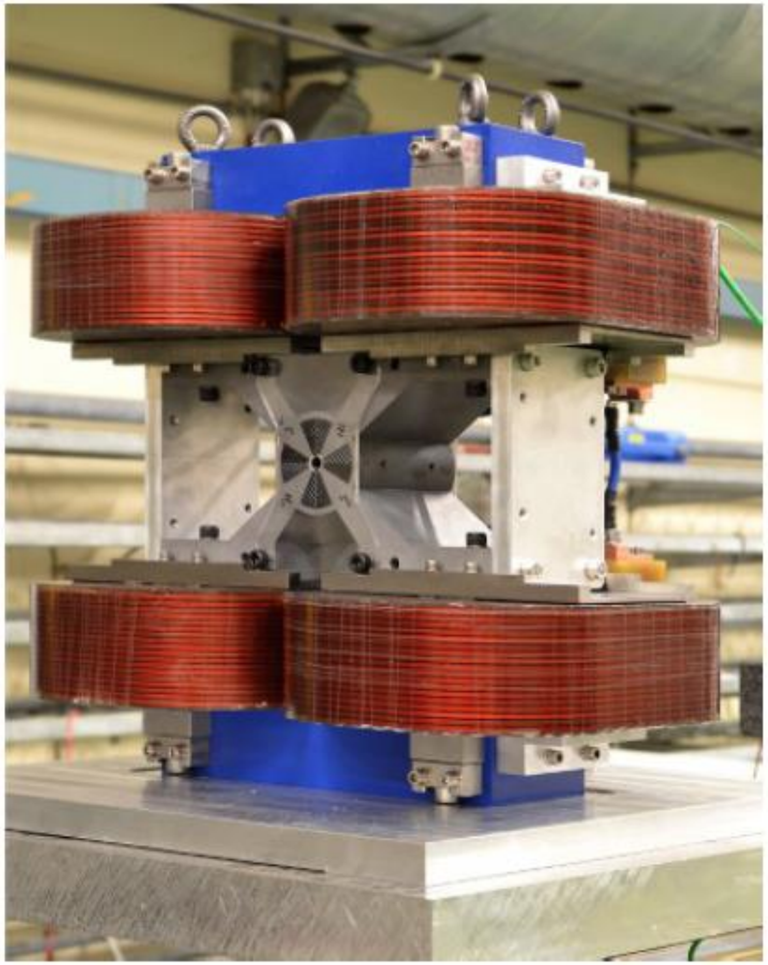


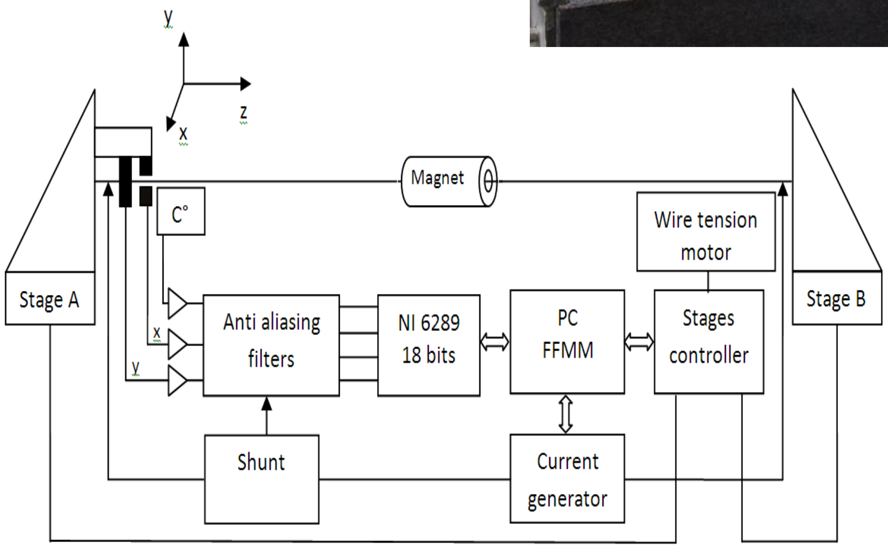


Always plot your results in logarithmic scale

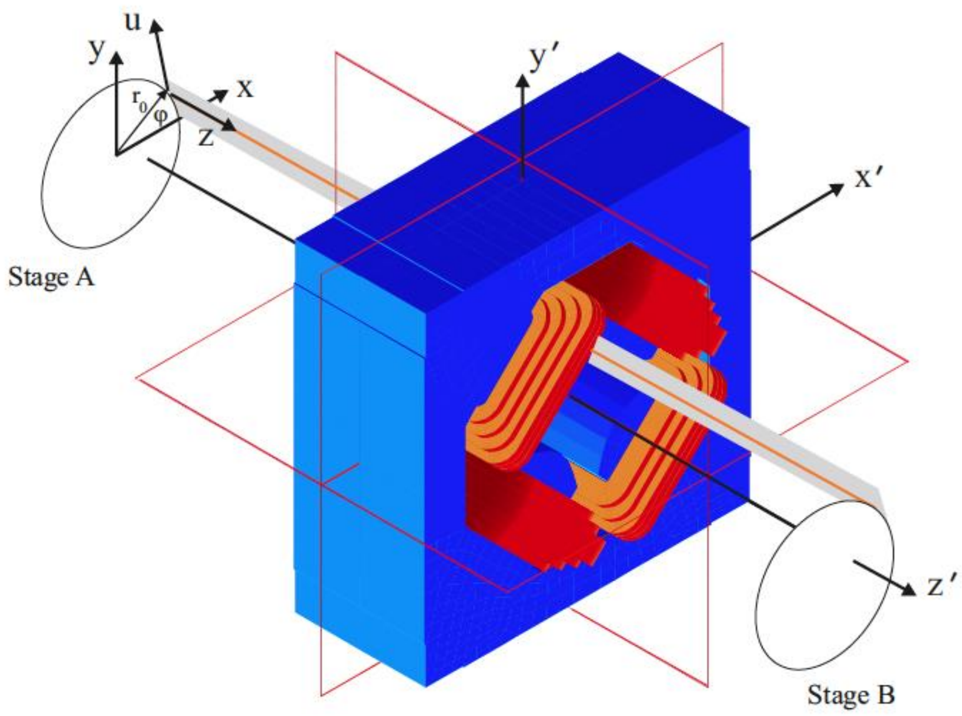
The Taut String

Oscillating Wire Measurements





Oscillating Wire Measurements



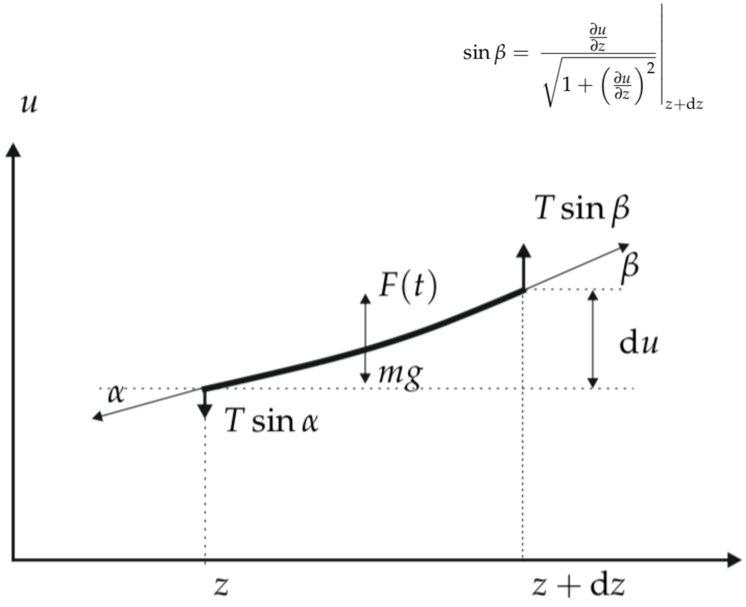
$$d_y^k(r_0) = \lambda_y \int_0^L B_x(r_0, \varphi_k)\mathrm{d}z$$

$$d_x^k(r_0) = \lambda_x \int_0^L B_y(r_0, \varphi_k)\mathrm{d}z\,.$$

Measure the oscillation amplitudes on K rulings of a cylindrical domain. Develop into a Fourier series.

$$\tilde{B}_n(r_0) = \frac{2}{K}\sum_{k=0}^{K-1} d_y^k(r_0)\sin n\varphi_k\,.$$

$$b_{n+1}(r_0) = \frac{\tilde{B}_n(r_0)}{\tilde{B}_N(r_0)}\,.$$



$$f(z+\mathrm{d}z) - f(z) = \left(\frac{\partial f}{\partial z}\right)\mathrm{d}z + O(\mathrm{d}z)^2$$

$$T\sin\beta - T\sin\alpha \approx T\left(\left(\frac{\partial u}{\partial z}\right)\Big|_{z+\mathrm{d}z} - \left(\frac{\partial u}{\partial z}\right)\Big|_{z}\right)$$

$$\lambda_\mathrm{m}\frac{\partial^2 u}{\partial t^2} + \alpha\frac{\partial u}{\partial t} - T\frac{\partial^2 u}{\partial z^2} = -I(t)B_\mathrm{n}(z)$$

$$\text{inertia} + \text{damping} + \text{spring} = \text{excitation force}$$

where λ_m is the mass per unit length $[\lambda_\mathrm{m}] = 1\,\mathrm{kg\,m^{-1}}$, α the damping coefficient $[\alpha] = 1\,\mathrm{kg\,m^{-1}\,s^{-1}}$. Note that the physical unit of the coefficient T/λ_m is $[T/\lambda_\mathrm{m}] = 1\,\mathrm{s^{-2}}$ and therefore

$$c := \sqrt{\frac{T}{\lambda_\mathrm{m}}}$$

CuBe: 9.7 N, 0.125 mm, 0.85 10-4 kg/m, 340 m/s

The One-dimensional Homogenous Wave Equation (no Damping)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Method of Separation

$$u(z,t) = U(z)Q(t)$$

$$U(z)\frac{\partial^2 Q(t)}{\partial t^2} = c^2 \frac{\partial^2 U(z)}{\partial z^2} Q(t)$$

$$\frac{\partial^2 U(z)}{\partial z^2} + \mu U(z) = 0$$

$$\frac{\partial^2 Q(t)}{\partial t^2} + c^2 \mu Q(t) = 0$$

Eigensolutions

$$U(z) = \mathcal{A} e^{\sqrt{-\mu} z} + \mathcal{B} e^{-\sqrt{-\mu} z},$$

$$Q(t) = \mathcal{C} e^{c\sqrt{-\mu} t} + \mathcal{D} e^{-c\sqrt{-\mu} t},$$

for $\mu > 0$:

$$U(z) = \mathcal{A} \sin(\sqrt{\mu} z) + \mathcal{B} \cos(\sqrt{\mu} z),$$

$$Q(t) = \mathcal{C} \sin(c\sqrt{\mu} t) + \mathcal{D} \cos(c\sqrt{\mu} t),$$

and for $\mu = 0$:

$$U(z) = \mathcal{A} + \mathcal{B} z,$$

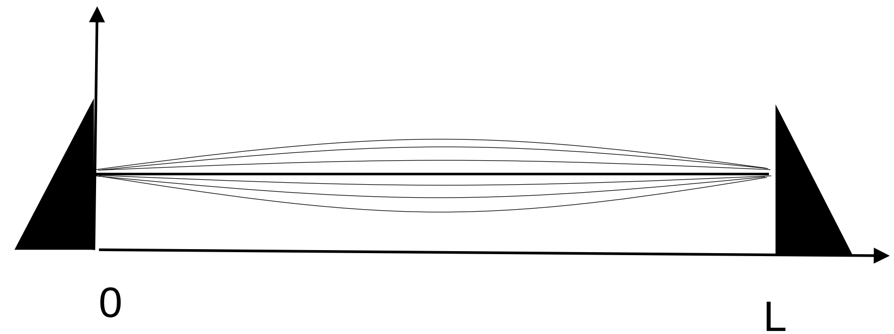
$$Q(t) = \mathcal{C} + \mathcal{D} t.$$

Boundary Conditions

Mode shape function

$$u(0,t) = 0 \qquad u(L,t) = 0$$

$$\mathcal{A}\sin(\sqrt{\mu}z) = 0 \qquad \sqrt{\mu} = \frac{n\pi}{L}$$



Nodal displacement (in time) function

$$\frac{\partial^2 Q(t)}{\partial t^2} + c^2\left(\frac{n\pi}{L}\right)^2 Q(t) = 0$$

$$Q_n(t) = \mathcal{A}_n \sin(\omega_n t) + \mathcal{B}_n \cos(\omega_n t)$$

c = 340 m/s, 2-m-long, $\omega_1 = 534$ Hz

Normal mode fequency

$$\omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\lambda_m}}$$

$$u(z,t) = \sum_n \mathcal{U}_n \sin\left(\frac{n\pi}{L}z\right)\sin(\omega_n t + \varphi_n)$$

Remark: Coefficients are still not know yet. It requires initial conditions – plucked string (Guitar) or struck string (piano)

More General Form of the 1D Wave Equation

$$a^2 \frac{\partial^2 u}{\partial z^2} + F(z,t) = \frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} + ku \, .$$

$$\text{spring} + \text{excitation} = \text{inertia} + \text{damping} + \text{elasticity}$$

Method of Separation

$$\frac{\partial^2 U(z)}{\partial z^2} + \mu U(z) = 0$$

$$\frac{\partial^2 Q(t)}{\partial t^2} + \alpha \frac{\partial Q(t)}{\partial t} + c^2 \mu Q(t) = 0 \quad \longleftarrow \quad Q = e^{rt}$$

Sturm-Liouville equation with linear, self-adjoint diff. operator $F = \langle Lf, g \rangle = \langle f, Lg \rangle$ Functional analysis on infinite dimensional vector-spaces. Preserving the vector space properties under the actions of the operator. Existence of a converging, orthogonal projection operator (spectral theorem)

- all the eigenvalues are real,
- to each eigenvalue there is one and only one linearly dependent eigenfunction,
- the eigenfunctions are orthogonal,
- if the function is continuously differentiable on the interval, the function can be expressed as a convergent series of these eigenfunctions.

Eigensolutions of the Sturm-Liouville Problem

$$r^2 + \alpha r + c^2\mu = 0$$

$$r = -\frac{\alpha}{2} \pm \sqrt{\left(\frac{\alpha}{2}\right)^2 - c^2\mu}$$

"Sufficiently" small damping

$$\frac{c2\pi}{\alpha L} > 1 \qquad \longleftarrow \qquad \left(\frac{\alpha}{2}\right)^2 < \min\{c^2\mu_n\} = \min\left\{c^2\left(\frac{n\pi}{L}\right)^2\right\}$$

$$\begin{aligned} Q_n(t) &= e^{-\frac{\alpha}{2}t}\left(\mathcal{C}_n e^{i\sqrt{c^2\mu-\left(\frac{\alpha}{2}\right)^2}t} + \mathcal{D}_n e^{-i\sqrt{c^2\mu-\left(\frac{\alpha}{2}\right)^2}t}\right) \\ &= e^{-\frac{\alpha}{2}t}\left((\mathcal{C}_n+\mathcal{D}_n)\cos\left(c^2\mu-\left(\frac{\alpha}{2}\right)^2 t\right) + i(\mathcal{C}_n-\mathcal{D}_n)\sin\left(c^2\mu-\left(\frac{\alpha}{2}\right)^2 t\right)\right) \\ &= e^{-\frac{\alpha}{2}t}\left(\mathcal{A}_n\cos\left(c^2\mu-\left(\frac{\alpha}{2}\right)^2 t\right) + \mathcal{B}_n\sin\left(c^2\mu-\left(\frac{\alpha}{2}\right)^2 t\right)\right) \end{aligned}$$

$$Q_n(t) = \mathcal{U}_n e^{-\frac{\alpha}{2}t}\sin(\tilde{\omega}_n t + \varphi_n)$$

$$\tilde{\omega}_n = \frac{n\pi}{L}\sqrt{\frac{T}{\lambda_m} - \left(\frac{\alpha}{2}\right)^2}$$

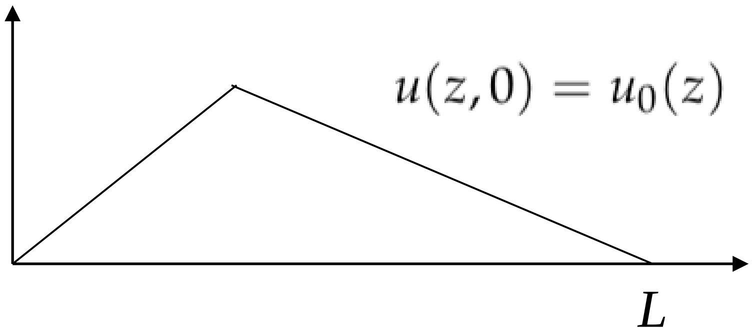
$$u(z,t) = \sum_n e^{-\frac{\alpha}{2}t}\sin\left(\frac{n\pi}{L}z\right)\left(\mathcal{A}_n\sin(\tilde{\omega}_n t) + \mathcal{B}_n\cos(\tilde{\omega}_n t)\right)$$

Nothing is "solved" yet

Initial Conditions (Again)

$$u(z,t) = \sum_n e^{-\frac{\alpha}{2}t} \sin\left(\frac{n\pi}{L}z\right) \left(\mathcal{A}_n \sin(\tilde{\omega}_n t) + \mathcal{B}_n \cos(\tilde{\omega}_n t)\right)$$

$$u_0(z) = \sum_n \mathcal{B}_n \sin\left(\frac{n\pi}{L}z\right)$$



$$\int_0^L \sin\left(\frac{m\pi}{L}z\right)\left(\sum_n \mathcal{B}_n \sin\left(\frac{n\pi}{L}z\right)\right) \mathrm{d}z = \int_0^L \sin\left(\frac{m\pi}{L}z\right) u_0(z)\mathrm{d}z$$

Unknown

$$\mathcal{B}_n = \frac{2}{L}\int_0^L u_0(z) \sin\left(\frac{n\pi}{L}z\right) \mathrm{d}z = C_n$$

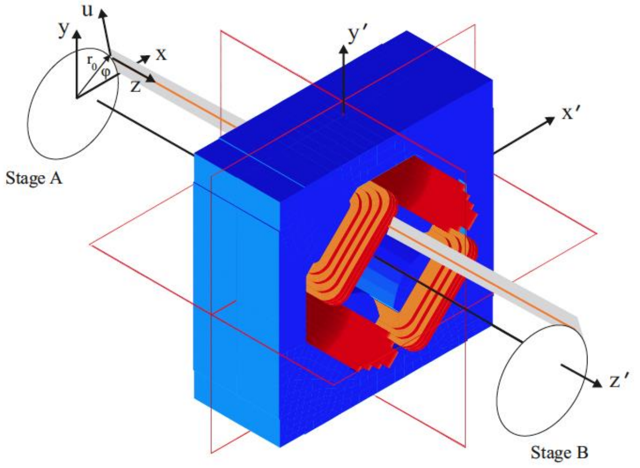
Known

Because of orthogonality: $\int_0^L \sin\left(\frac{m\pi}{L}z\right) \sin\left(\frac{n\pi}{L}z\right) \mathrm{d}z = L/2 \text{ for } m = n$

The Inhomogenous Equation (Variations of Variables - Guesstimations)

$$u(x,t) = \sum_n \mathcal{U} \sin\left(\frac{n\pi}{L} z\right) \sin(\omega t - \varphi_n)$$

$$F(z,t) = -B_\mathrm{n}(z) I_0 \sin(\omega t)$$



Lorentz Force Term on the Wire
Notice n = normal

$$\varphi_m = \arctan\left(\frac{\alpha\omega}{-\lambda_\mathrm{m}\omega^2 + T\left(\frac{m\pi}{L}\right)^2}\right)$$

Modal force

Nodal displacement

$$u(z,t) = \frac{2I_0}{L} \sum_m \frac{\int_0^L B_\mathrm{n}(z) \sin\left(\frac{m\pi}{L} z\right) \mathrm{d}z}{\sqrt{\left[T\left(\frac{m\pi}{L}\right)^2 - \lambda_\mathrm{m}\omega^2\right]^2 + (\alpha\omega)^2}} \sin\left(\frac{m\pi}{L} z\right) \sin(\omega t - \varphi_m)$$

Mode shape function

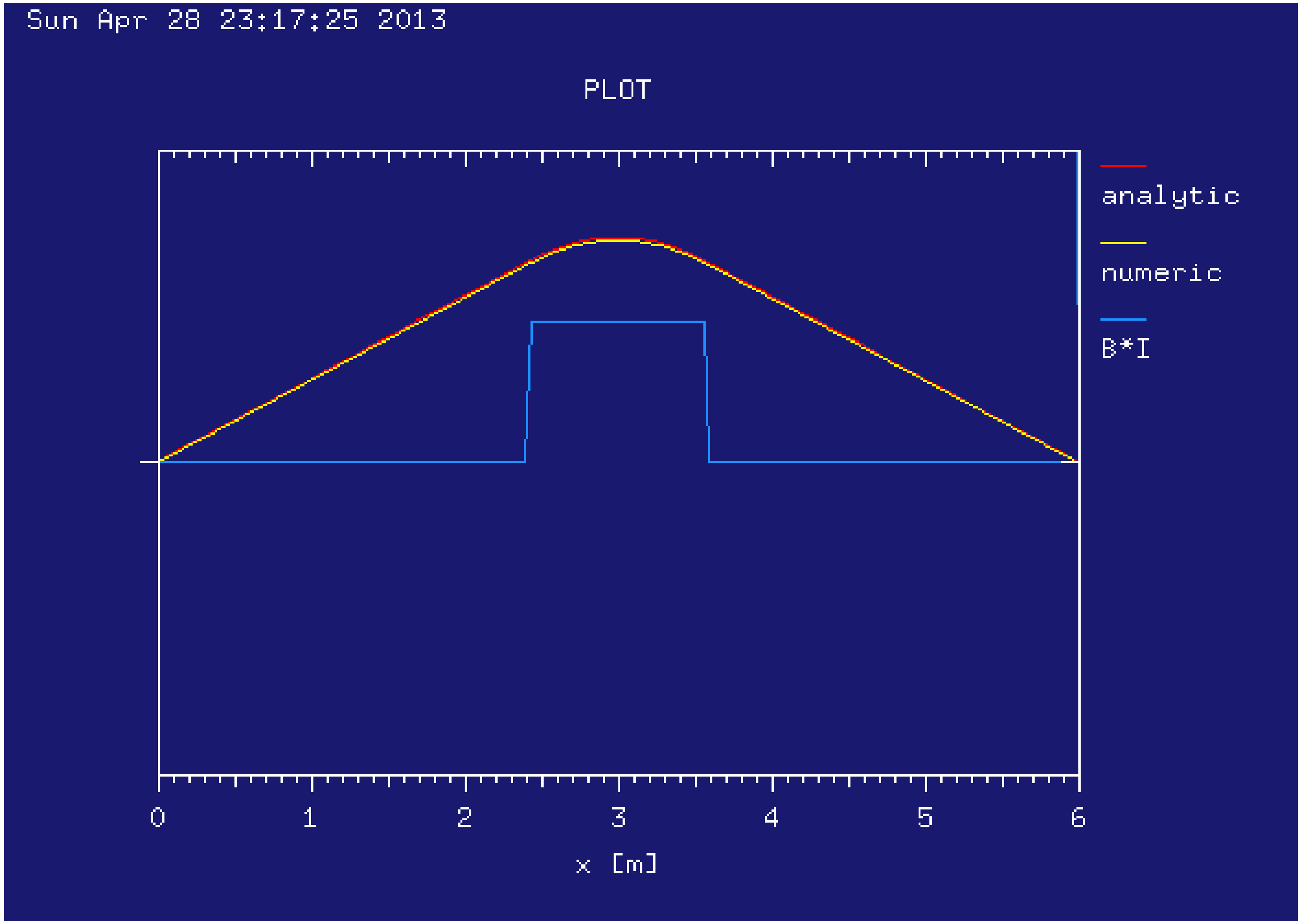
Check 2: Numerical simulation (FDTD) and the Steady State Solution

$$\frac{\partial^2 u}{\partial t^2}(t_i, z_n) \approx \frac{u(t_{i+1}, z_n) - 2u(t_i, z_n) + u(t_{i-1}, z_n)}{(\Delta t)^2} + O((\Delta t)^2),$$

$$\frac{\partial^2 u}{\partial z^2}(t_i, z_n) \approx \frac{u(t_i, z_{n+1}) - 2u(t_i, z_n) + u(t_i, z_{n-1})}{(\Delta z)^2} + O((\Delta z)^2).$$

$$u(t_i, z_{n+1}) = \sigma^2 u(t_i, z_{n+1}) + 2(1-\sigma^2) u(t_i, z_n) + \sigma^2 u(t_i, z_{n-1}) - u(t_{i-1}, z_n) - B_{\mathrm{n}}(z_n) I_0(t_i)$$

$$\sigma = \frac{c\Delta t}{\Delta z}$$

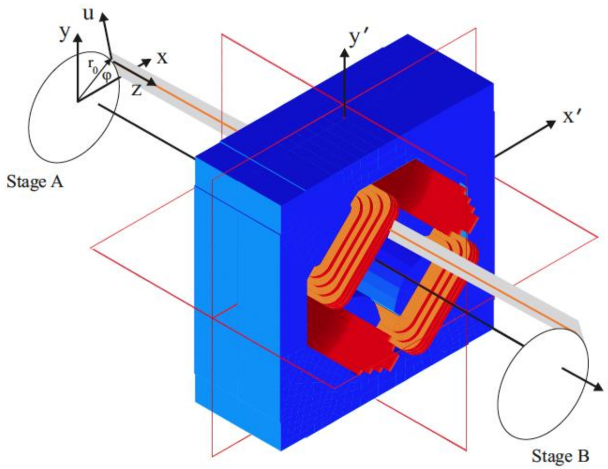


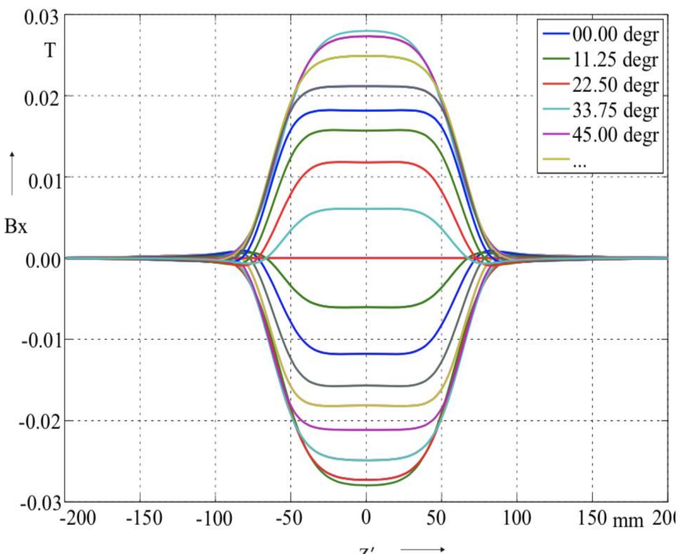
Why Have we not Started Numerically

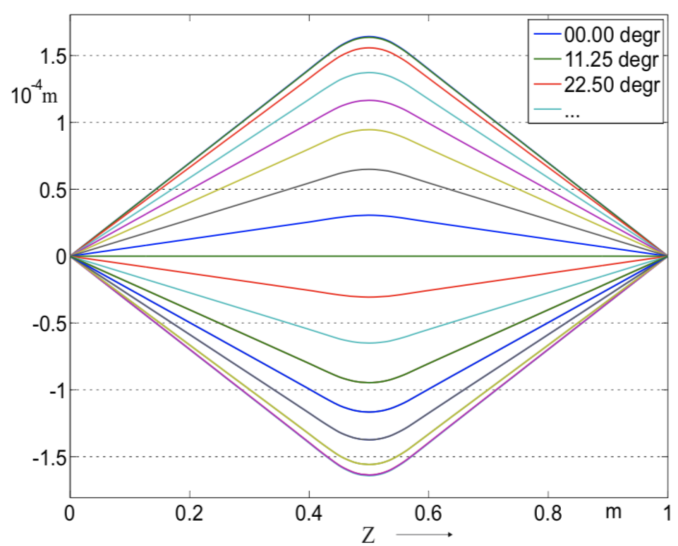
Remember: We wanted to proof that:

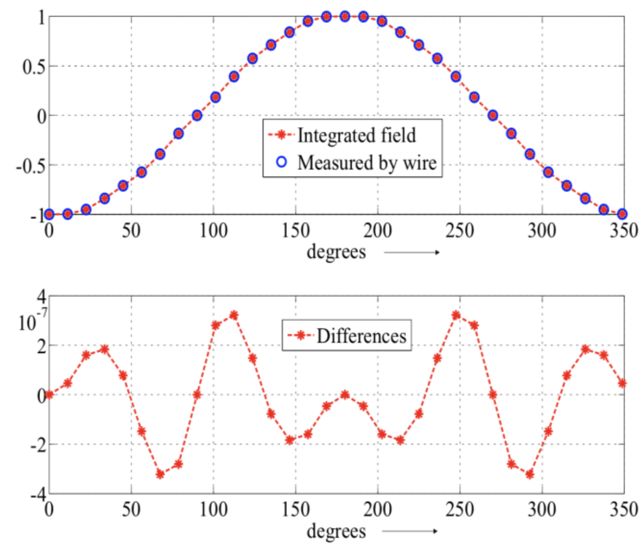
$$d_y^k(r_0) = \lambda_y \int_0^L B_x(r_0, \varphi_k)\mathrm{d}z$$

And we can only measure the amplitude at one position z_0



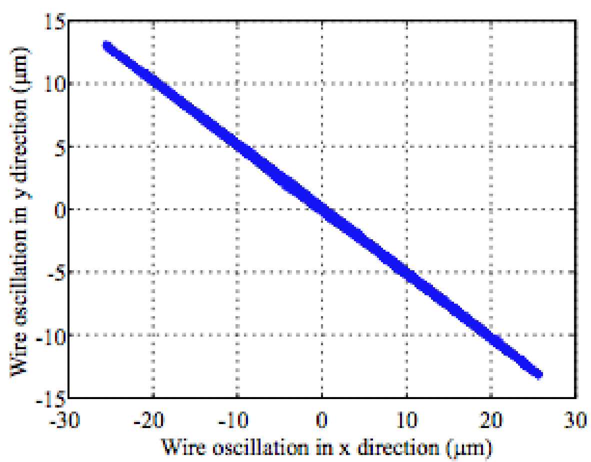


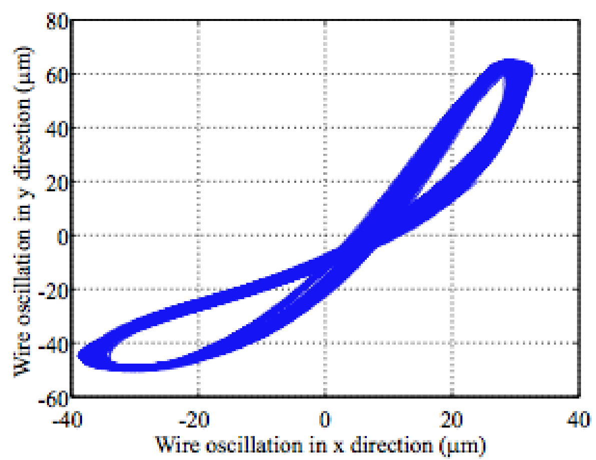




Solution of the Wave Equation (Assumptions and Check)

- Uniformity: The string has a constant mass density ρ.
- Planar oscillations: The string is suspended at the origin $z = 0$ and $z = L$. The string deflection $u(z, t)$ is caused by the distributed force, which is proportional to the the normal field to this plane: $f(z, t) = I(t)B_{\mathrm{n}}(z)$.
- Uniform tension: Each segment of the string pulls on its neighboring segments with the same magnitude of force T.
- The only force in the wire is its tension and the Lorentz force. No gravitational, frictional, or other external forces (wind) are considered.
- Small vibrations: The slope $\mathrm{d}u(z, t)/\mathrm{d}z$ remains small in the interval $[0, L]$.
- Steady state oscillations: After an initial setting time, the string oscillates in the form of a standing wave. Then there will be no energy flow along the string and no energy loss in the fixed suspensions.





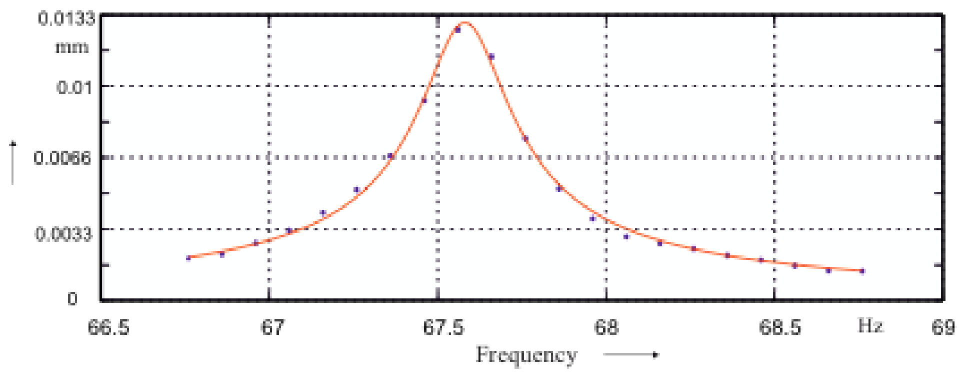
Check: Behavior around the first natural resonance: Are the fit parameters physically meaningful?

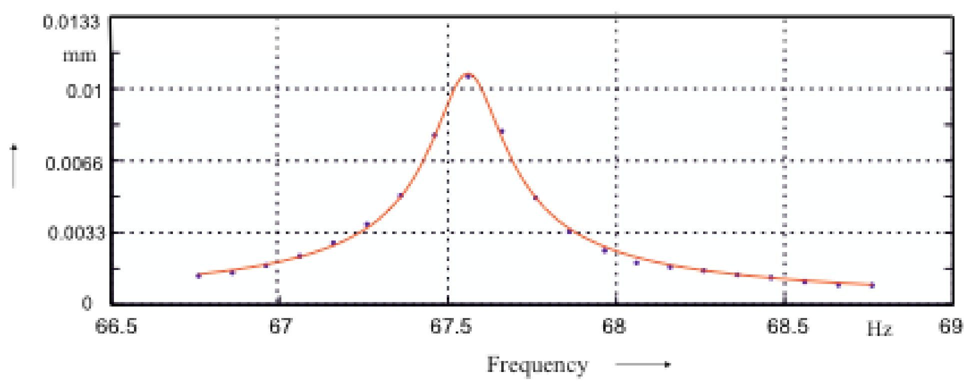
$$f(\omega) = \frac{a}{\sqrt{[(b^2 - \omega^2)]^2 + (c\omega)^2}}.$$

~ Tension

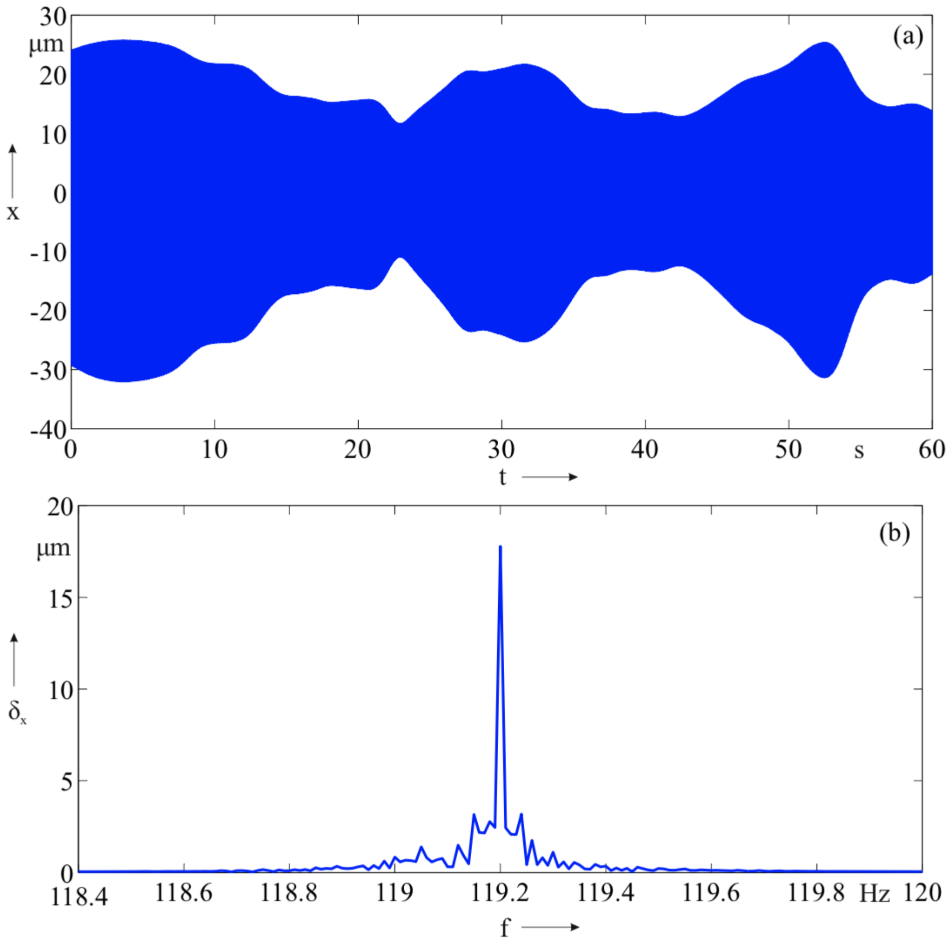
~ Mass density

~ Damping

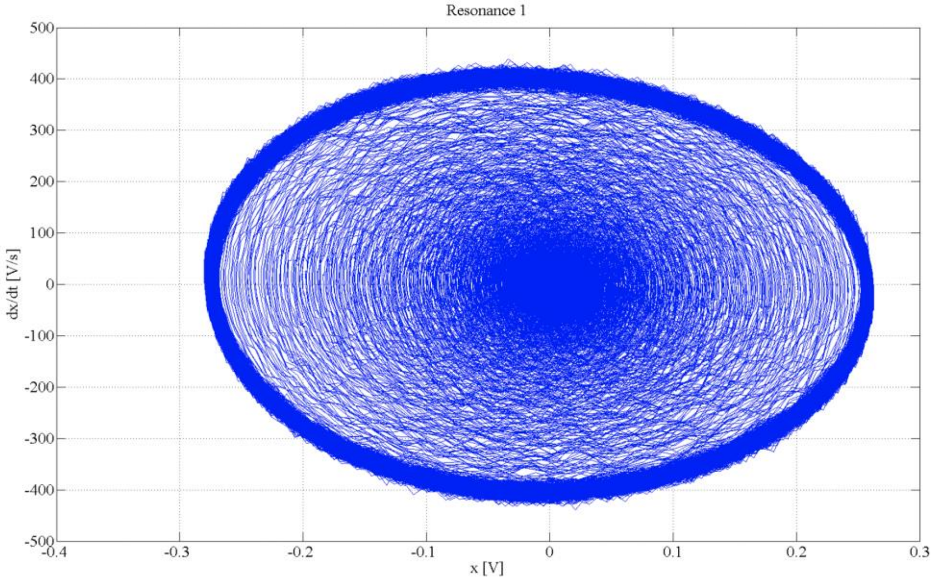




Nonlinearities and Overtones



Nonlinear stress-strain relations in the cable (we tension close to the Hook limit); results in a coupling of the planes of motion

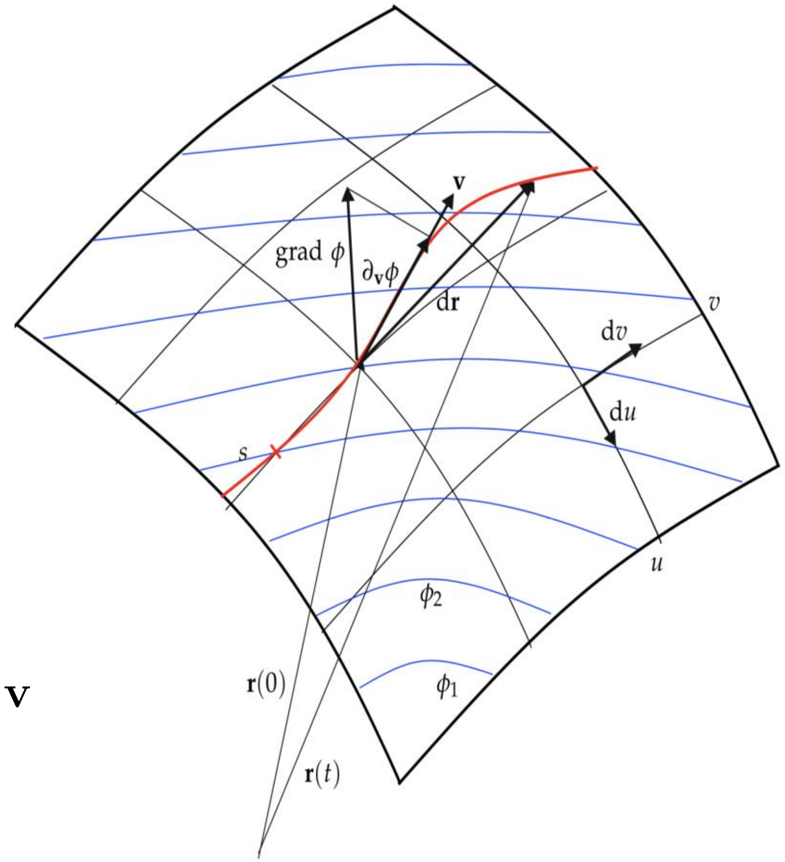


Foundations of Vector Analysis

Directional Derivative and the Total Differential

Space curve with $\mathbf{r}(t) = (x(t), y(t), z(t))$
parametrized such that $\mathbf{r}(0) = P$.

1-smooth scalar field $\phi : E_3 \to R : \mathbf{r} \mapsto \phi(\mathbf{r})$
expressed as $\phi(x,y,z)$, then $\phi(\mathbf{r}(t))$ at
parameter (time) t.



$$\partial_{\mathbf{v}}\phi = \frac{\partial \phi}{\partial v} = \frac{\mathrm{d}}{\mathrm{d}t}[\phi(\mathbf{r}+t\mathbf{v})]_{t=0} = \lim_{t\to 0}\frac{\phi(\mathbf{r}+t\mathbf{v}) - \phi(\mathbf{r})}{t}$$

$$\partial_{\mathbf{v}}\phi = \frac{\mathrm{d}}{\mathrm{d}t}\phi(\mathbf{r}(t)) = \frac{\partial \phi}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \phi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial \phi}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} = \mathrm{grad}\,\phi \cdot \mathbf{v}$$

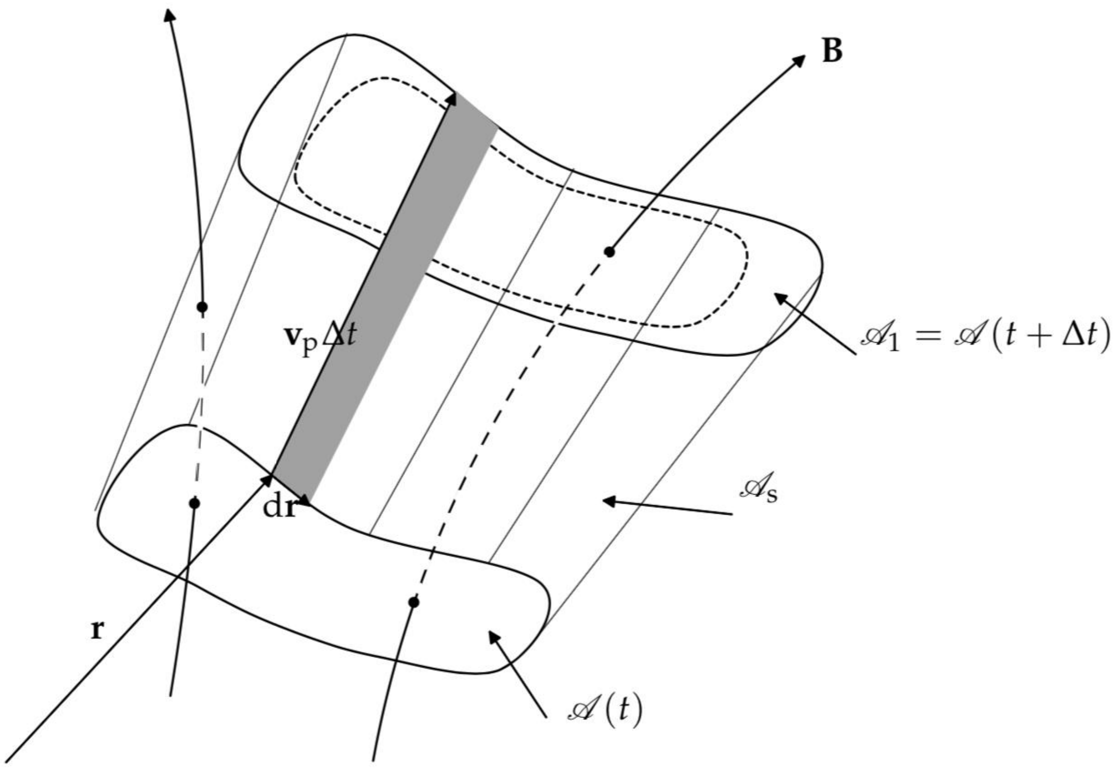
$$\mathrm{grad}\phi = \frac{\partial \phi}{\partial x}\mathbf{e}_x + \frac{\partial \phi}{\partial y}\mathbf{e}_y + \frac{\partial \phi}{\partial z}\mathbf{e}_z$$

Best linear approximation of ϕ over displacement distance d***r***

$$\mathrm{d}\mathbf{r} = \mathbf{v}\mathrm{d}t = \frac{\mathbf{v}}{v}v\mathrm{d}t = \mathbf{T}\,\mathrm{d}s \qquad \mathrm{d}\mathbf{a} = \mathbf{n}\,\mathrm{d}a = \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)\mathrm{d}u\mathrm{d}v \qquad \mathrm{d}\mathrm{f} = \frac{\partial \mathrm{f}}{\partial x}\mathrm{d}x + \frac{\partial \mathrm{f}}{\partial y}\mathrm{d}y + \frac{\partial \mathrm{f}}{\partial z}\mathrm{d}z$$

Precondition for the Differential under the Integral

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{A}} \mathbf{B}\cdot \mathrm{d}\mathbf{a} = \int_{\mathscr{A}} \left(\frac{\partial \mathbf{B}}{\partial t} + \mathbf{v}_{\mathrm{p}}\,\mathrm{div}\,\mathbf{B}\right)\cdot \mathrm{d}\mathbf{a} - \int_{\partial\mathscr{A}} (\mathbf{v}_{\mathrm{p}} \times \mathbf{B})\cdot \mathrm{d}\mathbf{r},$$



Ideal Pole Shape of Conventional Magnets

Remember the Cauchy Schwarz inequality $| \langle \mathbf{a}, \mathbf{b} \rangle | \leq \| \mathbf{a} \| \| \mathbf{b} \|,$

Thus for the directional derivative $|\partial_{\mathbf{v}} \phi| \leq |\operatorname{grad} \phi| \, |\mathbf{v}|$

This implies that the directional derivative takes its maximum when **v** points in the direction of the gradient. Therefore gradient points in the direction of the steepest ascent of Φ and is thus normal to the surface of equipotential.

The flux density **B** exits a highly permeable surface in normal direction. Therefore the pole shape of normal conducting magnets can be seen as an equipotential of the magnetic scalar potential.

Grad, Curl and Div in Cartesian Coordinates

$$\operatorname{grad}\phi := \frac{\partial \phi}{\partial x}\mathbf{e}_x + \frac{\partial \phi}{\partial y}\mathbf{e}_y + \frac{\partial \phi}{\partial z}\mathbf{e}_z$$

$$\operatorname{curl}\mathbf{g} = \left(\frac{\partial g_z}{\partial y} - \frac{\partial g_y}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial g_x}{\partial z} - \frac{\partial g_z}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial g_y}{\partial x} - \frac{\partial g_x}{\partial y}\right)\mathbf{e}_z .$$

$$\operatorname{div}\mathbf{g} = \frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z} .$$

The First Lemma of Poincare

$$
\begin{aligned}
\operatorname{curl}\operatorname{grad}\phi &= \operatorname{curl}\left[\frac{1}{h_1}\frac{\partial\phi}{\partial u^1}\mathbf{e}_{u^1} + \frac{1}{h_2}\frac{\partial\phi}{\partial u^2}\mathbf{e}_{u^2} + \frac{1}{h_3}\frac{\partial\phi}{\partial u^3}\mathbf{e}_{u^3}\right] \\
&= \frac{1}{h_2h_3}\left(\frac{\partial^2\phi}{\partial u^2\partial u^3} - \frac{\partial^2\phi}{\partial u^3\partial u^2}\right)\mathbf{e}_{u^1} \\
&\quad + \frac{1}{h_3h_1}\left(\frac{\partial^2\phi}{\partial u^3\partial u^1} - \frac{\partial^2\phi}{\partial u^1\partial u^3}\right)\mathbf{e}_{u^2} \\
&\quad + \frac{1}{h_1h_2}\left(\frac{\partial^2\phi}{\partial u^1\partial u^2} - \frac{\partial^2\phi}{\partial u^2\partial u^1}\right)\mathbf{e}_{u^3} = 0\,,
\end{aligned}
$$

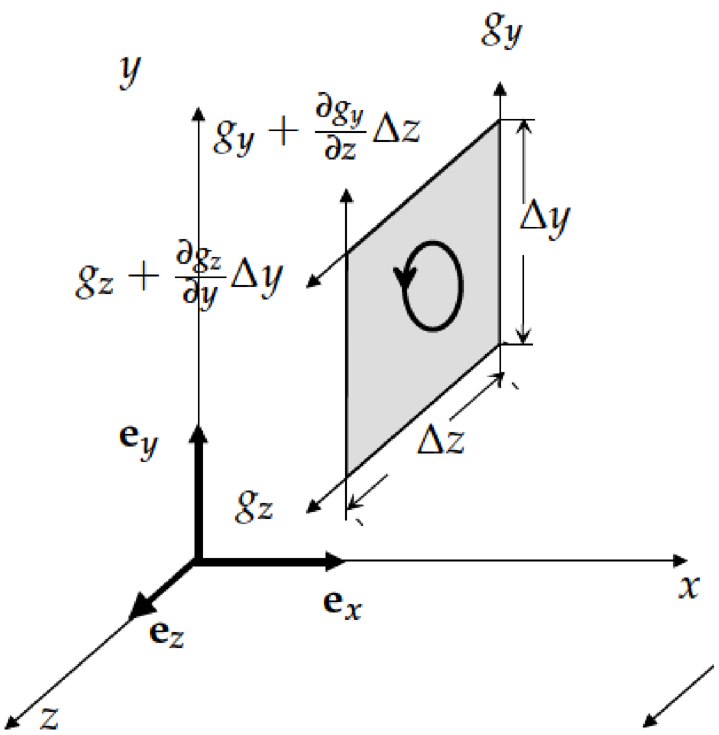
Ugly and not even a universal proof (orthogonality assumed)

Coordinate Free Definition of Grad, Curl, and Div

$$\int_{\mathscr{P}_1}^{\mathscr{P}_2} \mathbf{a} \cdot \mathrm{d}\mathbf{r} = \int_{\mathscr{P}_1}^{\mathscr{P}_2} \operatorname{grad} \phi \cdot \mathrm{d}\mathbf{r} = \int_{\mathscr{P}_1}^{\mathscr{P}_2} \mathrm{d}\phi = \phi(\mathscr{P}_2) - \phi(\mathscr{P}_1),$$

$$\mathbf{n} \cdot \operatorname{curl} \mathbf{g} = \lim_{a \to 0} \frac{\int_{\partial \mathscr{A}} \mathbf{g} \cdot \mathrm{d}\mathbf{r}}{a},$$

$$\operatorname{div} \mathbf{g} = \lim_{V \to 0} \frac{\int_{\partial \mathscr{V}} \mathbf{g} \cdot \mathrm{d}\mathbf{a}}{V},$$



The Boundary Operator

$$\partial(\partial\mathscr{V}) = \emptyset\,, \qquad \partial(\partial\mathscr{A}) = \emptyset\,,$$

$$\int_{\mathscr{V}} \operatorname{div}\,\operatorname{curl}\mathbf{g}\,\mathrm{d}V = \int_{\partial\mathscr{V}} \operatorname{curl}\mathbf{g}\cdot\mathrm{d}\mathbf{a} = \int_{\partial(\partial\mathscr{V})} \mathbf{g}\cdot\mathrm{d}\mathbf{r} = 0\,,$$

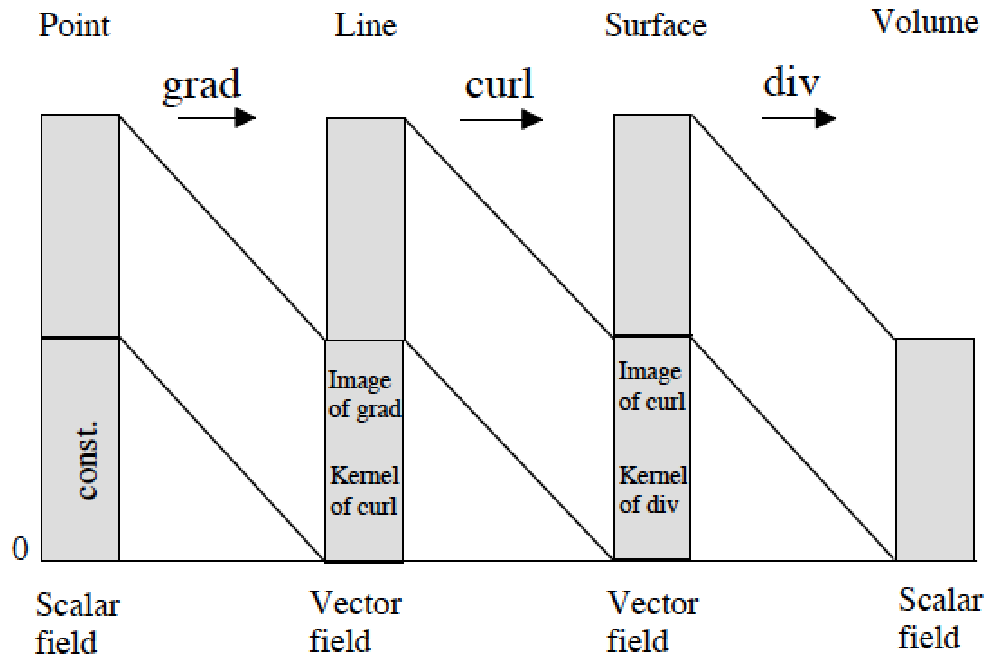
$$\int_{\mathscr{A}} \operatorname{curl}\,\operatorname{grad}\phi\cdot\mathrm{d}\mathbf{a} = \int_{\partial\mathscr{A}} \operatorname{grad}\phi\cdot\mathrm{d}\mathbf{r} = \phi|_{\partial(\partial\mathscr{A})} = 0\,,$$

Reversal of arguments yields two important statements (next slides):
Much nicer than writing it in coordinates

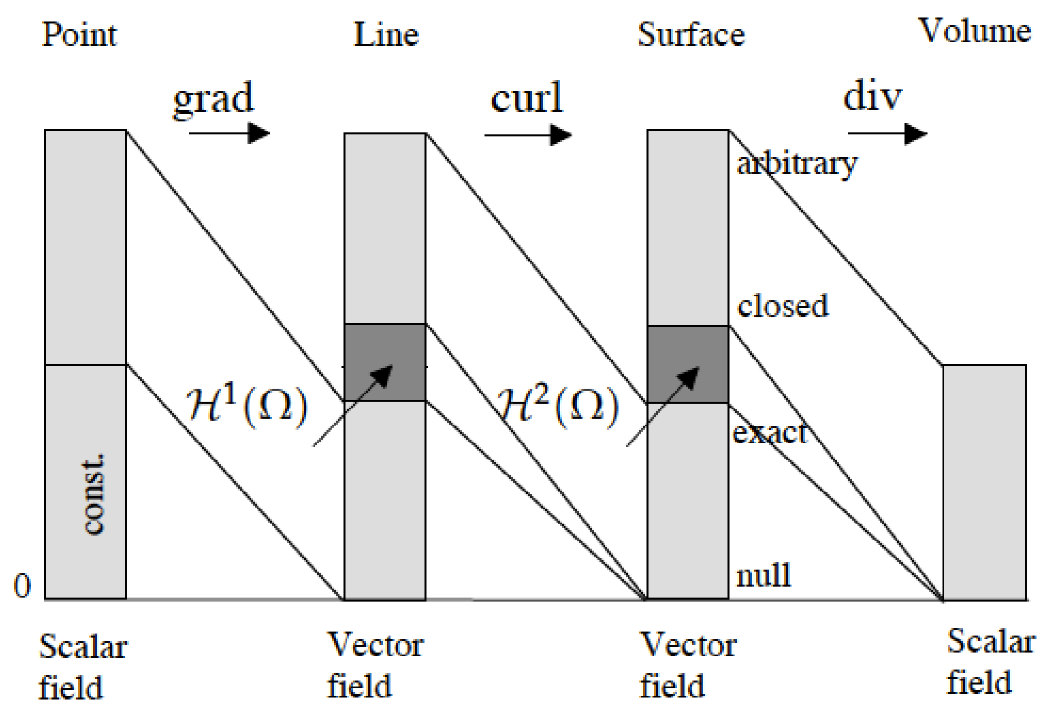
The second Lemma of Poincare (Contractible Domains)

$$\operatorname{div} \mathbf{b} = 0 \quad \rightarrow \quad \mathbf{b} = \operatorname{curl} \mathbf{a}.$$

$$\operatorname{curl} \mathbf{h} = 0 \quad \rightarrow \quad \mathbf{h} = \operatorname{grad} \phi.$$



Lemmata of Poincare (Non-Contractible Domains)



$$\mathcal{H}^1(\Omega) := \frac{\ker(\,\text{curl}\,)}{\text{im}(\,\text{grad}\,)}$$

$$\mathcal{H}^2(\Omega) := \frac{\ker(\text{div})}{\text{im}(\,\text{curl}\,)}$$

Toroidal domain Ω in a cylindrical coordinate system (r, φ, z):

$$H_\varphi = \frac{I}{2\pi r}$$

$$\text{curl}\,\mathbf{H} = \frac{1}{r}\frac{\partial}{\partial r}(r H_\varphi) = 0$$

But $\oint_C \mathbf{H} \cdot \mathsf{ds} = I$ and Ω, with $\oint_C \text{grad}\,\phi \cdot \mathsf{ds} = 0$

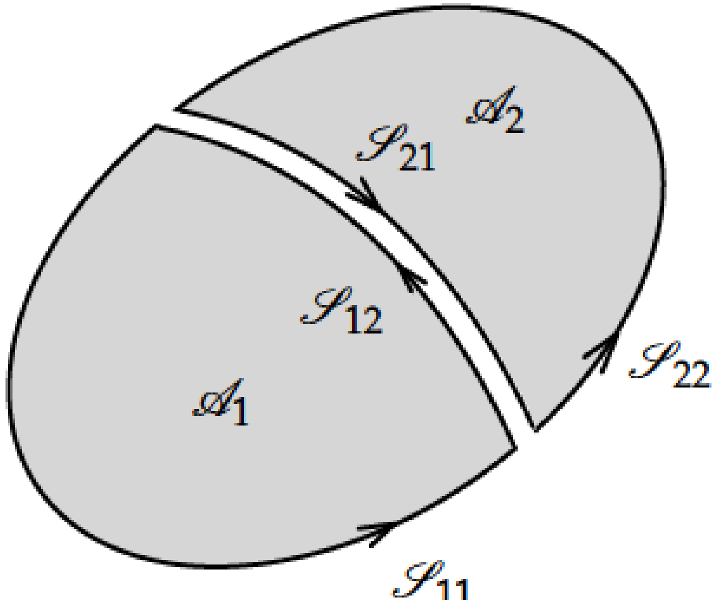
Domain Ω between two nested spheres centered at the origin.

$$D_R = \frac{Q}{4\pi R^2}\mathbf{e}_R$$

$$\text{div}\,\mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial R}(R^2 D_R) = 0$$

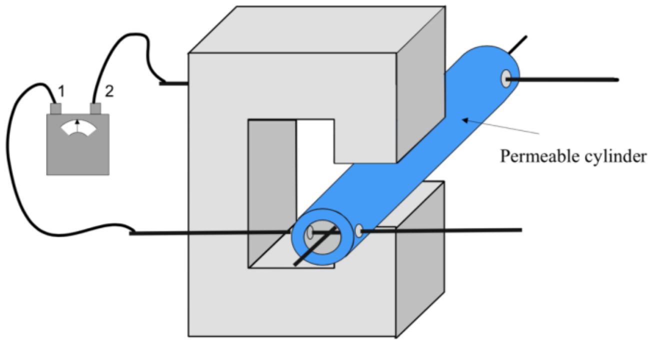
But $\oint_a \mathbf{D} \cdot \mathsf{da} = Q$ and $\oint_a \text{curl}\,\mathbf{A} \cdot \mathsf{da} = 0$

Kelvin-Stokes Theorem



Smooth vector fields, smooth surfaces with simply connected, closed, piecewise-smooth and consistently oriented boundaries, and volumes with piecewise-smooth, closed and consistently oriented surfaces.

No jump discontinuities (for example, co-moving shielding devices)

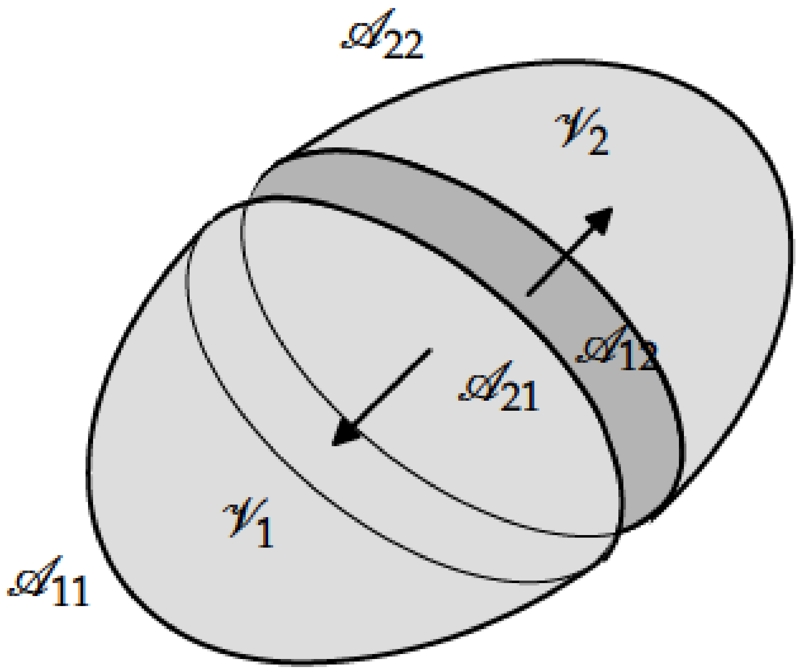


$$\int_{\partial\mathscr{A}} \mathbf{g}\cdot d\mathbf{r} = \int_{\mathscr{S}_1} \mathbf{g}\cdot d\mathbf{r} + \int_{\mathscr{S}_2} \mathbf{g}\cdot d\mathbf{r} = \int_{\mathscr{S}_{11}} \mathbf{g}\cdot d\mathbf{r} + \int_{\mathscr{S}_{22}} \mathbf{g}\cdot d\mathbf{r},$$

$$\int_{\partial\mathscr{A}} \mathbf{g}\cdot d\mathbf{r} = \lim_{I\to\infty}\sum_{i=1}^{I}\int_{\partial\mathscr{A}_i} \mathbf{g}\cdot d\mathbf{r} = \lim_{I\to\infty}\sum_{i=1}^{I}\Delta a_i \frac{1}{\Delta a_i}\int_{\partial\mathscr{A}_i} \mathbf{g}\cdot d\mathbf{r}$$

$$= \lim_{I\to\infty}\sum_{i=1}^{I}(\operatorname{curl}\mathbf{g})_i\cdot\mathbf{n}\,\Delta a_i = \int_{\mathscr{A}} \operatorname{curl}\mathbf{g}\cdot d\mathbf{a}\,.$$

Gauss' Theorem



Smooth vector fields, smooth surfaces with simply connected, closed, piecewise-smooth and consistently oriented boundaries, and volumes with piecewise-smooth, closed and consistently oriented surfaces.

$$\int_{\partial\mathscr{V}} \mathbf{g}\cdot\mathrm{d}\mathbf{a} = \lim_{I\to\infty}\sum_{i=1}^{I}\int_{\partial\mathscr{V}_i}\mathbf{g}\cdot\mathrm{d}\mathbf{a} = \lim_{I\to\infty}\sum_{i=1}^{I}\Delta V_i\frac{1}{\Delta V_i}\int_{\partial\mathscr{V}_i}\mathbf{g}\cdot\mathrm{d}\mathbf{a}$$

$$= \lim_{I\to\infty}\sum_{i=1}^{I}(\operatorname{div}\mathbf{g})_i\,\Delta V_i = \int_{\mathscr{V}}\operatorname{div}\mathbf{g}\,\mathrm{d}V\,.$$

More Integral Theorems

$$\int_a^b f(x)g'(x)\,\mathrm{d}x = \Big[g(x)f(x)\Big]_a^b - \int_a^b g(x)f'(x)\,\mathrm{d}x$$

Green's First

$$\int_{\mathcal{V}} \left(\operatorname{grad}\phi \cdot \operatorname{grad}\psi + \phi\nabla^2\psi\right)\,\mathrm{d}V = \int_{\partial\mathcal{V}} \phi\,\partial_{\mathbf{n}}\psi\,\mathrm{d}a$$

Green's Second

$$\int_{\Omega} \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)\,\mathrm{d}V = \int_{\Gamma} (\phi\partial_{\mathbf{n}}\psi - \psi\partial_{\mathbf{n}}\phi)\,\mathrm{d}a$$

Vector Form of Green's Second

$$\int_{\mathcal{V}} \mathbf{a}\cdot\operatorname{curl}\mathbf{b}\,\mathrm{d}V = \int_{\mathcal{V}} \mathbf{b}\cdot\operatorname{curl}\mathbf{a}\,\mathrm{d}V - \int_{\partial\mathcal{V}} \mathbf{a}\cdot(\mathbf{b}\times\mathbf{n})\,\mathrm{d}a.$$

Generalization of the Integration by Parts Rule

$$-\int_{\mathcal{V}} \mathbf{a}\cdot\operatorname{grad}\phi\,\mathrm{d}V = \int_{\mathcal{V}} \phi\operatorname{div}\mathbf{a}\,\mathrm{d}V - \int_{\partial\mathcal{V}} \phi(\mathbf{a}\cdot\mathbf{n})\,\mathrm{d}a.$$

Stratton #1 and #2

$$\int_{\mathcal{V}} \operatorname{div}(\mathbf{a}\times\operatorname{curl}\mathbf{b})\mathrm{d}V = \int_{\partial\mathcal{V}} (\mathbf{a}\times\operatorname{curl}\mathbf{b})\cdot\mathbf{n}\,\mathrm{d}a$$

$$\int_{\mathcal{V}} (\mathbf{a}\operatorname{curl}\operatorname{curl}\mathbf{b} - \mathbf{b}\operatorname{curl}\operatorname{curl}\mathbf{a})\,\mathrm{d}V = \int_{\partial\mathcal{V}} (\mathbf{b}\times\operatorname{curl}\mathbf{a} - \mathbf{a}\times\operatorname{curl}\mathbf{b})\cdot\mathbf{n}\,\mathrm{d}a\,.$$