B-physics prospects for lattice QCD

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April 23rd, 2019
Flavor anomalies

- **Flavor anomalies** = opportunity for BSM
- **QCD** = crucial for **confirming** significance and interpreting

\[
\text{experiment} = \text{SM} \times \text{perturbative QCD} \times (\text{non-perturbative QCD}) \\
+ \text{BSM} \times \text{perturbative QCD} \times (\text{non-perturbative QCD})
\]
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- QCD is complicated
- Difficult to extract non-perturbative predictions
Lattice QCD is a powerful tool for extracting QCD predictions

\[
\text{observable} = \int \mathcal{D}\phi \ e^{iS} \begin{bmatrix} \text{interpolator} \\ \text{for observable} \end{bmatrix}
\]
To proceed we have to make three modifications:

1. **nonzero lattice spacing**

2. **finite volume, \( L \)**

3. **Euclidean signature**

Also... Unphysical quark masses \( M_{\pi, \text{lattice}} > M_{\pi, \text{our universe}} \)

But, physical masses = increasingly common
Matrix elements and LQCD

- Single-hadron initial and final states
  - Calculated directly in LQCD
  - Euclidean irrelevant / lattice $\rightarrow 0$ / volume $\rightarrow \infty$
- New theory challenge = QED
- See FLAG averages

$B \rightarrow \pi \ell^+ \ell^-$
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- **Two-hadron final states**
  - Significantly more challenging
  - Subtle **finite volume** issues
  - Cannot treat resonances as stable particles
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- Multi-hadron states for $\sqrt{s} > 4M_\pi$
  - Volume mixes the two-, four-, six-particle contributions
  - All or nothing (must constrain the entire S-matrix for a prediction)
Single-hadron states

Three categories:

- **Decay constants**
  \[ \langle 0 | \mathcal{J} | 1 \rangle \]
  \[ f_\pi, f_K, f_B \]

- **Form factors**
  \[ \langle 1 | \mathcal{J} | 1' \rangle \]
  \[ f_{K^0\pi^-}(q^2), f_{B\rightarrow\pi}(q^2) \]

- **Mixing parameters**
  \[ \langle 1 | \mathcal{H}^{\Delta F=2} | 1 \rangle \]
  \[ B_{B_d}^{(n)}, B_{B_s}^{(n)} \]
Single-hadron states

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  - Decay constants
  - Form factors
  - Mixing parameters

\[
\begin{align*}
\langle 0 | \mathcal{J} | 1 \rangle & \quad f_\pi, f_K, f_B \\
\langle 1 | \mathcal{J} | 1' \rangle & \quad f_+^{K^0 \pi^-}(q^2), f_{B \rightarrow \pi}(q^2) \\
\langle \bar{1} | \mathcal{H}^{\Delta F=2} | 1 \rangle & \quad B_{B_d}^{(n)}, B_{B_s}^{(n)}
\end{align*}
\]

- Summary of the approach...
  - Importance sampling QCD gauge fields → correlators

\[
\langle A_{\mu}^{\text{bare}}(0) \pi_{p}(-\tau) \rangle_{T,L,m_q,a} = + + \cdots
\]
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\langle 0 | \mathcal{J} | 1 \rangle & \quad \langle 1 | \mathcal{J} | 1' \rangle \\
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B_{B_d}^{(n)}, B_{B_s}^{(n)}
\end{align*}
\]

- Summary of the approach...
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\[
\langle A_{\mu}^{\text{bare}} (0) \pi_p (-\tau) \rangle_{T,L,m_q,\alpha}
\]

- Temporal length
- Volume
- Quark masses
- Lattice spacing
Single-hadron states

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  - Decay constants
  - Form factors
  - Mixing parameters

\[ \langle 0 | \mathcal{J} | 1 \rangle \quad \langle 1 | \mathcal{J} | 1' \rangle \quad \langle 1 | \mathcal{H}^{\Delta F=2} | 1 \rangle \]

\[ f_\pi, f_K, f_B \quad f_+^{K^0\pi^-}(q^2), f_{B\to\pi}(q^2) \quad B_{B_d}^{(n)}, B_{B_s}^{(n)} \]

- Summary of the approach...
  - Importance sampling QCD gauge fields → correlators

\[ Z_{\text{renorm}} \langle A_{\mu}^{\text{bare}}(0) \pi_p(-\tau) \rangle_{T,L,m_q,a} + \cdots \]

- Renormalization of currents required (typically non-perturbative)

- Temporal length
- Volume
- Quark masses
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Single-hadron states

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  - Decay constants
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- Summary of the approach...
  - Importance sampling QCD gauge fields \( \rightarrow \) correlators

\[ Z^\text{renorm} \langle A_{\mu}^{\text{bare}}(0) \pi_p(-\tau) \rangle_{T,L,m_q,a} \rightarrow Z_\pi e^{-E_\pi \tau} i\rho_\mu f_\pi(T,L,m_q,a) \]

- Renormalization of currents required (typically non-perturbative)
- Large time separation filters excited states

- temporal length  
  - volume  
  - quark masses  
  - lattice spacing
Single-hadron states

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\[ Z^{\text{renorm}} \langle A_\mu^{\text{bare}}(0) \pi_p(-\tau) \rangle_{T,L,m_q,a} \quad \longrightarrow \quad Z_\pi e^{-E_\pi \tau} \quad \text{i} p_\mu f_\pi(T, L, m_q, a) \]

- temporal length
- volume
- quark masses
- lattice spacing
- Renormalization of currents required (typically non-perturbative)
- Large time separation filters excited states
- Extrapolation/interpolation to physical point

\[ \lim_{T,L \to \infty} \lim_{a \to 0} f_\pi(T, L, m_q^{\text{phys}}, a) = f_\pi^{\text{phys}} \]
Decay constants $\langle 0|\mathcal{J}|1 \rangle$

- **Summary (from Bazavov et. al. [Fermilab/MILC] 2018)**

  - $f_{D^+}$ and $f_{D_s}$
    - Fermilab/MILC 18
    - ETM 14
    - Fermilab/MILC 14
    - BES 3 + CKM unitarity
    - RBC/UKQCD 17
    - $\chi$QCD 14
    - HPQCD 12
    - Fermilab/MILC 11 (Clover c)
    - HPQCD 10

  - $f_{B^+}$ and $f_{B_s}$
    - Fermilab/MILC 18
    - HPQCD 17 (pseudoscalar current)
    - ETM 16
    - HPQCD 13 (NRQCD b)
    - RBC/UKQCD 14
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$\bar{f}_{D^+} f_{D^0}$

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**Current precision sufficient** for BES III, BELLE II

- Fermilab/MILC includes QED uncertainty (not yet rigorous)
- MILC quoting higher precision than any other 2+1(+1) calculation

Need comparable precision from other calculations to cross-check
lattice QCD + QED

- Relevant for sub-percent uncertainties

$$\alpha_{\text{QED}} \sim \frac{m_u - m_d}{\Lambda_{\text{QCD}}} \sim 1\%$$
lattice QCD + QED

- Relevant for sub-percent uncertainties
- Meaning of decay constants
- Pure QCD

\[ \Gamma(K^- \to \ell^- \bar{\nu}_\ell) = \frac{G_F^2 |V_{us}|^2 f_K^2}{8\pi} m_K m_\ell \left(1 - \frac{m_\ell^2}{m_K^2}\right)^2 \]

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  \]

- QCD + QED (GRS scheme)
  \[
  \Gamma(K^+ \rightarrow \mu^- \bar{\nu}_\mu [\gamma]) = (1.0032 \pm 0.0011) \Gamma^{(0)}(K^+ \rightarrow \mu^- \bar{\nu}_\mu)
  \]

C. Sachrajda (Durham flavour workshop) • Di Carlo et al. in preparation
lattice QCD + QED

- Relevant for sub-percent uncertainties

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- QED in a box
  - Periodicity incompatible with Gauss law
  - QED = long range
  - Require modification (vanishes as \( L \rightarrow \infty \))

- Different soft scales for different particles
  - Well-understood for pions and kaons
  - \( B \) and \( D \) = different soft scale → requires theory developments
Neutral meson mixing $\langle \tilde{1} | \mathcal{H}^{\Delta F=2} | 1 \rangle$

- B-mixing dominated by local matrix element

![Diagram showing $B_q$ and $\bar{B}_q$ connected through $O_{ji}^g$.]
Neutral meson mixing \( \langle 1 | \mathcal{H}^{\Delta F=2} | 1 \rangle \)

- B-mixing dominated by local matrix element

\[ B_q \rightarrow O^q_i \rightarrow \overline{B}_q \]

- Summary (from Bazavov et al. [Fermilab/MILC] 2016)

\[ \xi = \frac{f_{B_s} \sqrt{\hat{B}_{B_s}}}{f_{B_d} \sqrt{\hat{B}_{B_d}}} \]

- Lattice precision (~3-4%) is well behind even older experiments (~0.06 - 0.2%)
- Challenging to find optimal ‘discretization’ (lattice definition of quarks)
Neutral meson mixing $\langle 1 | \mathcal{H}^{\Delta F=2} | 1 \rangle$

- B-mixing dominated by local matrix element

\[ \begin{align*}
B_q & \quad \mathcal{O}_i^q \quad \bar{B}_q
\end{align*} \]

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- Extrapolate to heavy mass relativistic quarks

- RBC/UKQCD 2018

- No effective action for $b$ quark
Form factors $\langle 1|\mathcal{J}|1' \rangle$

- Significantly more information (functions vs numbers)
- Conformal mapping $\rightarrow$ $z$-expansion $\rightarrow$ wider kinematic range

Report $z$ coefficients + correlations

Bhattacharya, Hill, Paz (2011)
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Report $z$ coefficients + correlations

Joint fit to LQCD and experiment $\rightarrow$ CKM

Better precision needed for BES III, LHCb and BELLE II

Bhattacharya, Hill, Paz (2011)

Kronfeld (Durham workshop) (2019)
Form factors $\langle 1|\mathcal{J}|1' \rangle$

Example: $f^{B \to \pi}(q^2)$

See new FLAG report/website for details

Please cite original work (each figure has a .bib)
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Multi-hadron lattice quantities

- ‘On the lattice’ we calculate **finite-volume energies** and **matrix elements**

\[ \langle \mathcal{O}_j(\tau)\mathcal{O}_i^\dagger(0) \rangle = \sum_n \langle 0|\mathcal{O}_j(\tau)|E_n\rangle\langle E_n|\mathcal{O}_i^\dagger(0)|0\rangle = \sum_n e^{-E_n(L)\tau} Z_{n,j} Z_{n,i}^* \]

- Determine **optimized operators** by diagonalizing correlator matrix (GEVP)

\[ \langle \Omega_m(\tau)\Omega_m^\dagger(0) \rangle \sim e^{-E_m(L)\tau} + \ldots \]
\[ \langle \Omega_{m'}(\tau) J(0) \Omega_{m'}^\dagger(-\tau) \rangle \sim e^{-E_{m'}\tau} e^{-E_m\tau} \langle E_{m'}|J(0)|E_m\rangle + \ldots \]

- Our task is relate \( E_n(L) \) and \( \langle E_{m'}|J(0)|E_m\rangle \) to experimental observables
Multi-hadron processes from LQCD

**Key Idea:** We can use the finite volume as a **tool** to extract multi-hadron observables

- **Scattering (from finite-volume energies)**

- **Transitions (from finite-volume energies + matrix elements)**
Multi-hadron processes from LQCD

**Key Idea:** We can use the finite volume as a tool to extract multi-hadron observables.

- **Scattering (from finite-volume energies)**
  - $E_2(L)$
  - $E_1(L)$
  - $E_0(L)$

- **Transitions (from finite-volume energies + matrix elements)**
  - $B \rightarrow K \pi$
The finite-volume as a tool

Finite-volume set-up

- cubic, spatial volume (extent $L$)
- periodic boundary conditions
  \[ \vec{p} = \frac{2\pi}{L} \vec{n}, \quad \vec{n} \in \mathbb{Z}^3 \]

- $L$ is large enough to neglect $e^{-M_{\pi}L}$
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- Scattering observables leave an *imprint* on finite-volume quantities
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Scattering observables leave an **imprint** on finite-volume quantities

Consider a weakly-interacting, two-body system with no bound states

$$E_0 = 2M_\pi$$

Infinite-volume ground state

$$\mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a$$

Information is in the scattering amplitude
The finite-volume as a tool

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  - Consider a weakly-interacting, two-body system with no bound states

  \[
  E_0 = 2M_\pi \quad \text{Infinite-volume ground state}
  \]

  \[
  \mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a \\
  \text{Information is in the scattering amplitude}
  \]

  \[
  E_0(L) = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + \mathcal{O}(1/L^4)
  \]

  Huang, Yang (1958)
The finite-volume as a tool

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\]

Huang, Yang (1958)
General two-to-two scattering

- Lüscher’s formalism + extensions give a general mapping

\[
\det \left[ \mathcal{M}_2^{-1}(E^*_n) + F(E_n, \vec{P}, L) \right] = 0
\]

- All results are contained in a generalized quantization condition

- Matrices in angular momentum, spin and channel space

Using the result

- Simplest case is a single channel
  (e.g. for pions in a p-wave the relation reduces to)

\[ M_2(E^*_n) = -1/F(E_n, \vec{P}, L) \]

\[ M_2 \propto e^{2i\delta} - 1 \]

Using the result

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\[
\mathcal{M}_2\left( E_n^* \right) = -\frac{1}{F\left( E_n, \vec{P}, L \right)}
\]

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$\rho \rightarrow \pi \pi$

$I^G(J^{PC}) = 1^+(1--)$

Alexandrou et al. (2017)
$N_f=2+1$, $m_\pi=316$ MeV

$am_\rho = 0.4609(16)(14)$
$g_{\rho\pi\pi} = 5.69(13)(16)$

Andersen et al. (2018)
$N_f=2+1$, $m_\pi=220$ MeV

Guo et al. (2016)
$N_f=2$, $m_\pi=226$ MeV

Andersen et al. (2018)
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$I^G(J^{PC}) = 1^+(1--)$
Coupled channels

- The cubic volume mixes different partial waves...
  
  e.g. \( K\pi \to K\pi \)
  \[ \bar{P} \neq 0 \quad \Rightarrow \quad \det \left[ \begin{pmatrix} M_s^{-1} & 0 \\ 0 & M_p^{-1} \end{pmatrix} + \begin{pmatrix} F_{ss} & F_{sp} \\ F_{ps} & F_{pp} \end{pmatrix} \right] = 0 \]

  ...as well as different flavor channels...
  
  e.g. \( a = \pi\pi \), \( b = K\bar{K} \)
  \[ \det \left[ \begin{pmatrix} M_a \to a & M_a \to b \\ M_b \to a & M_b \to b \end{pmatrix}^{-1} + \begin{pmatrix} F_a & 0 \\ 0 & F_b \end{pmatrix} \right] = 0 \]

  MTH, Sharpe (2012) • Briceño, Davoudi (2012)
Coupled channels

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...as well as different flavor channels...

e.g. $a = \pi\pi$

$b = K\bar{K}$

$\Rightarrow \quad \det \left[ \begin{pmatrix} \mathcal{M}_{a\rightarrow a} & \mathcal{M}_{a\rightarrow b} \\ \mathcal{M}_{b\rightarrow a} & \mathcal{M}_{b\rightarrow b} \end{pmatrix}^{-1} + \begin{pmatrix} F_a & 0 \\ 0 & F_b \end{pmatrix} \right] = 0$

The road to physics...

Calculate a matrix of correlators with a large & varied operator basis

$\langle \mathcal{O}_a(\tau)\mathcal{O}_b^\dagger(0) \rangle$

Diagonalize (GEVP) to reliably extract finite-volume energies

$\langle \Omega_m(\tau)\Omega_m^\dagger(0) \rangle \sim e^{-E_m(L)\tau}$

Vary $L$ and $P$ to recover a dense set of energies

$[000], A_1$

$[001], A_1$

$[011], A_1$

$E_n(L)$

MTH, Sharpe (2012) • Briceño, Davoudi (2012)
**Coupled channels**

- The cubic volume mixes different partial waves...
  
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- The road to physics...

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  *Diagonalize (GEVP) to reliably extract finite-volume energies*
  
  \[ \langle \Omega_{m}(\tau)\Omega_{m}^\dagger(0) \rangle \sim e^{-E_m(L)\tau} \]

  *Identify a broad list of K-matrix parametrizations*
  
  polynomials and poles  
  EFT based  
  dispersion theory based

  *Vary L and P to recover a dense set of energies*
  
  \[ \begin{align*}
  &[000], A_1 & \bullet & \bullet & \bullet & \bullet & \bullet \\
  &[001], A_1 & \bullet & \bullet & \bullet & \bullet & \bullet \\
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  \end{align*} \]

  \[ E_n(L) \]
Coupled channels

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  - EFT based
  - dispersion theory based

  Perform global fits to the finite-volume spectrum

  Calculate a matrix of correlators with a large & varied operator basis
  \[ \langle O_a(\tau)O_b^\dagger(0) \rangle \]

  Diagonalize (GEVP) to reliably extract finite-volume energies
  \[ \langle \Omega_m(\tau)\Omega_m^\dagger(0) \rangle \sim e^{-E_m(L)\tau} \]

  Vary \( L \) and \( P \) to recover a dense set of energies
  \[ [000], A_1 \quad [001], A_1 \quad [011], A_1 \]
  \[ E_n(L) \]

MTH, Sharpe (2012) • Briceño, Davoudi (2012)
$I^G(J^{PC}) = 0^+(0^{++})$

**Coupled-channel scattering**

\[ \rho_i \rho_j |t_{ij}|^2 \]

$m_\pi \sim 390\text{MeV}$

$\pi\pi \rightarrow \pi\pi$

$K\bar{K} \rightarrow K\bar{K}$

$\pi\pi \rightarrow K\bar{K}$

$\sigma$ \hspace{1cm} $f_0$ \hspace{1cm} $|\frac{c_{K\bar{K}}}{c_{\pi\pi}}|^2 = 1.4(3)$

Multi-hadron processes from LQCD

**Key Idea:** We can use the finite volume as a **tool** to extract multi-hadron observables

- **Scattering (from finite-volume energies)**
  - $E_2(L)$
  - $E_1(L)$
  - $E_0(L)$

- **Transitions (from finite-volume energies + matrix elements)**
  - $\langle 2|J^1 \rangle_L$
  - $\langle 2|J^2 \rangle_L$
  - $B \rightarrow K\pi$
Multi-hadron processes from LQCD

**Key Idea:** We can use the finite volume as a tool to extract multi-hadron observables

- **Scattering (from finite-volume energies)**
  
  \[
  E_2(L) \rightarrow \langle 2 | J | 1 \rangle_L \\
  E_1(L) \rightarrow \langle 2 | J | 2 \rangle_L \\
  E_0(L)
  \]

- **Transitions (from finite-volume energies + matrix elements)**
  
  \[
  \langle J \rangle | 1 \rangle \rightarrow \langle J \rangle | 2 \rangle \\
  B \rightarrow K \pi
  \]

Recent Review: *Lattice QCD and Three-particle Decays of Resonances*  
MTH & Sharpe [1901.00483]
Multi-hadron processes from LQCD

**KEY IDEA:** We can use the finite volume as a tool to extract multi-hadron observables

- **Scattering (from finite-volume energies)**
  - $E_2(L)$
  - $E_1(L)$
  - $E_0(L)$

- **Transitions (from finite-volume energies + matrix elements)**
  - $\langle 2 | \mathcal{J} | 1 \rangle$
  - $\langle 2 | \mathcal{J} | 2 \rangle$
  - $B \rightarrow K\pi$
Multi-hadron matrix elements

- Theoretical method established for many observables

- Weak decay
  \[ \langle \pi \pi, \text{out} | \mathcal{H} | K \rangle \equiv \text{Diagram} \]


- Time-like form factors
  \[ \langle \pi \pi, \text{out} | J_\mu | 0 \rangle \equiv \text{Diagram} \]

Meyer (2011)

- Resonance form factors
  \[ \langle K \pi, \text{out} | J_{\alpha \beta} | B \rangle \equiv \text{Diagram} \]


- Particles with spin
  \[ \langle N \pi, \text{out} | J_\mu | N \rangle \equiv \text{Diagram} \]
Pion photo-production

- Formal relation

\[ \langle \pi \pi, \text{out} | J_\mu | \pi \rangle \equiv \]

get this from the lattice

\[ | \langle n, L | J_\mu | \pi \rangle |^2 = \langle \pi | J_\mu | \pi \pi, \text{in} \rangle \mathcal{R}(E_n, L) \langle \pi \pi, \text{out} | J_\mu | \pi \rangle \]

experimental observable

Briceño, MTH, Walker-Loud (2015)

- Numerical implementation

\( m_\pi \approx 400 \text{MeV} \)

\( m_\pi \approx 320 \text{MeV} \)


Take home messages…

- **Single hadron states**
  - Decay constants = competitive with experiment
  - Mixing matrix elements = less competitive
  - Sub-percent uncertainty → QED

- **Multi-hadron states**
  - Resonance form-factor calculations beginning
  - $B \to K \pi \ell^+ \ell^-$ underway (Meinel et al.)

Thanks!
Backup Slides
Lots of activity…

\[ \rho \rightarrow \pi \pi \]

- CP-PACS/PACS-CS 2007, 2011
- ETMC 2010
- Lang et al. 2011
- Pellisier 2012
- RQCD 2015
- Guo et al. 2016
- Fu et al. 2016
- Bulava et al. 2016
- Alexandrou et al. 2017

\[ \kappa \rightarrow K \pi \]

\[ K^* \rightarrow K \pi \]

- Lang et al. 2012
- Prelovsek et al. 2013
- Wilson et al. 2015
- RQCD 2015
- Brett et al. 2018

\[ \sigma \rightarrow \pi \pi \]

- Prelovsek et al. 2010
- Fu 2013
- Wakayama 2015
- Howarth and Giedt 2017
- Briceño et al. 2017

\[ \kappa(700) \]

\[ J^P = 1^- \]

\[ a_0(980) \rightarrow \pi \eta, K \bar{K} \]

- Dudek et al. 2016

\[ \sigma, f_0, f_2 \rightarrow \pi \pi, K \bar{K}, \eta \eta \]

- Briceño et al. 2017

See the recent review by Briceño, Dudek and Young
Weak decays...

get this from the lattice

$$|\langle n, L | \mathcal{H}_W | K \rangle|^2 = R(E_n, L)|\langle \pi\pi, \text{out} | \mathcal{H}_W | K \rangle|^2$$

depends on scattering phase shift

Lellouch, Lüscher (2001)

Three steps to lattice weak decay

- Calculate finite-volume energies $\rightarrow$ $\pi\pi$ scattering phase $\rightarrow R(E_n, L)$
- Calculate renormalized finite-volume matrix elements
- Combine $R(E_n, L)$ with f.v. matrix elements $\rightarrow$ decay amplitudes

Complete numerical calculation by RBC/UKQCD

RBC/UKQCD, e.g. PRL 2015, (1505.07863)
$D$ decays...

get this from the lattice

$$\langle n, L | \mathcal{H}_W | D \rangle = \begin{pmatrix} C_{\pi\pi} & C_{K\bar{K}} & C_{\eta\eta} \end{pmatrix}$$

depends on scattering matrix

- Coupled channels mix in the finite volume
- Three steps to $D$ decays
  - Calculate many finite-volume energies
    → coupled scattering → $C_{xy}$
  - Calculate many renormalized finite-volume matrix elements
  - Extract amplitudes in a global fit

- Important caveat: The relation ignores $\pi\pi\pi\pi$ states

Experimental observables

$$\begin{pmatrix}
\langle \pi\pi, \text{out} | \mathcal{H}_W | D \rangle \\
\langle K\bar{K}, \text{out} | \mathcal{H}_W | D \rangle \\
\langle \eta\eta, \text{out} | \mathcal{H}_W | D \rangle
\end{pmatrix}$$

MTH, Sharpe (2012)
Three-hadron scattering

- Formalism is complete for two and three (identical) scalars

- Currently exploring utility through numerical toy examples

Volume effects on an Efimov state

Model of a 3-particle resonance

Briceño, MTH, Sharpe (2017)

Recent review for Annual Review of Nuclear and Particle Physics

MTH, Sharpe (2019) [1901.00483]