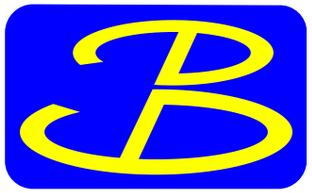
B-physics prospects for lattice QCD

Maxwell T. Hansen

April 23rd, 2019

Flavor anomalies

- **Flavor anomalies** = opportunity for BSM
- **QCD** = crucial for **confirming significance** and **interpreting**



<u>*experiment*</u> = **SM** x *perturbative QCD* x **(non-perturbative QCD)**

+ **BSM** x *perturbative QCD* x **(non-perturbative QCD)**

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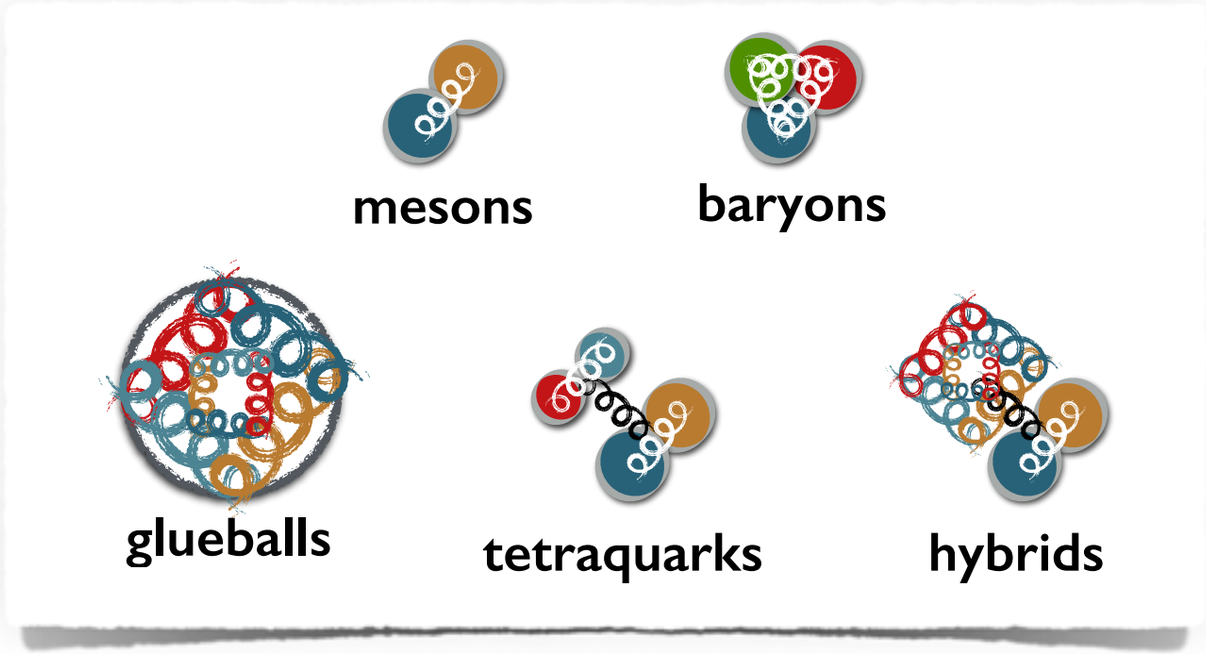
- **QCD** is complicated
- Difficult to extract non-perturbative predictions



Lattice QCD is a powerful tool for extracting QCD predictions

$$\text{observable} = \int \mathcal{D}\phi \; e^{iS} \left[\begin{array}{c} \text{interpolator} \\ \text{for observable} \end{array} \right]$$

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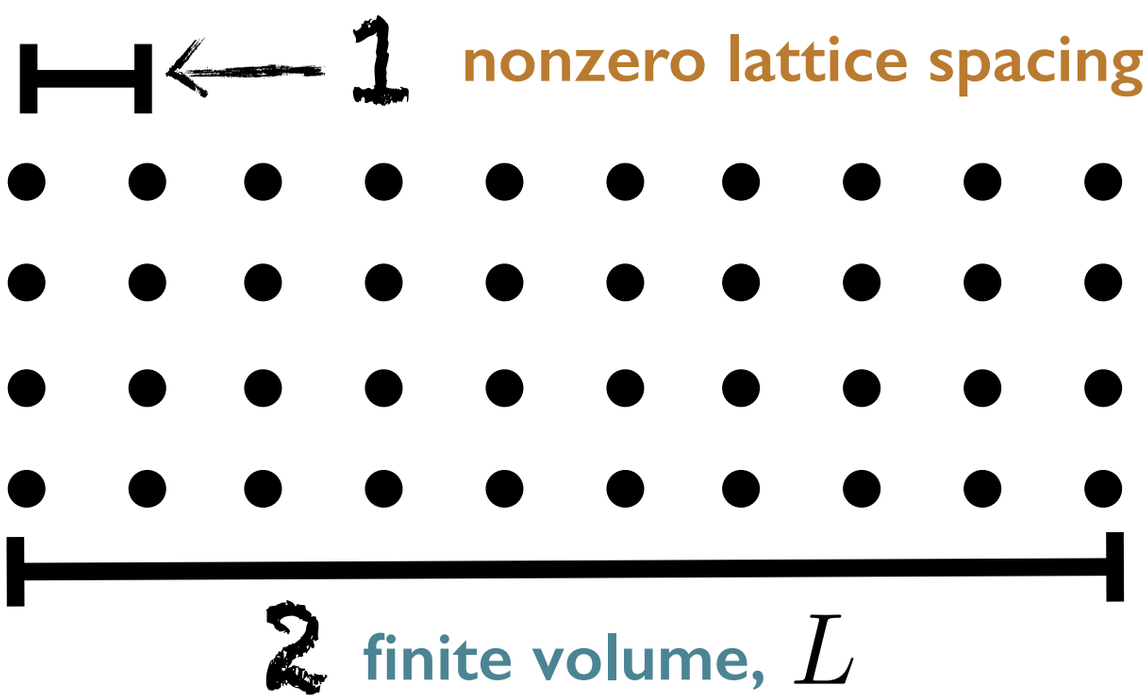
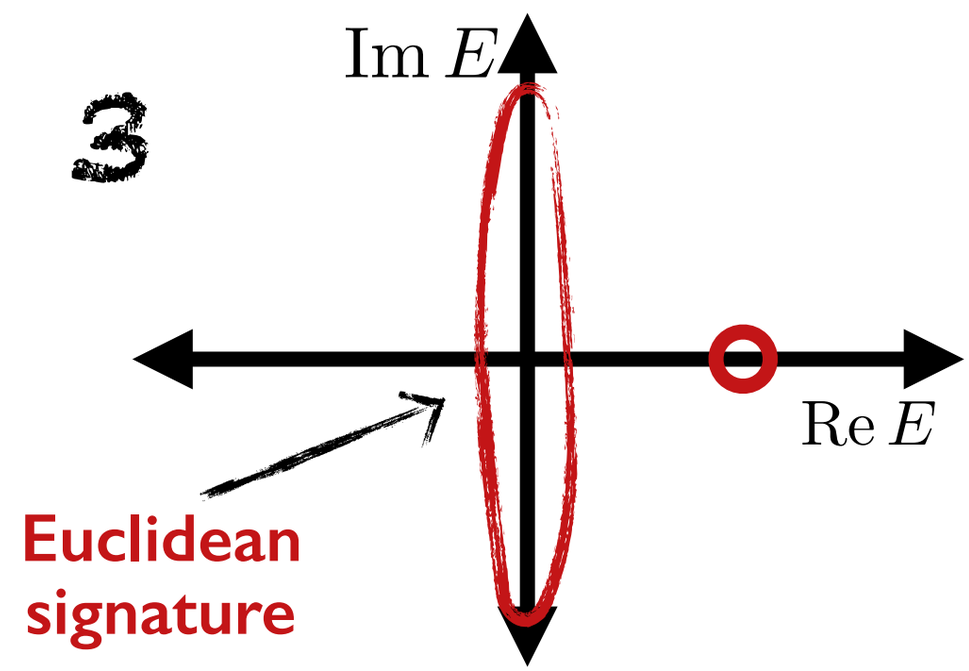
$$\text{observable?} = \int \prod_i^N d\phi_i \; e^{-S} \left[\begin{array}{c} \text{interpolator} \\ \text{for observable} \end{array} \right]$$

To proceed we have to make **three modifications**

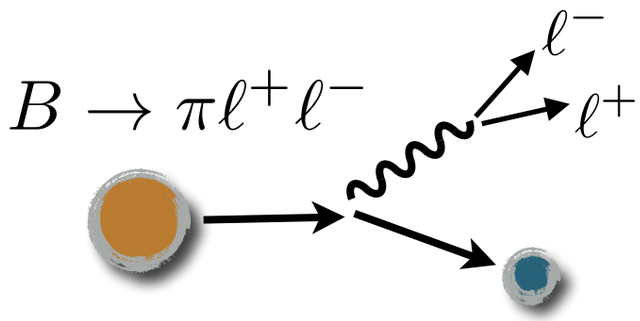




Also… Unphysical quark masses $M_{\pi,\mathrm{lattice}} > M_{\pi,\mathrm{our\ universe}}$

But, physical masses = increasingly common

Matrix elements and LQCD



- **Single-hadron initial and final states**
  - **Calculated directly in LQCD**
    - **Euclidean irrelevant / lattice → 0 / volume → ∞**
  - **New theory challenge = QED**
  - **See FLAG averages**

Matrix elements and LQCD

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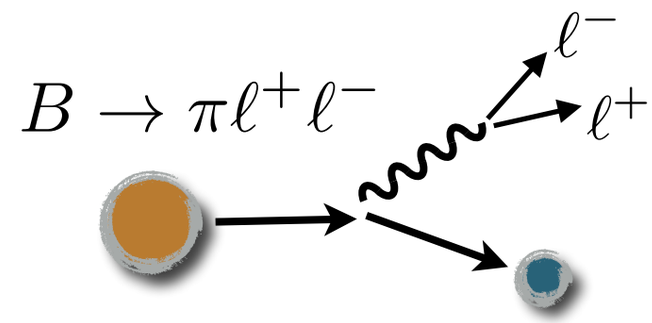

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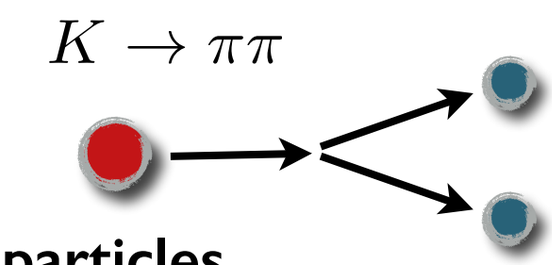
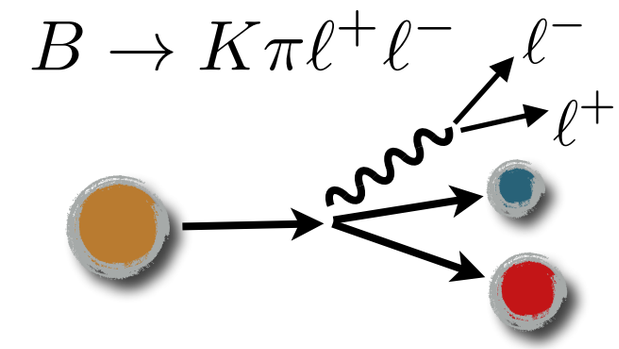
  - Significantly more challenging
  - Subtle **finite volume** issues
  - Cannot treat resonances as stable particles





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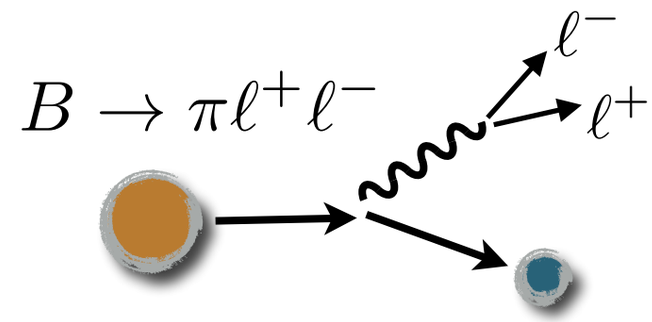

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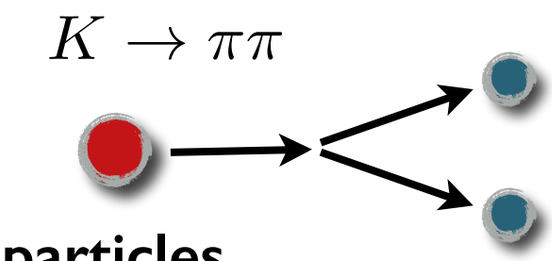

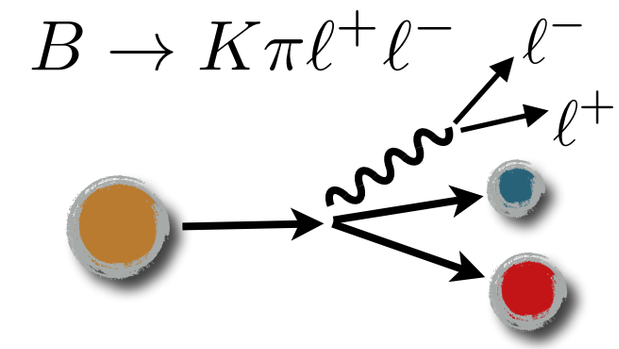

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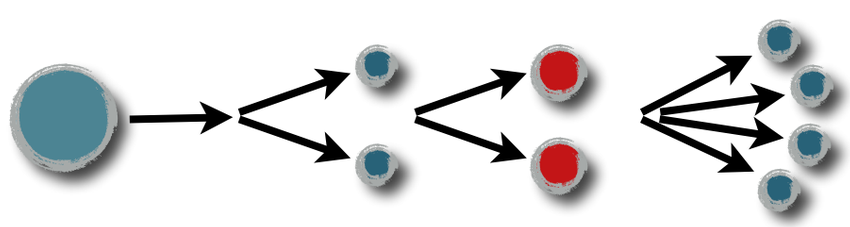
- **Multi-hadron states for** $\sqrt{s} > 4M_\pi$
  - Volume mixes the two-, four-, six-particle contributions
  - **All or nothing** (must constrain the entire S-matrix for a prediction)



$D \to \pi\pi, K\overline{K}$

Single-hadron states

- **Three categories:**
  - **Decay constants**
  - **Form factors**
  - **Mixing parameters**

$$\langle 0|\mathcal{J}|\mathbf{1}\rangle \qquad \langle \mathbf{1}|\mathcal{J}|\mathbf{1}'\rangle \qquad \langle \overline{\mathbf{1}}|\mathcal{H}^{\Delta F=2}|\mathbf{1}\rangle$$

$$f_\pi, f_K, f_B \qquad f_+^{K^0\pi^-}(q^2),\ f_{B\to\pi}(q^2) \qquad B_{B_d}^{(n)},\ B_{B_s}^{(n)}$$

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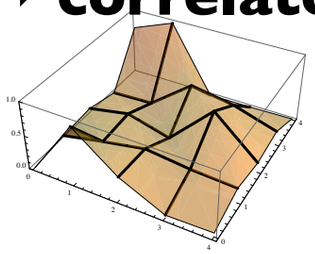
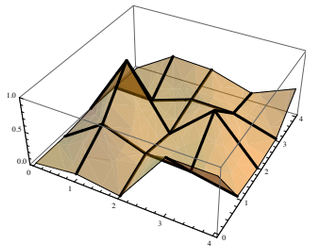


$$+$$



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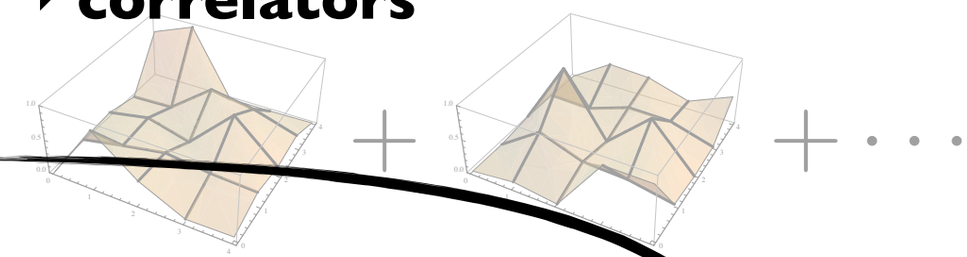

- **temporal length**
- **volume**
- **quark masses**
- **lattice spacing**

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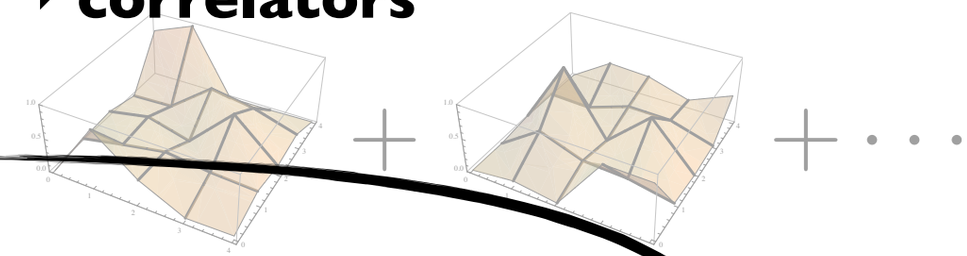

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  - **Large time** separation filters excited states
  - Extrapolation/interpolation to physical point

$$\lim_{T,L\to\infty}\lim_{a\to 0} f_\pi(T,L,m_q^{\text{phys}},a) = f_\pi^{\text{phys}}$$

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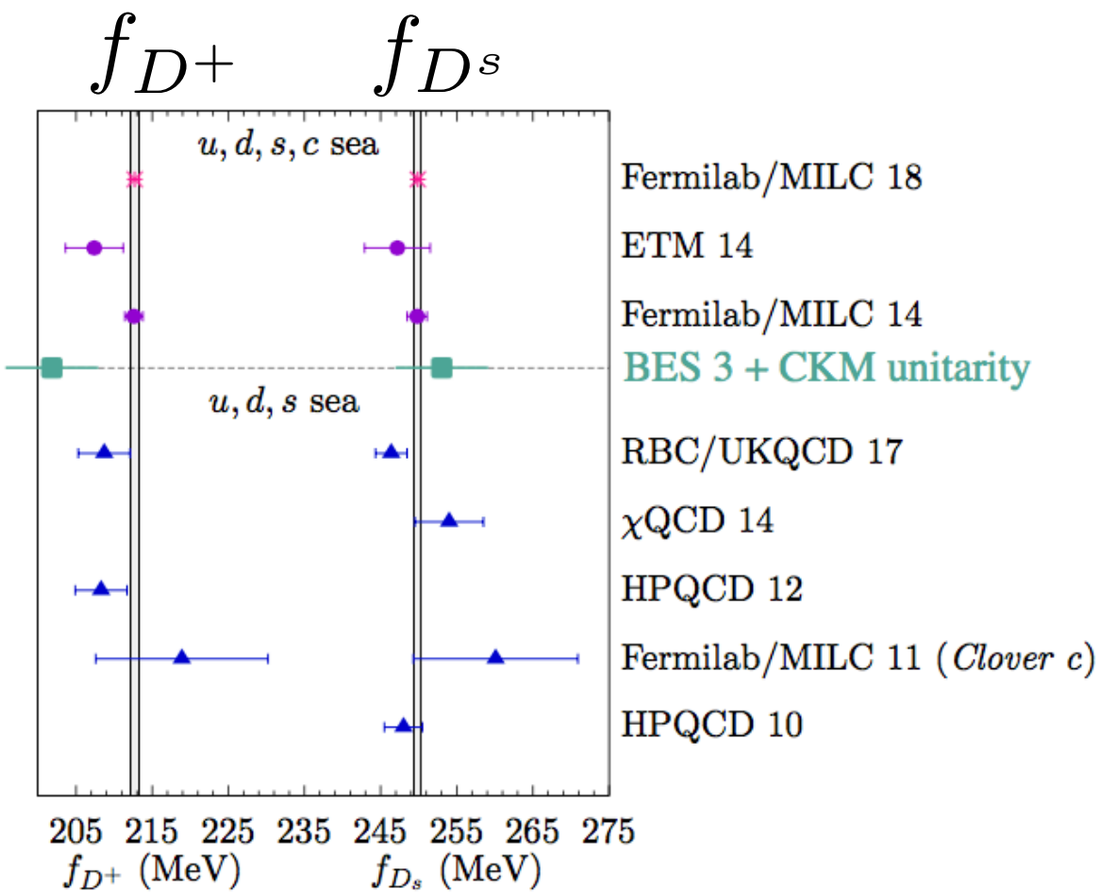
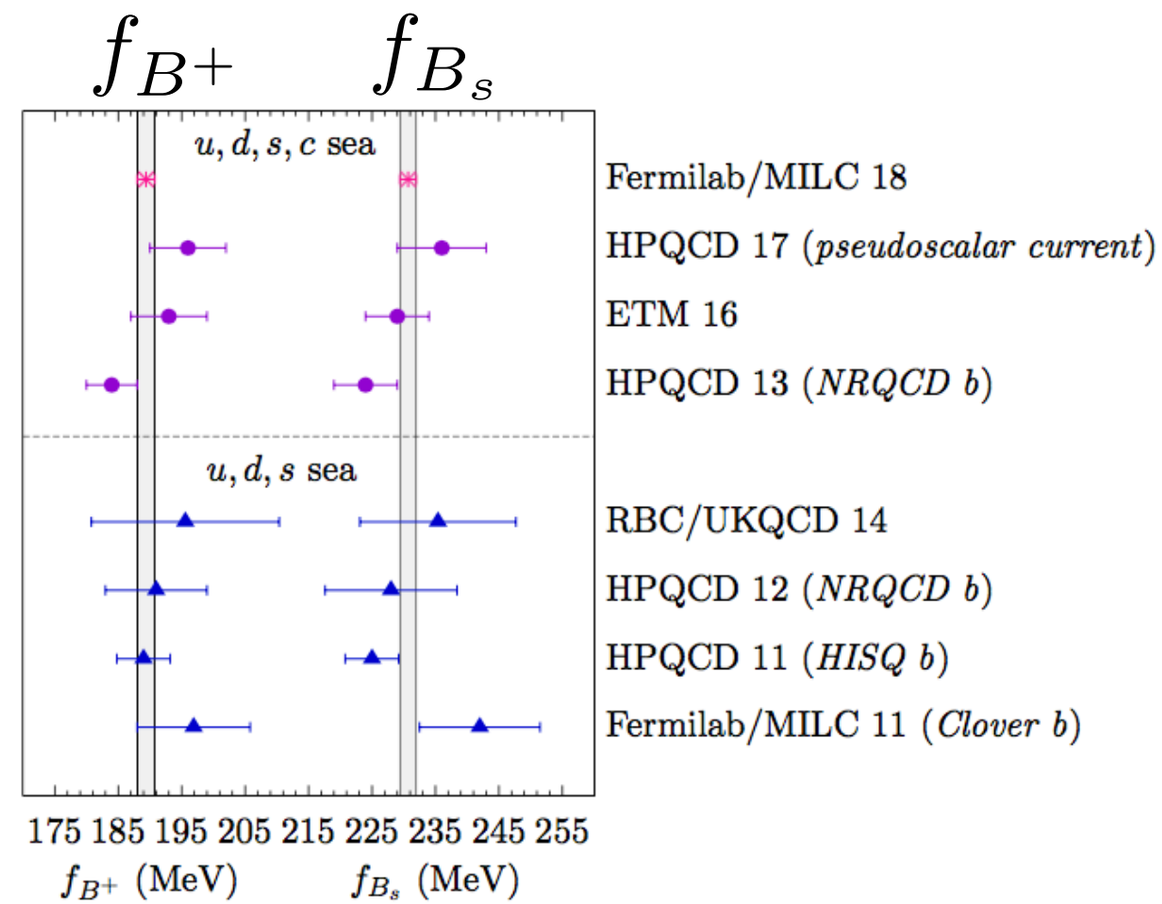
- **Summary (from Bazavov et. al. [Fermilab/MILC] 2018)**





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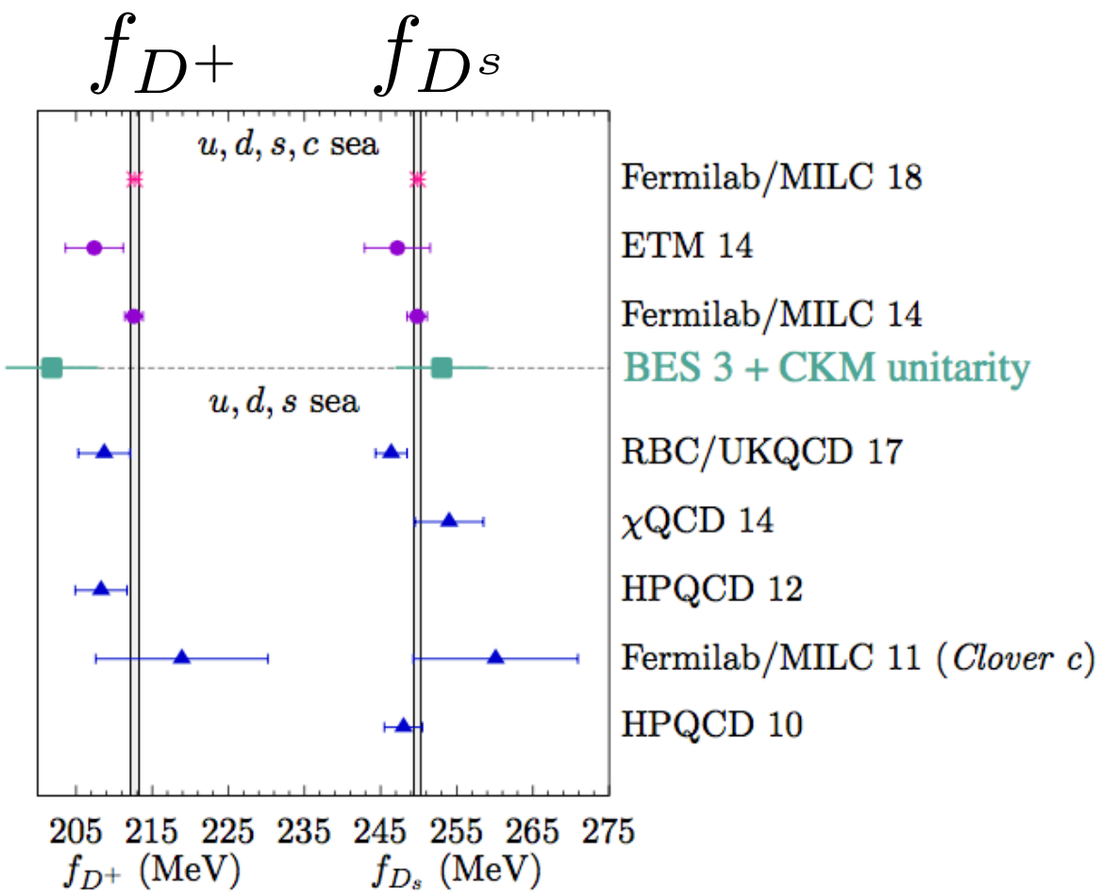

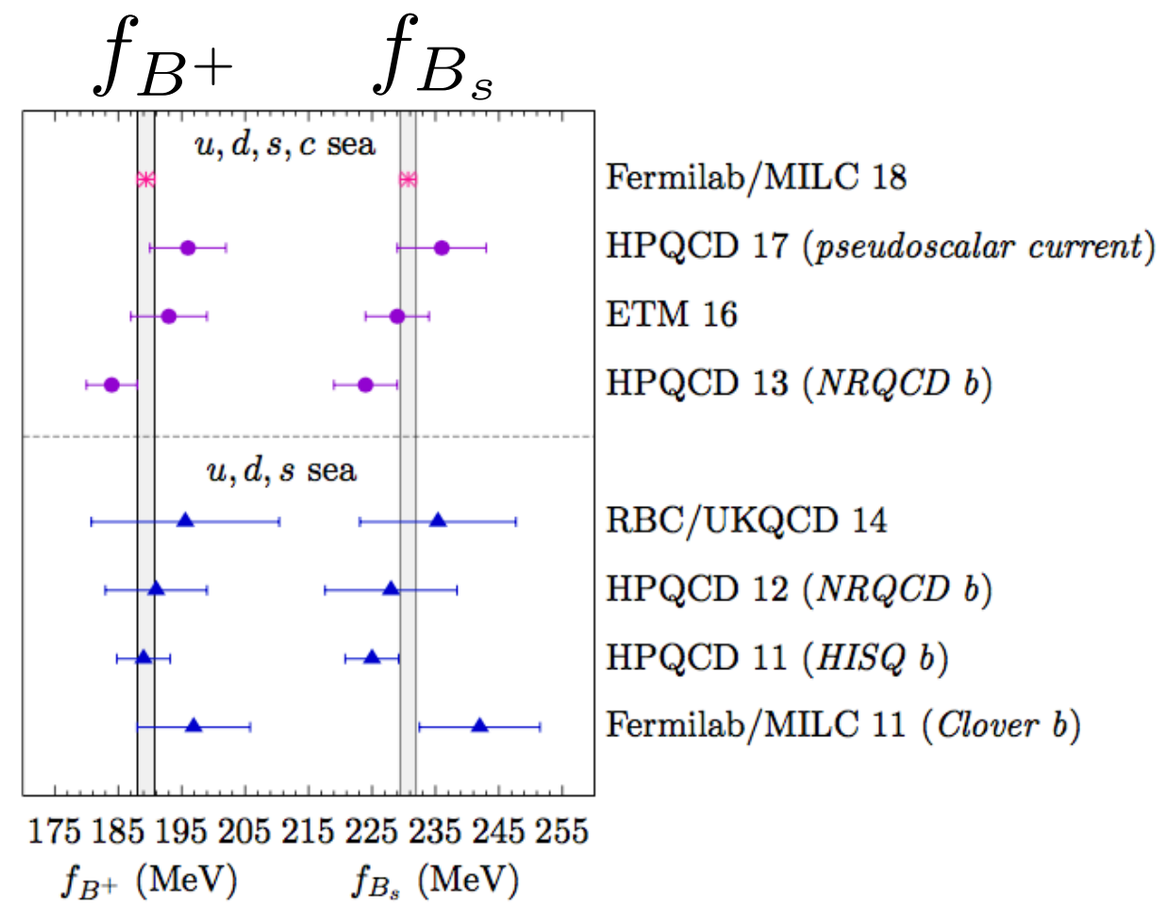

- **Current precision sufficient** for BES III, BELLE II
- Fermilab/MILC includes **QED** uncertainty (not yet rigorous)
- MILC quoting higher precision than any other 2+1(+1) calculation

Need comparable precision from other calculations to **cross-check**

lattice QCD + **QED**

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  - **Pure QCD**

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  - **QCD + QED (GRS scheme)**

$$\Gamma(K^- \to \mu^- \overline{\nu}_\mu [\gamma]) = (1.0032 \pm 0.0011)\Gamma^{(0)}(K^- \to \mu^- \overline{\nu}_\mu)$$

C. Sachrajda *(Durham flavour workshop)* • Di Carlo et al. *in preparation*

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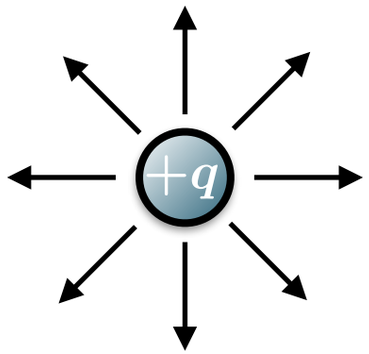
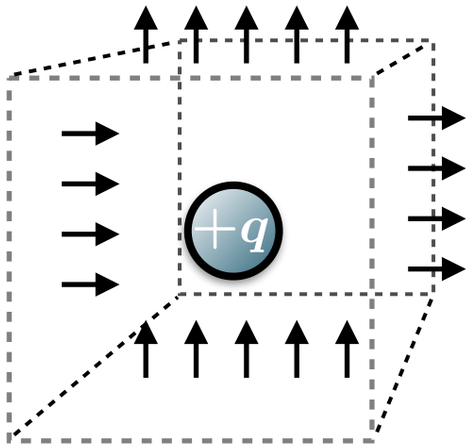
- QED in a box
  - Periodicity incompatible with Gauss law
  - **QED = long range**
  - Require modification (vanishes as $L \to \infty$)

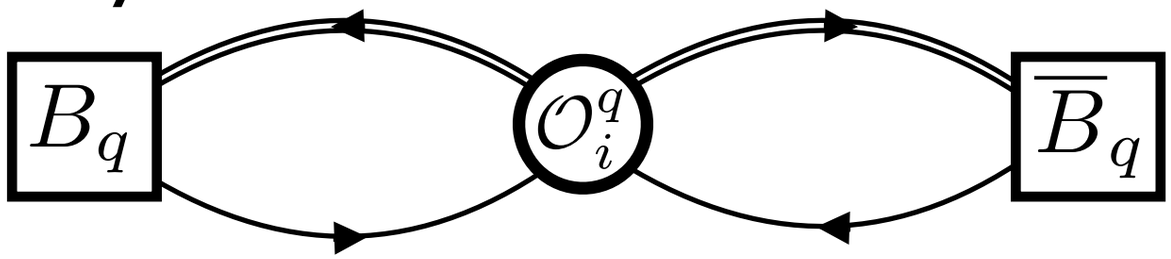




- Different soft scales for different particles
  - Well-understood for pions and kaons
  - *B* and *D* = different soft scale → **requires theory developments**

Neutral meson mixing $\langle \overline{\mathbf{1}} | \mathcal{H}^{\Delta F=2} | \mathbf{1} \rangle$

- **B-mixing dominated by local matrix element**



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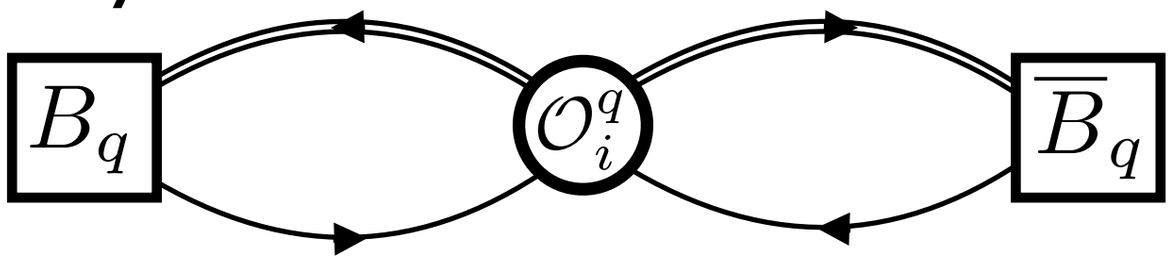

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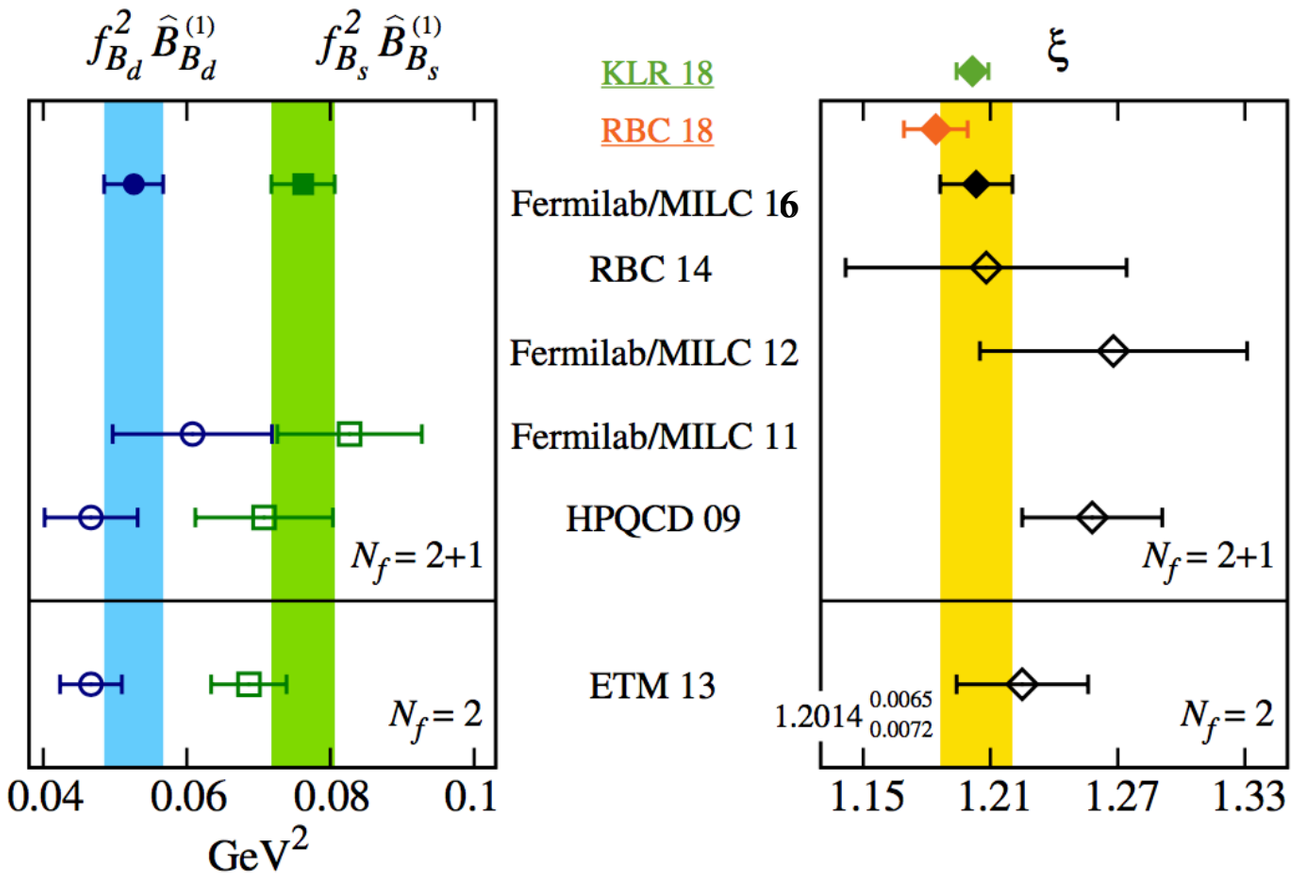

$$\xi = \frac{f_{B_s}\sqrt{\hat{B}_{B_s}}}{f_{B_d}\sqrt{\hat{B}_{B_d}}}$$

KLR 18 = King, Lenz, Rauh *(2018)* *(QCD sum rules)*

plot from Kronfeld (Durham workshop 2019)

- **Lattice precision (~3-4%) is well behind even older experiments (~0.06 - 0.2%)**
- **Challenging to find optimal 'discretization' (lattice definition of quarks)**

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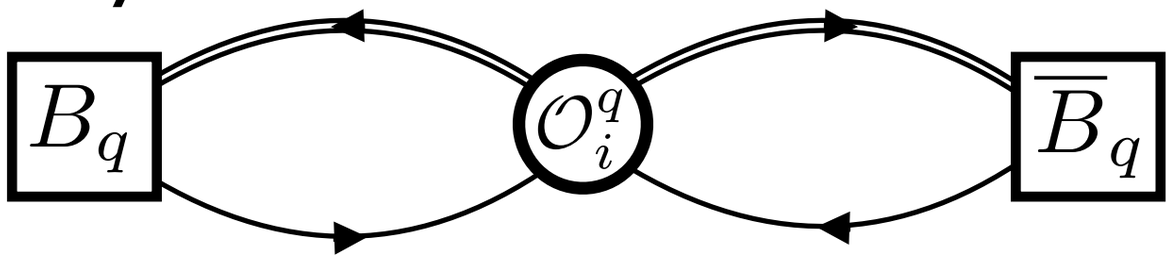

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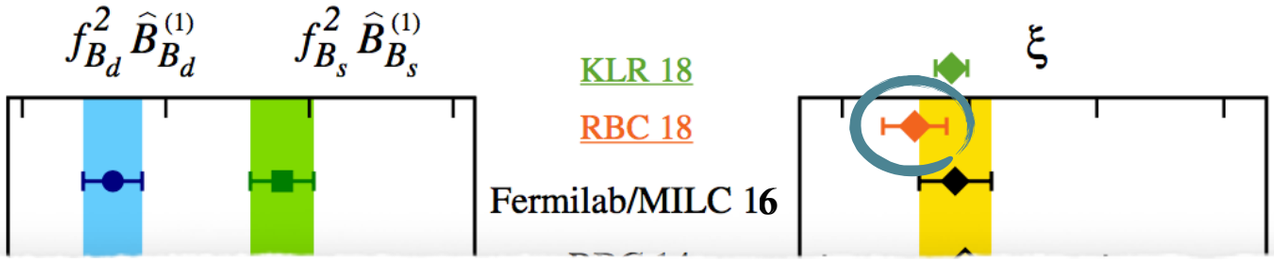

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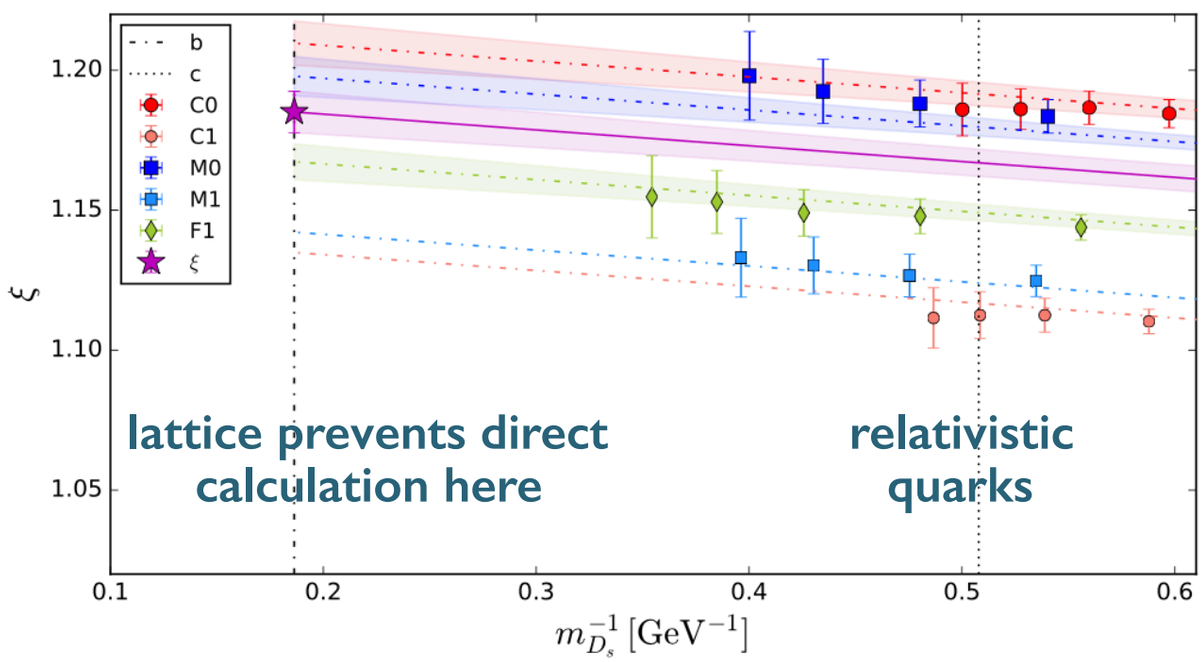
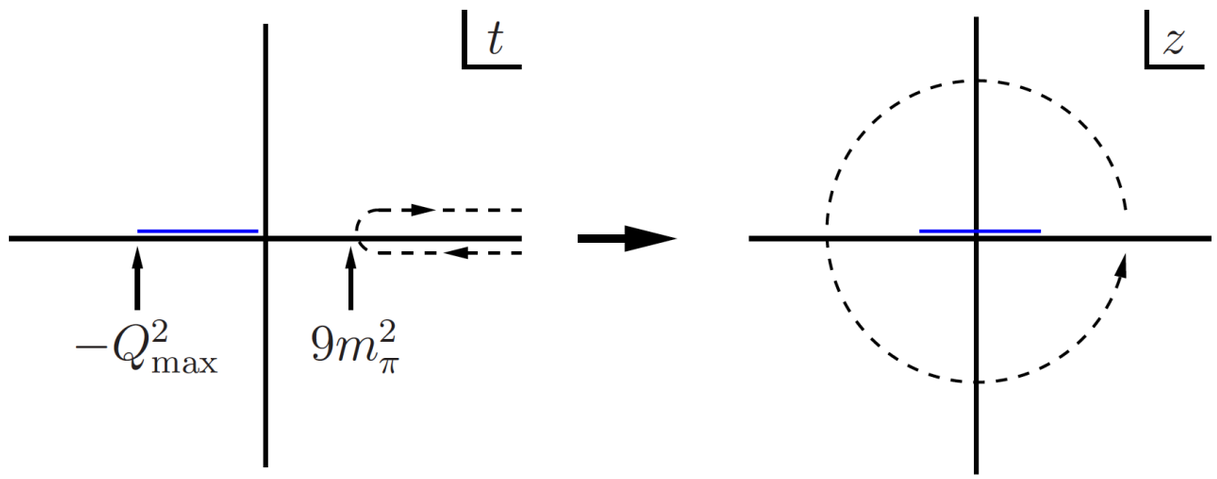
$$\xi(a, m_\pi, m_H)$$



- **RBC/UKQCD 2018**
- **No effective action for *b* quark**
- **Extrapolate** to heavy mass

Form factors $\langle \mathbf{1}|\mathcal{J}|\mathbf{1}'\rangle$

- Significantly more information (functions vs numbers)
- Conformal mapping → z-expansion → wider kinematic range



Bhattacharya, Hill, Paz *(2011)*

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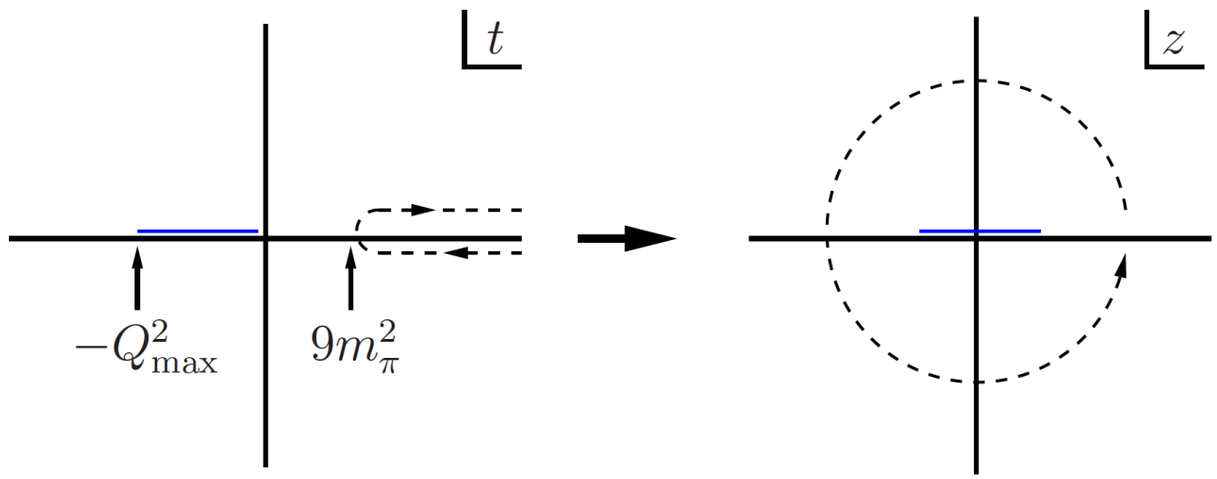


Bhattacharya, Hill, Paz *(2011)*

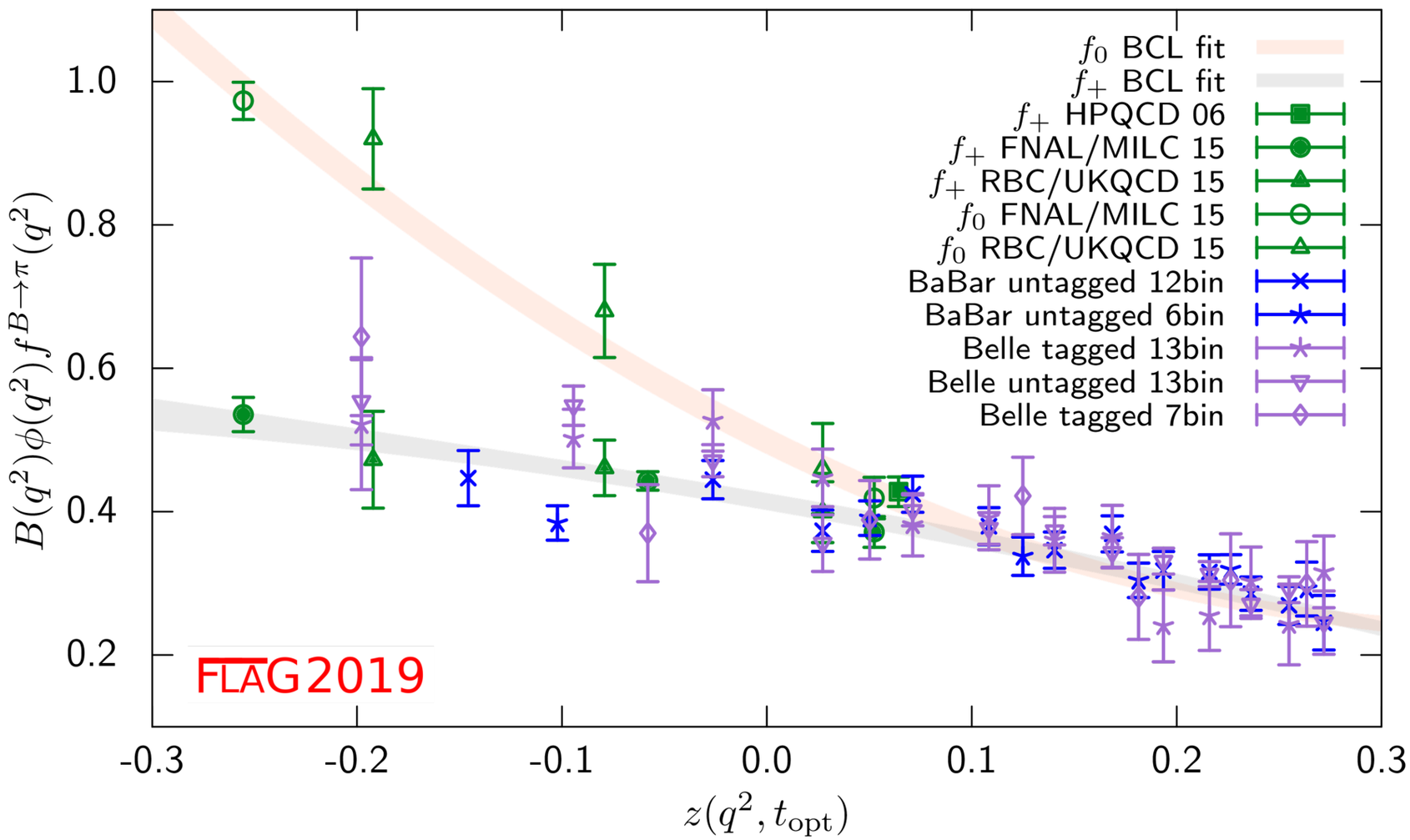
- Report z coefficients + correlations
- Joint fit to LQCD and experiment → CKM
- Better precision needed for BES III, LHCb and BELLE II

$$\begin{pmatrix} |V_{ud}| & |V_{us}| & |V_{ub}| \\ \pi^+\to l^+\nu & K^+\to l^+\nu & B^+\to\tau^+\nu \\ \pi^+\to\pi^0 e^+\nu & K\to\pi l^+\nu & B\to\pi l^+\nu \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ D^+\to l^+\nu & D_s^+\to l^+\nu & B_c^+\to\tau^+\nu \\ D\to\pi l^+\nu & D\to K l^+\nu & B\to\pi l^+\nu \\ |V_{td}| & |V_{ts}| & |V_{tb}| \\ B^0\to\pi^0 l^+ l^- & B^0\to K^0 l^+ l^- & \\ B^0\leftrightarrow\bar{B}^0 & B_s^0\leftrightarrow\bar{B}_s^0 & \end{pmatrix}$$

Kronfeld (Durham workshop) *(2019)*

Form factors $\langle \mathbf{1} | \mathcal{J} | \mathbf{1}' \rangle$

- **Example:** $f^{B\to\pi}(q^2)$

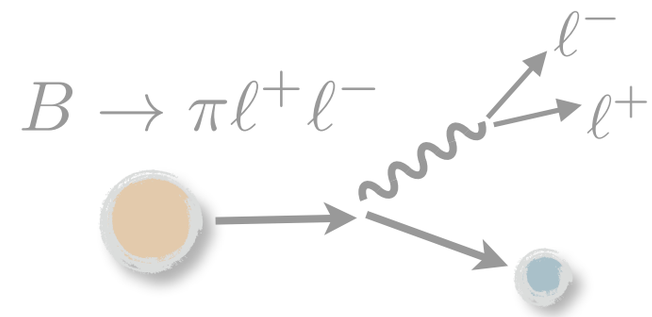


- **See new FLAG report/website for details**
- **Please cite original work *(each figure has a .bib)***

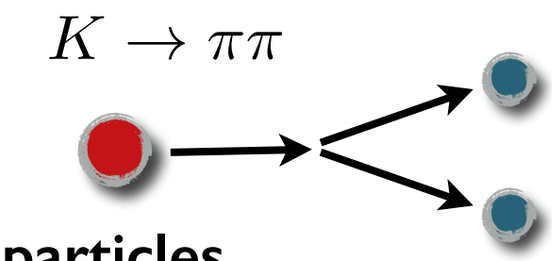
FLAG 2019
Flavour Lattice Averaging Group

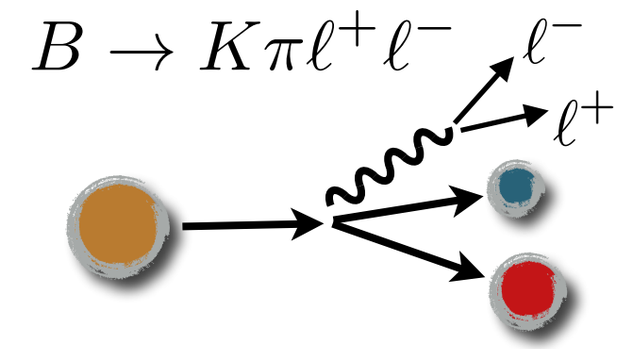
Matrix elements and LQCD

- ✓ Single-hadron initial and final states
  - Calculated directly in **LQCD**
    - Euclidean irrelevant / lattice → 0 / volume → ∞
  - New theory challenge = **QED**
  - See FLAG averages

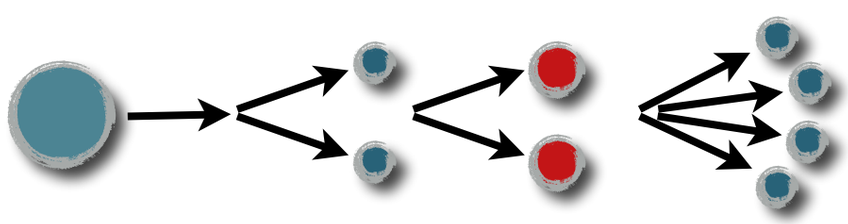


- Two-hadron final states
  - Significantly more challenging
  - Subtle **finite volume** issues
  - Cannot treat resonances as stable particles





- Multi-hadron states for $\sqrt{s} > 4M_\pi$
  - Volume mixes the two-, four-, six-particle contributions
  - **All or nothing** (must constrain the entire S-matrix for a prediction)



$D \to \pi\pi, K\overline{K}$

Multi-hadron lattice quantities

- **'On the lattice' we calculate finite-volume energies and matrix elements**

$$\langle \mathcal{O}_j(\tau)\mathcal{O}_i^\dagger(0)\rangle = \sum_n \langle 0|\mathcal{O}_j(\tau)|E_n\rangle\langle E_n|\mathcal{O}_i^\dagger(0)|0\rangle = \sum_n e^{-E_n(L)\tau} Z_{n,j} Z^*_{n,i}$$

- **Determine optimized operators by diagonalizing correlator matrix (GEVP)**

$$\left\langle \Omega_m(\tau)\Omega_m^\dagger(0)\right\rangle \sim e^{-E_m(L)\tau} + \cdots$$

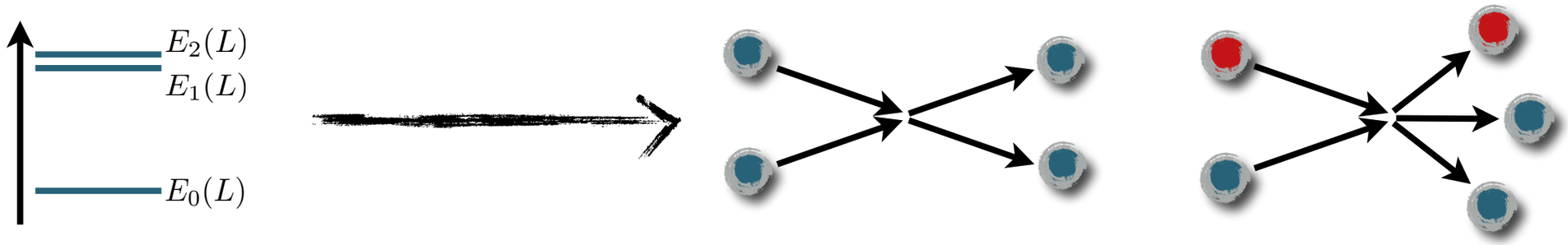
$$\left\langle \Omega_{m'}(\tau)\ \mathcal{J}(0)\ \Omega_m^\dagger(-\tau)\right\rangle \sim e^{-E_{m'}\tau}\ e^{-E_m\tau}\ \langle E_{m'}|\mathcal{J}(0)|E_m\rangle + \cdots$$

- **Our task is relate $E_n(L)$ and $\langle E_{m'}|\mathcal{J}(0)|E_m\rangle$ to experimental observables**

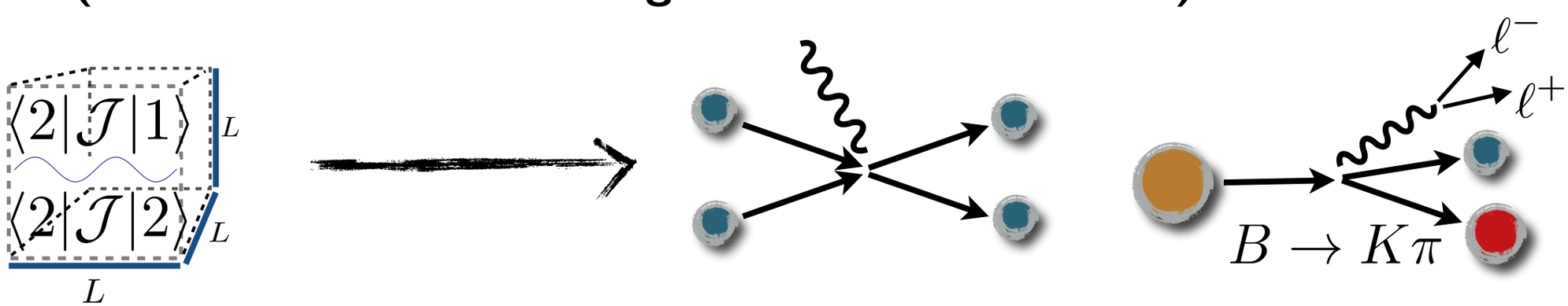
Multi-hadron processes from LQCD

KEY IDEA: **We can use the finite volume as a tool to extract multi-hadron observables**

- **Scattering (from finite-volume energies)**



- **Transitions (from finite-volume energies + matrix elements)**



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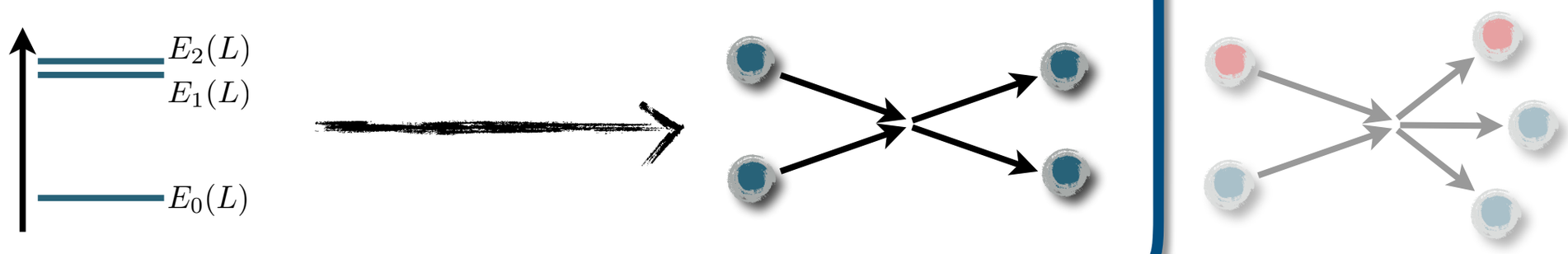

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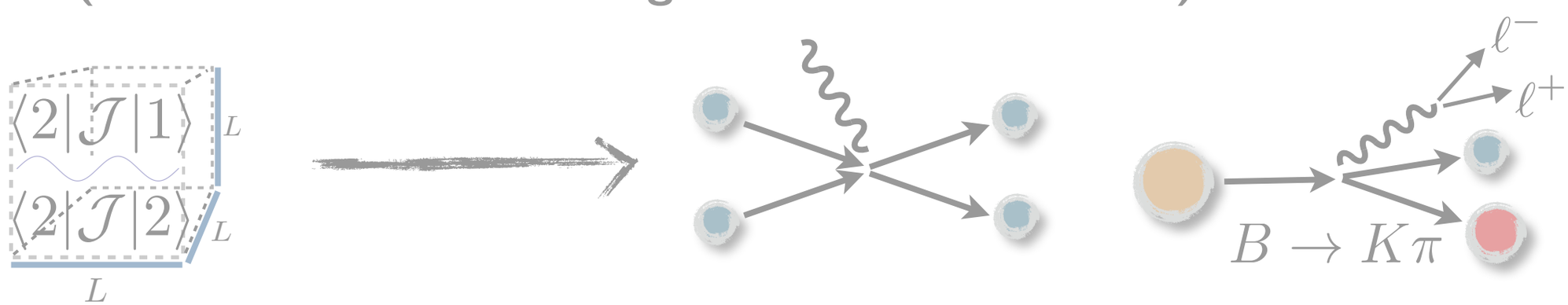
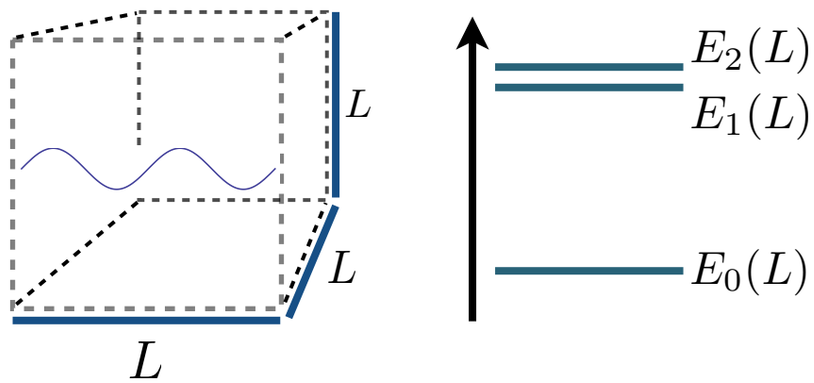


The finite-volume as a tool

- **Finite-volume set-up**



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- **periodic boundary conditions**

$$\vec{p} = \frac{2\pi}{L}\vec{n}\,, \quad \vec{n} \in \mathbb{Z}^3$$

- ***L* is large enough to neglect** $e^{-M_\pi L}$

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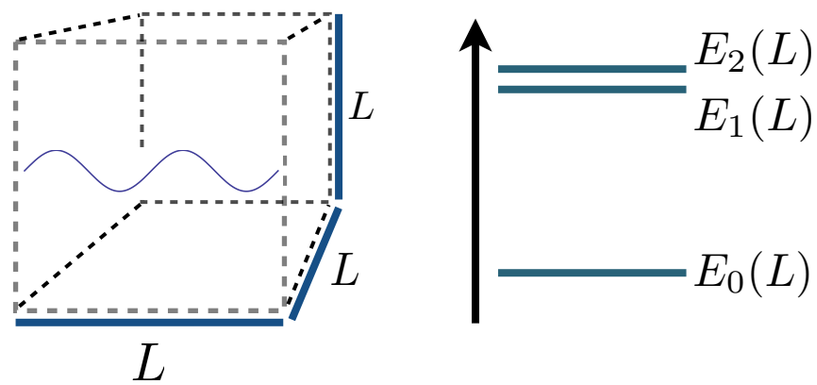

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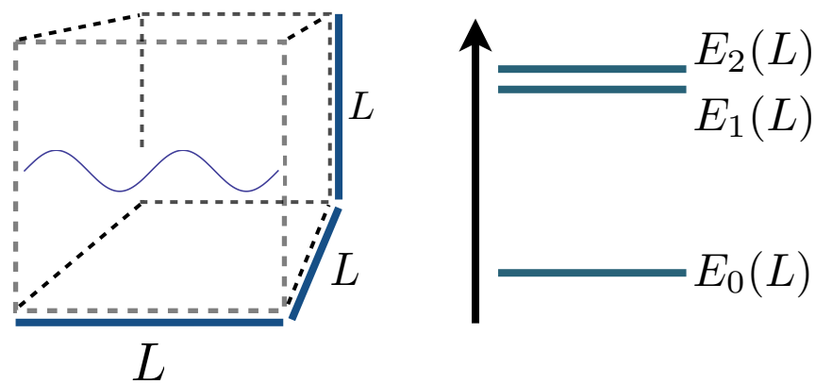

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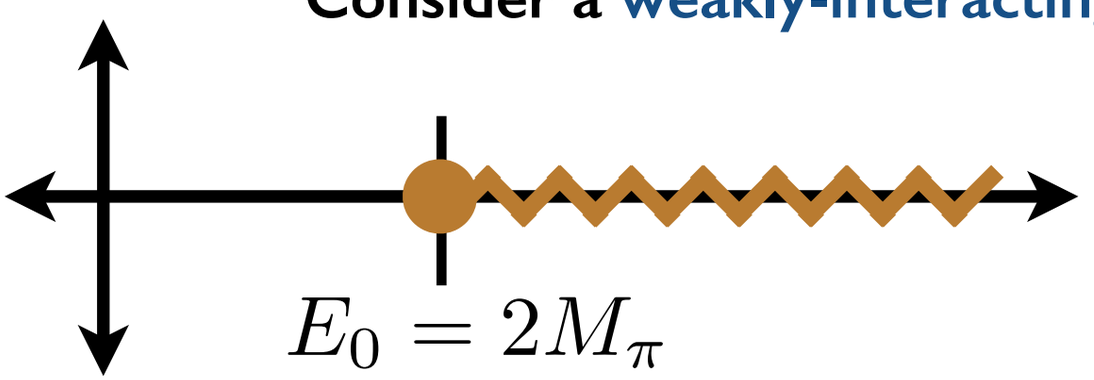

$$E_0 = 2M_\pi$$

Infinite-volume ground state

$$\mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a$$

Information is in the scattering amplitude

The finite-volume as a tool

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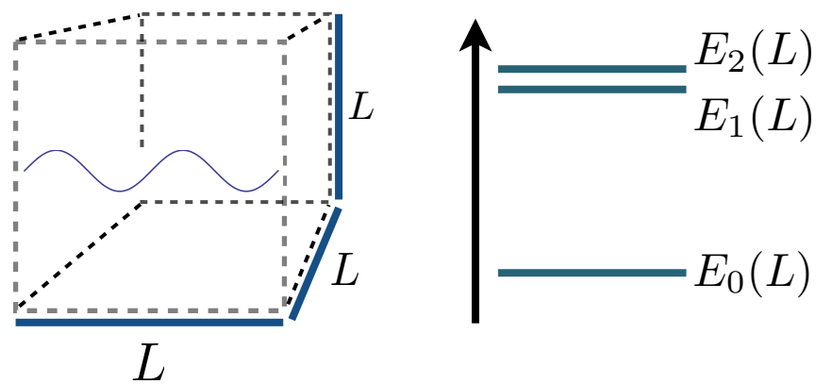

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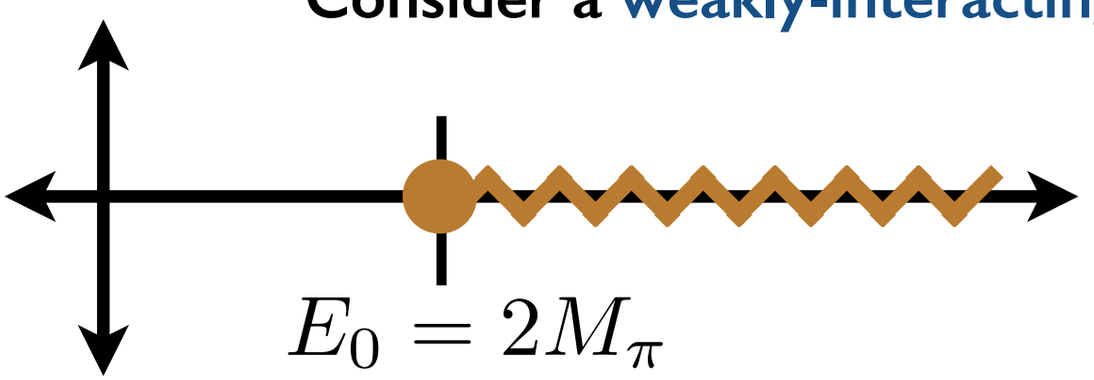

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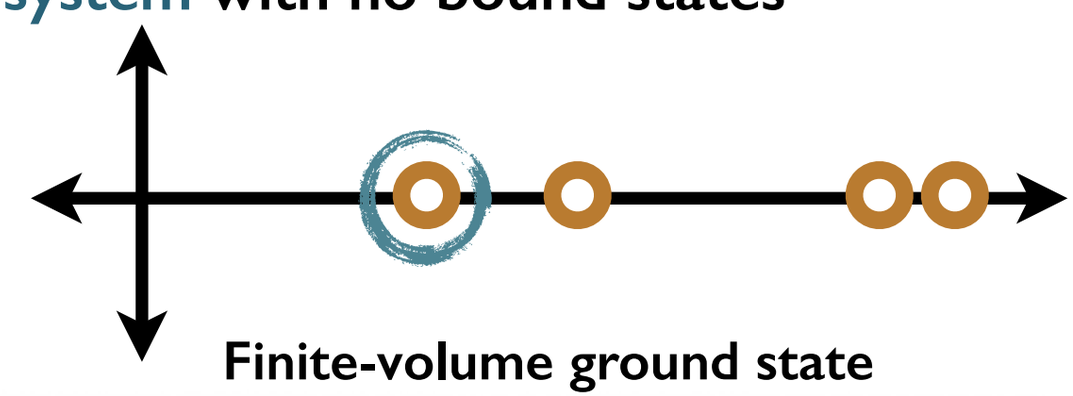

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Huang, Yang (1958)

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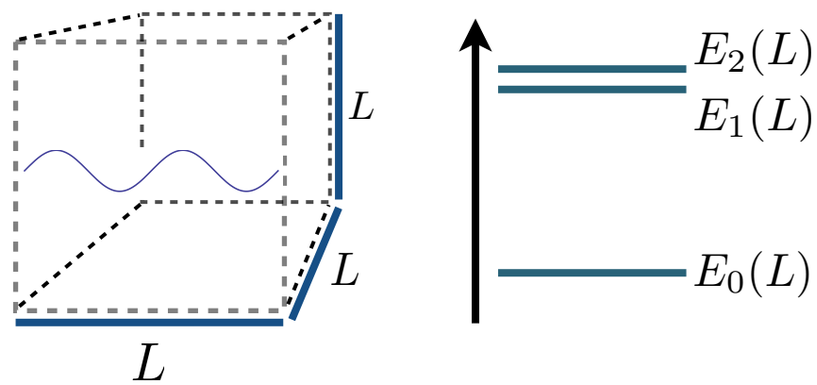

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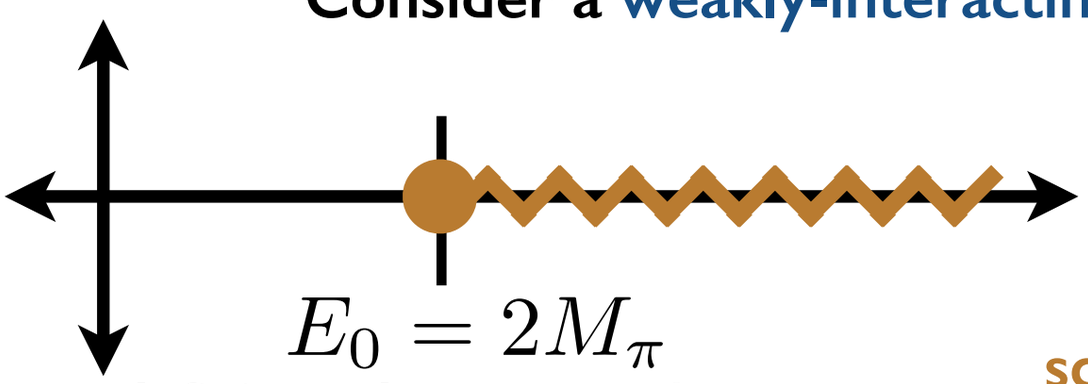

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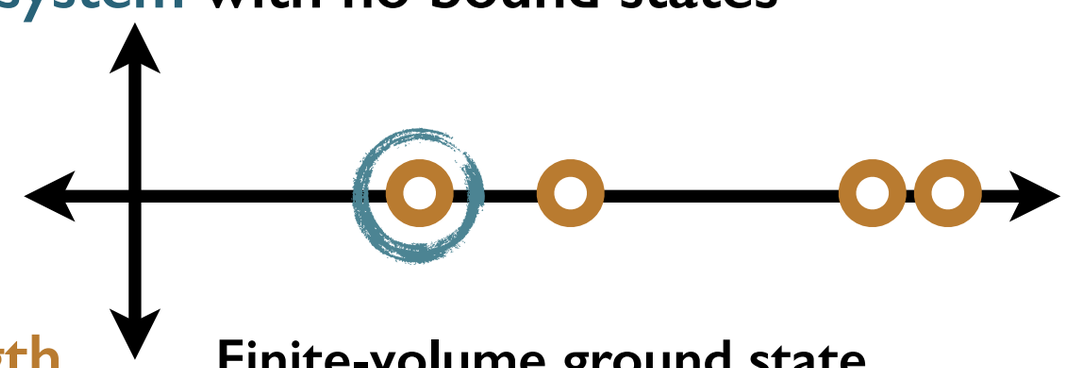

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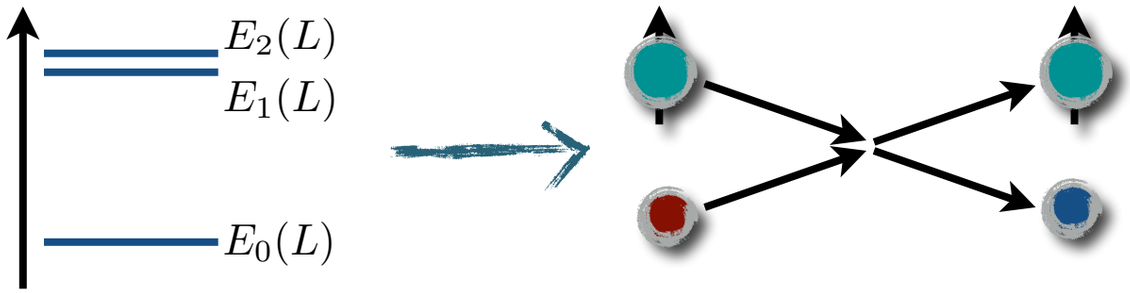
Huang, Yang (1958)

General two-to-two scattering

- **Lüscher's formalism + extensions give a general mapping**



- **All results are contained in a generalized quantization condition**

$$\det\left[\mathcal{M}_2^{-1}(E_n^*) + F(E_n, \vec{P}, L)\right] = 0$$

scattering amplitude **known geometric function**

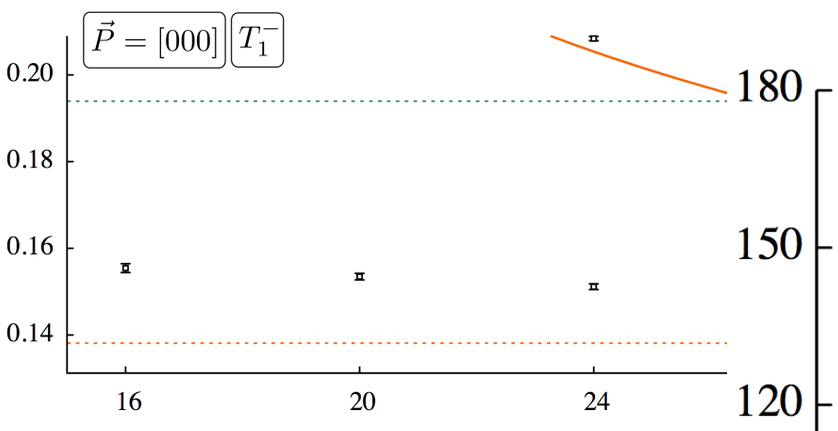
- **Matrices in angular momentum, spin and channel space**

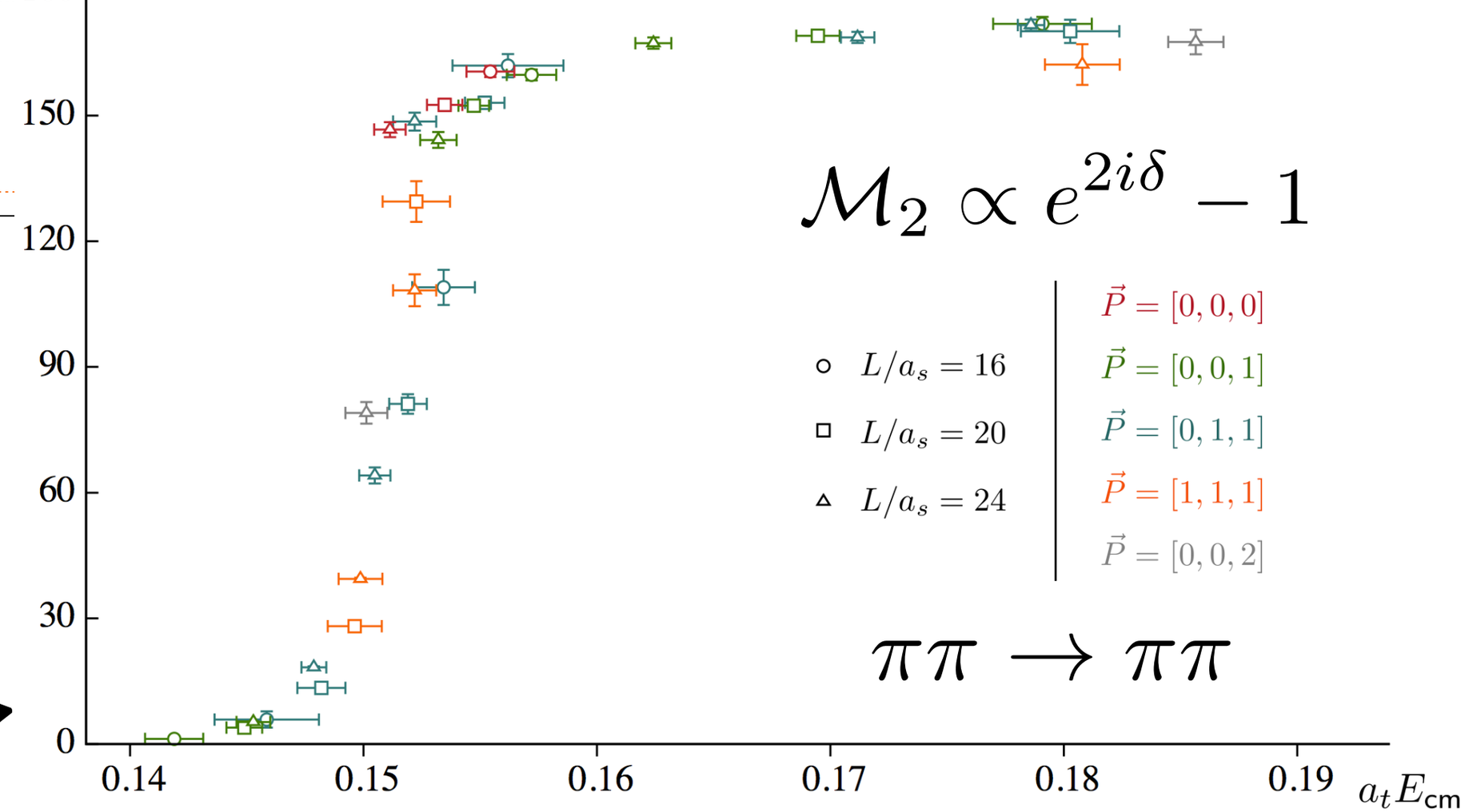
Huang, Yang (1958) • Lüscher (1986, 1991) • Rummukainen, Gottlieb (1995)
Kim, Sachrajda, Sharpe (2005) • Christ, Kim, Yamazaki (2005) • He, Feng, Liu (2005)
Beane, Detmold, Savage (2007) • Tan (2008) • Leskovec, Prelovsek (2012) • Bernard *et. al.* (2012)
MTH, Sharpe (2012) • Briceño, Davoudi (2012) • Li, Liu (2013) • Briceño (2014)

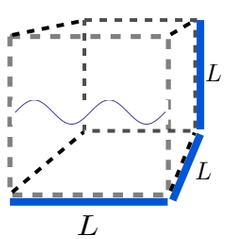
Using the result

- **Simplest case is a single channel (e.g. for pions in a p-wave the relation reduces to)**

$$\mathcal{M}_2(E_n^*) = -1/F(E_n, \vec{P}, L)$$







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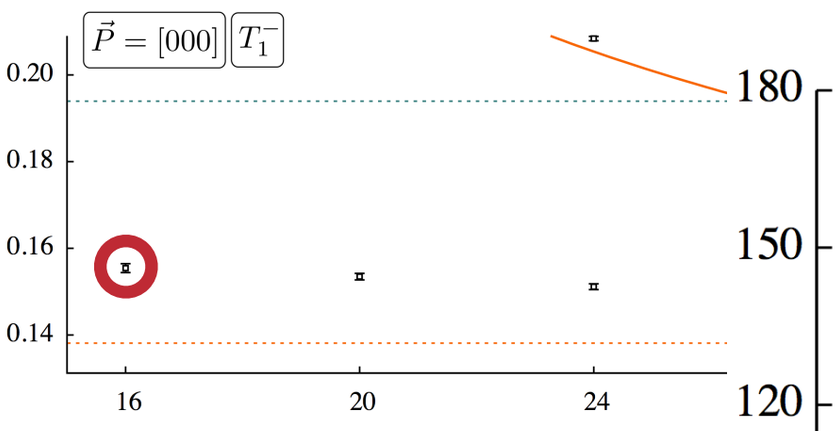

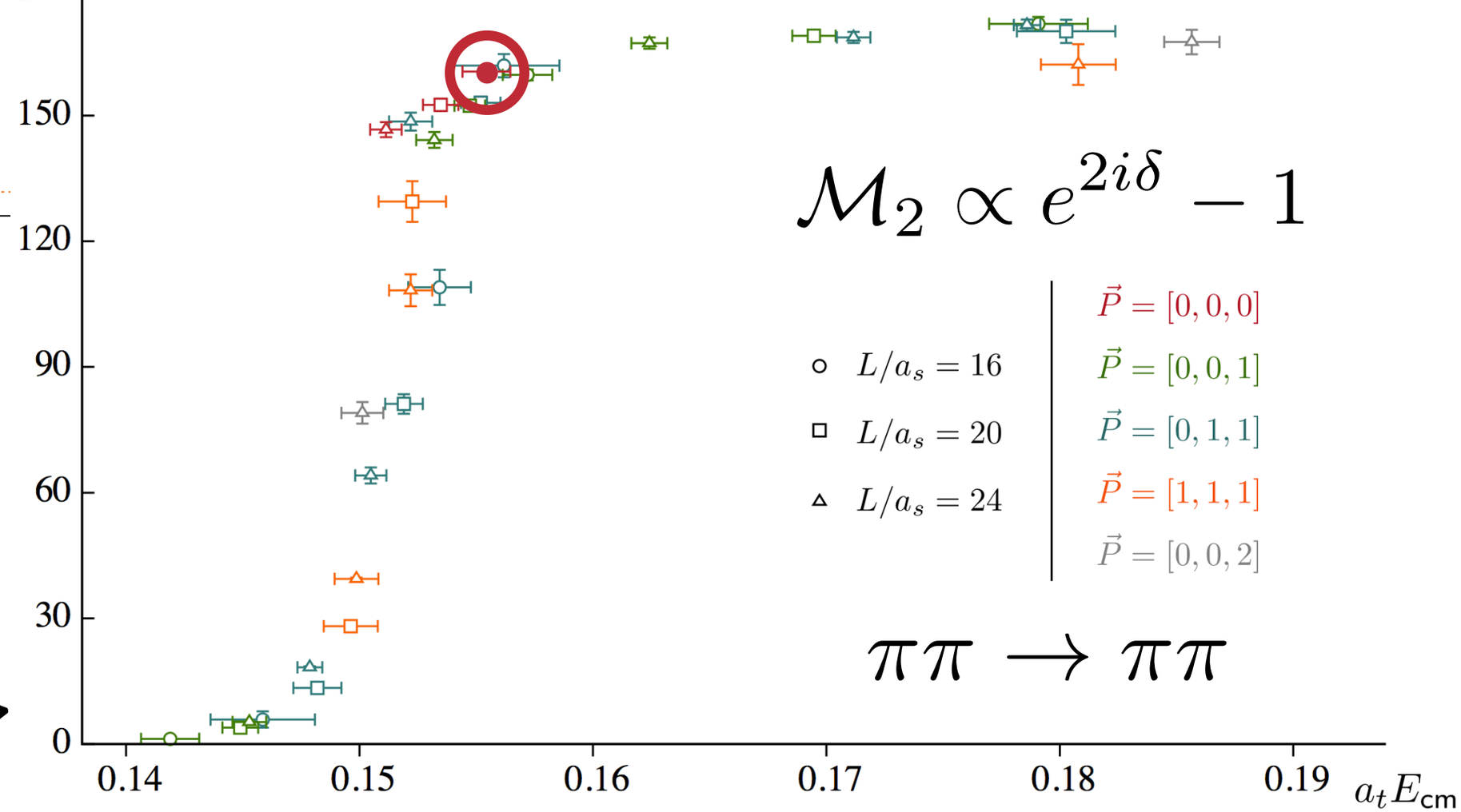

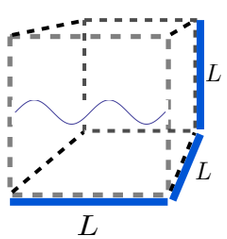

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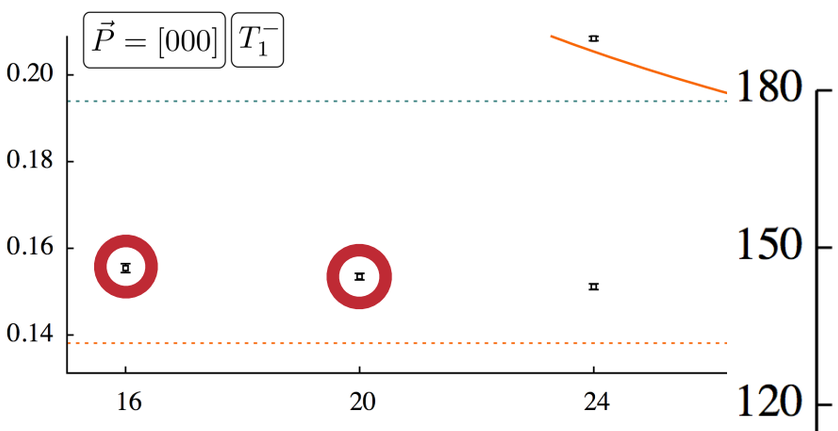

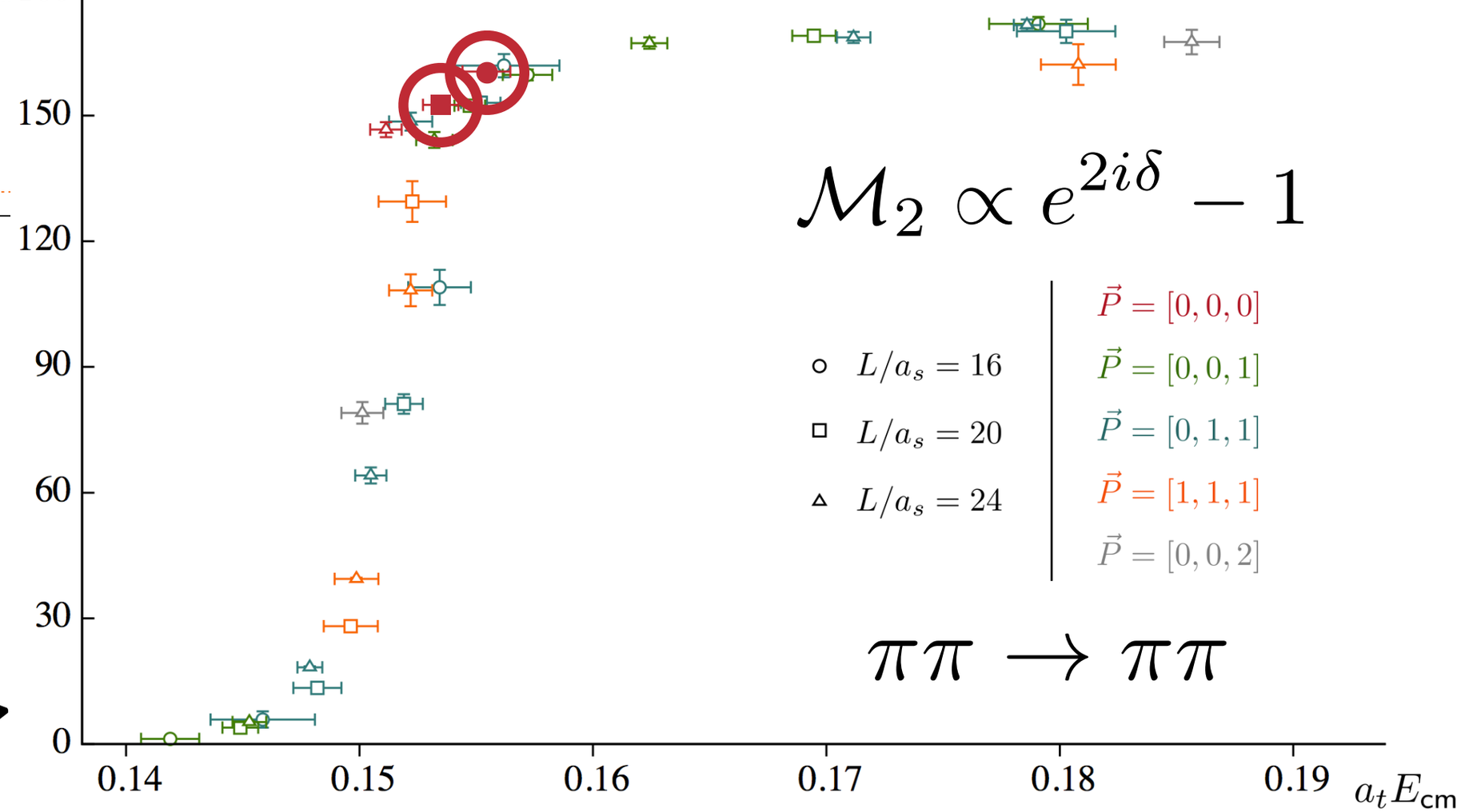

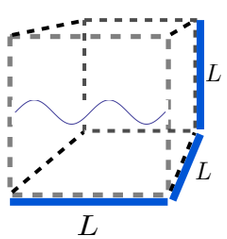

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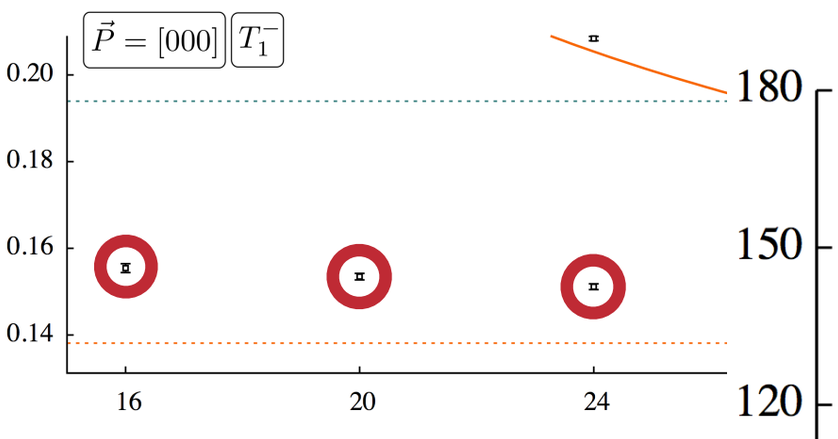

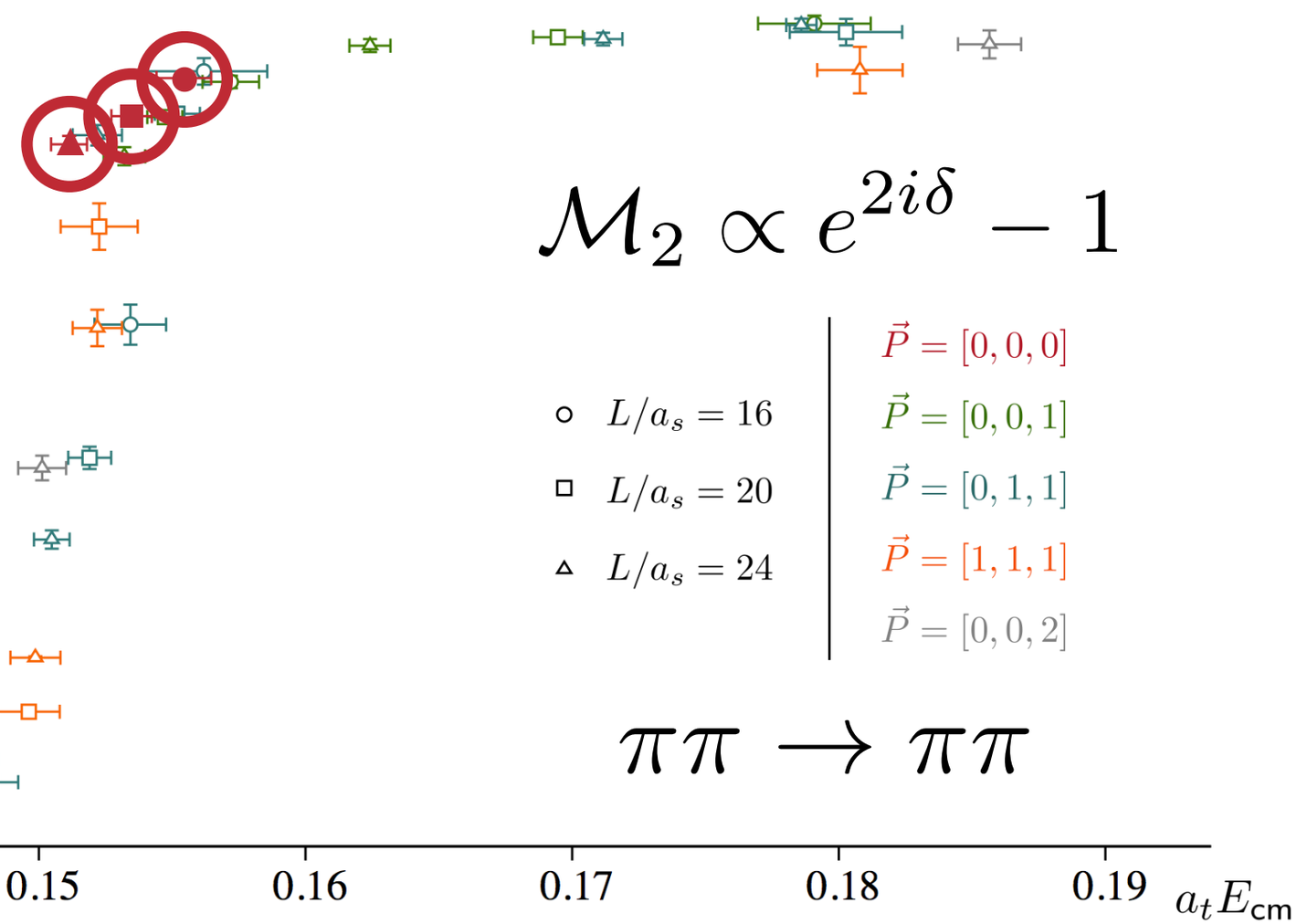

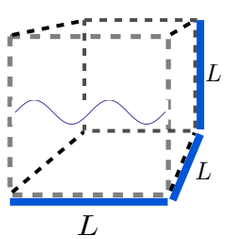

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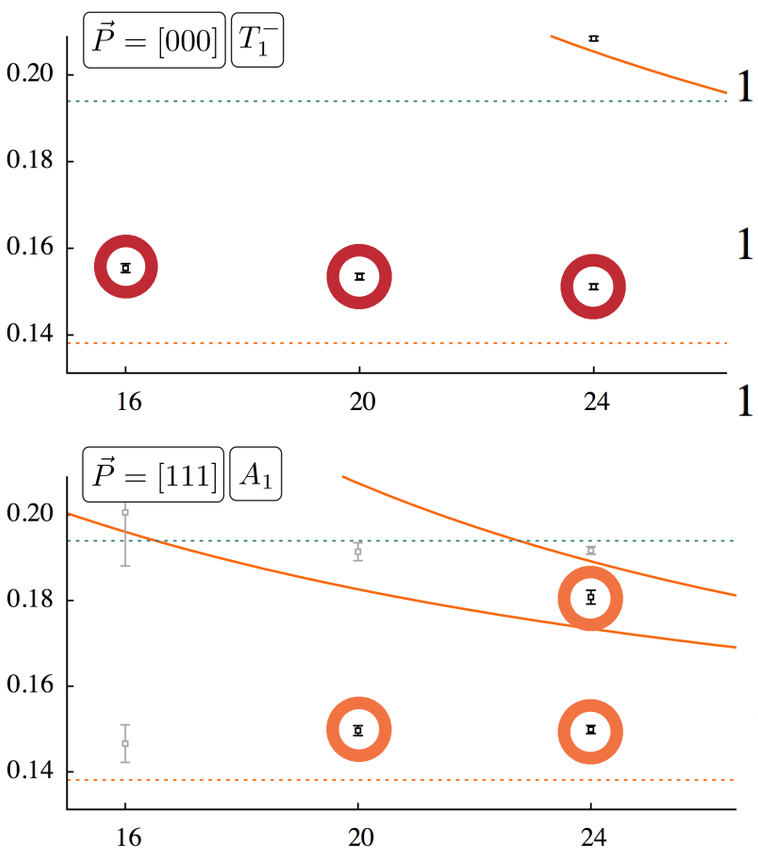

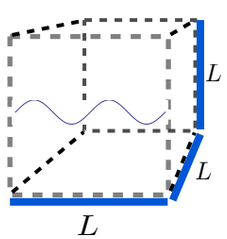

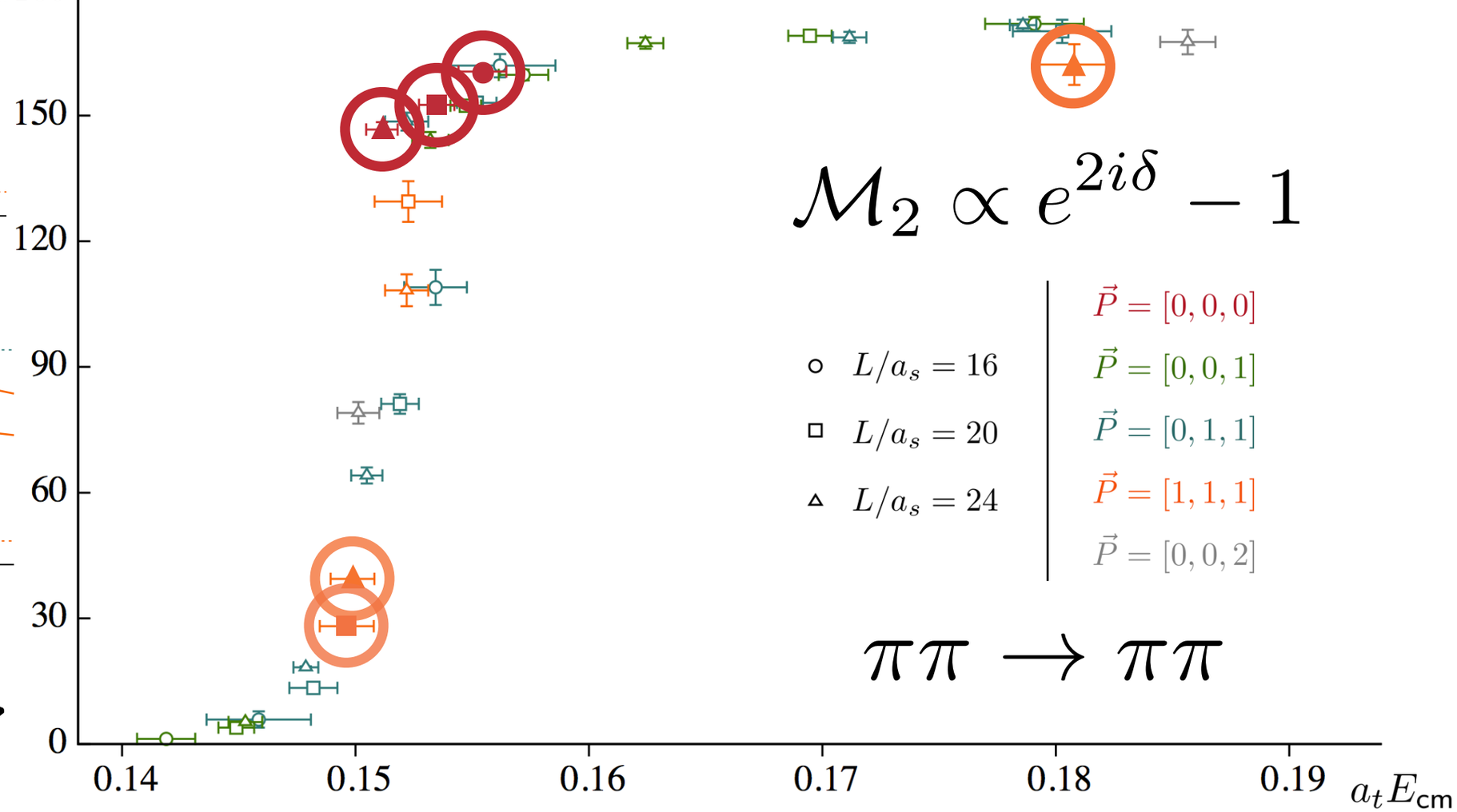

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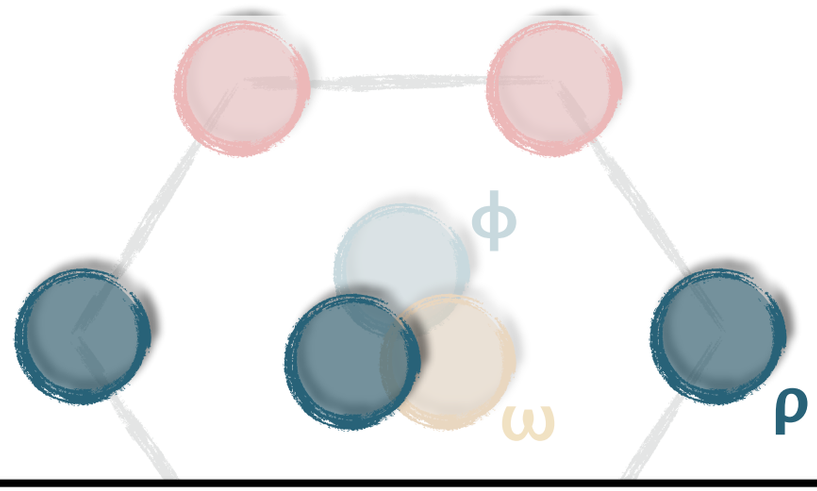
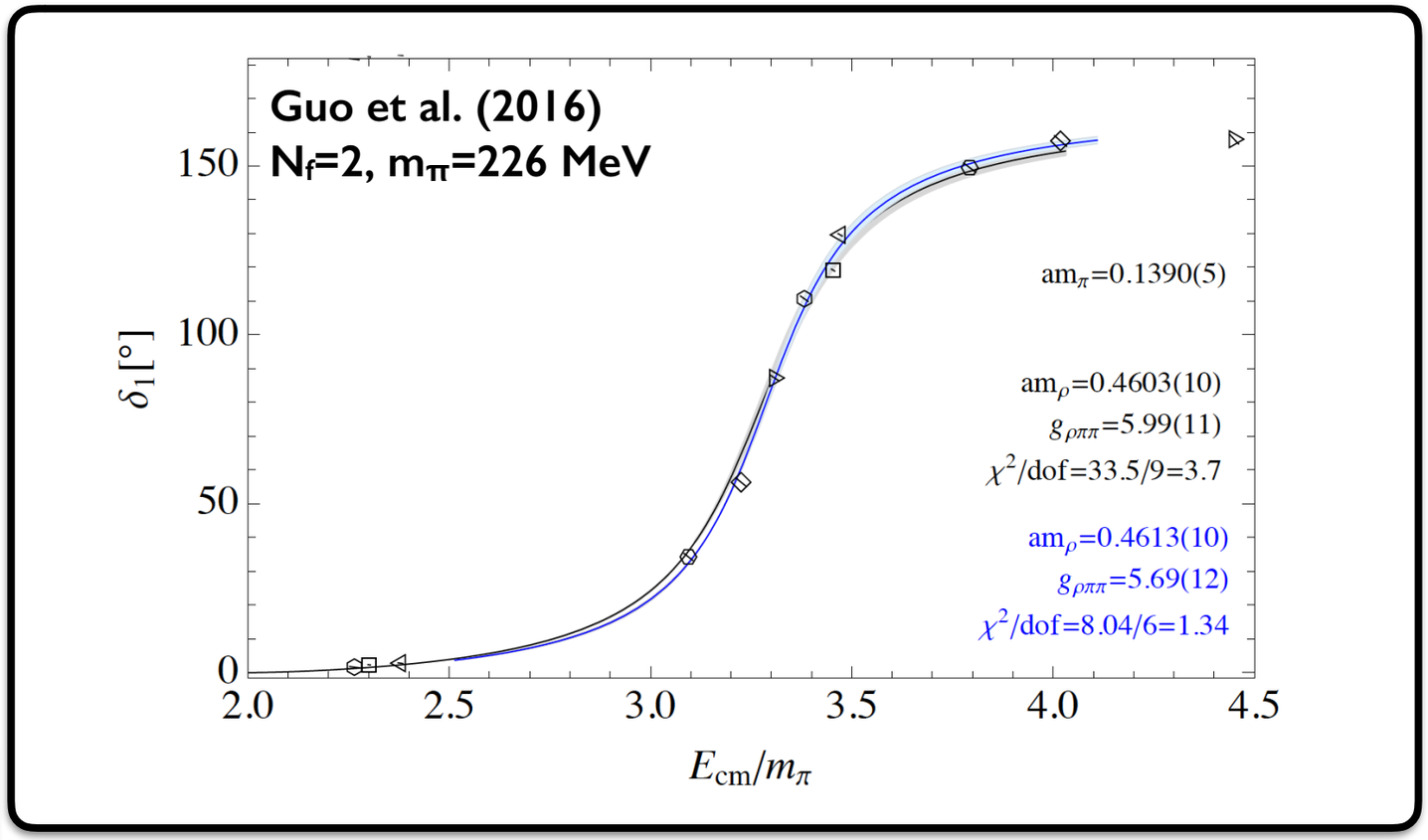
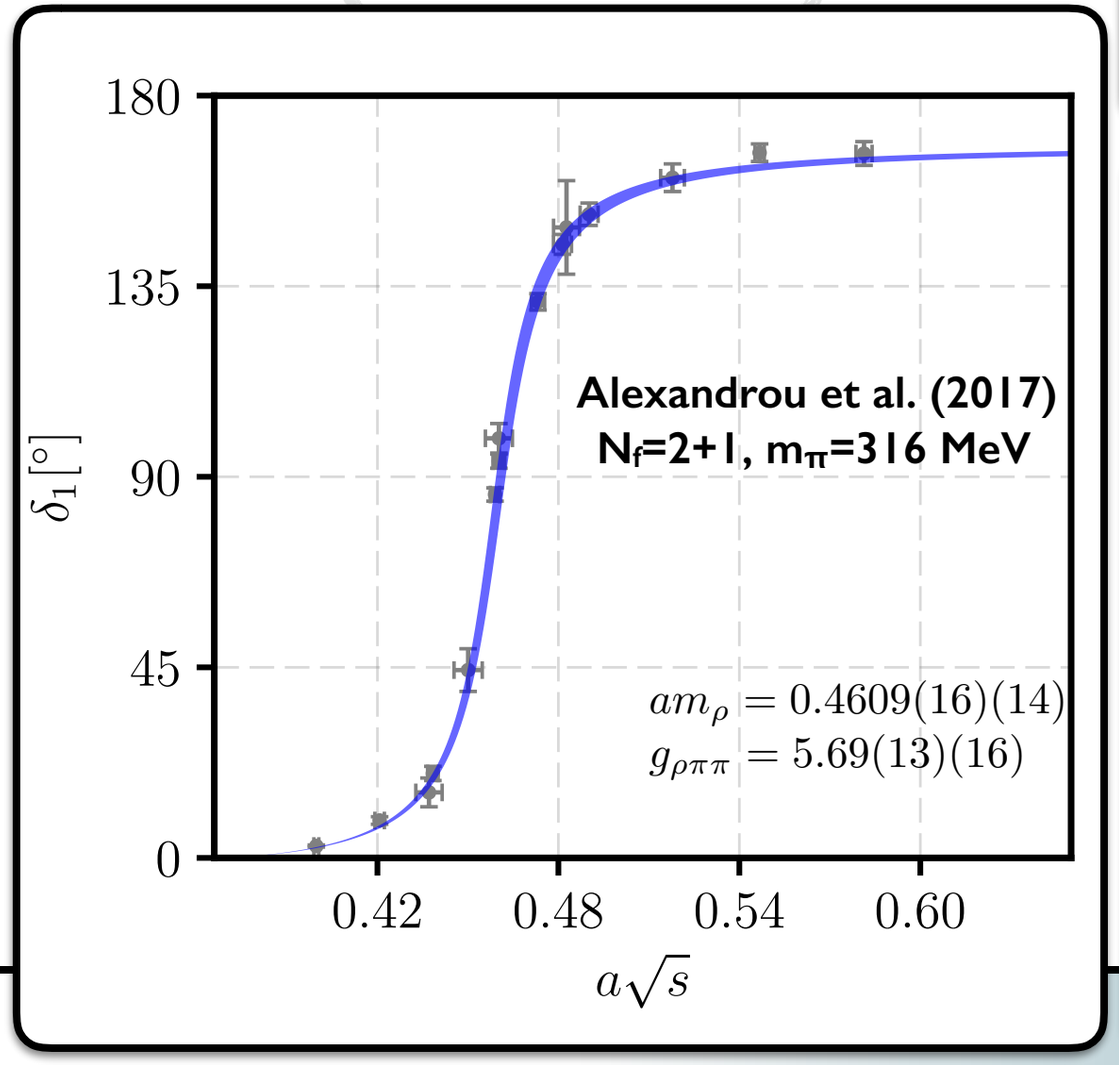
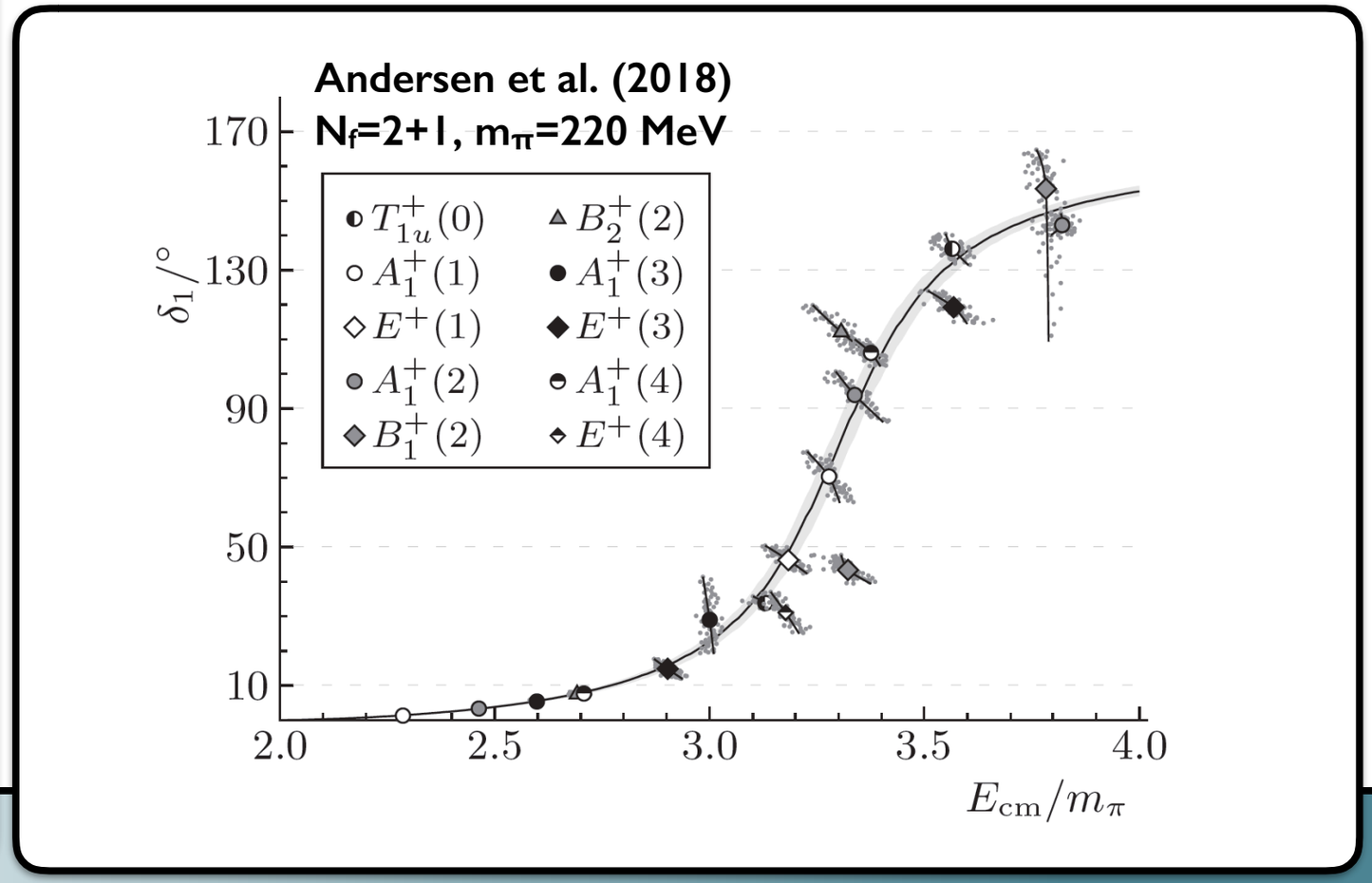
$$\rho \to \pi\pi$$

$$I^G(J^{PC}) = 1^+(1^{--})$$





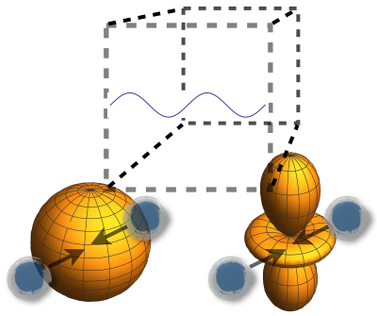




Coupled channels

- **The cubic volume mixes different partial waves…**

e.g. $K\pi \to K\pi$, $\vec{P} \neq 0$ $\longrightarrow$ $$\det\left[\begin{pmatrix}\mathcal{M}_s^{-1} & 0\\ 0 & \mathcal{M}_p^{-1}\end{pmatrix} + \begin{pmatrix}F_{ss} & F_{sp}\\ F_{ps} & F_{pp}\end{pmatrix}\right] = 0$$



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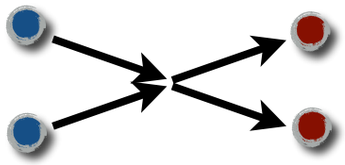

MTH, Sharpe (2012) • Briceño, Davoudi (2012)

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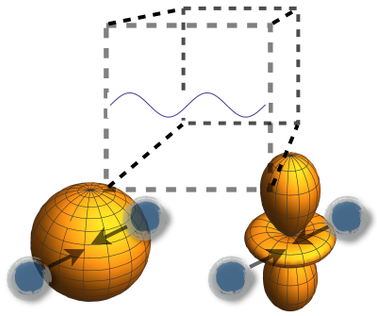

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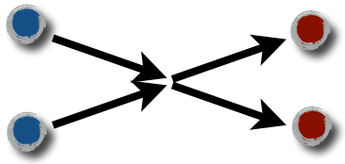


- **The road to physics…**

MTH, Sharpe (2012) • Briceño, Davoudi (2012)

Calculate a matrix of correlators with a large & varied operator basis

$$\langle \mathcal{O}_a(\tau)\mathcal{O}_b^\dagger(0)\rangle$$

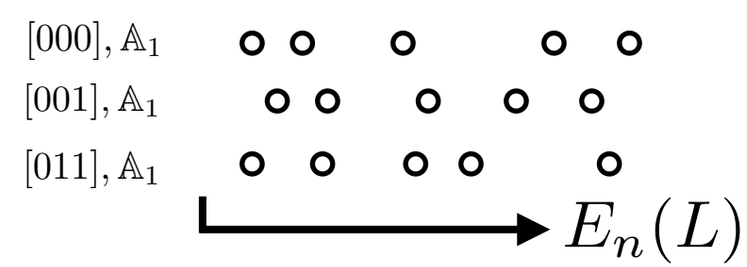
→

Diagonalize (GEVP) to reliably extract finite-volume energies

$$\langle \Omega_m(\tau)\Omega_m^\dagger(0)\rangle \sim e^{-E_m(L)\tau}$$

→

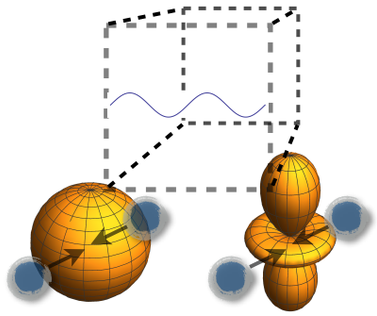
Vary *L* and *P* to recover a dense set of energies



Coupled channels

- **The cubic volume mixes different partial waves…**

e.g. $K\pi \to K\pi$, $\vec{P} \neq 0$ $\longrightarrow$ $\det\left[\begin{pmatrix}\mathcal{M}_s^{-1} & 0\\ 0 & \mathcal{M}_p^{-1}\end{pmatrix} + \begin{pmatrix}F_{ss} & F_{sp}\\ F_{ps} & F_{pp}\end{pmatrix}\right] = 0$



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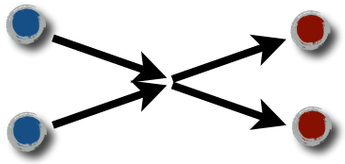

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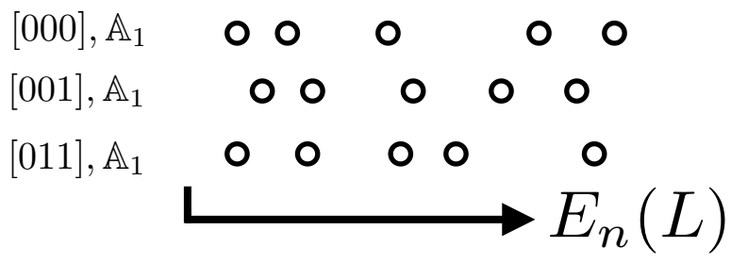

Identify a broad list of K-matrix parametrizations

polynomials and poles

EFT based

dispersion theory based

Coupled channels

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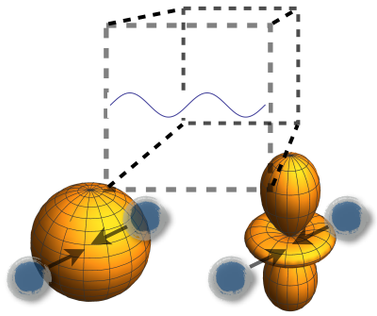

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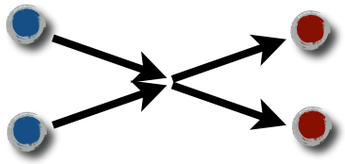

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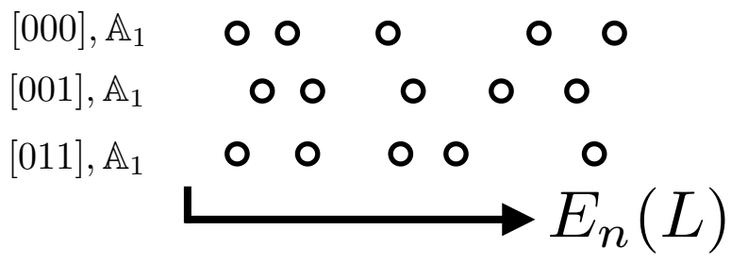


↓

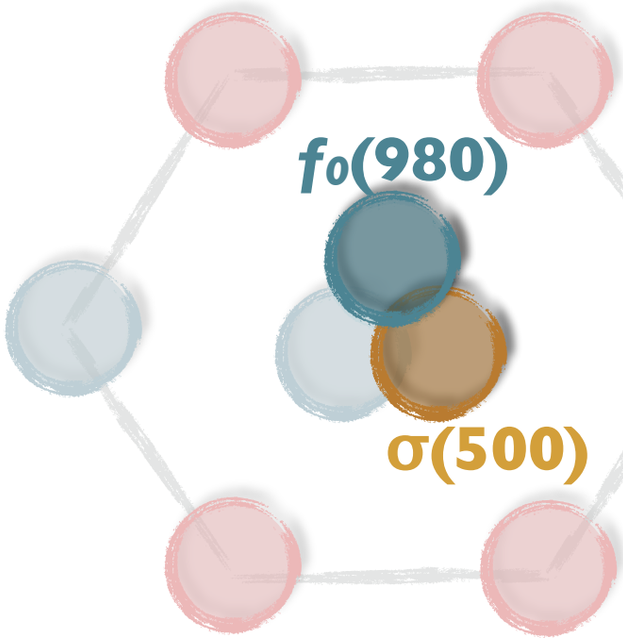
Identify a broad list of K-matrix parametrizations

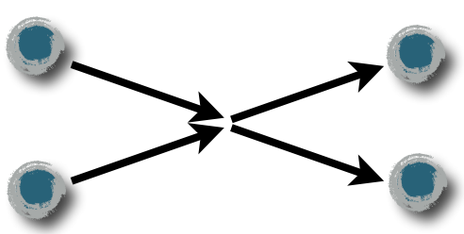
polynomials and poles | EFT based | dispersion theory based

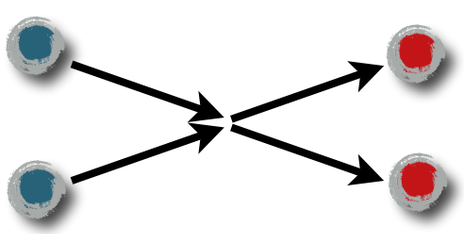
→

Perform global fits to the finite-volume spectrum

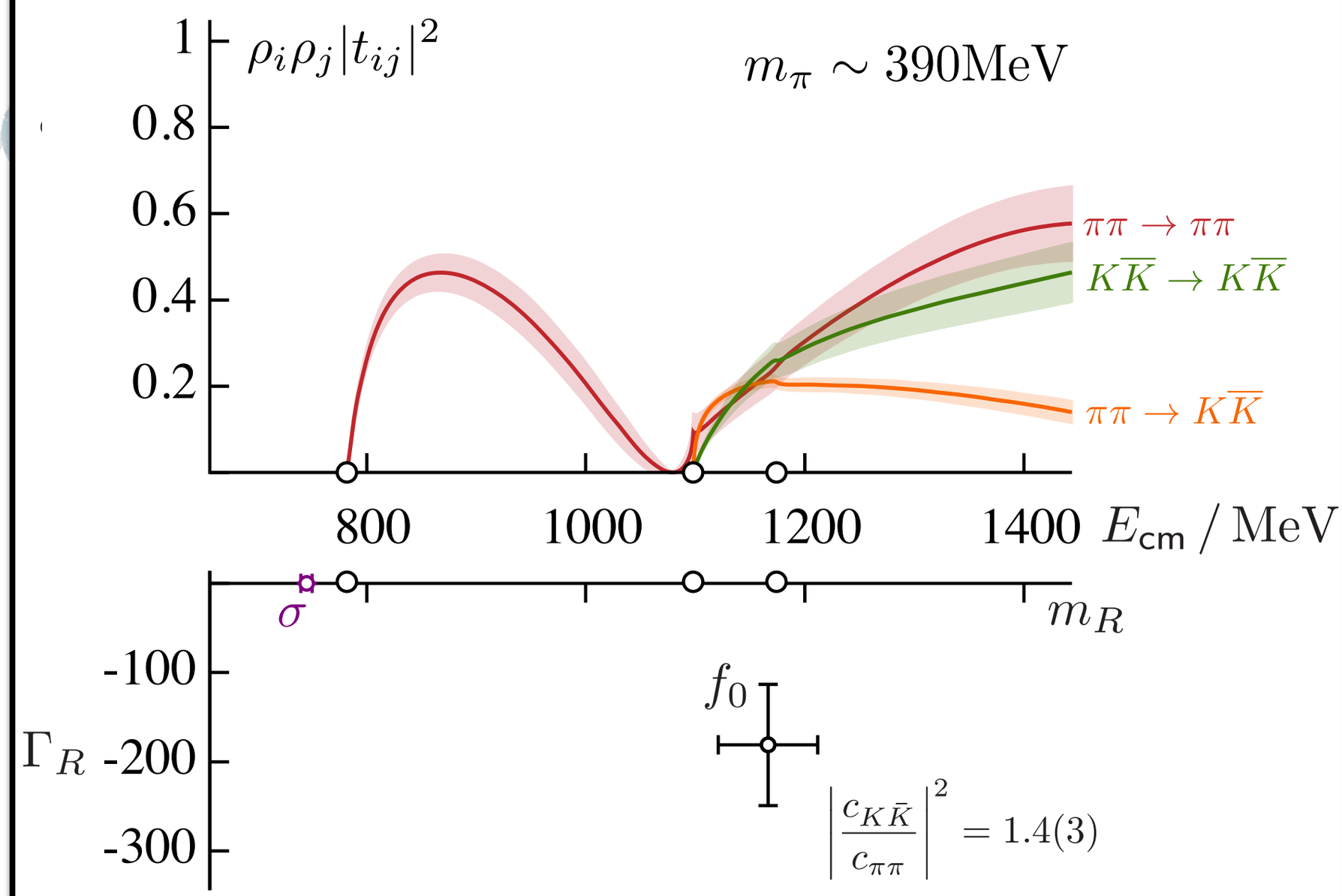
$$I^G(J^{PC}) = 0^+(0^{++})$$







Coupled-channel scattering

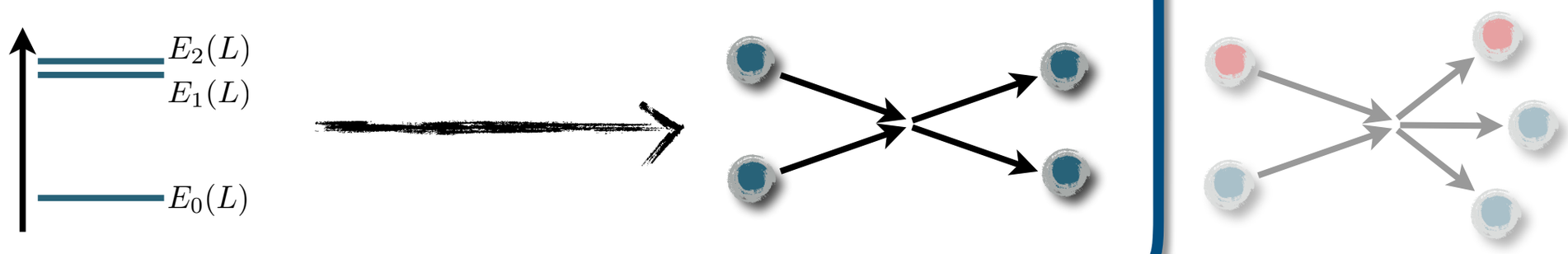


Briceño, Dudek, Edwards & Wilson (2017)

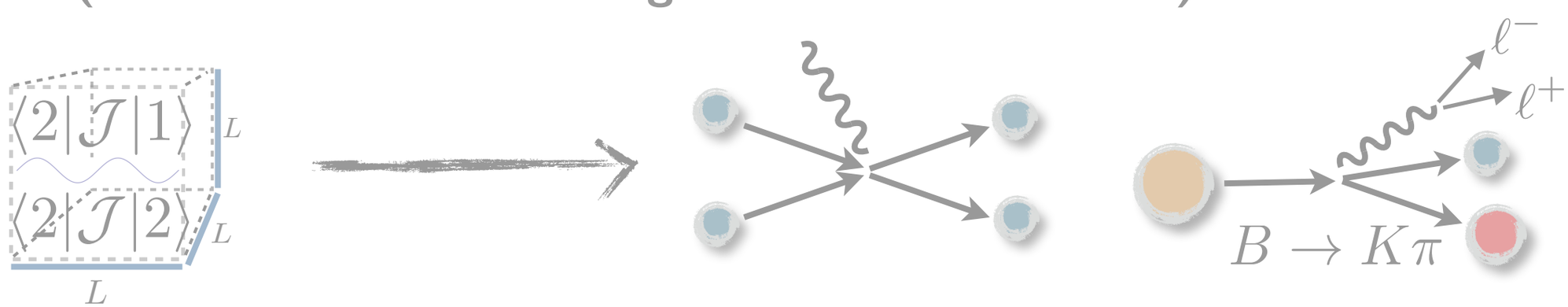
Multi-hadron processes from LQCD

KEY IDEA: We can use the finite volume as a **tool** to extract multi-hadron observables

☑ **Scattering (from finite-volume energies)**



☐ **Transitions (from finite-volume energies + matrix elements)**



Multi-hadron processes from LQCD

KEY IDEA: **We can use the finite volume as a tool to extract multi-hadron observables**

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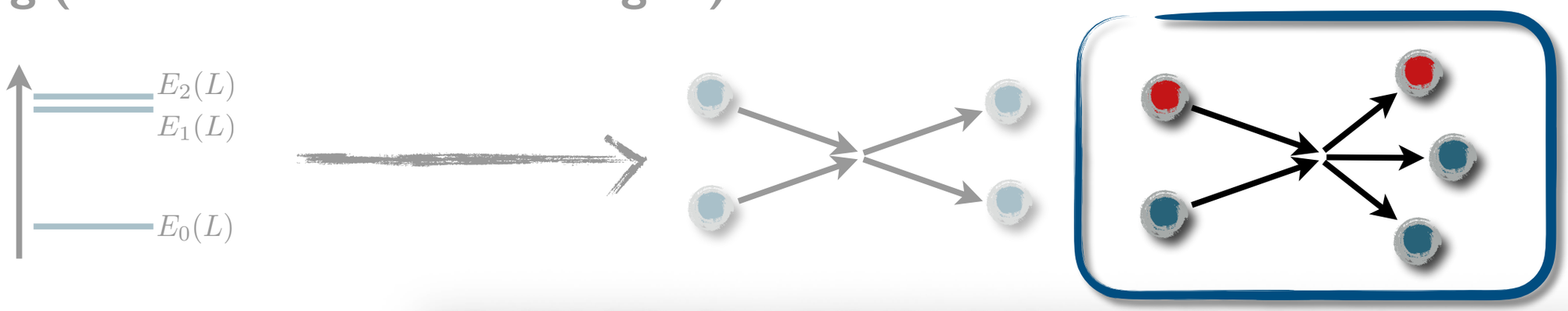
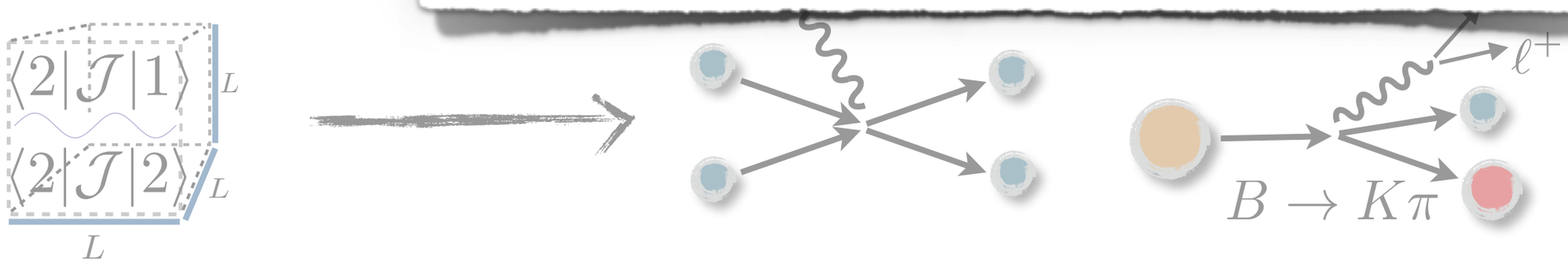


☐ Transitions (from finite-vol

Recent Review: ***Lattice QCD and Three-particle Decays of Resonances***
MTH & Sharpe *[1901.00483]*



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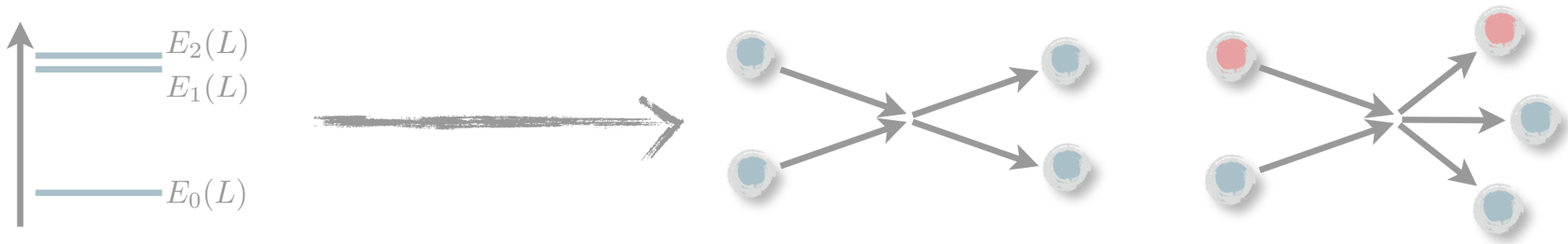
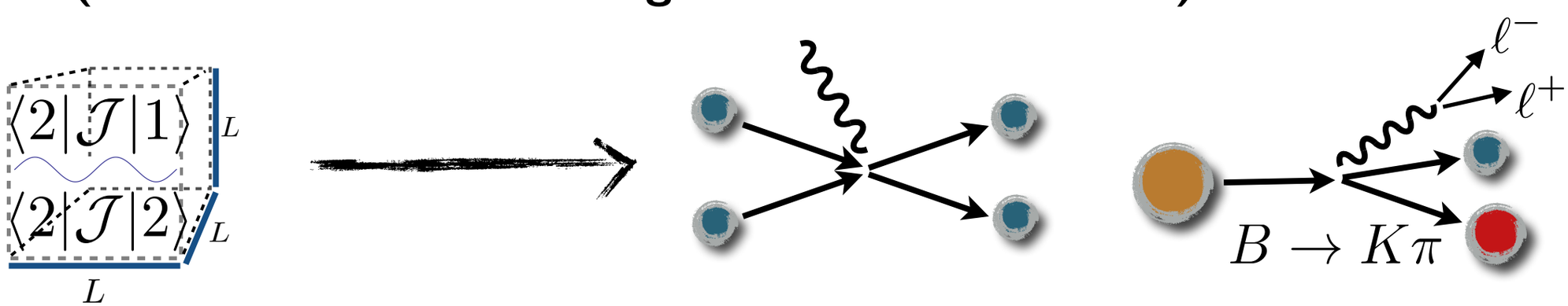
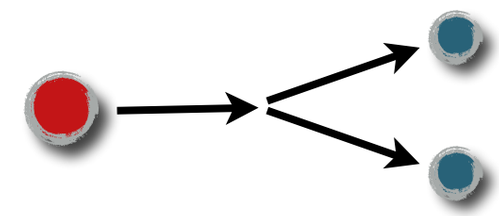


☐ **Transitions (from finite-volume energies + matrix elements)**

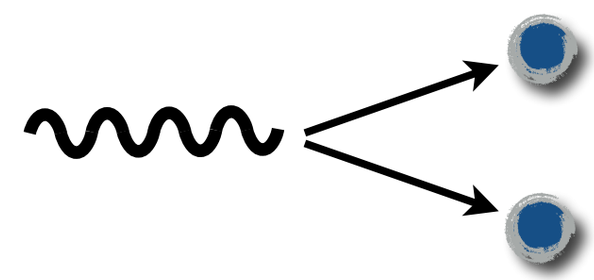


Multi-hadron matrix elements

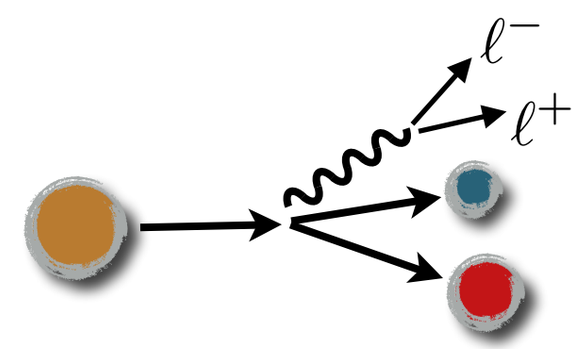
- **Theoretical method established for many observables**

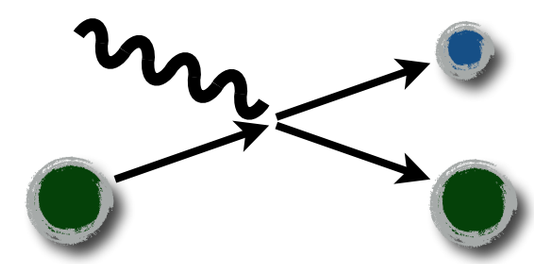
  - **Weak decay** $\langle \pi\pi, \mathrm{out} | \mathcal{H} | K \rangle \equiv$ 

Lellouch, Lüscher *(2001)* • Kim, Sachrajda, Sharpe *(2005)* • Christ, Kim, Yamazaki *(2005)* • MTH, Sharpe *(2012)*

  - **Time-like form factors** $\langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu | 0 \rangle \equiv$ 

Meyer *(2011)*

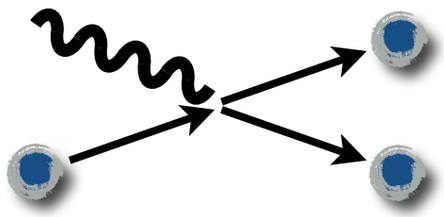
  - **Resonance form factors** $\langle K\pi, \mathrm{out} | \mathcal{J}_{\alpha\beta} | B \rangle \equiv$ 

  - **Particles with spin** $\langle N\pi, \mathrm{out} | \mathcal{J}_\mu | N \rangle \equiv$ 

Agadjanov *et al. (2014)* • Briceño, MTH, Walker-Loud *(2015)* • Briceño, MTH *(2016)*

Pion photo-production

$\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle \equiv$



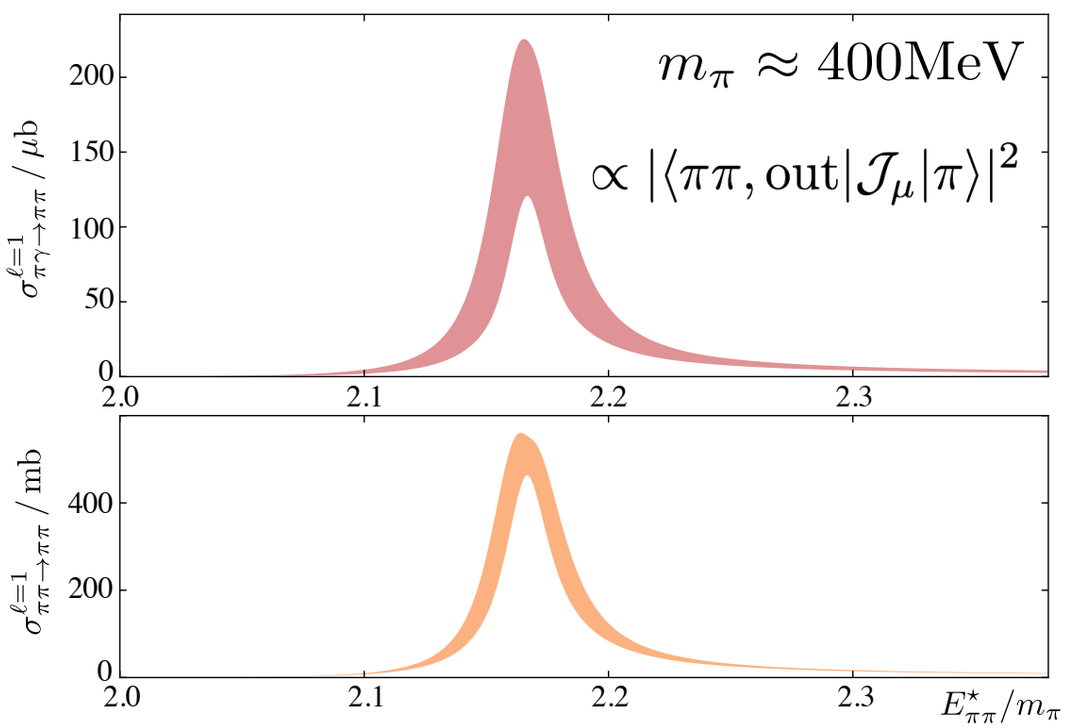
❑ Formal relation

get this from the lattice — **experimental observable**

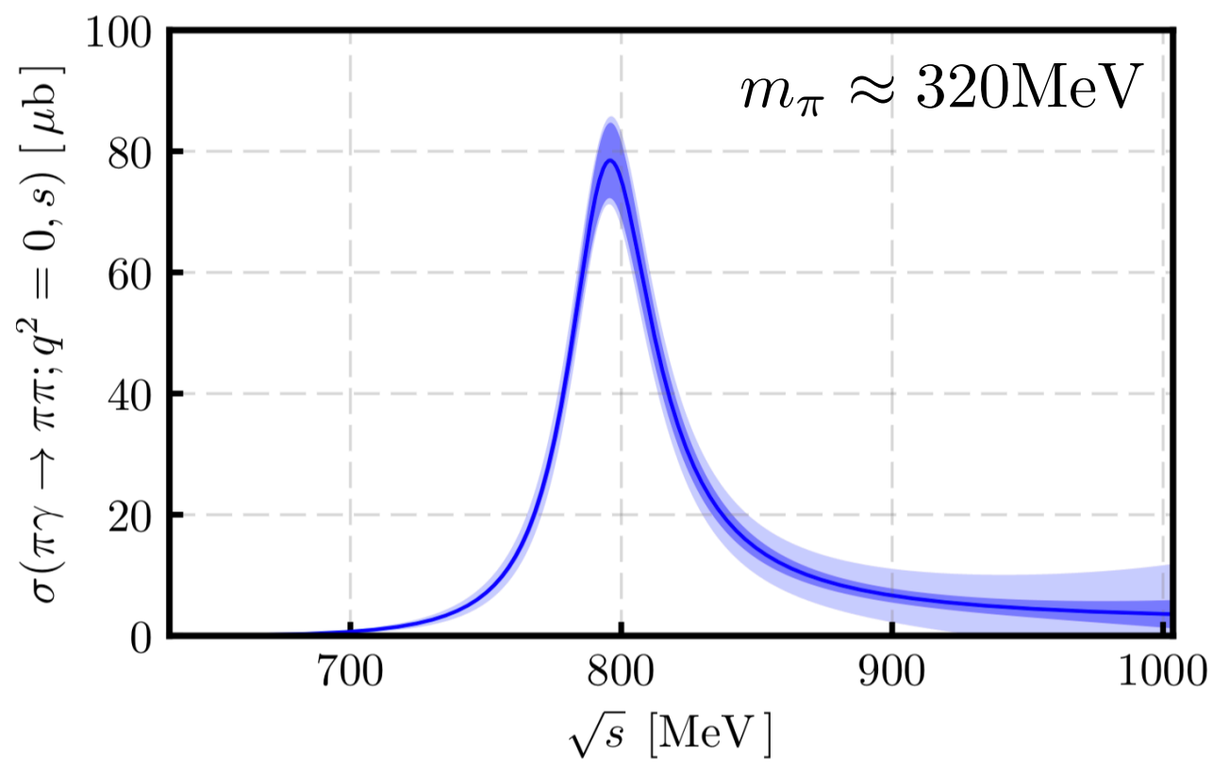
$$|\langle n, L|\mathcal{J}_\mu|\pi\rangle|^2 = \langle \pi|\mathcal{J}_\mu|\pi\pi, \mathrm{in}\rangle \mathcal{R}(E_n, L) \langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle$$

Briceño, MTH, Walker-Loud (2015)

❑ Numerical implementation

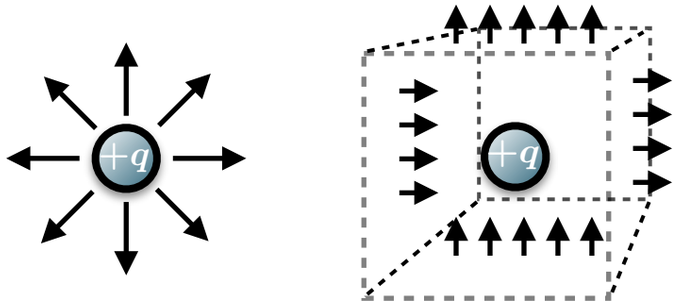


Briceño et. al., Phys. Rev. D93, 114508 (2016)



Alexandrou et. al., Phys. Rev. D98, 074502 (2018)

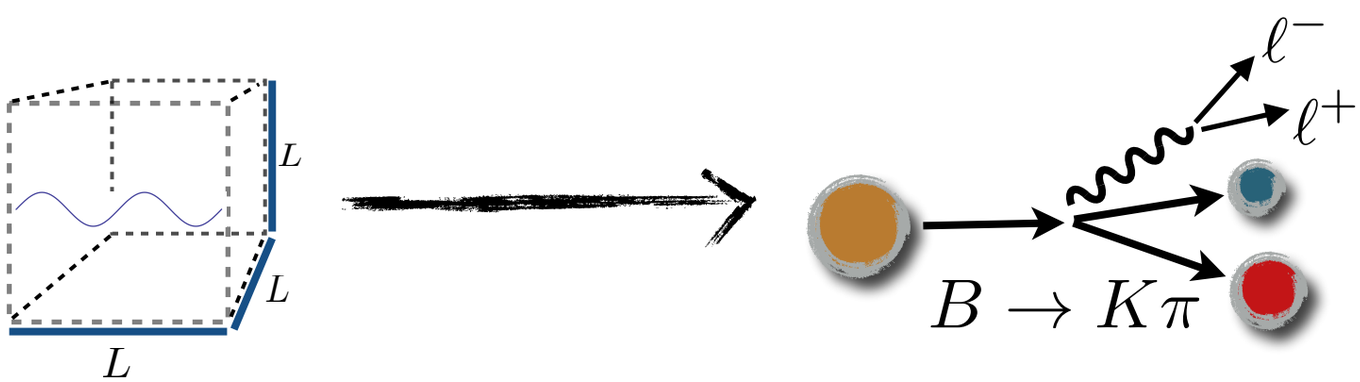
Take home messages…

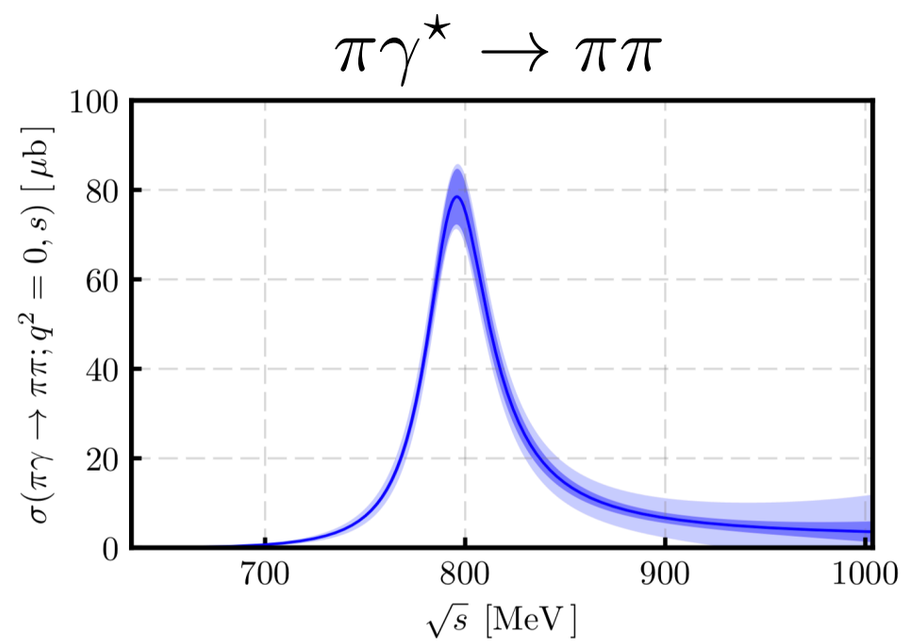


- **Single hadron states**
  - Decay constants = competitive with experiment
  - Mixing matrix elements = less competitive
  - Sub-percent uncertainty → **QED**

FLAG 2019
Flavour Lattice Averaging Group

- **Multi-hadron states**
  - **Resonance form-factor** calculations beginning
  - $B \to K\pi\ell^+\ell^-$ underway *(Meinel et al.)*





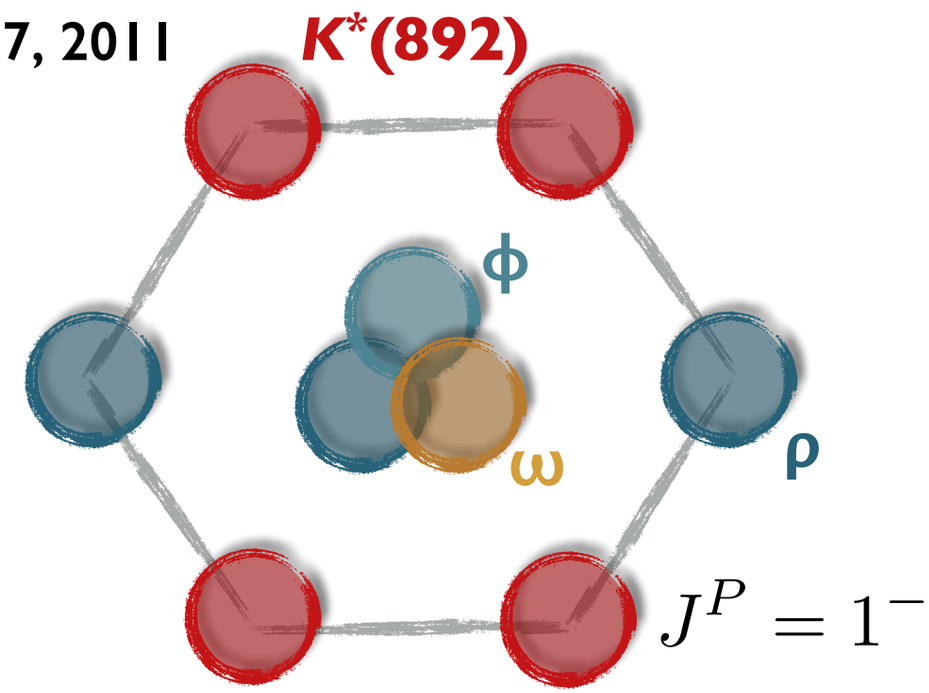
Thanks!

Backup Slides

Lots of activity…

$$\rho \to \pi\pi$$

- CP-PACS/PACS-CS 2007, 2011
- ETMC 2010
- Lang et al. 2011
- HadSpec 2012, 2016
- Pellisier 2012
- RQCD 2015
- Guo et al. 2016
- Fu et al. 2016
- Bulava et al. 2016
- Alexandrou et al. 2017

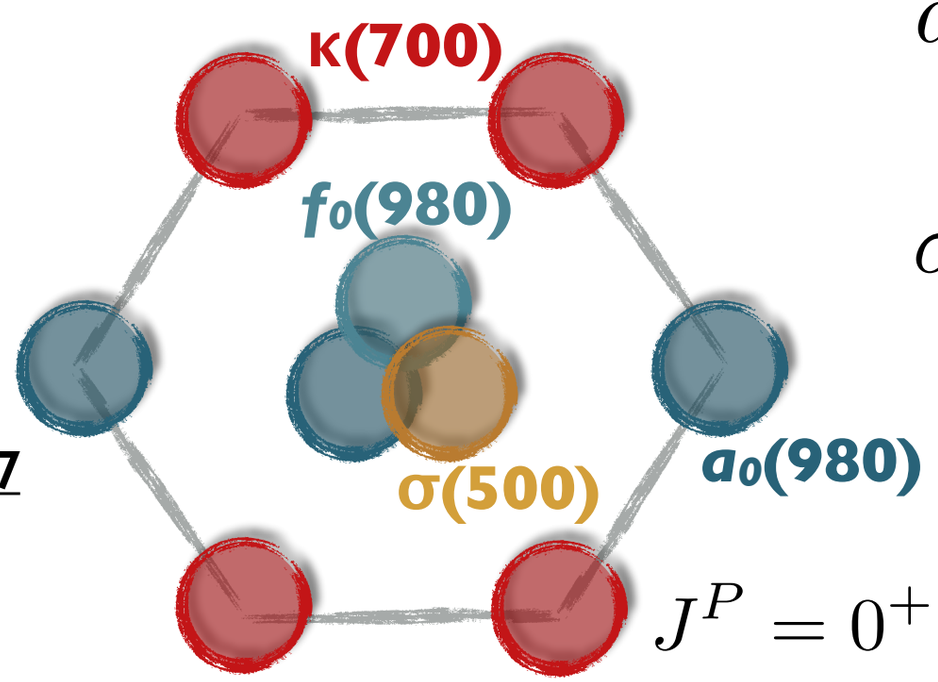


$$\kappa \to K\pi$$
$$K^* \to K\pi$$

- Lang et al. 2012
- Prelovsek et al. 2013
- Wilson et al. 2015
- RQCD 2015
- Brett et al. 2018

$$\sigma \to \pi\pi$$

- Prelovsek et al. 2010
- Fu 2013
- Wakayama 2015
- Howarth and Giedt 2017
- Briceño et al. 2017



$$a_0(980) \to \pi\eta, K\overline{K}$$

- Dudek et al. 2016

$$\sigma, f_0, f_2 \to \pi\pi, K\overline{K}, \eta\eta$$

- Briceño et al. 2017

See the recent review by Briceño, Dudek and Young

Weak decays…

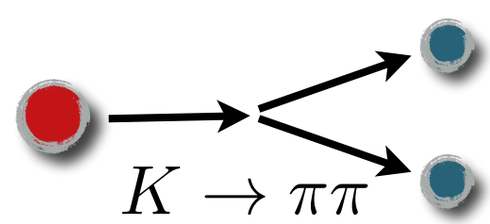
get this from the lattice — experimental observable

$$|\langle n, L|\mathcal{H}_W|K\rangle|^2 = \mathcal{R}(E_n, L)|\langle \pi\pi, \mathrm{out}|\mathcal{H}_W|K\rangle|^2$$

depends on scattering phase shift

Lellouch, Lüscher (2001)

- ☐ **Three steps to lattice weak decay**
  - ☐ Calculate finite-volume energies → **ππ** scattering phase → $R(E_n, L)$
  - ☐ Calculate renormalized finite-volume matrix elements
  - ☐ Combine $R(E_n, L)$ with f.v. matrix elements → **decay amplitudes**



- ☑ **Complete numerical calculation by RBC/UKQCD**

RBC/UKQCD, e.g. PRL 2015, ***(1505.07863)***

D decays…

get this from the lattice

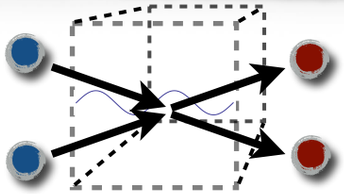
experimental observables

$$\langle n, L|\mathcal{H}_W|D\rangle = \begin{pmatrix} \mathcal{C}_{\pi\pi} & \mathcal{C}_{K\overline{K}} & \mathcal{C}_{\eta\eta} \end{pmatrix} \begin{pmatrix} \langle \pi\pi, \text{out}|\mathcal{H}_W|D\rangle \\ \langle K\overline{K}, \text{out}|\mathcal{H}_W|D\rangle \\ \langle \eta\eta, \text{out}|\mathcal{H}_W|D\rangle \end{pmatrix}$$

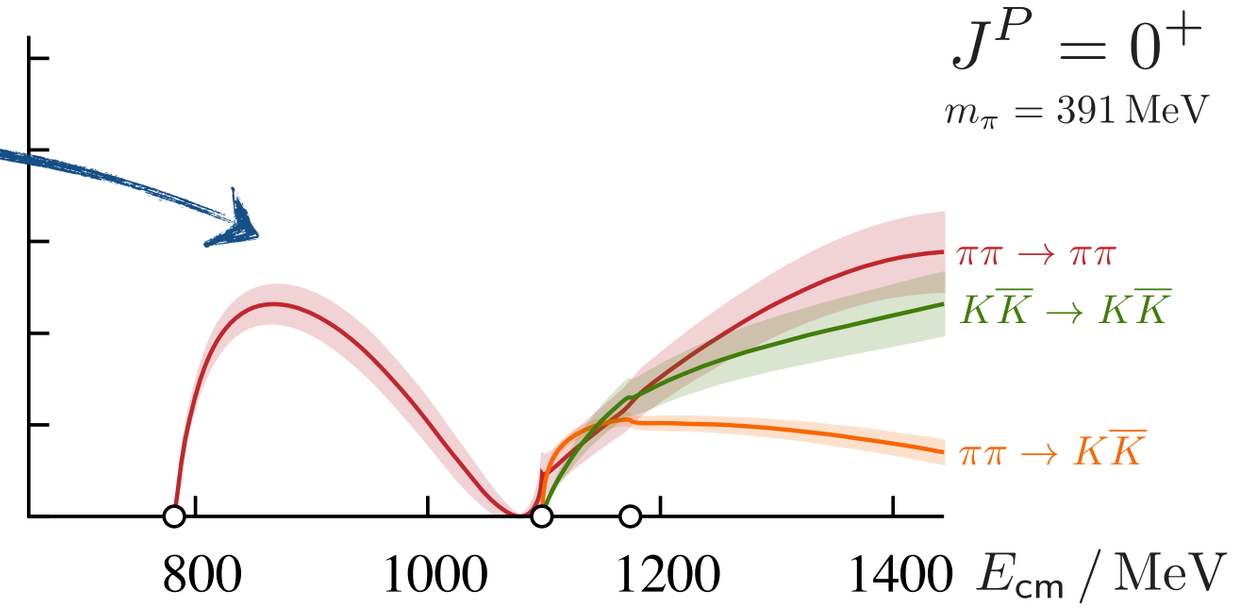
depends on scattering matrix

MTH, Sharpe (2012)

- **Coupled channels mix in the finite volume**



- **Three steps to D decays**
  - **Calculate *many* finite-volume energies → coupled scattering → C_{xy}**
  - **Calculate *many* renormalized finite-volume matrix elements**
  - **Extract amplitudes in a global fit**



- **Important caveat: The relation ignores ππππ states**

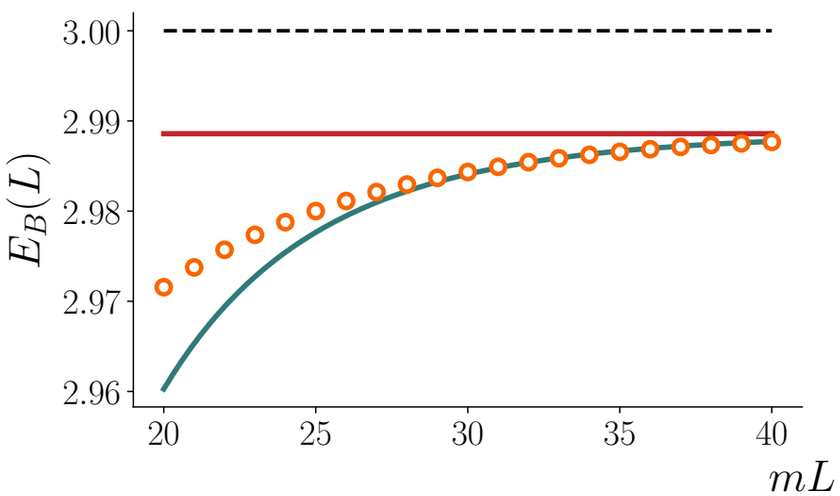
Three-hadron scattering

- **Formalism is complete for two and three (identical) scalars**

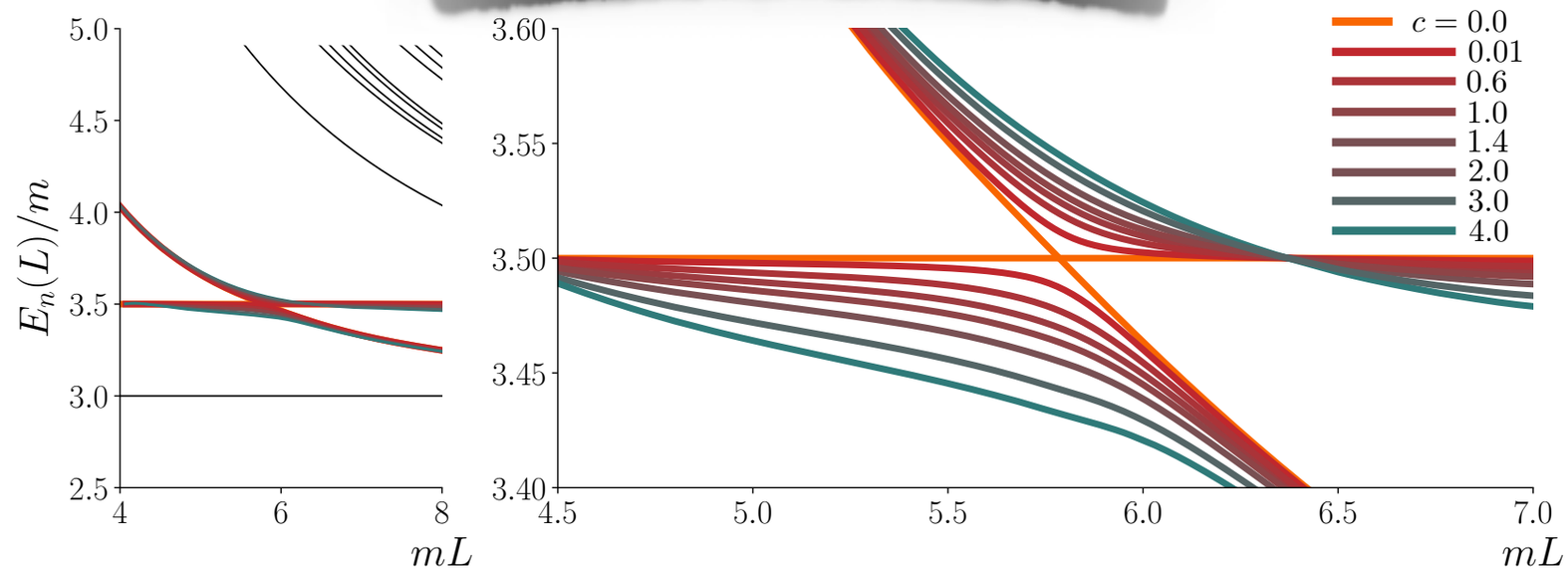
MTH, Sharpe (2014-2016) ○ Briceño, MTH, Sharpe (2017, 2018)

- **Currently exploring utility through numerical toy examples**

Volume effects on an Efimov state



Model of a 3-particle resonance



Briceño, MTH, Sharpe (2017)

- **Recent review for *Annual Review of Nuclear and Particle Physics***

MTH, Sharpe (2019) *[1901.00483]*