Direct Vlasov solvers – part II

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Direct Vlasov solvers

Part I

- Introduction: collective effects
- Motivation for Vlasov solvers
- Vlasov equation historically, and in the context of accelerators
- Transverse impedance and instabilities
- Building of a simple Vlasov solver for impedance instabilities

Part II

- Compact way to present the theory: Hamiltonians & Poisson brackets
- Upgrade of part I theory to obtain Sacherer integral equation
- Solving Sacherer equation – convergence
- Benchmarks & examples of application of Vlasov solvers

Introducing Hamiltonians

- Some of the analytical work shown in part I can be made simpler by using Hamiltonians: the (conservative) system under study is governed by the Hamiltonian

$$H(x, x', y, y', z, \delta; t)$$

- Coordinates and momenta go in pair, and obey Hamilton's equations: for example in the vertical plane

$$\frac{dy}{dt} = \frac{\partial H}{\partial y'} \quad \text{and} \quad \frac{dy'}{dt} = -\frac{\partial H}{\partial y}$$

- This does not introduce any additional physics, it just makes part of the derivation easier, more efficient and more elegant.

- For more details on Hamiltonians, see W. Herr's lecture in this CAS (14/11): https://indico.cern.ch/event/759124/contributions/3148186/attachments/1748350/2838297/ham1.pdf

Vlasov equation with Hamiltonians

- Going back to our simple Vlasov solver in 2D:

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial y}\frac{dy}{dt} + \frac{\partial\psi}{\partial y'}\frac{dy'}{dt}$$
$$= \frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial y}\frac{\partial H}{\partial y'} - \frac{\partial\psi}{\partial y'}\frac{\partial H}{\partial y}$$

This is simply $[\psi, H]$!

Reminder: Poisson brackets (x_i = positions, p_i = momenta)

$$[f, g] = \sum_i \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x_i}$$

(see W. Herr's CAS lecture on 14/11/2018)

- Vlasov equation $\frac{d\psi}{dt} = 0$ then becomes

$$\frac{\partial\psi}{\partial t} + [\psi, H] = 0$$

… and this is completely general.

Perturbation

- In any Vlasov solver using perturbation theory we look for a distribution function of the form

$$\psi = \psi_0 + \Delta\psi$$

Stationary distribution for the Hamiltonian without impedance.

Perturbation of the stationary distribution, of first order

- ψ is the solution of the perturbed Hamiltonian

$$H = H_0 + \Delta H$$

Unperturbed Hamiltonian

First order perturbation of the Hamiltonian, here from impedance.

Linearized Vlasov equation using Poisson brackets

$$\frac{\partial \psi}{\partial t} + [\psi, H] = 0$$

$$\Leftrightarrow \frac{\partial (\psi_0 + \Delta\psi)}{\partial t} + [\psi_0 + \Delta\psi, H_0 + \Delta H] = 0$$

$$\Leftrightarrow \cancel{\frac{\partial \psi_0}{\partial t} + [\psi_0, H_0]} + \frac{\partial \Delta\psi}{\partial t} + [\Delta\psi, H_0] + [\psi_0, \Delta H] + \cancel{[\Delta\psi, \Delta H]} = 0$$

=0 since ψ_0 is solution of Vlasov eq. for H_0

Only remaining terms

Second order

Isn't that exactly what we did – somewhat more painfully – during part I?

$$\frac{\partial \Delta\psi}{\partial t} + \cancel{\left(\frac{\partial \psi_0}{\partial y} v y' - \frac{\partial \psi_0}{\partial y'} v y \left(\frac{Q_y}{R}\right)^2 \right)}$$

$$+ \left(\frac{\partial \Delta\psi}{\partial y} v y' - \frac{\partial \Delta\psi}{\partial y'} v y \left(\frac{Q_y}{R}\right)^2 + \frac{\partial \psi_0}{\partial y'} \frac{F_y^{imp}}{m_0 \gamma v} \right) + \cancel{\frac{\partial \Delta\psi}{\partial y'} \frac{F_y^{imp}}{m_0 \gamma v}} = 0$$

Linearized Vlasov equation using Poisson brackets

$$\frac{\partial \Delta\psi}{\partial t} + [\Delta\psi, H_0] + [\psi_0, \Delta H] = 0$$

- This is completely general for any Hamiltonian system within linear perturbation theory, up to the first order in the perturbation.
- Poisson brackets are conserved within any canonical transformation of coordinates $(x_i, p_i) \to (X_i, P_i)$, i.e. any transformation for which there is:
  - preservation of Hamilton's equations,
  - equivalently, symplecticity of the Jacobian $\mathcal{J}$: $\boxed{\mathcal{J}^T \cdot S \cdot \mathcal{J} = S}$

with $\mathcal{J} = \begin{pmatrix} \frac{\partial X_1}{\partial x_1} & \cdots & \frac{\partial X_1}{\partial x_n} & \frac{\partial X_1}{\partial p_1} & \cdots & \frac{\partial X_n}{\partial p_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial X_n}{\partial x_1} & \cdots & \frac{\partial X_n}{\partial x_n} & \frac{\partial X_n}{\partial p_1} & \cdots & \frac{\partial X_n}{\partial p_n} \\ \frac{\partial P_1}{\partial x_1} & \cdots & \frac{\partial P_1}{\partial x_n} & \frac{\partial P_1}{\partial p_1} & \cdots & \frac{\partial P_1}{\partial p_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial P_n}{\partial x_1} & \cdots & \frac{\partial P_n}{\partial x_n} & \frac{\partial P_n}{\partial p_1} & \cdots & \frac{\partial P_n}{\partial p_n} \end{pmatrix}$, $S = \begin{pmatrix} 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \\ -1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -1 & 0 & \cdots & 0 \end{pmatrix}$

  - Note that symplecticity entails $\det(\mathcal{J}) = 1 \longrightarrow \iint dX_i dP_i = \iint dx_i dp_i$

Application to the derivation of part I

The transformation $(y, y') \rightarrow (J_y, \theta_y)$ is symplectic:

$$y = \sqrt{\frac{2J_y R}{Q_y}} \cos\theta_y, \qquad y' = \sqrt{\frac{2J_y Q_y}{R}} \sin\theta_y$$

$$J_y = \frac{1}{2}\left[y^2 \frac{Q_y}{R} + y'^2 \frac{R}{Q_y}\right], \quad \theta_y = \operatorname{atan}\left(\frac{Ry'}{Q_y y}\right)$$

$$\text{so} \quad \mathcal{J} = \begin{pmatrix} \frac{\partial J_y}{\partial y} & \frac{\partial J_y}{\partial y'} \\ \frac{\partial \theta_y}{\partial y} & \frac{\partial \theta_y}{\partial y'} \end{pmatrix} = \begin{pmatrix} \frac{y\,Q_y}{R} & \frac{y' R}{Q_y} \\ -\sqrt{\frac{Q_y}{2J_y R}} \sin\theta_y & \sqrt{\frac{R}{2J_y Q_y}} \cos\theta_y \end{pmatrix}$$

and we get (see appendix)

$$\mathcal{J}^T \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Hamiltonian of our simple Vlasov solver (part I)

For our simple Vlasov equation in part I we had

$$\frac{\partial H}{\partial y'} = \frac{dy}{dt} = v \cdot y'$$

$$-\frac{\partial H}{\partial y} = \frac{dy'}{dt} = \frac{F_y^{imp}}{m_0\, \gamma v} - v y \left(\frac{Q_y}{R}\right)^2$$

which corresponds to the Hamiltonian

$$H = H_0 + \Delta H = \frac{v}{2}\left[y'^2 + y^2\left(\frac{Q_y}{R}\right)^2\right] - \frac{y}{m_0 \gamma v} F_y^{imp}$$

$$= \frac{v Q_y}{R} J_y - \sqrt{\frac{2 J_y R}{Q_y}} \cos\theta_y \frac{F_y^{imp}}{m_0 \gamma v}$$

Application to our simple Vlasov solver (part I)

$$H_0 = \omega_0 Q_y J_y$$

$$\omega_0 = \frac{v}{R}$$

$$\Delta H = -\sqrt{\frac{2J_y R}{Q_y}} \cos\theta_y \frac{F_y^{imp}}{m_0\gamma v}$$

$$\psi = \psi_0 + \Delta\psi$$

In (J_y, θ_y) coordinates, the linearized Vlasov equation

$$\frac{\partial\Delta\psi}{\partial t} + [\Delta\psi, H_0] + [\psi_0, \Delta H] = 0$$

gives immediately

Reminder: $[f, g] = \frac{\partial f}{\partial J_y}\frac{\partial g}{\partial\theta_y} - \frac{\partial f}{\partial\theta_y}\frac{\partial g}{\partial J_y}$

$$\boxed{\frac{\partial\Delta\psi}{\partial t} - \frac{\partial\Delta\psi}{\partial\theta_y}\omega_0 Q_y + \psi_0'(J_y)\sqrt{\frac{2J_y R}{Q_y}} \sin\theta_y \frac{F_y^{imp}}{m_0\gamma v} = 0}$$

Building a Vlasov solver: method outline

New outline

1. Write Hamiltonian
2. Choose coordinates
3. Write stationary distribution
4. Write linearized Vlasov equation
5. Decompose perturbation
6. Reduce number of variables
7. Write impedance force
8. Final equation

A more elaborate Vlasov solver

- Let's try to relieve some assumptions of the Vlasov solver of part I:
  - Impedance $Z_y(\omega)$ is the only source of instability considered, and gives the EM force arising from the interaction of the beam with the resistive or geometric elements around it,
  - only vertical plane, with position and "momentum" $\left(y, y' = \frac{dy}{ds}\right)$ (using for convenience y' rather than p_y)
  - purely linear, uncoupled optics in transverse, within smooth approximation,
  - ~~no longitudinal motion, i.e. essentially rigid bunches in z,~~
  - ~~chromaticity~~ $\cancel{Q'_y = \frac{dQ_y}{d\delta} = 0}$,
  - Phase space distribution function is then

$$\psi = \psi\left(y, y', z, \delta \equiv \frac{p_z}{m_0 \gamma v}; t\right)$$

But we still neglect any effect from the transverse plane on the longitudinal motion.

Hamiltonian

Hamiltonian
Coordinates
Stationary distribution
Linearized Vlasov eq.
Perturbation decomp.
Reduction variables
Impedance force
Final equation

We add linear longitudinal motion (see A. W. Chao, *Physics of Collective Beam Instabilities in High Energy Accelerators*, John Wiley & Sons (1993), chap. 6):

$$H_0 = \omega_0 Q_y J_y - \frac{v}{2\eta}\left(\frac{\omega_s}{v}\right)^2 z^2 - \frac{\eta}{2} v \delta^2$$

$Q_y = Q_{y0} + Q'_y \delta$

Slippage factor

Synchrotron angular frequency

$$\Delta H = -\sqrt{\frac{2J_y R}{Q_y}} \cos\theta_y \frac{F_y^{imp}(z;t)}{m_0 \gamma v}$$

J_y remains as defined previously

$$J_y = \frac{1}{2}\left[y^2 \frac{Q_{y0}}{R} + y'^2 \frac{R}{Q_{y0}}\right]$$

and is still assumed to be an invariant, despite the (y, z) coupling introduced by chromaticity → approximation (typically done in textbooks).

Transformation of coordinates

In transverse: $J_y = \frac{1}{2}\left[y^2 \frac{Q_{y0}}{R} + y'^2 \frac{R}{Q_{y0}}\right],\ \theta_y = \text{atan}\left(\frac{Ry'}{Q_{y0}y}\right)$

In longitudinal: $z = \sqrt{\frac{2J_z v\eta}{\omega_s}}\cos\phi, \qquad \delta = \sqrt{\frac{2J_z\omega_s}{v\eta}}\sin\phi,$

$$J_z = \frac{1}{2}\left(\frac{\omega_s}{v\eta}z^2 + \frac{v\eta}{\omega_s}\delta^2\right),\ \phi = \text{atan}\left(\frac{v\eta\delta}{\omega_s z}\right)$$

which is a canonical transformation (symplecticity checked in appendix).

Then the Hamiltonian reads:

$$H_0 = \omega_0 Q_y J_y - \omega_s J_z$$

$$\Delta H = -\sqrt{\frac{2J_y R}{Q_y}}\cos\theta_y \frac{F_y^{imp}}{m_0\gamma v}$$

Stationary distribution

- Hamiltonian
- Coordinates
- **Stationary distribution**
- Linearized Vlasov eq.
- Perturbation decomp.
- Reduction variables
- Impedance force
- Final equation

The new unperturbed Hamiltonian

$$H_0 = \omega_0 Q_y J_y - \omega_s J_z$$

admits as stationary distribution

$$\psi_0(y, y', z, \delta; t) = f_0(J_y) g_0(J_z)$$

Linearized Vlasov equation

Hamiltonian
Coordinates
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$$\frac{\partial \Delta\psi}{\partial t} + [\Delta\psi, H_0] + [\psi_0, \Delta H] = 0$$

with $H_0 = \omega_0 Q_y J_y - \omega_s J_z$ $\qquad \psi_0(y, y', z, \delta; t) = f_0(J_y) g_0(J_z)$

$$\Delta H = -\sqrt{\frac{2J_y R}{Q_y}} \cos\theta_y \frac{F_y^{imp}(z;t)}{m_0 \gamma v}$$

We get: $\quad f_0{}' = \frac{df_0}{dJ_y}$ $\qquad$ Reminder: $[f, g] = \frac{\partial f}{\partial J_y}\frac{\partial g}{\partial \theta_y} - \frac{\partial f}{\partial \theta_y}\frac{\partial g}{\partial J_y} + \frac{\partial f}{\partial r}\frac{\partial g}{\partial \phi} - \frac{\partial f}{\partial \phi}\frac{\partial g}{\partial r}$

$$\boxed{\frac{\partial \Delta\psi}{\partial t} - \frac{\partial \Delta\psi}{\partial \theta_y}\omega_0 Q_y + \frac{\partial \Delta\psi}{\partial \phi}\omega_s + f_0'(J_y) g_0(J_z)\sqrt{\frac{2J_y R}{Q_y}} \sin\theta_y \frac{F_y^{imp}}{m_0\gamma v} = 0}$$

Note: from our initial assumption that the transverse plane does not affect the longitudinal one, we have neglected $\frac{\partial \Delta H}{\partial z}$, as in Chao's book.

Writing the perturbation

- Hamiltonian
- Coordinates
- Stationary distribution
- Linearized Vlasov eq.
- Perturbation decomp.
- Reduction variables
- Impedance force
- Final equation

We assume again a single mode of angular frequency $\Omega \approx Q_{y0}\omega_0$, and we introduce for convenience (no need to be a canonical transform at this stage)

$$r = \sqrt{\frac{2J_z v\eta}{\omega_s}}, \qquad z = r\cos\phi\,, \qquad \frac{v\eta}{\omega_s}\delta = r\sin\phi$$

such that $\qquad \Delta\psi(J_y, \theta_y, J_z, \phi; t) = \Delta\psi_1(J_y, \theta_y, r, \phi)e^{j\Omega t}$

Then we decompose this mode using a Fourier series of the angle θ_y and another one for the angle ϕ:

$$\Delta\psi(J_y, \theta_y, r, \phi; t) = e^{j\Omega t}\sum_{p=-\infty}^{+\infty} f_p(J_y)e^{jp\theta_y} \cdot e^{-\frac{jpQ'_y z}{\eta R}} \cdot \sum_{l=-\infty}^{+\infty} R_l(r)e^{-jl\phi}$$

Additional phase factor (that we are allowed to put here without loss of generality) – will appear later to be very convenient
→ headtail phase factor

Reducing the number of variables

- Hamiltonian
- Coordinates
- Stationary distribution
- Linearized Vlasov eq.
- Perturbation decomp.
- **Reduction variables**
- Impedance force
- Final equation

Injecting the perturbation into Vlasov equation, we can simplify it even more:

$$\frac{\partial \Delta\psi}{\partial t} - \frac{\partial \Delta\psi}{\partial \theta_y} Q_y + \frac{\partial \Delta\psi}{\partial \phi}\omega_s + f_0'(J_y) g_0(r) \sqrt{\frac{2J_y R}{Q_y}} \sin\theta_y \frac{F_y^{imp}}{m_0 \gamma v} = 0$$

$$\Leftrightarrow e^{j\Omega t} \sum_{p=-\infty}^{+\infty} f_p(J_y) e^{jp\theta_y} e^{-\frac{jpQ_y' z}{\eta R}} \sum_{l=-\infty}^{+\infty} R_l(r) e^{-jl\phi} \left(j\Omega - jpQ_{y0}\omega_0 - jl\omega_s\right) =$$

$$-f_0'(J_y) g_0(r) \sqrt{\frac{2J_y R}{Q_y}} \frac{e^{j\theta_y} - \cancel{e^{-j\theta_y}}}{2j} \frac{F_y^{imp}}{m_0 \gamma v}$$

This is where we use this factor to simplify the term in brackets.

As in part I, term by term identification leads to

$$f_p(J_y) = 0 \text{ for any } p \neq \pm 1$$

and the assumption $\Omega \approx Q_{y0}\omega_0$, gives

$$f_{-1}(J_y) \approx 0$$

Reducing the number of variables

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Pushing further the computation gives:

$$e^{j\Omega t} f_1(J_y) e^{\frac{-jQ'_y z}{\eta R}} \sum_{l=-\infty}^{+\infty} R_l(r) e^{-jl\phi} \left(\Omega - Q_{y0}\omega_0 - l\omega_s\right)$$

$$= f_0'(J_y) g_0(r) \sqrt{\frac{2J_y R}{Q_y}} \frac{F_y^{imp}(z;t)}{2m_0\gamma v}$$

Depends only on J_y, θ_y $\Rightarrow$ the ratio must be a constant.

$$\Leftrightarrow \sum_{l=-\infty}^{+\infty} R_l(r) e^{-jl\phi} \left[\frac{f_1(J_y)\left(\Omega - Q_{y0}\omega_0 - l\omega_s\right)}{f_0'(J_y)\sqrt{\frac{2J_y R}{Q_y}}} \right]$$

$$= e^{-j\Omega t} e^{\frac{jQ'_y z}{\eta R}} g_0(r) \frac{F_y^{imp}(z;t)}{2m_0\gamma v}$$

Depends only on r, ϕ and t

Reducing the number of variables

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This gives the transverse shape of the perturbative distribution as in part I :

$$f_1(J_y) \propto f_0'(J_y)\sqrt{\frac{J_y R}{2Q_y}}$$

Putting the proportionality constant inside $R_l(r)$:

$$\Rightarrow \Delta\psi(J_y, \theta_y; t) = e^{j\Omega t} e^{j\theta_y} f_0'(J_y)\sqrt{\frac{J_y R}{2Q_y}} \cdot e^{-\frac{jQ_y' z}{\eta R}} \cdot \sum_{l=-\infty}^{+\infty} R_l(r) e^{-jl\phi}$$

Only the r and ϕ dependencies remain to be dealt with.

Force from impedance

Hamiltonian
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Compared to part I, one "simply" puts the additional longitudinal dependence:

$$F_y^{imp} = \frac{e^2}{2\pi R} \sum_{k=-\infty}^{+\infty} \iint d\tilde{z}\, d\delta\, W_y(\tilde{z} + 2\pi k R - z)$$

$$\times \iint dJ_y\, d\theta_y\, \Delta\psi\left(J_y, \theta_y, r, \phi; t - k\frac{2\pi R}{v}\right) \sqrt{\frac{2J_y R}{Q_y}} \cos\theta_y$$

and one can simplify this as in part I, using in addition:

$$\iint d\tilde{z} d\delta = \iint dJ_z d\phi = \frac{\omega_s}{v\eta} \iint r dr d\phi$$

Bessel function

$$\int_0^{2\pi} d\phi\, e^{-jl\phi} e^{\frac{-jQ_y' r\cos\phi}{\eta R}} = 2\pi j^{-l} J_l\left(\frac{Q_y' r}{\eta R}\right)$$

Force from impedance

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In the end, defining the coherent tune of the mode $Q_{coh} = \frac{\Omega}{\omega_0}$, we get:

$$F_y^{imp} = e^{j\Omega t} \frac{jN\omega_0 e^2}{2\pi Q_{y0}} \sum_{k=-\infty}^{+\infty} Z_y[(Q_{coh}+k)\omega_0] e^{\frac{-j(Q_{coh}+k)r\cos\phi}{R}}$$
$$\times \sum_{l'=-\infty}^{+\infty} j^{l'} \int_0^{+\infty} \tilde{r} d\tilde{r} R_{l'}(\tilde{r}) J_{l'}\left[\left(Q_{coh}+k-\frac{Q_y'}{\eta}\right)\frac{\tilde{r}}{R}\right]$$

Sacherer integral equation

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Plugging everything back into Vlasov equation:

$$\sum_{l'=-\infty}^{+\infty} R_{l'}(r)e^{-jl'\phi}\left(\Omega - Q_{y0}\omega_0 - l'\omega_s\right) = \frac{jN\omega_0 e^2}{4\pi Q_{y0} m_0 \gamma v} g_0(r)$$

$$\times \sum_{k=-\infty}^{+\infty} Z_y[(Q_{coh}+k)\omega_0]\, e^{\frac{-j\left(Q_{coh}+k-\frac{Q'_y}{\eta}\right) r\cos\phi}{R}}$$

$$\times \sum_{l'=-\infty}^{+\infty} j^{l'} \int_0^{+\infty} \tilde{r}d\tilde{r}R_{l'}(\tilde{r})\, J_{l'}\left[\left(Q_{coh}+k-\frac{Q'_y}{\eta}\right)\frac{\tilde{r}}{R}\right]$$

We can get rid of ϕ by integrating both sides with $\frac{1}{2\pi}\int_0^{+\infty} d\phi e^{jl\phi}$, and using again (here α is any constant)

$$\int_0^{2\pi} d\phi\, e^{jl\phi} e^{-j\alpha\cos\phi} = 2\pi j^{-l}\, J_l(\alpha)$$

Sacherer integral equation

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In the end, doing as in part I the approximation $Q_{coh} \approx Q_{y0}$ (smoothness of impedance and Bessel functions), we get the famous equation:

$$
\begin{aligned}
(\Omega - Q_{y0}\omega_0 - l\omega_s)R_l(r) &= \frac{jN\omega_0 e^2}{4\pi\gamma m_0 v Q_{y0}} g_0(r) \sum_{l'=-\infty}^{+\infty} j^{l'-l} \\
&\times \sum_{k=-\infty}^{+\infty} \int_0^{+\infty} \tilde{r} d\tilde{r} R_{l'}(\tilde{r}) J_{l'}\left[\left(Q_{y0} + k - \frac{Q_y'}{\eta}\right)\frac{\tilde{r}}{R}\right] \\
&\times \quad Z_y\left((Q_{y0}+k)\omega_0\right) J_l\left[\left(Q_{y0} + k - \frac{Q_y'}{\eta}\right)\frac{r}{R}\right]
\end{aligned}
$$

Solving Sacherer integral equation

They are various options to solve the integral equation:

- Consider a simple and easy to solve longitudinal distribution $g_0(r)$, e.g. an airbag model (see A. Chao's book).
- Discretize $g_0(r)$ as a superposition of airbag models (as in the NHTVS).
- Integrate with $\int_0^{+\infty} rdr\, J_l\left[\left(Q_{y0} + k - \frac{Q'_y}{\eta}\right)\frac{r}{R}\right]$ and solve for $\sigma_{lk} = \int_0^{+\infty} rdr\, J_l\left[\left(Q_{y0} + k - \frac{Q'_y}{\eta}\right)\frac{r}{R}\right] R_l(r)$ (as in Laclare's approach).
- Decompose $R_l(r)$ and $g_0(r)$ over a basis of orthogonal polynomials such as Laguerre polynomials and compute the integrals involving Bessel functions analytically, as in MOSES and DELPHI:

$$R_l(r) = A\left(\frac{r}{B}\right)^{|l|} e^{-\kappa r^2} \sum_{n=0}^{+\infty} c_l^n L_n^{|l|}(\kappa r^2)$$

κ, A and B constants to be adjusted

Solving Sacherer integral equation

In the end one typically obtains an eigenvalue problem:

Eigenvalue looked for: angular frequency shift of the mode

Kronecker delta: $\delta_{ll'} = 0$ if $l \neq l'$, 1 otherwise

$$\underbrace{\left(\Omega - Q_{y0}\omega_0 - l\omega_s\right)}\alpha_{ln} = \sum_{l'=-\infty}^{+\infty} \sum_{n'=0}^{+\infty} \left(\delta_{ll'}\delta_{nn'} + \mathcal{M}_{ln,l'n'}\right)\alpha_{l'n'}$$

Eigenvectors: e.g. Laclare's σ_{lk}, or coefficients c_l^n of the decomposition over Laguerre polynomials, or coefficients of the discretization over airbag rings, etc.

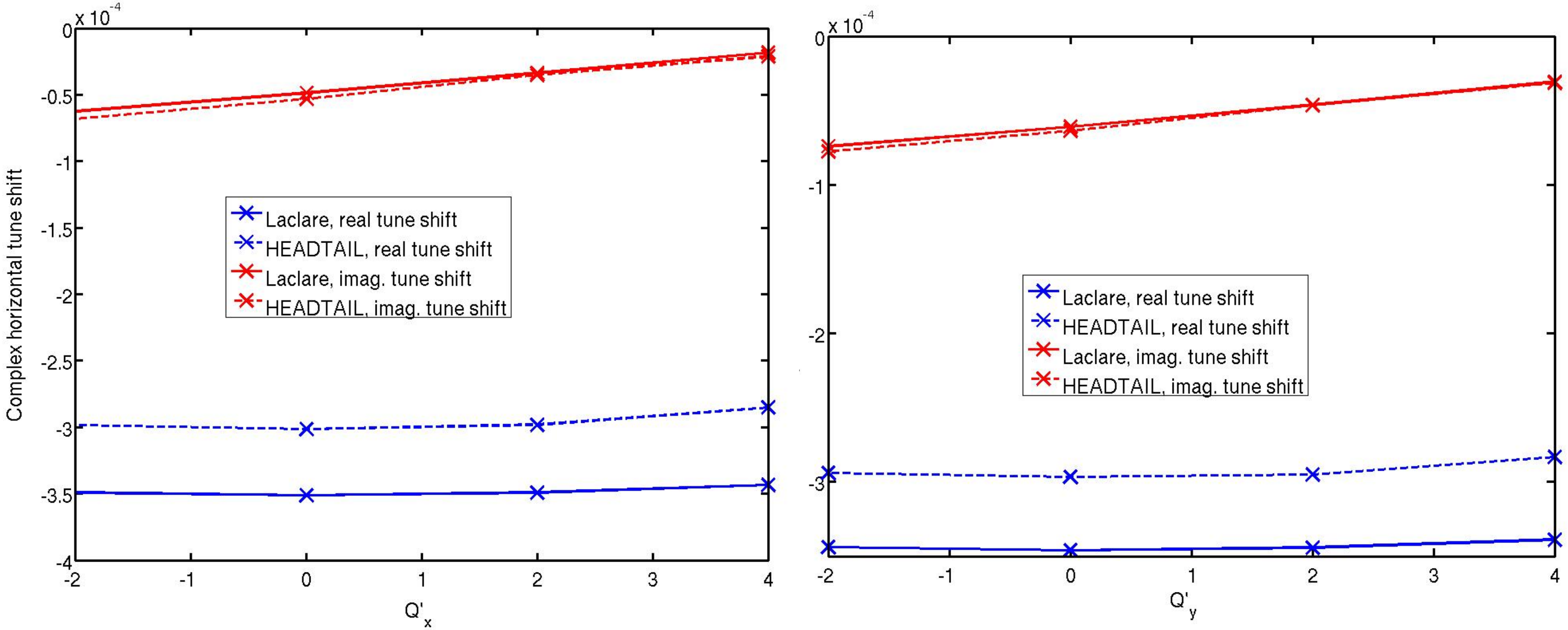
Matrix, to be computed analytically

⇒ In the end one needs to diagonalize the matrix $\mathcal{M}$, which can be done numerically in many ways (e.g. Python, MATLAB®, Mathematica® , C, etc.)

⇒ The matrix being infinite in principle, the problem of truncation is the most important (and essentially the only) numerical issue: truncation sets the number of possible modes considered, and convergence has to be checked for each case.

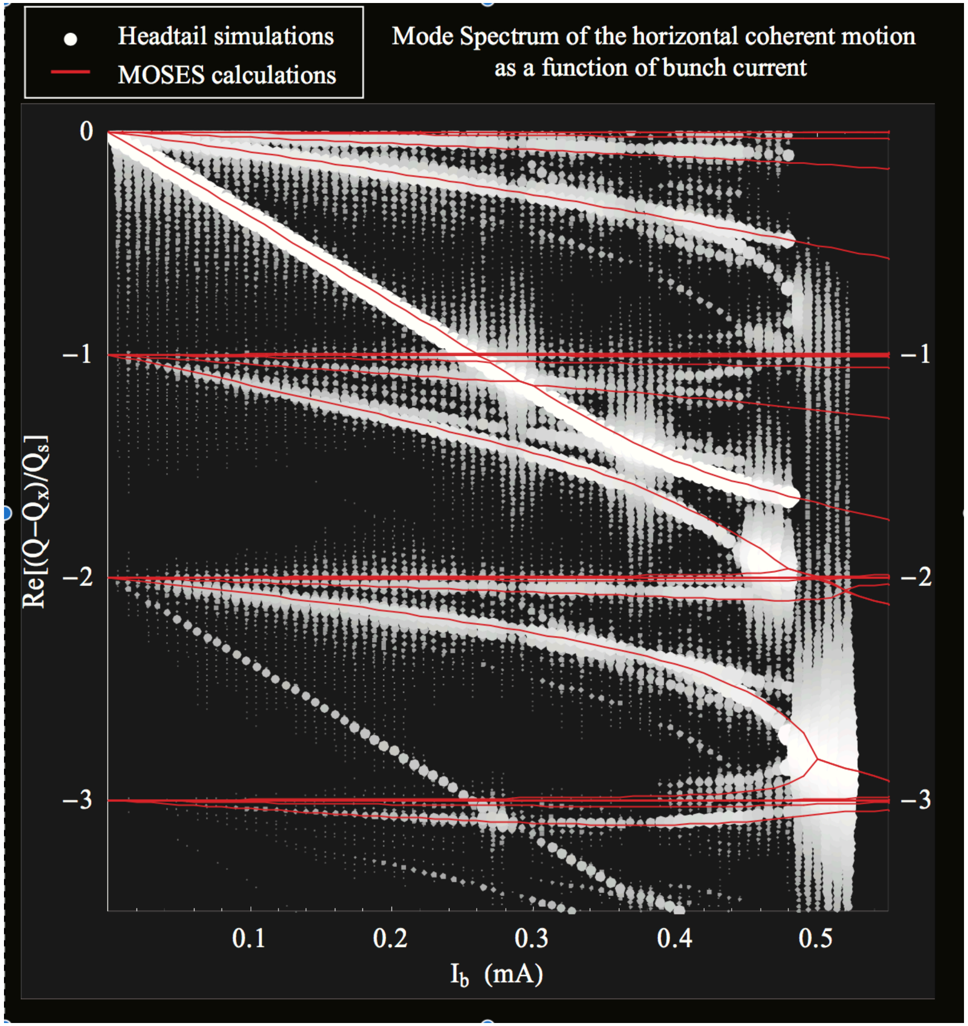
Benchmarks

Vlasov solvers have been heavily benchmarked w.r.t. multi-particle simulations: here HEADTAIL (multi-particle simulation) vs. Laclare's Vlasov approach, for LHC coupled-bunch instabilities vs. chromaticity



Benchmarks

HEADTAIL vs MOSES (Vlasov solver), for the SPS transverse mode coupling instability:



From **B. Salvant**'s PhD thesis [*EPFL nº4585 (2010)*]

⇒ Vlasov solvers and multi-particle compare very well, provided they are used in the same situation (and are well converged!)

Applications – LEP TMCI with damper

D Brandt et al

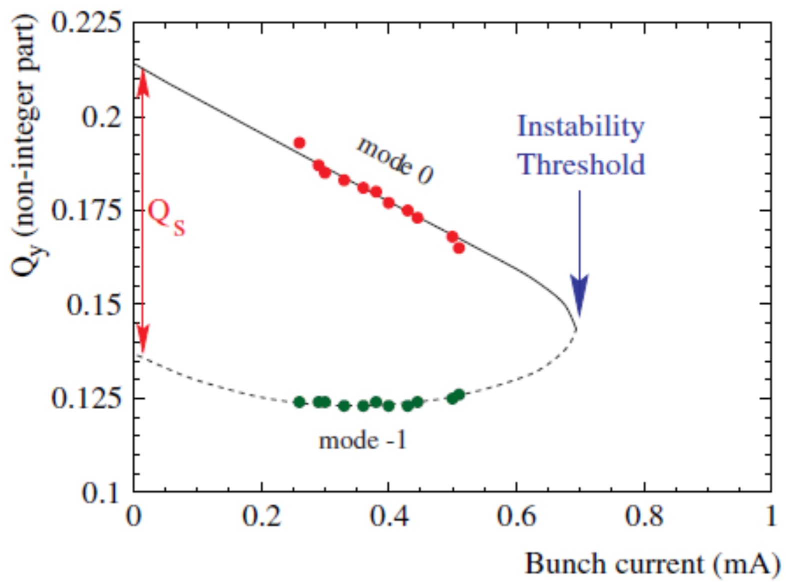


Figure 12. Measurement of the 0 and −1 modes of oscillation as a function of the bunch current at LEP for $Q_s = 0.082$. As the current increases the two modes approach until they merge at the instability threshold.

Impedance model: two broad-band resonators (RF cavities + bellows), the rest is relatively small (<10%) [G. Sabbi, 1995].

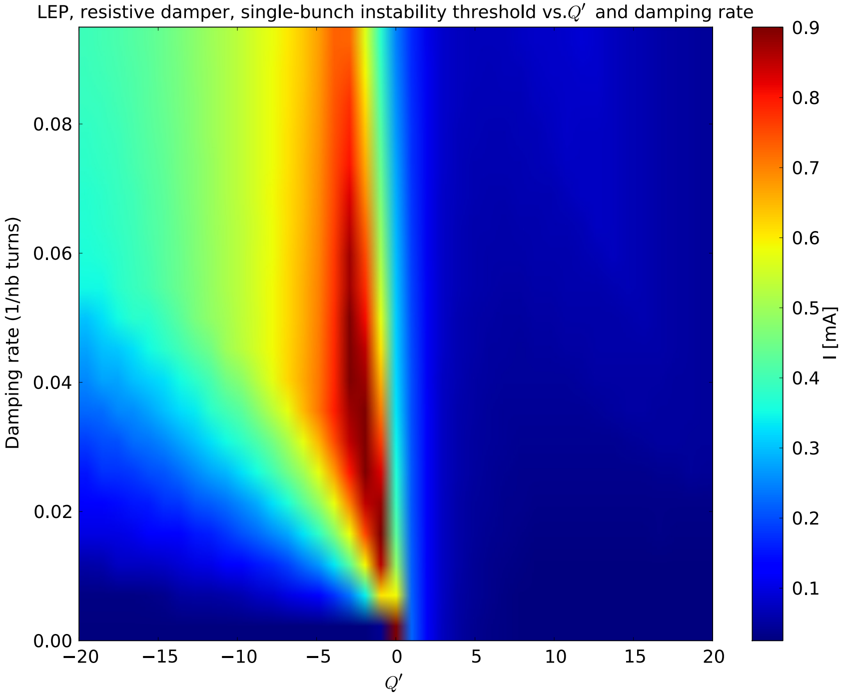
- experimental tune shifts and TMCI threshold (from simple formula) well reproduced,
- TMCI threshold slightly less than 1mA.

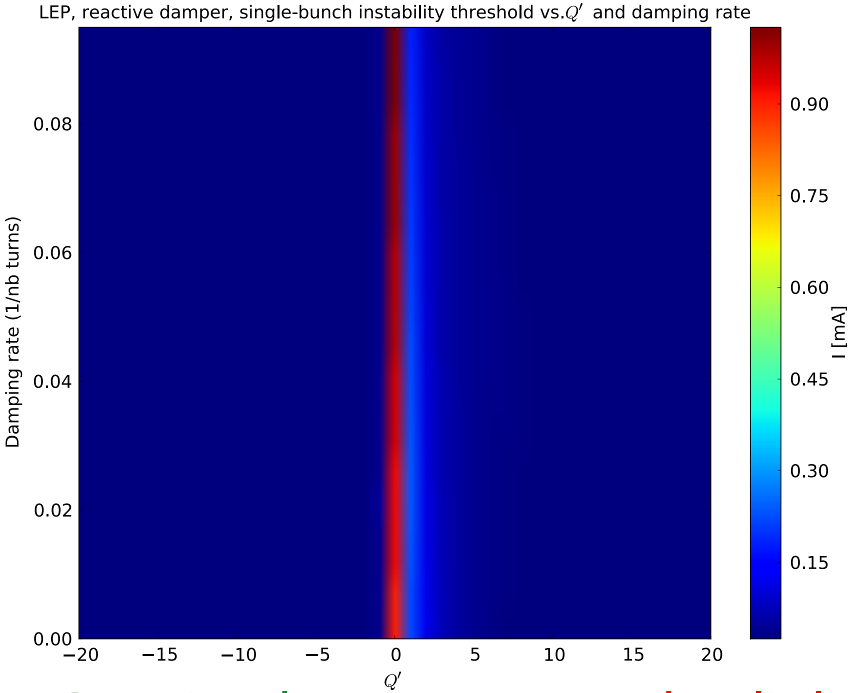
Transverse feedback damper: several ideas and trials in LEP

- reactive feedback (prevent mode 0 to shift down and couple with mode -1) → not more than 5-10 % increase in threshold, despite several attempts and models developed [Danilov-Perevedentsev 1993, Sabbi 1996, Brandt et al 1995],
- resistive feedback, first found ineffective [Ruth 1983], tried at LEP but never used in operation.

Applications – LEP TMCI with damper

Instability threshold vs. chromaticity Q' and damper gain (up to 10 turns) with DELPHI Vlasov solver:



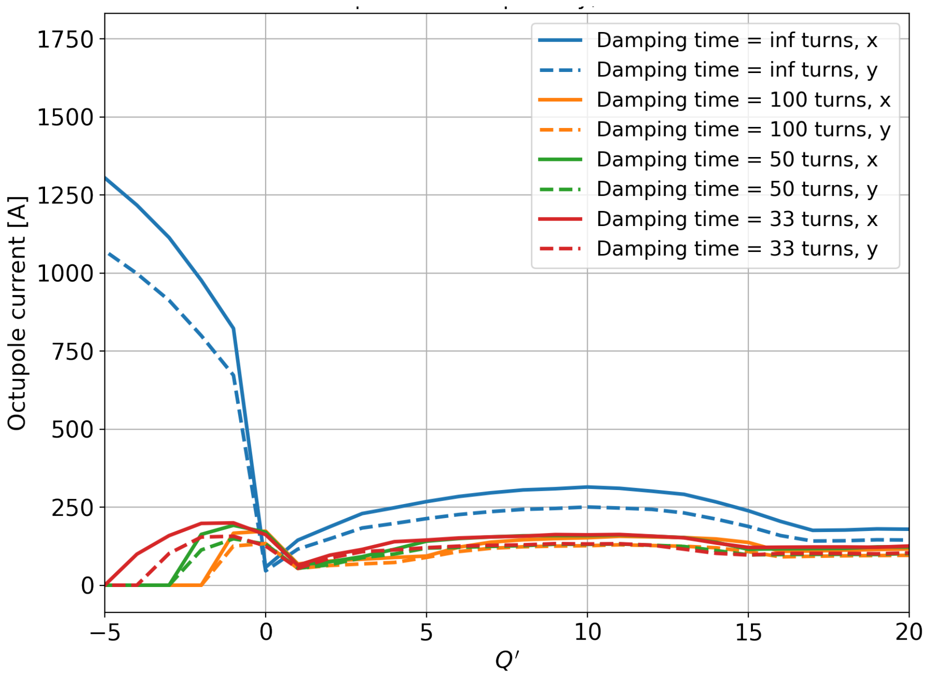


Resistive damper: one cannot do better than the "natural" (i.e. without damper) TMCI threshold.

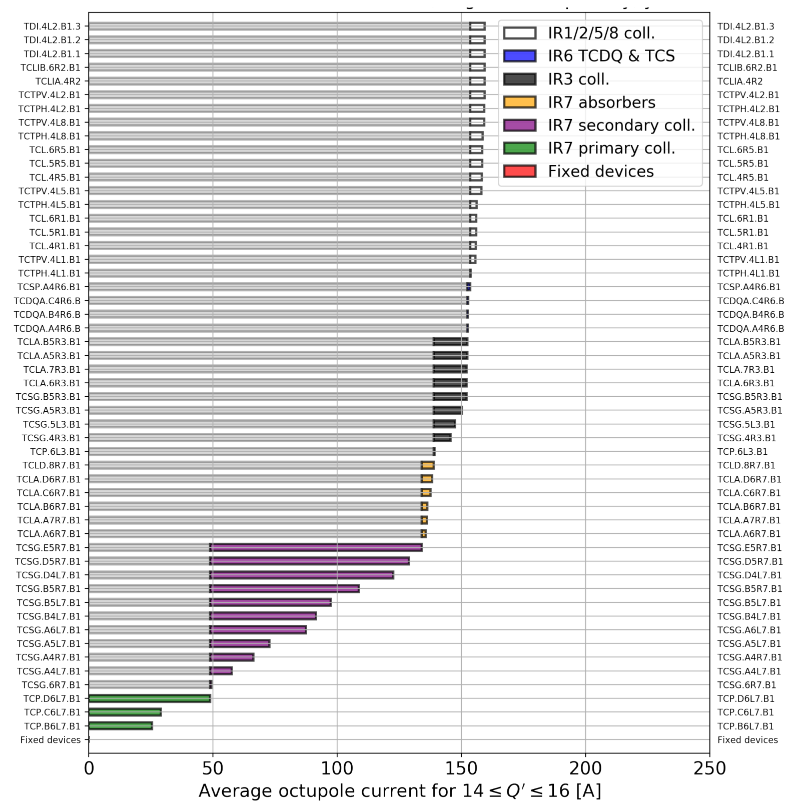
Reactive damper: one can do a little better than the "natural" TMCI.
→ seems to match (qualitatively) LEP observations.

Applications – LHC

Predicting the octupole instability threshold vs. chromaticity Q' and damper gain, with DELPHI:



… and we can also plot the respective contributions of each machine elements (essentially collimators):



Direct Vlasov solvers – summary part II

- We have revisited the theory exposed in part I, introducing Hamiltonians and Poisson brackets to ease up the analytical work.
- We have derived Sacherer integral equation within this framework, re-introducing the longitudinal plane.
- We went through a few ways to solve Sacherer integral equation, and how to deal with the associated eigenvalue problem.
- Finally we have shown benchmarks and applications of Vlasov solvers in CERN synchrotrons (LEP, SPS, LHC).

Appendix

Symplectic transformations

The transformation $(y, y') \rightarrow (J_y, \theta_y)$ is symplectic:

$$y = \sqrt{\frac{2J_y R}{Q_y}} \cos\theta_y, \;\; y' = \sqrt{\frac{2J_y Q_y}{R}} \sin\theta_y, \; J_y = \frac{1}{2}\left[y^2 \frac{Q_y}{R} + y'^2 \frac{R}{Q_y}\right], \theta_y = \operatorname{atan}\left(\frac{Ry'}{Q_y y}\right)$$

$$\text{so} \quad \mathcal{J} = \begin{pmatrix} \frac{\partial J_y}{\partial y} & \frac{\partial J_y}{\partial y'} \\ \frac{\partial \theta_y}{\partial y} & \frac{\partial \theta_y}{\partial y'} \end{pmatrix} = \begin{pmatrix} \frac{y\, Q_y}{R} & \frac{y' R}{Q_y} \\ -\sqrt{\frac{Q_y}{2J_y R}} \sin\theta_y & \sqrt{\frac{R}{2J_y Q_y}} \cos\theta_y \end{pmatrix}$$

and we get

$$\begin{pmatrix} \frac{y\, Q_y}{R} & -\sqrt{\frac{Q_y}{2J_y R}} \sin\theta_y \\ \frac{y' R}{Q_y} & \sqrt{\frac{R}{2J_y Q_y}} \cos\theta_y \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{y\, Q_y}{R} & \frac{y' R}{Q_y} \\ -\sqrt{\frac{Q_y}{2J_y R}} \sin\theta_y & \sqrt{\frac{R}{2J_y Q_y}} \cos\theta_y \end{pmatrix} =$$

$$\begin{pmatrix} \sqrt{\frac{Q_y}{2J_y R}} \sin\theta_y & \frac{y\, Q_y}{R} \\ -\sqrt{\frac{R}{2J_y Q_y}} \cos\theta_y & \frac{y' R}{Q_y} \end{pmatrix} \cdot \begin{pmatrix} \frac{y\, Q_y}{R} & \frac{y' R}{Q_y} \\ -\sqrt{\frac{Q_y}{2J_y R}} \sin\theta_y & \sqrt{\frac{R}{2J_y Q_y}} \cos\theta_y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Symplectic transformations

The transformation $(z,\delta) \rightarrow (J_z,\phi)$ is symplectic:

$$z = \sqrt{\frac{2J_z v\eta}{\omega_s}}\cos\phi, \quad \delta = \sqrt{\frac{2J_z\omega_s}{v\eta}}\sin\phi, \quad J_z = \frac{1}{2}\left(\frac{\omega_s}{v\eta}z^2 + \frac{v\eta}{\omega_s}\delta^2\right), \quad \phi = \mathrm{atan}\left(\frac{v\eta\delta}{\omega_s z}\right)$$

$$\text{so} \quad \mathcal{J} = \begin{pmatrix} \frac{\partial J_z}{\partial z} & \frac{\partial J_z}{\partial \delta} \\ \frac{\partial \phi}{\partial z} & \frac{\partial \phi}{\partial \delta} \end{pmatrix} = \begin{pmatrix} \frac{\omega_s}{\eta v} z & \frac{v\eta}{\omega_s}\delta \\ -\frac{\delta}{2J_z} & \frac{z}{2J_z} \end{pmatrix}$$

and we get

$$\begin{pmatrix} \frac{\omega_s}{\eta v} z & -\frac{\delta}{2J_z} \\ \frac{v\eta}{\omega_s}\delta & \frac{z}{2J_z} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\omega_s}{\eta v} z & \frac{v\eta}{\omega_s}\delta \\ -\frac{\delta}{2J_z} & \frac{z}{2J_z} \end{pmatrix} = \begin{pmatrix} \frac{\delta}{2r} & \frac{\omega_s}{\eta v} z \\ -\frac{z}{2J_z} & \frac{v\eta}{\omega_s}\delta \end{pmatrix} \cdot \begin{pmatrix} \frac{\omega_s}{\eta v} z & \frac{v\eta}{\omega_s}\delta \\ -\frac{\delta}{2J_z} & \frac{z}{2J_z} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$