Linear Theory

$$a = \exp(A \cdot \nabla) I. \tag{3.28}$$

The map I is the identity map of phase space: it is made of two trivial projection functions in 1-d-f, in other words

$$I(\mathbf{z}) = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \tag{3.29}$$

If the vector field A is symplectic (or Hamiltonian), it means that a Hamiltonian H_a can achieve the same result:

$$A \cdot \nabla = \frac{\partial H_a}{\partial z_1} \frac{\partial}{\partial z_2} - \frac{\partial H_a}{\partial z_2} \frac{\partial}{\partial z_1} =: H_a : . \tag{3.30}$$

It can be shown easily, in the Hamiltonian case, that Eq. (3.26) can be written as:

$$K^{new} = K \circ a = \exp(: H_a :) K. \tag{3.31}$$


COMING UP NEXT: AN AUTHENTIC RECREATION OF THE 2016 U.S. ELECTION. SOME SCENES MAY BE DISTURBING.

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \Longrightarrow M = ARA^{-1}$$

$$1 = ad - bc$$

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{M} = \underbrace{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}}_{A} \begin{pmatrix} \cos(\mu) & \sin(\mu) \\ -\sin(\mu) & \cos(\mu) \end{pmatrix} \underbrace{\begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}}_{A^{-1}}$$

A is also assumed to have det(A)=1.

Invariant (nonlinear OK)

$$\begin{pmatrix} x_n \\ p_n \end{pmatrix} = A^{-1} \begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}$$

$$M = ARA^{-1}$$

$$\begin{aligned} \text{Invariant} &= x_n^2 + p_n^2 = (A_{22}x - A_{12}p)^2 + (-A_{21}x + A_{11}p)^2 \\ &= \underbrace{\left(A_{22}^2 + A_{21}^2\right)}_{\gamma} x^2 + 2\underbrace{\left(-A_{12}A_{22} - A_{21}A_{11}\right)}_{\alpha} xp + \underbrace{\left(A_{11}^2 + A_{12}^2\right)}_{\beta} p^2 \end{aligned}$$

Lattice Functions (linear only)

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{M} = \underbrace{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}}_{A} \begin{pmatrix} \cos(\mu) & \sin(\mu) \\ -\sin(\mu) & \cos(\mu) \end{pmatrix} \underbrace{\begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}}_{A^{-1}}$$

$$M = \cos(\mu) I + \sin(\mu) \begin{pmatrix} \underbrace{-A_{11}A_{21} - A_{12}A_{22}}_{\alpha} & \underbrace{A_{11}^2 + A_{12}^2}_{\beta} \\ -\underbrace{A_{21}^2 - A_{22}^2}_{-\gamma} & \underbrace{A_{11}A_{21} + A_{12}A_{22}}_{-\alpha} \end{pmatrix}$$

$$\Rightarrow \quad M = \begin{pmatrix} \cos(\mu) + \alpha \sin(\mu) & \beta \sin(\mu) \\ -\gamma \sin(\mu) & \cos(\mu) - \alpha \sin(\mu) \end{pmatrix}$$

$1 + \alpha^2 = \beta\gamma$ ← Consequence of det(A)=1

2.2.4 *Lattice functions as coefficients of the moments* $\langle z_i z_j \rangle$

Here let us use the matrix A directly and compute the three quadratic moments of the 1-d-f linear theory.

$$\begin{aligned}\langle z_1^2 \rangle &= \left\langle (A_{11} z_1^{new} + A_{12} z_2^{new})^2 \right\rangle = A_{11}^2 \left\langle z_1^{new2} \right\rangle + A_{12}^2 \left\langle z_2^{new2} \right\rangle + 2A_{11}A_{12} \langle z_1^{new} z_2^{new} \rangle \\ &= \underbrace{(A_{11}^2 + A_{12}^2)}_{\beta} \frac{\langle r^2 \rangle}{2}\end{aligned} \tag{2.34}$$

In Eq. (2.34), if we are dealing with a single particle and performing a time average, then the average can be removed in the final expression. In Eq. (2.34) I use the fact that that z_1^{new} and z_2^{new} move on circles and thus the value $z_1^{new2} + z_2^{new2}$ is constant on a trajectory. Other averages can be computed as well:

$$\begin{aligned}\langle z_2^2 \rangle &= \left\langle (A_{21} z_1^{new} + A_{22} z_2^{new})^2 \right\rangle = A_{21}^2 \left\langle z_1^{new2} \right\rangle + A_{22}^2 \left\langle z_2^{new2} \right\rangle + 2A_{21}A_{22} \langle z_1^{new} z_2^{new} \rangle \\ &= \underbrace{(A_{21}^2 + A_{22}^2)}_{\gamma} \frac{\langle r^2 \rangle}{2}\end{aligned} \tag{2.35}$$

and more interestingly,

$$\begin{aligned}\langle z_1 z_2 \rangle &= \langle (A_{11} z_1^{new} + A_{12} z_2^{new})(A_{21} z_1^{new} + A_{22} z_2^{new}) \rangle \\ &= A_{11}A_{21} \left\langle z_1^{new2} \right\rangle + A_{12}A_{22} \left\langle z_2^{new2} \right\rangle + \{A_{11}A_{22} + A_{12}A_{21}\} \langle z_1^{new} z_2^{new} \rangle \\ &= \underbrace{(A_{11}A_{21} + A_{12}A_{22})}_{-\alpha} \frac{\langle r^2 \rangle}{2}\end{aligned} \tag{2.36}$$

$$M = \cos(\mu) H + \sin(\mu) B \tag{3.1}$$

where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \text{ with } \mathrm{B}^2 = -\mathrm{H}$$

Using this formula, which is a representation of the unit complex numbers, we immediately see that

$$M^N = \cos(N\mu) H + \sin(N\mu) B \tag{3.2}$$

Eq. (3.2) is the famous de Moivre formula known to accelerator physicists. The notation H (as in capital η), for what appears to be the identity, anticipates the results in more than 1-d-f. We will see that in 3-d-f, the matrix H is not the identity but corresponds to generalised dispersions of reference [4].

The matrix B (as in beta functions) contains the regular lattice functions as defined by Ripken. In fact we have the following results:

$$B = SK \tag{3.3}$$

$$B = ES \tag{3.4}$$

$$K = SES^{\mathrm{T}} \tag{3.5}$$

where $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

The matrix K of Eq. (3.3) defines the Courant-Snyder invariant

$$K = \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} \Longrightarrow \varepsilon(z) = z^{\mathrm{T}} K z = \gamma z_1^2 + 2\alpha z_1 z_2 + \beta z_2^2 \tag{3.6}$$

The matrix E of Eq. (3.4) gives us the quadratic moments:

$$E = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} \Longrightarrow \langle z_i z_j \rangle = E_{ij} \left\langle \frac{z_1^2 + z_2^2}{2} \right\rangle = E_{ij} \langle J \rangle$$

$$\text{for example } \langle z_1^2 \rangle = \beta \langle J \rangle \tag{3.7}$$

Finally, the one-turn map M can be written with a single exponent and we will see that all the results of this section can be extended to many dimensions.

$$M = \exp(\mu B) = \exp(\mu S K) \tag{3.8}$$

From Eq. (3.8) one can derive two interesting corollaries. Firstly the invariant of Eq. (3.6), the so called Courant-Snyder invariant, is a pseudo-Hamiltonian for the matrix M. The Hamiltonian $\mathscr{H}$,

$$\mathscr{H} = \frac{\mu}{2} z^{\mathrm{T}} K z \tag{3.9}$$

4. De Moivre's formula in many degrees of freedom

The derivation of this formula only requires the existence of a normal form. We might as well do it in the non-symplectic case. In 3-d-f, when classical radiation is present, the map M can be factorised as

$$M = A\Lambda R A^{-1} \tag{4.1}$$

$$R = \begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix} \quad r_i = \begin{pmatrix} \cos\mu_i & \sin\mu_i \\ -\sin\mu_i & \cos\mu_i \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & \Lambda_3 \end{pmatrix} \quad \Lambda_i = \begin{pmatrix} \exp(-\alpha_i) & 0 \\ 0 & \exp(-\alpha_i) \end{pmatrix} \tag{4.2}$$

---

[3] It was reported to me by Frank Schmidt of CERN, that Dr. Ripken referred to the dispersion-like lattice functions of reference [4] as "the only thing in linear Hamiltonian perturbation theory" he had been unaware. I surmise that he would have like the cute result of this paper— that his theory contained an exact form of Sands' approximate methods.

Applying Eq. (4.2) to Eq. (4.1) we get

$$M = A\left(\sum_i e^{-\alpha_i}\left\{\cos(\mu_i) I^i + \sin(\mu_i) SI^i\right\}\right) A^{-1} \tag{4.3}$$

$$= \sum_i e^{-\alpha_i}\left\{\cos(\mu_i)\underbrace{AI^iA^{-1}}_{H^i} + \sin(\mu_i)\underbrace{ASI^iA^{-1}}_{B^i}\right\} \tag{4.4}$$

where I^i is the identity only in the i^{th} plane. For example,

$$I^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.5}$$

It is easy to show that the matrices H^i and B^i form three independent representations of the complex numbers 1 and i:

$$H^iH^j = \delta_{ij}H^j \tag{4.6}$$

$$H^iB^j = B^jH^i = \delta_{ij}B^j \tag{4.7}$$

$$B^iB^j = -\delta_{ij}H^j \tag{4.8}$$

De Moivre's general formula follows from the three previous equations.

$$M^N = \sum_i e^{-N\alpha_i} \left\{\cos\left(N\mu_i\right) H^i + \sin\left(N\mu_i\right) B^i\right\} \tag{4.9}$$

Moreover, we can again rewrite the map M in terms of a single exponent:

$$M = \exp\left(\sum_i -\alpha_i H^i + \mu_i B^i\right) = \exp\left(\sum_i -\alpha_i H^i + \mu_i S K^i\right) \tag{4.10}$$

There are a few comments we need to make concerning the results of this section.

1. As shown by Eq. (4.10), the lattice functions H^i cannot be removed easily from any treatment of a non-symplectic system. Of course, using Eq. (4.8), we can replace them by quadratic polynomials in the B^i's.

2. Forest, in [4], has shown that analytical tune shift formulas for an arbitrary force depend on the B^i's while the damping shifts depend on the H^i's.

3. In a symplectic system, if perturbation theory is done on the Hamiltonian, the lattice functions H^i will not naturally appear since the exponent in Eq. (4.10) only depends of the B^i's (when the damping is zero). This is why they never appear in Ripken's extensive literature on the subject.

5.2 Other Useful Properties

Properties equivalent those of Eqs. (3.3), (3.4) and (3.5) can be proved in the multidimensional case:

$$B^i = SK^i \tag{5.11}$$

$$B^i = E^i S \tag{5.12}$$

$$K^i = SE^i S^{\mathrm{T}} \tag{5.13}$$

For example, the pseudo-Hamiltonian which generates the matrix M is in the sum of the three invariants weighted by the tunes:

$$\mathscr{H} = \sum_{i=1,3} \frac{\mu_i}{2} z^{\mathrm{T}} K^i z = \sum_{i=1,3} \frac{\mu_i}{2} \varepsilon_i (z) \tag{5.14}$$

$$\varepsilon_i (z) = z^{\mathrm{T}} K^i z \quad \leftarrow \quad \text{Coupled Courant – Snyder Invariants} \tag{5.15}$$

As for the quadratic moments, they are given by E:

$$\langle z_a z_b \rangle = \sum_{i=1,3} E^i_{ab} \langle J_i \rangle \tag{5.16}$$

There are many more interesting corollaries but that would take our story into an unwanted tangent. Let us return to our main discovery: the strange and unexpected equivalence between Sands and Chao.

Possible A: select $A_{11}>0$ and $A_{12}=0$

$$A_{11} = \left| \frac{M_{12}}{\sin(\mu)} \right|^{1/2}$$

$$A_{22} = A_{11}^{-1}$$

$$A_{21} = \frac{\left(M_{21} + \sin(\mu) A_{11}^{-2}\right) A_{11}}{\cos(\mu) - M_{22}}$$

A in terms of lattice functions

$$A_{11} = A_{22}^{-1} = \left| \frac{M_{12}}{\sin(\mu)} \right|^{1/2} = \left| \frac{\beta \sin(\mu)}{\sin(\mu)} \right|^{1/2} = \sqrt{\beta}$$

$$A_{21} = \frac{\left(M_{21} + \sin(\mu) A_{11}^{-2}\right) A_{11}}{\cos(\mu) - M_{22}} = \frac{\left(-\gamma \sin(\mu) + \frac{\sin(\mu)}{\beta}\right)\sqrt{\beta}}{\cos(\mu) - (\cos(\mu) - \alpha \sin(\mu))}$$
$$= \frac{\left(-\gamma + \frac{1}{\beta}\right)\sqrt{\beta}}{\alpha} = -\frac{\alpha}{\sqrt{\beta}}$$

$$\Longrightarrow A = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}$$

Evolution of invariant (nonlinear OK)

$$H = \frac{H_{22}(s)}{2}p^2 + \frac{H_{11}(s)}{2}x^2 + H_{12}(s)xp$$

$$\begin{aligned} H_{11} &= k(s) \equiv \text{Focusing quadrupole strength} \\ H_{22} &= 1 \\ H_{12} &= 0 \,. \end{aligned}$$

$$\begin{aligned} \mathcal{M}_{s+ds} &= \mathcal{M}^{-1}_{s\to s+ds}\mathcal{M}_s\mathcal{M}_{s\to s+ds} \\ &\stackrel{\text{Eq. (2.89)}}{=} \exp\left(ds : H :\right)\mathcal{M}_s \exp\left(ds : -H :\right) \\ &\stackrel{\text{Eqs. (2.43,2.44,2.45)}}{=} \exp\left(ds : H :\right)\exp\left(: -\frac{\mu}{2}\varepsilon_s :\right)\exp\left(ds : -H :\right) \\ &\stackrel{\text{See exercise 2-9}}{=} \exp\left(: -\frac{\mu}{2}e^{ds:H:}\varepsilon_s :\right) \\ &\Downarrow \\ \varepsilon_{s+ds} &= e^{ds:H_2:}\varepsilon_s = \varepsilon_s + ds\,[H_2, \varepsilon_s]\cdots \end{aligned} \qquad (2.91)$$

$$\frac{d\varepsilon_s}{ds} = [H_2, \varepsilon_s]$$

$$= 2(\alpha\ H_{11} - \gamma\ H_{12})x^2 + 2(\beta H_{11} - \gamma H_{22})xp$$

$$+2(\beta H_{12} - \alpha H_{22})p^2$$

$$\Rightarrow \begin{cases} \frac{d\gamma}{ds} = 2(\alpha\ H_{11} - \gamma\ H_{12}) \\ \frac{d\alpha}{ds} = \quad \beta H_{11} - \gamma H_{22} \\ \frac{d\beta}{ds} = 2(\beta H_{12} - \alpha H_{22}) \end{cases} .$$

If the Hamiltonian is the usual one :

$$\frac{d\gamma}{ds} = 2\alpha\ k(s)$$

$$\frac{d\alpha}{ds} = \beta k(s) - \gamma$$

$$\frac{d\beta}{ds} = -2\alpha.$$

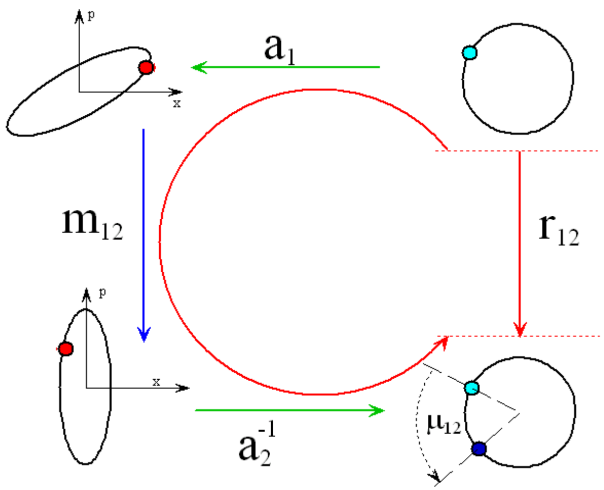
Nonlinear works

$$\frac{dI_s}{ds} = [H, I_s]. \tag{2.94}$$

With Maps: In general, consider a quantity I_s which is invariant under $\mathcal{M}_s$:

$$\begin{aligned} I_s &= \mathcal{M}_s I_s \\ \mathcal{M}^{-1}_{s\to s+\Delta} I_s &= \underbrace{\mathcal{M}^{-1}_{s\to s+\Delta}\mathcal{M}_s\mathcal{M}_{s\to s+\Delta}}_{\mathcal{M}_{s+ds}}\mathcal{M}^{-1}_{s\to s+\Delta} I_s \\ &\Downarrow \\ I_{s+\Delta} &= \mathcal{M}^{-1}_{s\to s+\Delta} I_s. \end{aligned} \tag{2.95}$$

In Equation (2.95), the time shift Δ can be finite. The result of Equation (2.94) is gotten by replacing Δ by ds.



Writing $\mathcal{R}_{s\to s+ds}$ as $\exp(-d\mu : J :)$, we then compute the map $\mathcal{A}_{s+ds}$ to first order in ds:

$$\exp(-d\mu : J :) = \mathcal{A}_s e^{ds:-H:} \mathcal{A}_{s+ds}^{-1}$$
$$\Downarrow$$
$$\mathcal{A}_{s+ds} = \mathcal{A}_s - ds\mathcal{A}_s : H : + d\mu : J : \mathcal{A}_s. \qquad (2.99)$$

Up to this point, we have not used the fact that we interested in the Courant-Snyder transformation. This transformation is characterized by the vanishing A_{12} entry of its associated transfer map. To get this entry, we act on the function "x" with $\mathcal{A}_{s+ds}$ and we require the coefficient of the function "p" to be zero:

$$\begin{aligned}
\mathcal{A}_{s+ds}x &= \mathcal{A}_s x - ds\mathcal{A}_s[H,x] + d\mu : J : \mathcal{A}_s x \\
&= \sqrt{\beta_s}x - ds\mathcal{A}_s(-H_{22}p - H_{12}x)] + d\mu\sqrt{\beta_s}[J,x] \\
&= \sqrt{\beta_s}x + dsH_{22}\left(\frac{p}{\sqrt{\beta_s}} - \frac{\alpha_s x}{\sqrt{\beta_s}}\right) + dsH_{12}\sqrt{\beta_s}x - d\mu\sqrt{\beta_s}p \\
&= \left(\sqrt{\beta_s} + ds\frac{-H_{22}\ \alpha_s + H_{12}\beta_s}{\sqrt{\beta_s}}\right)x + \underbrace{\left(\frac{dsH_{22}}{\sqrt{\beta_s}} - d\mu\sqrt{\beta_s}\right)}_{\text{Must be zero for the Courant-Snyder Functional}} p
\end{aligned}$$

$$\Downarrow$$
$$\frac{d\mu}{ds} = \frac{H_{22}}{\beta_s}. \qquad (2.100)$$

Why Courant-Snyder choice?

$$\begin{aligned} z^1(n) &= M_{01} M_0^n z^0(0) \\ &= A_1 R_{01} A_0^{-1} A_0 R_0^n A_0^{-1} z^0(0) \\ &= A_1 R_{01} R_0^n A_0^{-1} z^0(0) \\ &= A_1 A_0^{-1} \underbrace{A_0 R_{01} R_0^n A_0^{-1}}_{\text{De Moivre applicable}} z^0(0) \end{aligned} \quad (4.14)$$

We can apply the result of Sec. (3.4.1) to re-express Eq. (4.14) as a function of the lattice functions at $s = 0$:

$$\begin{aligned} z^1(n) &= A_1 A_0^{-1} \begin{pmatrix} \cos(\Phi_n^c) + \alpha_0 \sin(\Phi_n^c) & \beta_0 \sin(\Phi_n^c) \\ -\gamma_0 \sin(\Phi_n^c) & \cos(\Phi_n^c) - \alpha_0 \sin(\Phi_n^c) \end{pmatrix} z^0 \quad (4.15) \\ &= \begin{pmatrix} \sqrt{\frac{\beta_1}{\beta_0}} & 0 \\ -\frac{(\alpha_1 - \alpha_2)}{\sqrt{\beta_1 \beta_0}} & \sqrt{\frac{\beta_0}{\beta_1}} \end{pmatrix} \begin{pmatrix} \cos(\Phi_n^c) + \alpha_0 \sin(\Phi_n^c) & \beta_0 \sin(\Phi_n^c) \\ -\gamma_0 \sin(\Phi_n^c) & \cos(\Phi_n^c) - \alpha_0 \sin(\Phi_n^c) \end{pmatrix} z^0 \end{aligned}$$

where $\Phi_n^c = n\mu + \mu_{01}^c$ here μ_{01}^c results from $Eq.$ (4.12)

The final is step is simply to write the position q at $s = 1$:

$$q^1(n) = \sqrt{\frac{\beta_1}{\beta_0}} \left((\cos(\Phi_n^c) + \alpha_0 \sin(\Phi_n^c)) q_0 + \beta_0 \sin(\Phi_n^c) p_0 \right) \quad (4.16)$$

Extends to coupled system if A_{12}=0 and A_{34}=0