

# Lattice-based Equation of State of QCD matter with a critical point



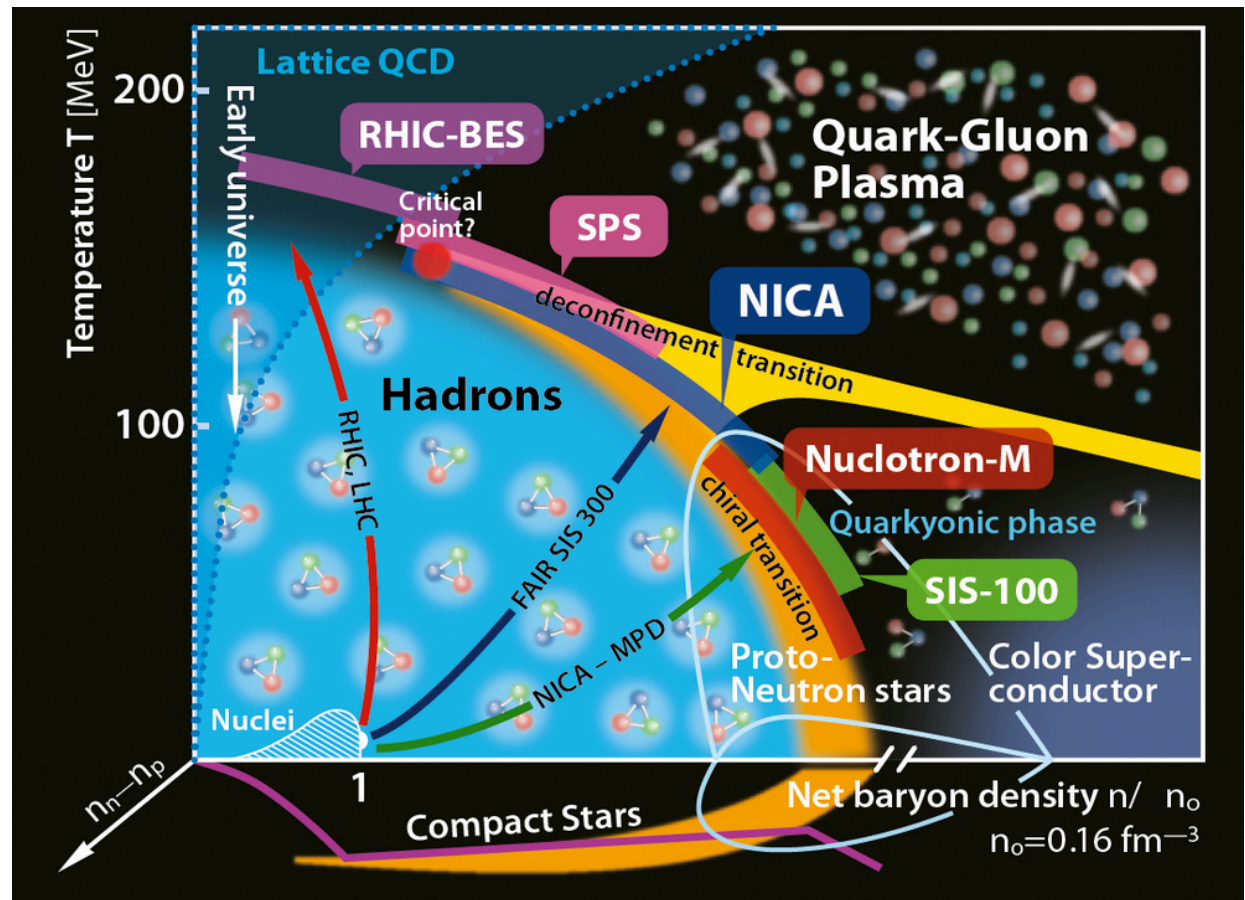
**CLAUDIA RATTI**  
**UNIVERSITY OF HOUSTON**

P. Parotto, M. Bluhm, D. Mroczek, M. Nahrgang, J. Noronha-Hostler, K. Rajagopal, C. Ratti, T. Schaefer, M. Stephanov: hep-ph/1805.05249



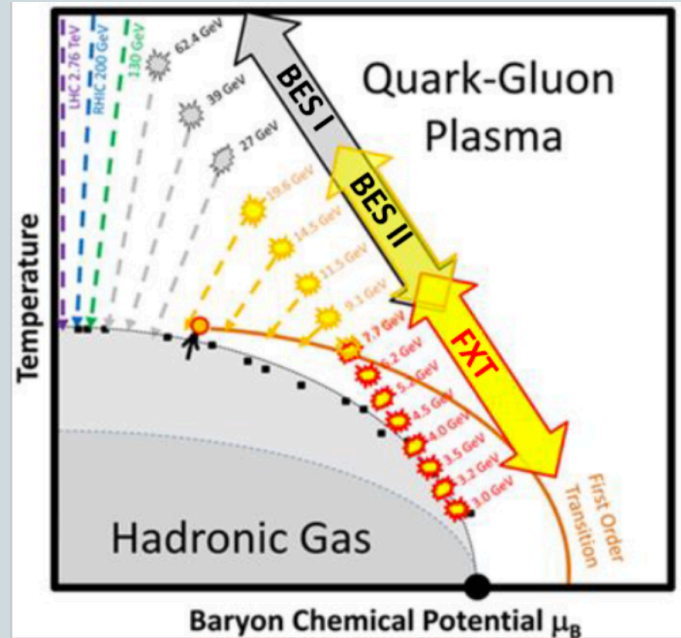
# Open Questions

- Is there a critical point in the QCD phase diagram?
- What are the degrees of freedom in the vicinity of the phase transition?
- Where is the transition line at high density?
- What are the phases of QCD at high density?
- Are we creating a thermal medium in experiments?



# Second Beam Energy Scan (BESII) at RHIC

- Planned for 2019-2020
- 24 weeks of runs each year
- Beam Energies have been chosen to keep the  $\mu_B$  step  $\sim 50$  MeV
- Chemical potentials of interest:  $\mu_B/T \sim 1.5 \dots 4$



$\sqrt{s}$ (GeV)	19.6	14.5	11.5	9.1	7.7	6.2	5.2	4.5
$\mu_B$ (MeV)	205	260	315	370	420	487	541	589
# Events	400M	300M	230M	160M	100M	100M	100M	100M

Collider

Fixed Target 3/33

# Comparison of the facilities

Compilation by D. Cebra

Facility	RHIC BESII	SPS	NICA	SIS-100 SIS-300	J-PARC HI
Exp.:	STAR +FXT	NA61	MPD + BM@N	CBM	JHITS
Start:	2019-20 2018	2009	2020 2017	2022	2025
Energy:	7.7– 19.6	4.9-17.3	2.7 - 11	2.7-8.2	2.0-6.2
$v_{sNN}$ (GeV)	2.5-7.7		2.0-3.5		
Rate:	100 HZ	100 HZ	<10 kHz	<10 MHZ	100 MHZ
At 8 GeV	2000 Hz				
Physics:	CP&OD	CP&OD	OD&DHM	OD&DHM	OD&DHM
	Collider Fixed target	Fixed target Lighter ion collisions	Collider Fixed target	Fixed target	Fixed target

CP=Critical Point    OD= Onset of Deconfinement    DHM=Dense Hadronic Matter

# Purpose of this work



- **Build an equation of state which:**
  - Reproduces the one from lattice QCD up to  $O(\mu_B^4)$
  - Contains a critical point in the 3D Ising model universality class
  - Can be used as input for hydrodynamic simulations to test the effect of the critical point on observables
- **Future hydro simulations and comparison with BESII data might help to constrain the position of the critical point**

# QCD Equation of State at finite density from the lattice



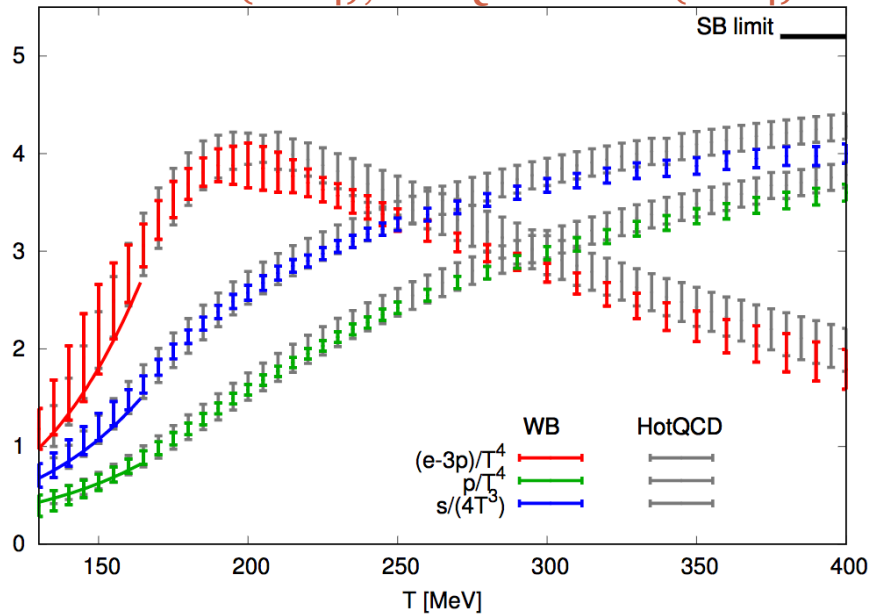
**TAYLOR EXPANSION**

**ANALYTICAL CONTINUATION FROM  
IMAGINARY CHEMICAL POTENTIAL**

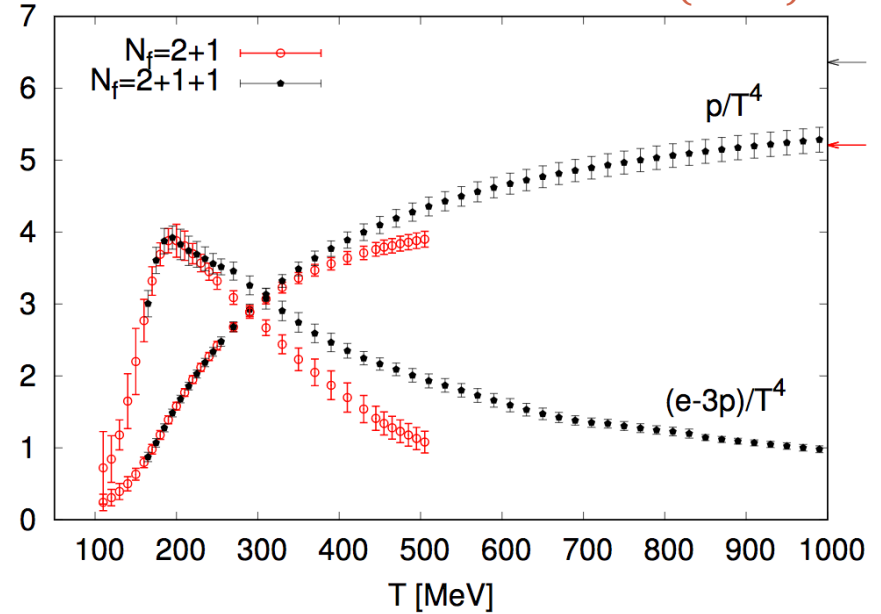
# QCD EoS at $\mu_B=0$



WB: PLB (2014); HotQCD: PRD (2014)

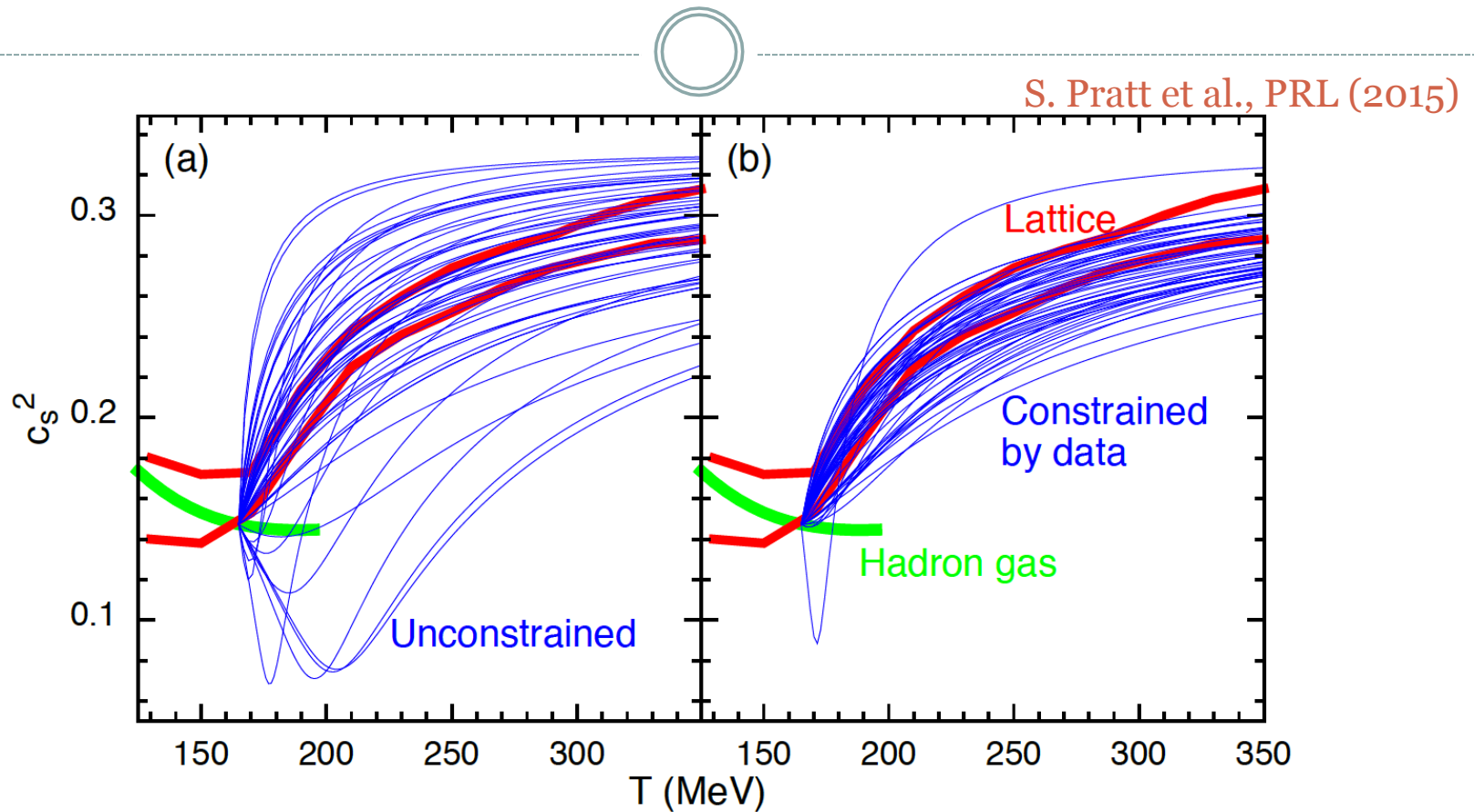


WB: Nature (2016)



- EoS for  $N_f=2+1$  known in the continuum limit since 2013
- Good agreement with the HRG model at low temperature
- Charm quark relevant degree of freedom already at  $T \sim 250$  MeV

# Constraints on the EoS from the experiments



- Comparison of data from RHIC and LHC to theoretical models through Bayesian analysis
- The posterior distribution of EoS is consistent with the lattice QCD one



# Taylor expansion of EoS



- Taylor expansion of the pressure:

$$\frac{p(T, \mu_B)}{T^4} = \frac{p(T, 0)}{T^4} + \sum_{n=1}^{\infty} \frac{1}{(2n)!} \left. \frac{d^{2n}(p/T^4)}{d(\mu_B/T)^{2n}} \right|_{\mu_B=0} \left( \frac{\mu_B}{T} \right)^{2n} = \sum_{n=0}^{\infty} c_{2n}(T) \left( \frac{\mu_B}{T} \right)^{2n}$$

- Two ways of extracting the Taylor expansion coefficients:
  - Direct simulation
  - Simulations at imaginary  $\mu_B$
- Two physics choices:
  - $\mu_B \neq 0, \mu_S = \mu_Q = 0$
  - $\mu_S$  and  $\mu_Q$  are functions of  $T$  and  $\mu_B$  to match the experimental constraints:

$$\langle n_S \rangle = 0$$

$$\langle n_Q \rangle = 0.4 \langle n_B \rangle$$

# Pressure coefficients: direct simulation



## Direct simulation:

- Calculate derivatives of  $\ln Z$ , where  $Z$  in the staggered formulation is given by:

$$Z = \int \mathcal{D}U e^{-S_g} (\det M_1)^{1/4} (\det M_2)^{1/4} (\det M_3)^{1/4} = \int \mathcal{D}U e^{-S_{\text{eff}}}$$

where  $M_i$  is the fermionic determinant of flavor  $i$  and  $S_g$  the gauge action

- The derivatives with respect to the chemical potential of flavor  $i$  are

$$\begin{aligned} A_j &= \frac{d}{d\mu_j} (\det M_j)^{1/4} = \tilde{\text{tr}} M_j^{-1} M'_j, \\ B_j &= \frac{d^2}{(d\mu_j)^2} (\det M_j)^{1/4} = \tilde{\text{tr}} \left( M''_j M_j^{-1} - M'_j M_j^{-1} M'_j M_j^{-1} \right), \\ C_j &= \frac{d^3}{(d\mu_j)^3} (\det M_j)^{1/4} = \tilde{\text{tr}} \left( M'_j M_j^{-1} - 3M''_j M_j^{-1} M'_j M_j^{-1} \right. \\ &\quad \left. + 2M'_j M_j^{-1} M'_j M_j^{-1} M'_j M_j^{-1} \right), \\ D_j &= \frac{d^4}{(d\mu_j)^4} \log(\det M_j)^{1/4} = \tilde{\text{tr}} \left( M''_j M_j^{-1} - 4M'_j M_j^{-1} M'_j M_j^{-1} - 3M''_j M_j^{-1} M'_j M_j^{-1} \right. \\ &\quad \left. + 12M''_j M_j^{-1} M'_j M_j^{-1} M'_j M_j^{-1} \right. \\ &\quad \left. - 6M'_j M_j^{-1} M'_j M_j^{-1} M'_j M_j^{-1} M'_j M_j^{-1} \right), \end{aligned}$$

**From which:**

$$\begin{aligned} \partial_i^4 \log Z &= \langle A_i^4 \rangle - 3 \langle A_i^2 \rangle^2 + 3 \left( \langle B_i^2 \rangle - \langle B_i \rangle^2 \right) \\ &\quad + 6 \left( \langle A_i^2 B_i \rangle - \langle A_i^2 \rangle \langle B_i \rangle \right) + 4 \langle A_i C_i \rangle + \langle D_i \rangle \end{aligned}$$

**and so on...**

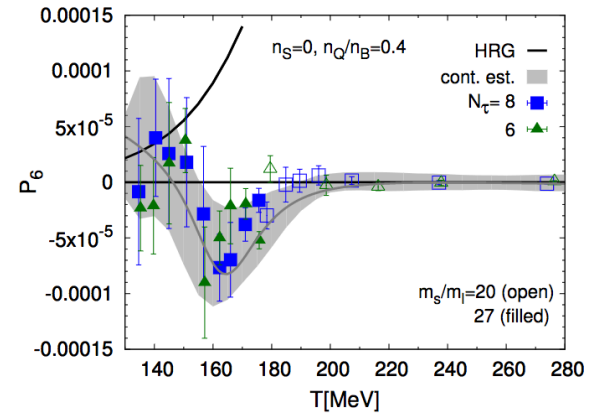
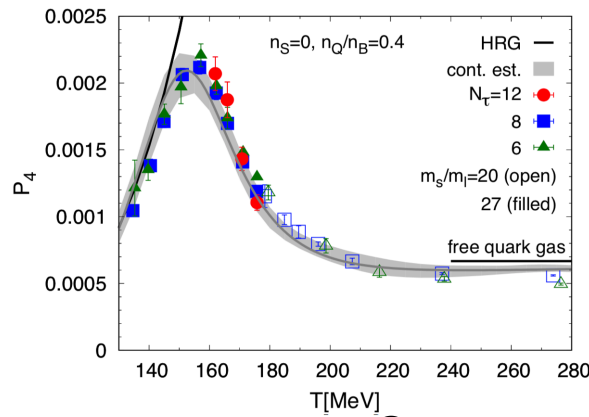
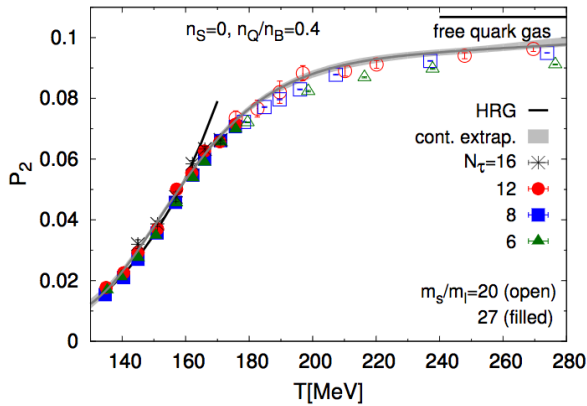
# Pressure coefficients



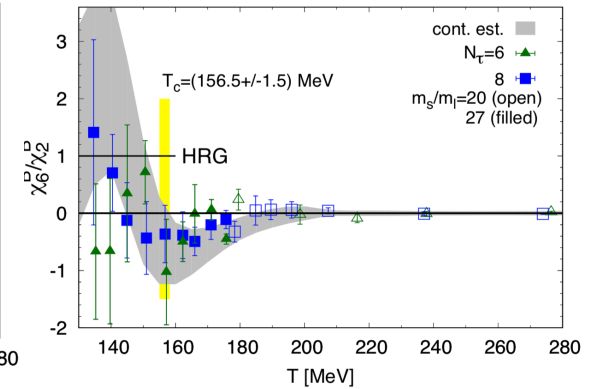
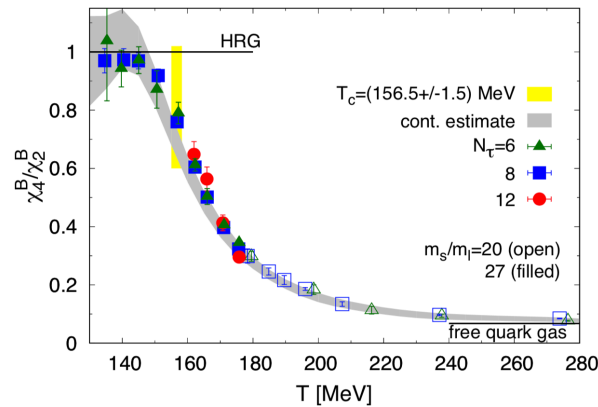
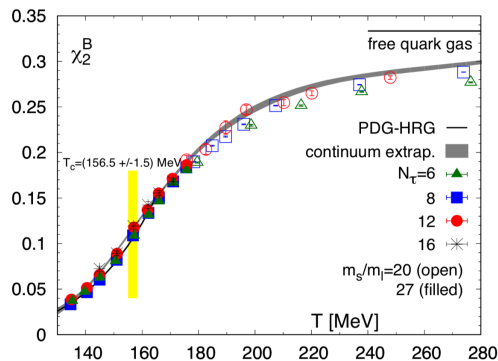
## Direct simulation:

O( $10^5$ ) configurations (hotQCD: PRD (2017) and update 06/2018)

## Strangeness neutrality



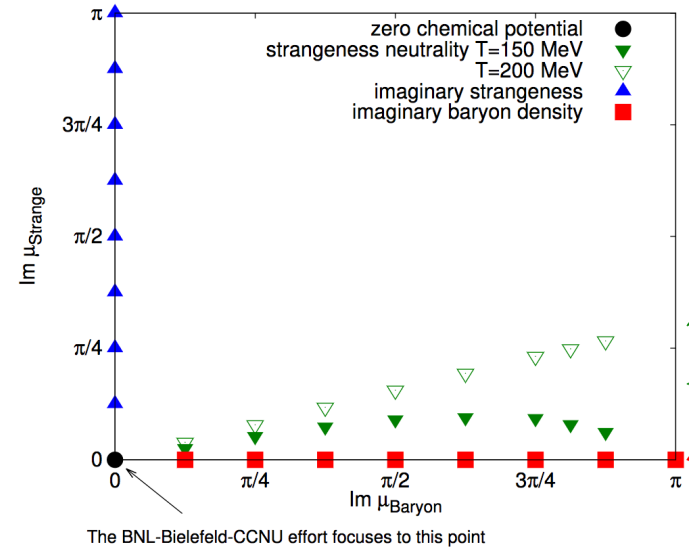
$$\mu_S = \mu_Q = 0$$



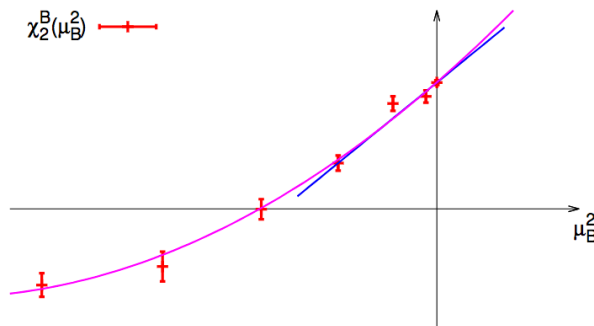
# Pressure coefficients: simulations at imaginary $\mu_B$

## Simulations at imaginary $\mu_B$ :

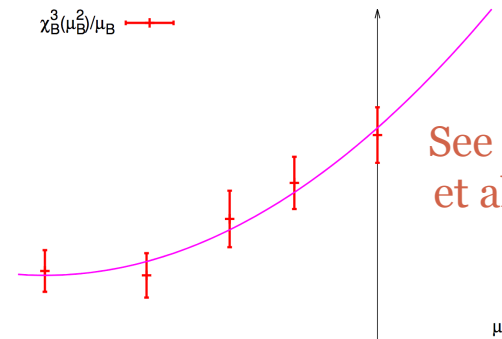
Common technique: [de Forcrand, Philipsen (2002)], [D'Elia and Lombardo, (2002)], [Bonati et al., (2015), (2018)], [Cea et al., (2015)]



Strategy: simulate lower-order fluctuations and use them in a combined, correlated fit



$$\chi_2^B(\mu_B^2) \approx \chi_2^B(0) + \frac{1}{2}\mu_B^2\chi_4^B(0) + \frac{1}{24}\mu_B^4\chi_6^B(0) + \dots$$



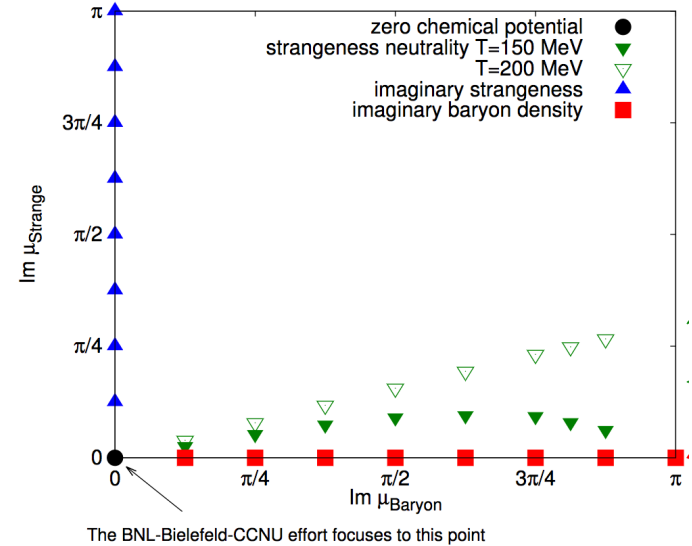
$$\frac{\chi_3^B(\mu_B^2)}{\mu_B} \approx \chi_4^B(0) + \frac{1}{6}\mu_B^2\chi_6^B(0) + \frac{1}{120}\mu_B^4\chi_8^B(0)$$

See also M. D'Elia et al., PRD (2017)

# Pressure coefficients: simulations at imaginary $\mu_B$

## Simulations at imaginary $\mu_B$ :

Common technique: [de Forcrand, Philipsen (2002)], [D'Elia and Lombardo, (2002)], [Bonati et al., (2015), (2018)], [Cea et al., (2015)]



Strategy: simulate lower-order fluctuations and use them in a combined, correlated fit

$$\chi_1^B(\hat{\mu}_B) = 2c_2\hat{\mu}_B + 4c_4\hat{\mu}_B^3 + 6c_6\hat{\mu}_B^5 + \frac{4!}{7!}c_4\epsilon_1\hat{\mu}_B^7 + \frac{4!}{9!}c_4\epsilon_2\hat{\mu}_B^9$$

$$\chi_2^B(\hat{\mu}_B) = 2c_2 + 12c_4\hat{\mu}_B^2 + 30c_6\hat{\mu}_B^4 + \frac{4!}{6!}c_4\epsilon_1\hat{\mu}_B^6 + \frac{4!}{8!}c_4\epsilon_2\hat{\mu}_B^8$$

$$\chi_3^B(\hat{\mu}_B) = 24c_4\hat{\mu}_B + 120c_6\hat{\mu}_B^3 + \frac{4!}{5!}c_4\epsilon_1\hat{\mu}_B^5 + \frac{4!}{7!}c_4\epsilon_2\hat{\mu}_B^7$$

$$\chi_4^B(\hat{\mu}_B) = 24c_4 + 360c_6\hat{\mu}_B^2 + c_4\epsilon_1\hat{\mu}_B^4 + \frac{4!}{6!}c_4\epsilon_2\hat{\mu}_B^6.$$

See also M. D'Elia et al., PRD (2017)

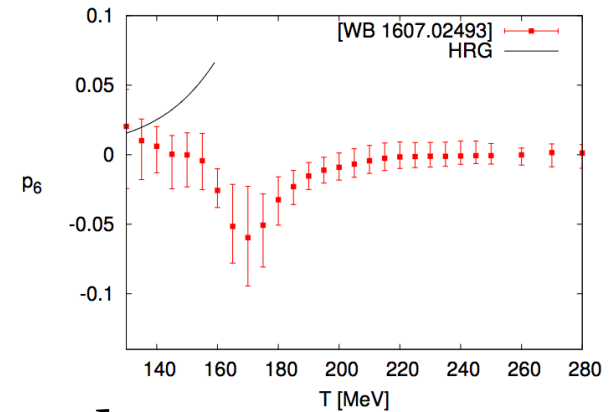
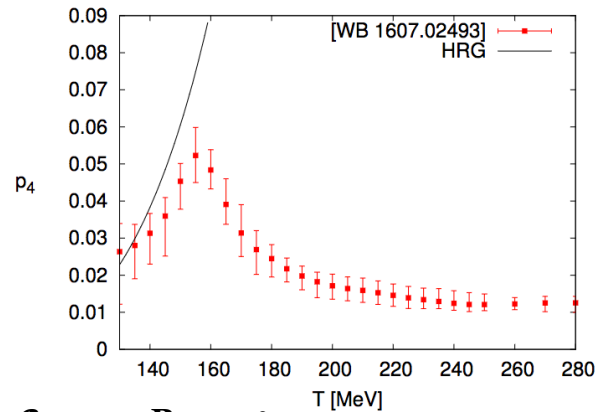
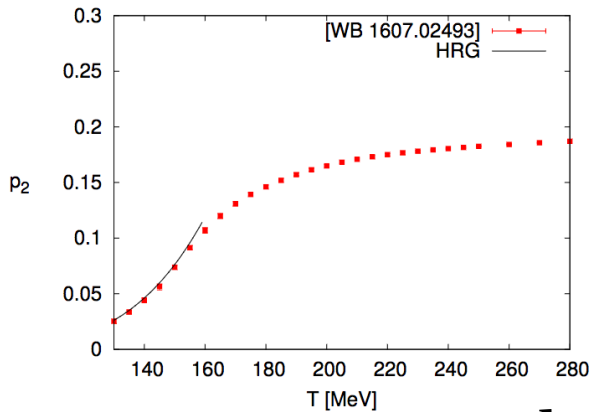
# Pressure coefficients



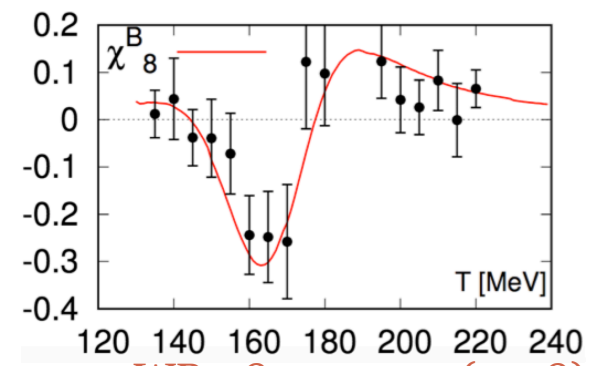
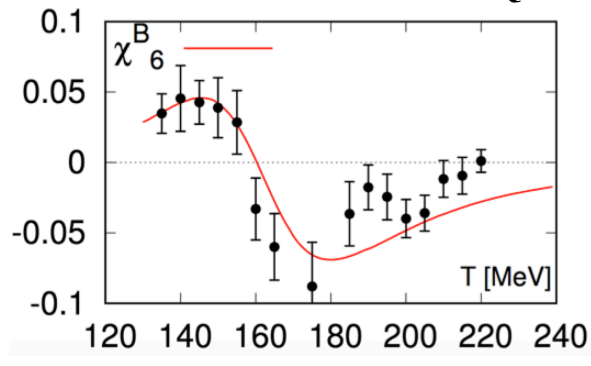
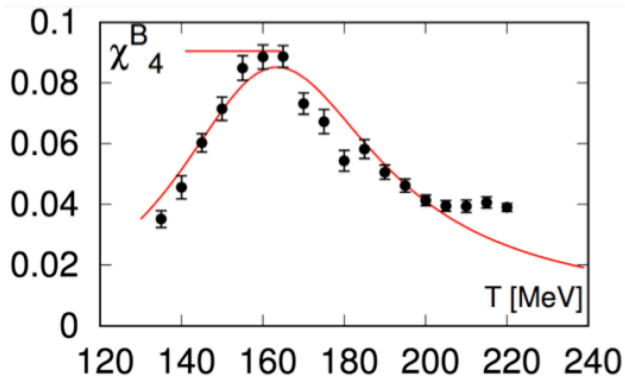
Simulations at imaginary  $\mu_B$ :

Continuum,  $O(10^4)$  configurations, errors include systematics (WB: NPA (2017))

## Strangeness neutrality



**New results for  $\chi_n^B = n!c_n$  at  $\mu_S = \mu_Q = 0$  and  $Nt=12$**



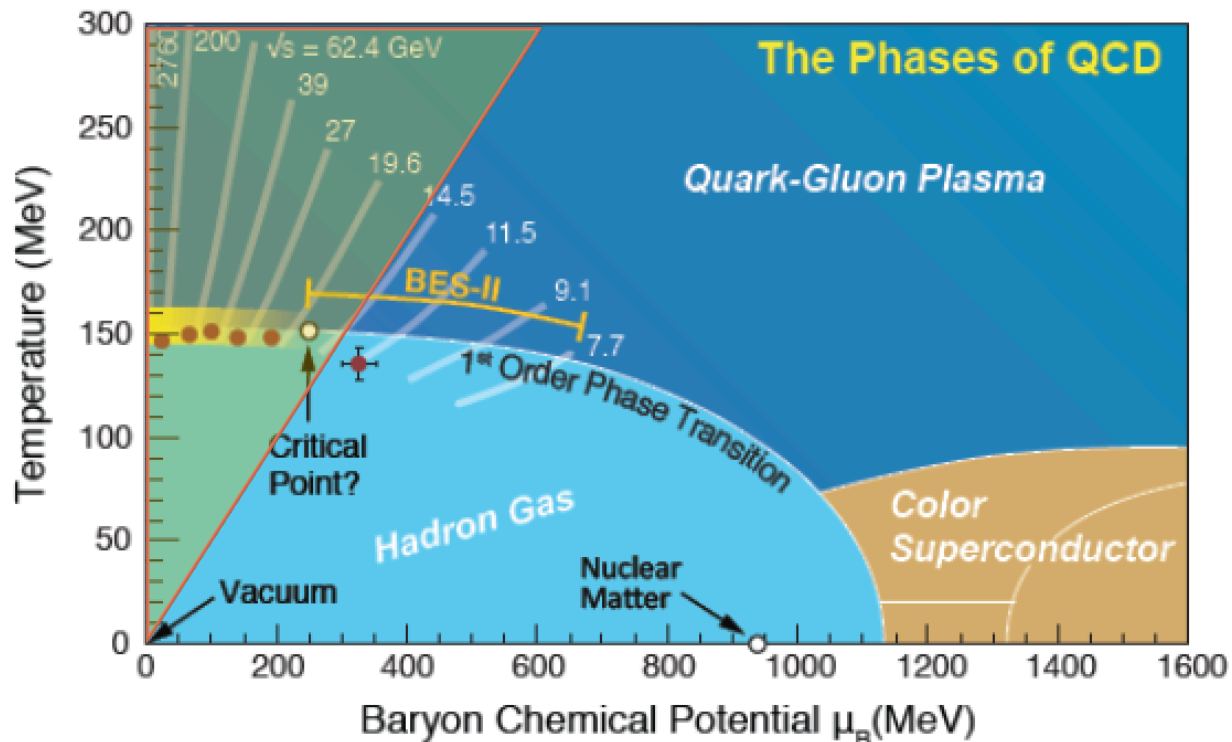
WB, 1805.04445 (2018)

# Range of validity of equation of state



- We now have the equation of state for  $\mu_B/T \leq 2$  or in terms of the RHIC energy scan:

$$\sqrt{s} = 200, 62.4, 39, 27, 19.6, 14.5 \text{ GeV}$$



# Introducing a 3D-Ising critical point



**IMPLEMENT THE SCALING BEHAVIOR OF  
THE 3D ISING MODEL EOS**

**MAP THE 3D ISING MODEL PHASE  
DIAGRAM ONTO THE QCD ONE**

**ESTIMATE THE CONTRIBUTION FROM  
THE CRITICAL POINT**

**RECONSTRUCT THE FULL PRESSURE  
OVER THE WHOLE PHASE DIAGRAM**



# Implement the scaling EoS for 3D Ising



Parametrization of the scaling form of the EoS can be given for magnetization  $M$ , magnetic field  $h$  and reduced temperature  $r = (T - T_c)/T_c$  in 3D Ising model:

$(\mathbf{R}, \theta) \mapsto (\mathbf{r}, \mathbf{h}) :$

$$M = M_0 R^\beta \theta$$

$$h = h_0 R^{\beta\delta} \tilde{h}(\theta)$$

$$r = R(1 - \theta^2)$$

where:

- ▶  $M_0 \simeq 0.605$ ,  $h_0 \simeq 0.394$  are normalization constants;
- ▶  $\tilde{h}(\theta) = \theta(1 + a\theta^2 + b\theta^4)$  with  $(a = -0.76201, b = 0.00804)$ ;
- ▶  $R \geq 0$  and  $|\theta| \leq 1.154$  (second zero of  $\tilde{h}(\theta)$ );
- ▶  $\beta \simeq 0.326$ ,  $\delta \simeq 4.80$  are critical exponents.

# Implement the scaling EoS for 3D Ising



Construct (Helmoltz) and thus Gibbs free energy densities:

$$F(M, r) = h_0 M_0 R^{2-\alpha} g(\theta) \quad \longrightarrow \quad G(r, h) = F(M, r) - Mh$$

Thanks to the map:

$$(\mathbf{R}, \theta) \longmapsto (\mathbf{r}, \mathbf{h}) \longleftrightarrow (\mathbf{T}, \mu_B)$$

we can write the pressure in QCD as:

$$P_{\text{Ising}}(T, \mu_B) = -G(T(R, \theta), \mu_B(R, \theta)) = h_0 M_0 R^{2-\alpha} \left[ g(\theta) - \theta \tilde{h}(\theta) \right]$$

**NOTE:** Explicit functional form of  $G(T(R, \theta), \mu_B(R, \theta))$  ONLY as a function of  $(R, \theta)$ . Evaluation will require numerical inversion of :

$$T(R, \theta) = T^* \quad \mu_B(R, \theta) = \mu_B^*$$

# Map the phase diagram

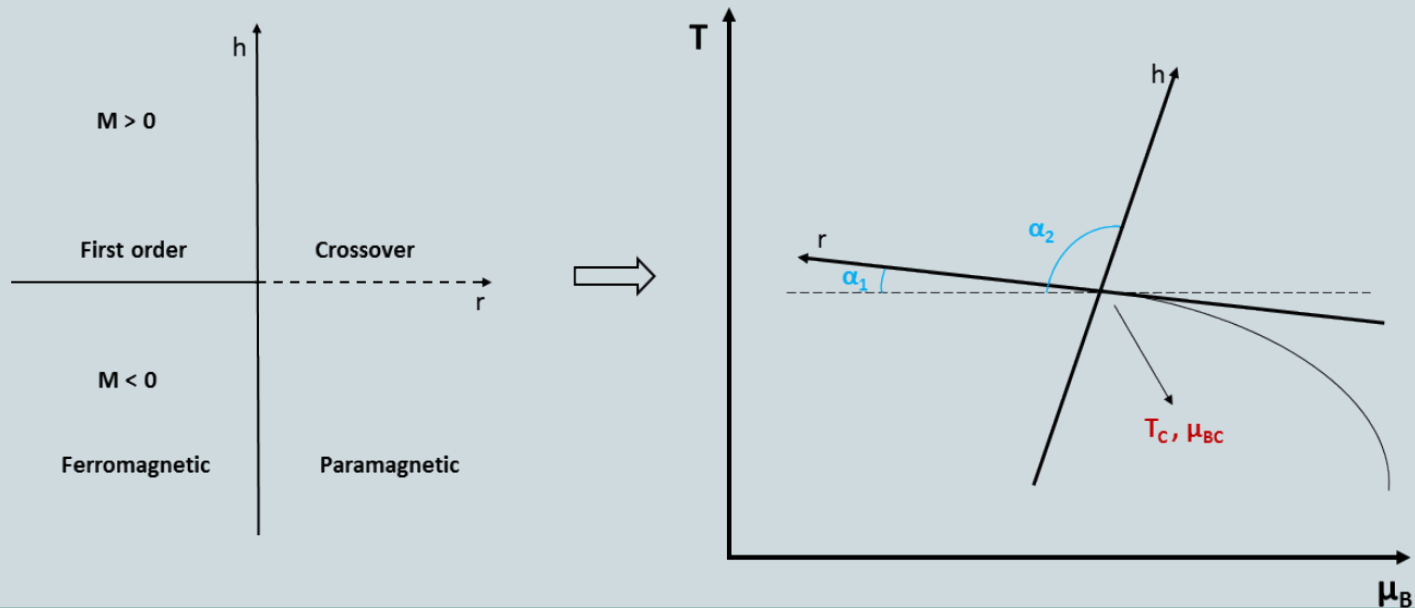


The relation between the Ising model scaling variables  $(h, r)$  and the QCD thermodynamic coordinates  $(T, \mu_B)$ , can be expressed in linear form, with the use of six parameters:

$$(\mathbf{r}, \mathbf{h}) \longleftrightarrow (\mathbf{T}, \mu_B)$$

$$\frac{T - T_C}{T_C} = w (r \rho \sin \alpha_1 + h \sin \alpha_2)$$

$$\frac{\mu_B - \mu_{BC}}{T_C} = w (-r \rho \cos \alpha_1 - h \cos \alpha_2)$$



# Map the phase diagram



- Exploit the parametric nature of the EoS to constrain the values of the parameters:
  - Theoretical (a priori) arguments (i.e. require **thermodynamic stability and causality**)
  - Comparison (a posteriori) to future BES-II program experimental data
- How is the choice of the parameters driven?
  - From Lattice QCD calculations:  $T_C < 150 \text{ MeV}$ ,  $\mu_{BC} > 2T_C$
  - place the critical point in the region of the phase diagram accessible to the BESII
  - we want to investigate the role of **w** and  **$\rho$**

# Map the phase diagram



- The number of free parameters is reduced:
  - Assume the shape of transition line is a parabola (good approximation at BES-like energies)  $\rightarrow$  reduce to four parameters:

$$\frac{T_C}{T_C(\mu_B = 0)} = 1 + \kappa \left( \frac{\mu_B}{T_C(\mu_B = 0)} \right)^2 + \mathcal{O}(\mu_B^4)$$

with the values  $T_C(\mu_B=0)=155 \text{ MeV}$ ,  $\kappa=-0.0149$ .

- For a chosen value of  $\mu_{BC}$ , one gets

$$T_C = T_0 + \frac{\kappa}{T_0} \mu_{BC}^2 \qquad \alpha_1 = \tan^{-1} \left( 2 \frac{\kappa}{T_0} \mu_{BC} \right)$$

- In the following:

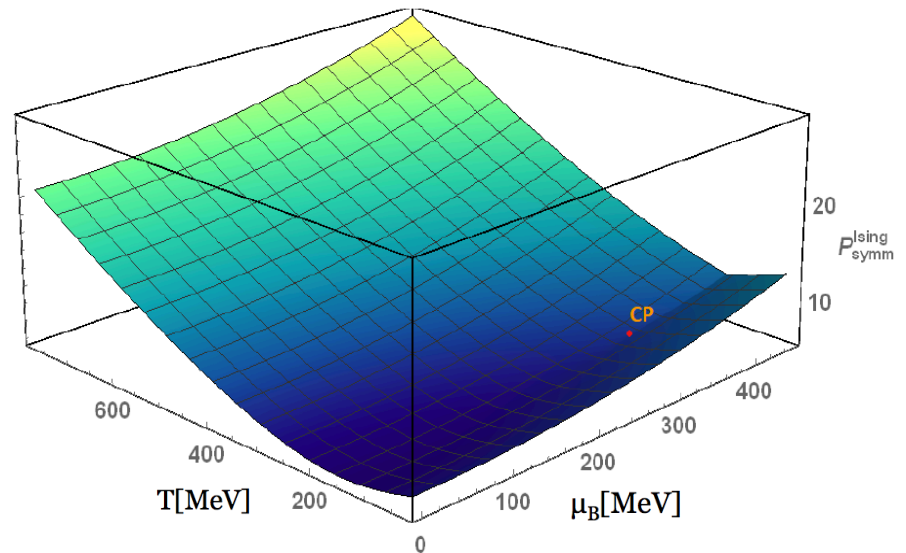
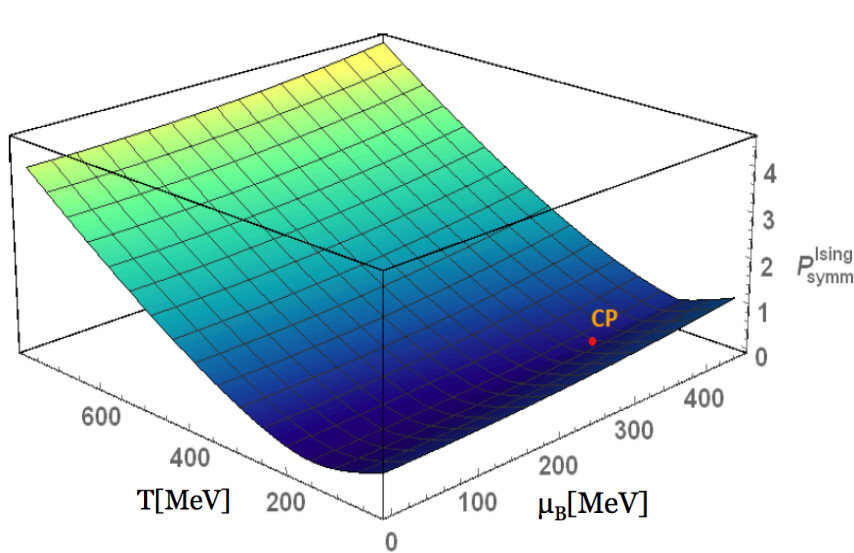
$$\begin{array}{lll} \mu_{BC} = 350 \text{ MeV} & w = 1 & T_C \simeq 143.2 \text{ MeV} \\ \alpha_2 - \alpha_1 = \pi/2 & \rho = 2 & \alpha_1 \simeq 4^\circ \end{array}$$

# Critical pressure



The critical pressure for this parameter choice (left) and for a smaller value of  $w = 0.25$  (right)

⇒ smaller  $w$  results in larger critical contribution



**NOTE:** the critical pressure is symmetrized around  $\mu_B = 0$  to ensure  $c^n(T) = 0, \forall n$  odd

# Expansion coefficients



The Taylor coefficients from Lattice QCD contain an “**Ising**” contribution from the critical point and a “**Non-Ising**” one:

$$T^4 c_n^{\text{LAT}}(T) = T^4 c_n^{\text{Non-Ising}}(T) + f(T, \mu_B) c_n^{\text{Ising}}(T)$$

where  $f(T, \mu_B)$  is a regular function of  $T$  and  $\mu_B$ , with dimension 4. In the following, we will choose  $f(T, \mu_B) = T_C^4$ .

The Ising coefficients are just:

$$c_{\text{Ising}}^n(T) = \frac{1}{n!} \chi_B^n(T, \mu_B = 0) = -\frac{1}{n!} T^n \left. \frac{\partial^n G}{\partial \mu_B^n} \right|_{\mu_B=0}$$

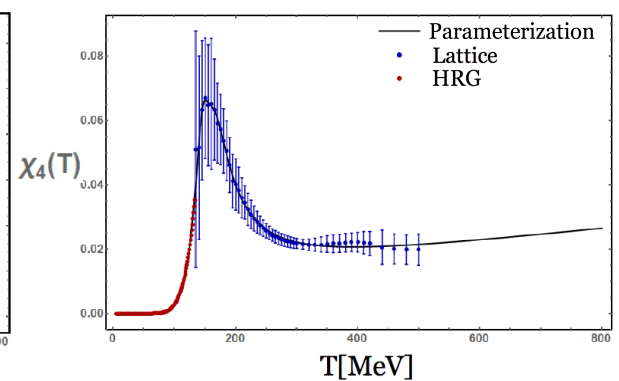
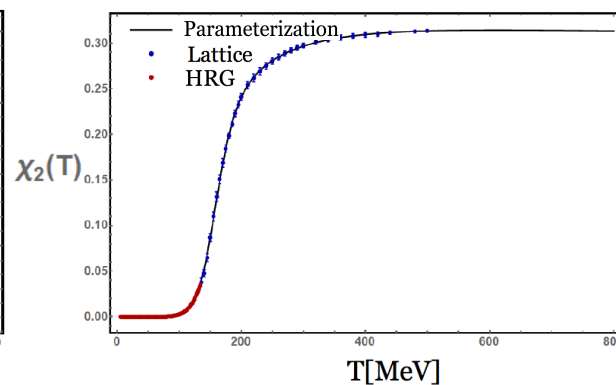
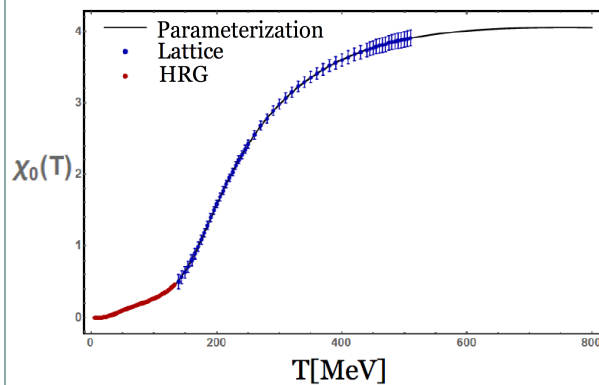
and their explicit expression can be derived with multiple use of the chain rule.

# Parameterization of lattice coefficients



In order to be used in our procedure, Lattice QCD coefficients need to:

- Be extended to low temperatures
  - extend with HRG model
- Be smooth enough to allow for derivatives
  - parameterization



Lattice results from: WB: JHEP (20110), PRD (2015)

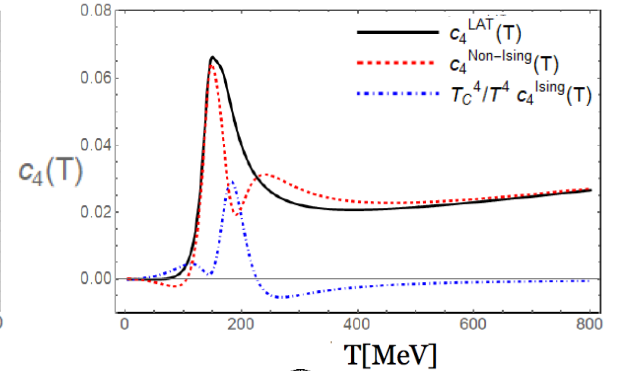
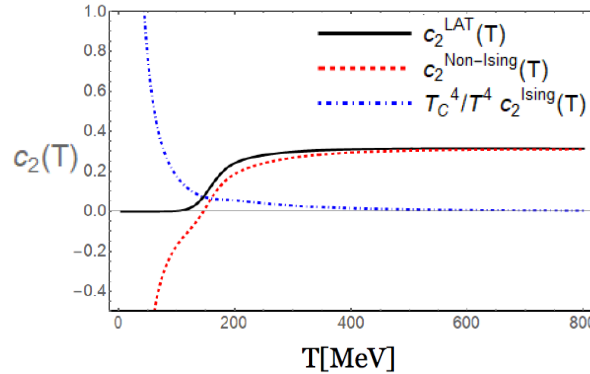
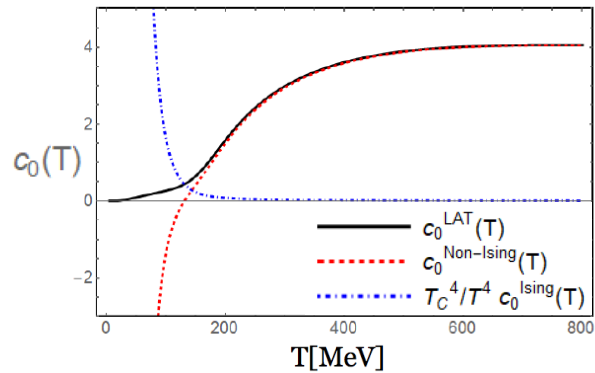


# Expansion coefficients



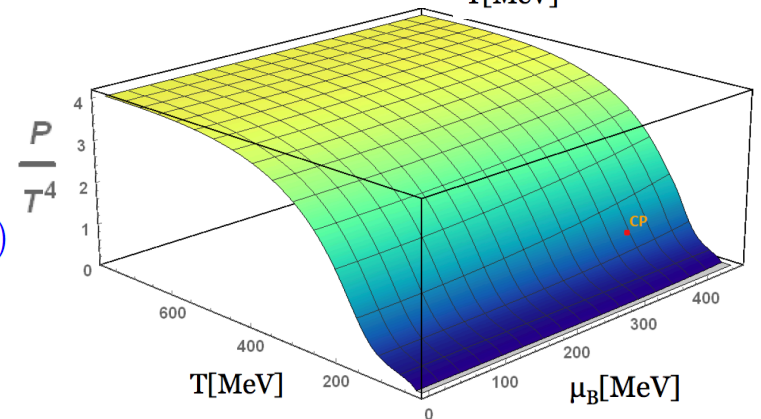
Extract the “regular” contribution as the difference between the lattice and Ising ones

$$T^4 c_n^{\text{LAT}}(T) = T^4 c_n^{\text{Non-Ising}}(T) + T_C^4 c_n^{\text{Ising}}(T)$$



Total pressure becomes:

$$P(T, \mu_B) = T^4 \sum_n c_{\text{Non-Ising}}^n(T) \left(\frac{\mu_B}{T}\right)^n + T_C^4 P_{\text{Ising}}(T, \mu_B)$$



# Merging with HRG model at low T



⇒ Smooth merging with Hadron Resonance Gas (HRG) model through:

$$\frac{P_{\text{Final}}(T, \mu_B)}{T^4} = \frac{P(T, \mu_B)}{T^4} \frac{1}{2} \left[ 1 + \tanh \left( \frac{T - T'(\mu_B)}{\Delta T} \right) \right] + \frac{P_{\text{HRG}}(T, \mu_B)}{T^4} \frac{1}{2} \left[ 1 - \tanh \left( \frac{T - T'(\mu_B)}{\Delta T} \right) \right]$$

where:

- ▶  $T'(\mu_B)$  is the “transition” temperature, depending on  $\mu_B$ :

$$T'(\mu_B) = T_0 + \frac{\kappa}{T_0} \mu_B^2 - T^*$$

- ▶  $\Delta T$  is a measure of the overlap region size

⇒ In the following:  $T^* = 23 \text{ MeV}$  ,  $\Delta T = 17 \text{ MeV}$

# Final EoS: Thermodynamic quantities



Once the pressure is determined, all thermodynamic quantities can be calculated:

**Entropy density :** 
$$\frac{S}{T^3} = \frac{1}{T^3} \frac{\partial P}{\partial T}$$

**Baryon density :** 
$$\frac{n_B}{T^3} = \frac{1}{T^3} \frac{\partial P}{\partial \mu_B}$$

**Energy density :** 
$$\frac{\epsilon}{T^4} = \frac{S}{T^3} - \frac{P}{T^4} + \frac{\mu_B}{T} \frac{n_B}{T^3}$$

**Speed of sound :** 
$$c_s^2 = \left( \frac{\partial P}{\partial \epsilon} \right)_{s/n_B} = \frac{n_B^2 \partial_T^2 P - 2S n_B \partial_T \partial_{\mu_B} P + S^2 \partial_{\mu_B}^2 P}{(\epsilon + P) (\partial_T^2 P \partial_{\mu_B}^2 P - (\partial_T \partial_{\mu_B} P)^2)}$$

**Baryon susceptibilities :** 
$$\chi_n^B = \frac{\partial(P/T^4)}{\partial(\mu_B/T)} \quad (\text{only the second one in the paper})$$

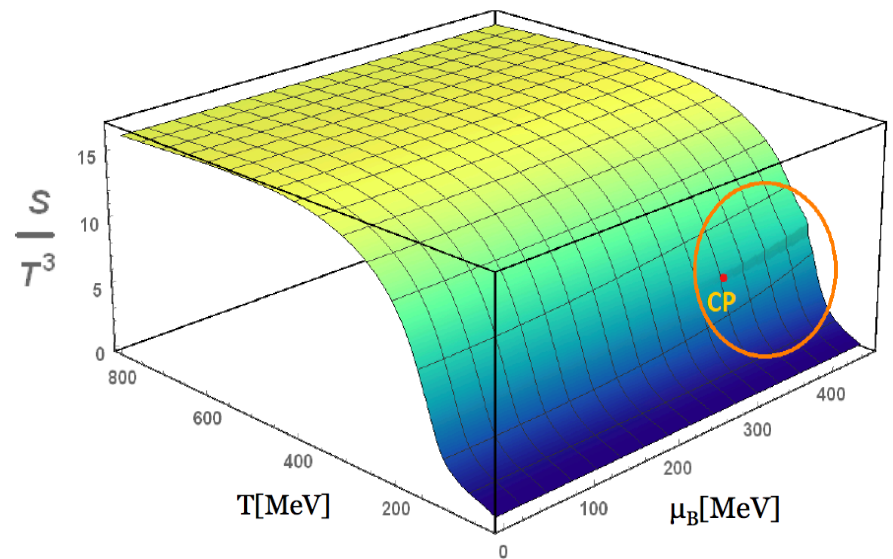
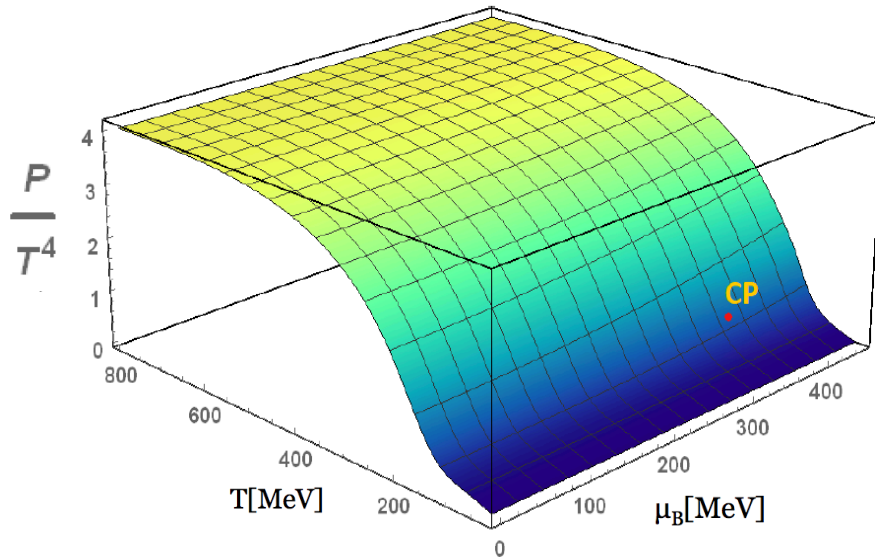
# Final EoS: Pressure and Entropy Density

P. Parotto et al.,: hep-ph/1805.05249

The final EoS covers the range:

$$T = 30 - 800 \text{ MeV}$$

$$\mu_B = 0 - 450 \text{ MeV}$$



Although the effect is barely visible in the pressure, the entropy density shows a discontinuity in the first order transition region.

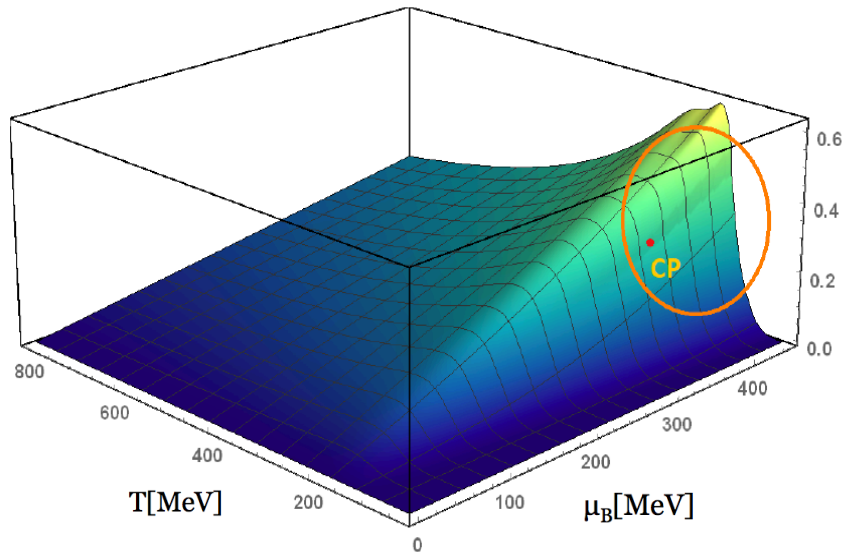
# Final EoS: Baryon and Energy Density

P. Parotto et al.: hep-ph/1805.05249

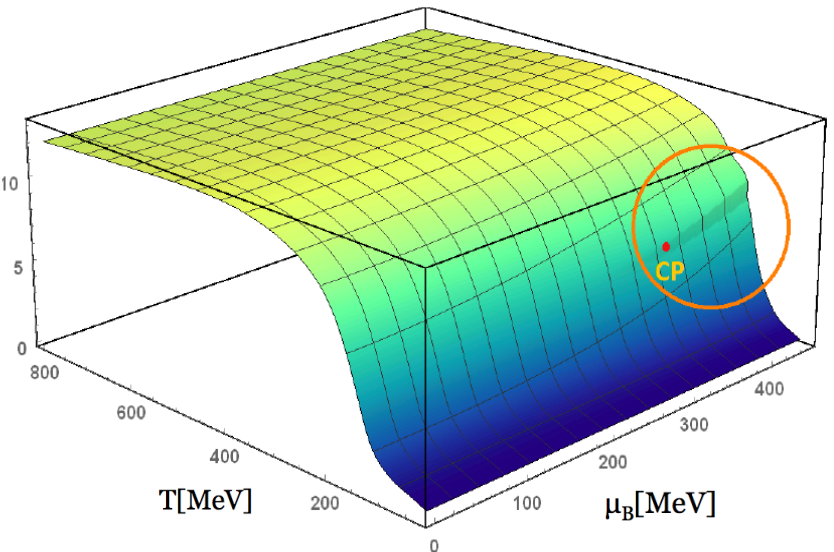
The final EoS covers the range:

$$T = 30 - 800 \text{ MeV}$$

$$\mu_B = 0 - 450 \text{ MeV}$$



$$\frac{n_B}{T^3} \quad \frac{\epsilon}{T^4}$$



Baryon and energy density also show a discontinuity in the first order transition region.

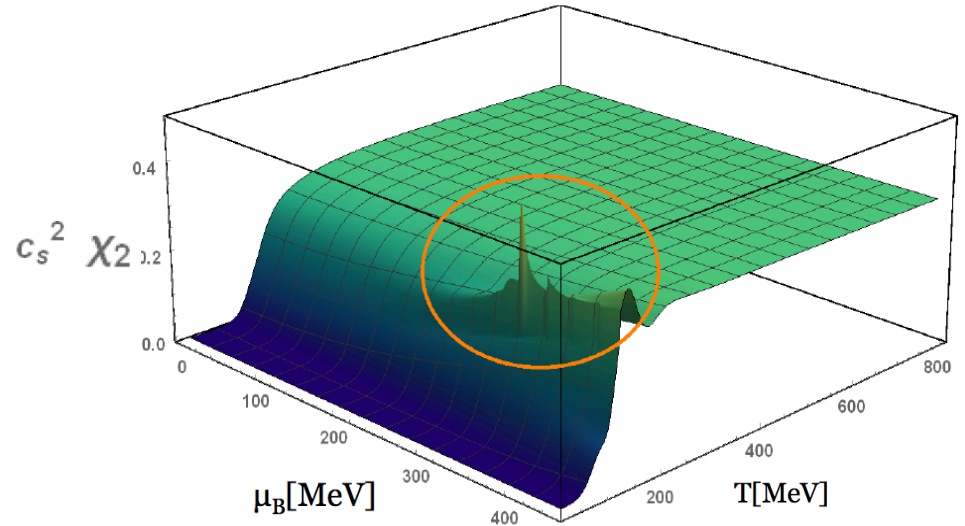
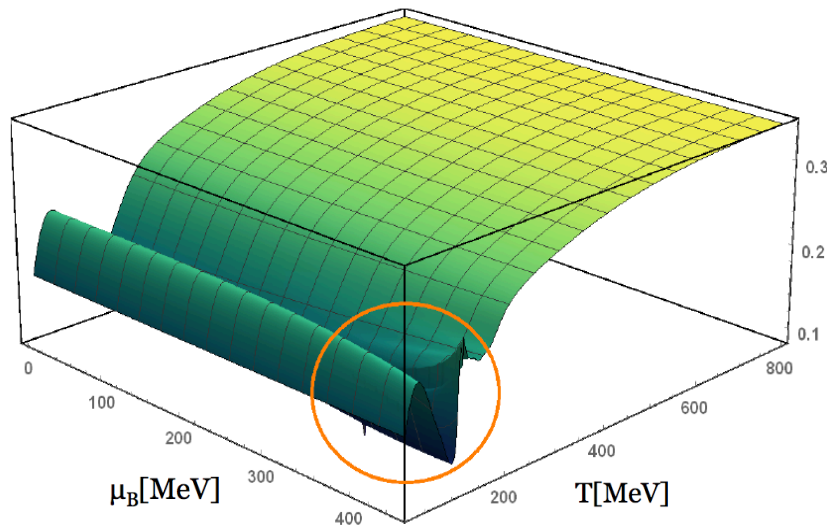
# Final EoS: Speed of sound and baryon number $\chi_2$

P. Parotto et al.,: hep-ph/1805.05249

The final EoS covers the range:

$$T = 30 - 800 \text{ MeV}$$

$$\mu_B = 0 - 450 \text{ MeV}$$

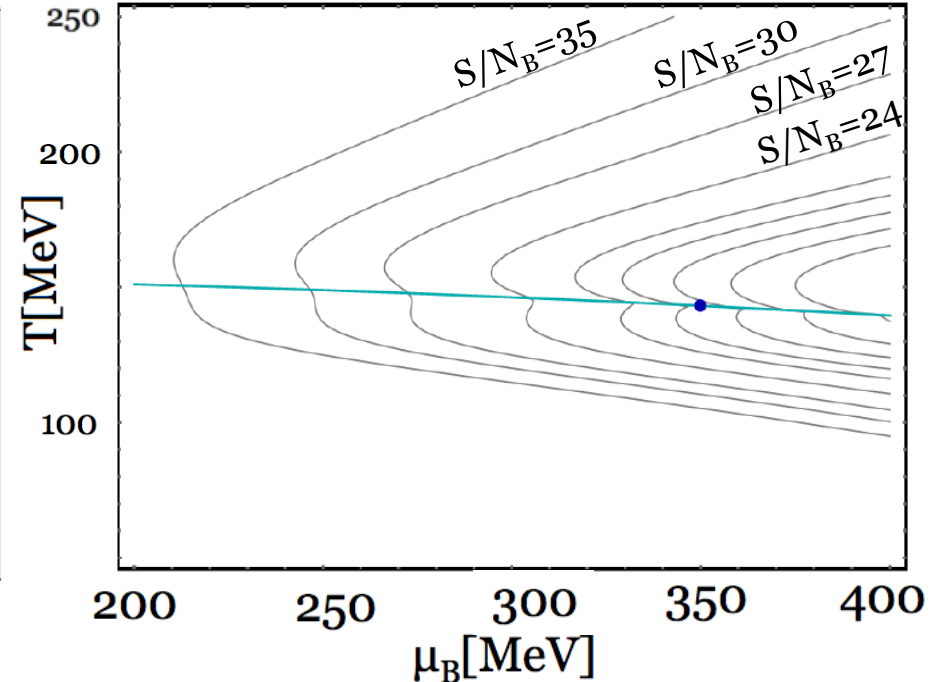
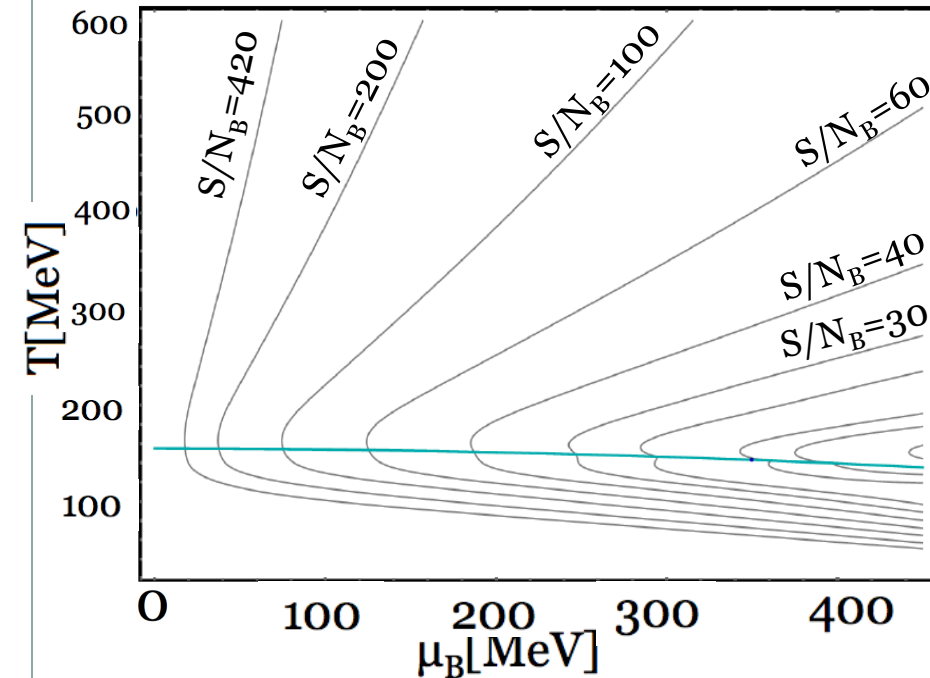


The speed of sound and the second baryon number cumulant show a (weak) dip and a (strong) peak at the critical point respectively.

# Final EoS: Isentropic trajectories

P. Parotto et al., hep-ph/1805.05249

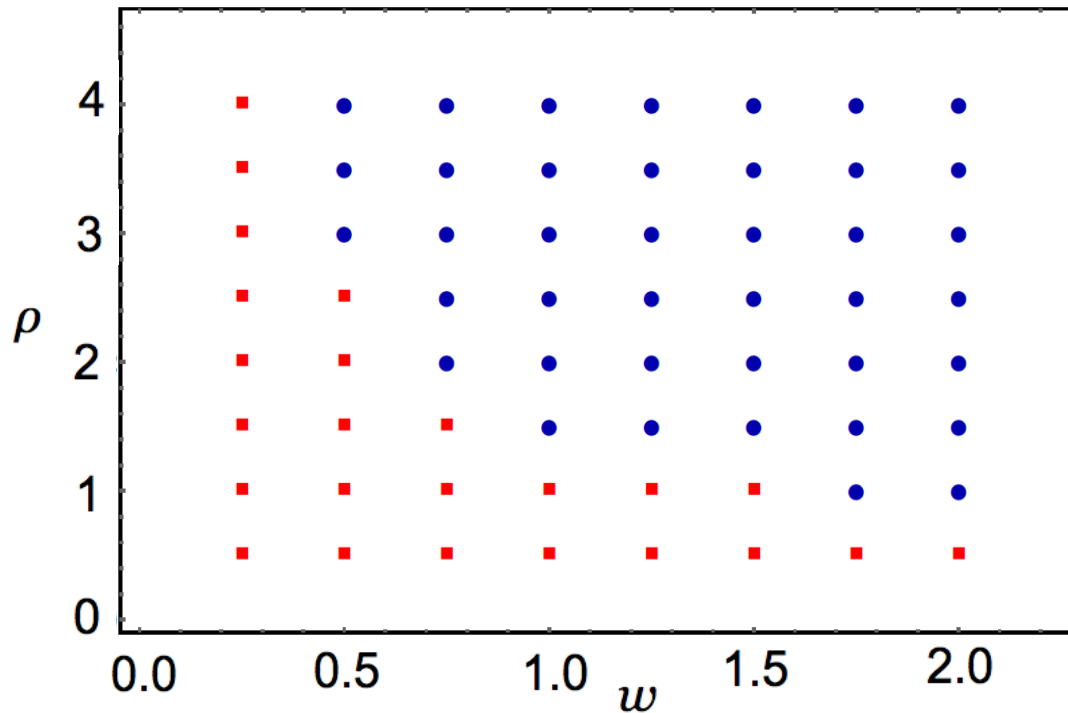
- ▶ Relevant for hydrodynamic evolution are the lines of  $s/n_B = \text{const}$ :
  - ▶ Low- $\mu_B$ : match behavior from Lattice QCD
  - ▶ Close to the CP: some structure appears



# Final EoS: explore parameter space



Keeping the position of the critical point fixed, as well as the orientation of the axes mapped from the 3D Ising model phase diagram, we varied the parameters  $w$ ,  $\rho$ , and required thermodynamic stability ( $P, S, n_B, \epsilon, c_s^2 > 0$ ) and causality ( $c_s^2 \leq 1$ )



In blue (dots) the values corresponding to acceptable choices, in red (squares) the values leading to pathological EoS's



# Conclusions



- Need for quantitative results at finite-density to support the experimental programs
- Current lattice results for thermodynamics up to  $\mu_B/T \leq 2$
- We created a family of EoS matching lattice QCD calculations up to  $O(\mu_B^4)$
- These equations of state can be readily used in hydrodynamic simulations
- The code is available at:  
<https://www.bnl.gov/physics/best/resources.php>

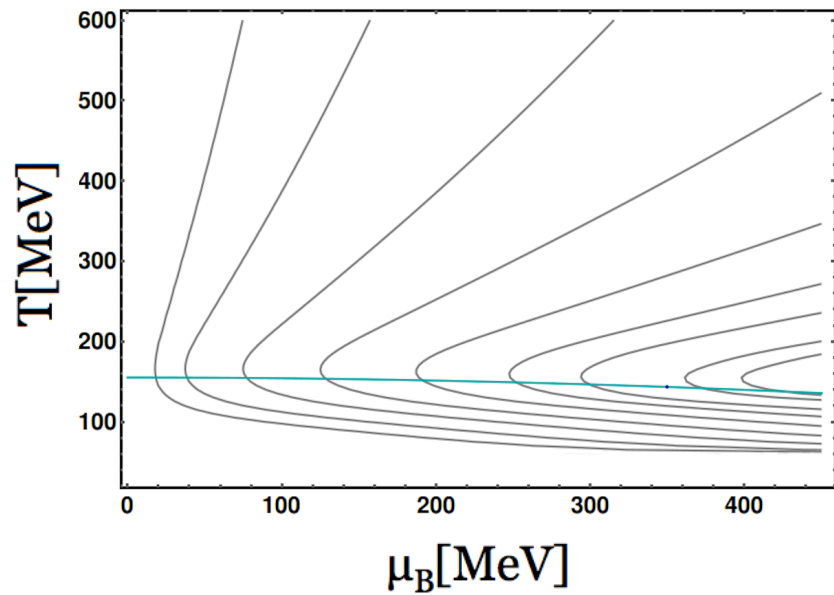
# Backup slides



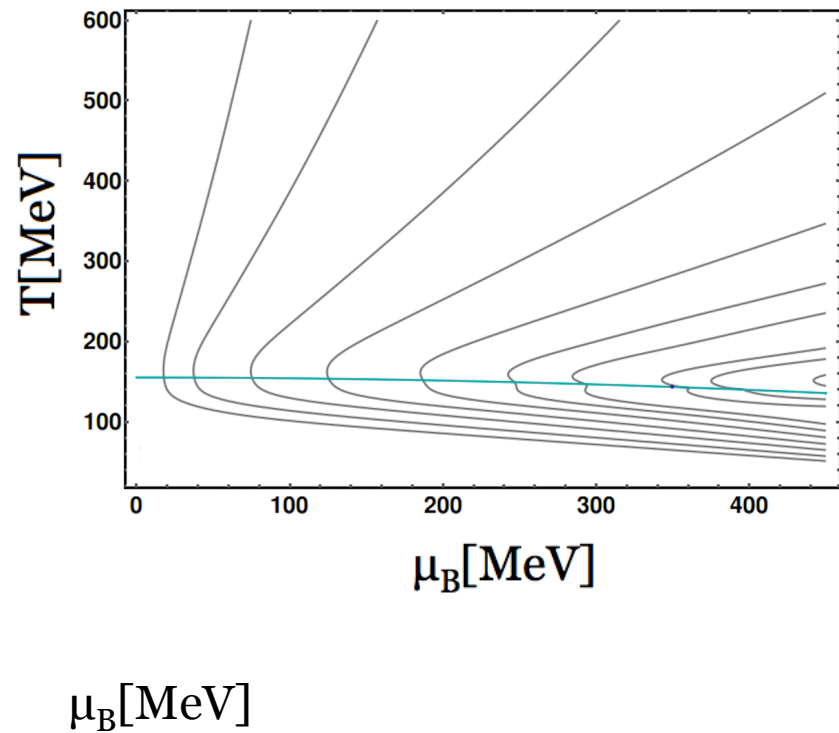
# Final EoS: Isentropic trajectories

P. Parotto et al., hep-ph/1805.05249

Isentropic trajectories from lattice QCD



Our isentropic trajectories



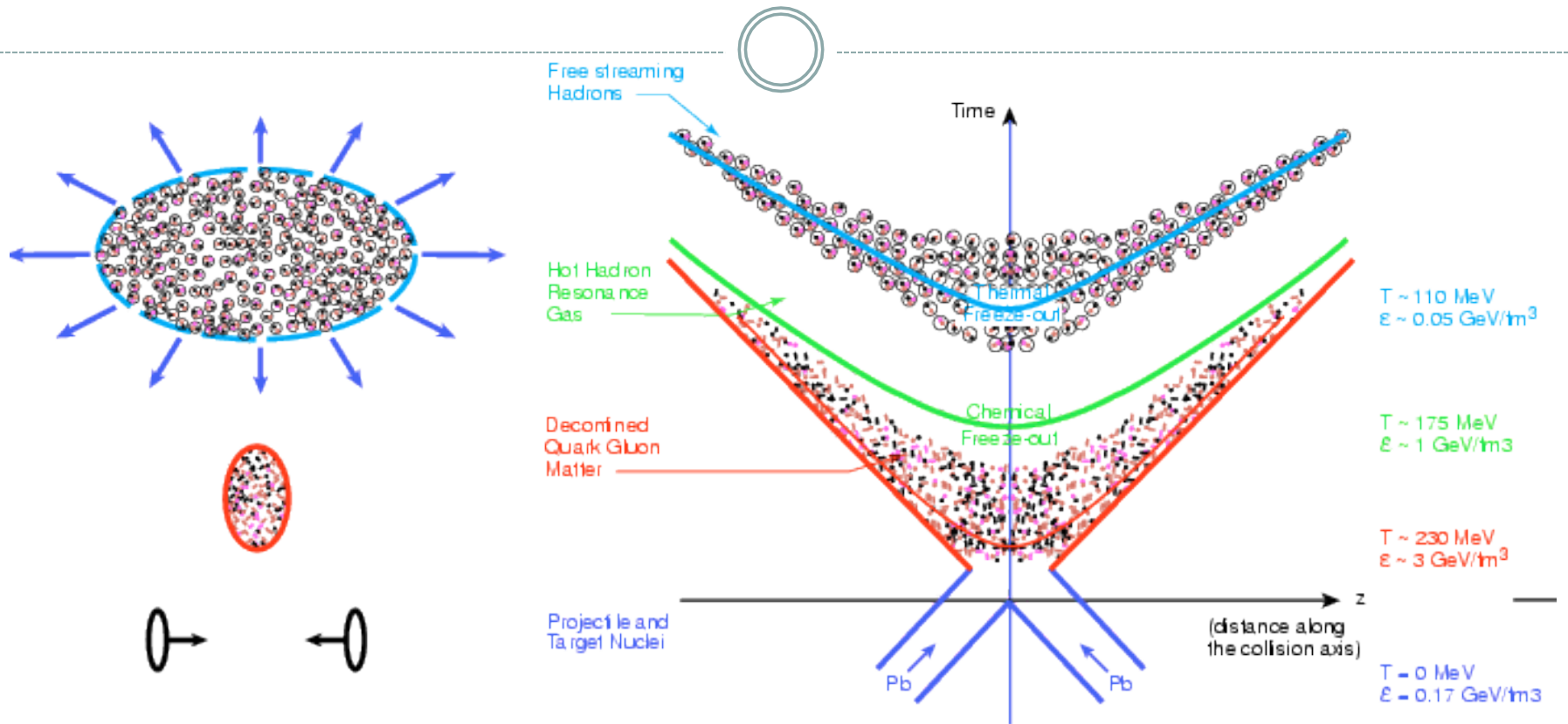
# Fluctuations of conserved charges



**COMPARISON TO EXPERIMENT:  
CHEMICAL FREEZE-OUT PARAMETERS**

**COMPARISON TO HRG MODEL:  
SEARCH FOR THE CRITICAL POINT**

# Evolution of a heavy-ion collision

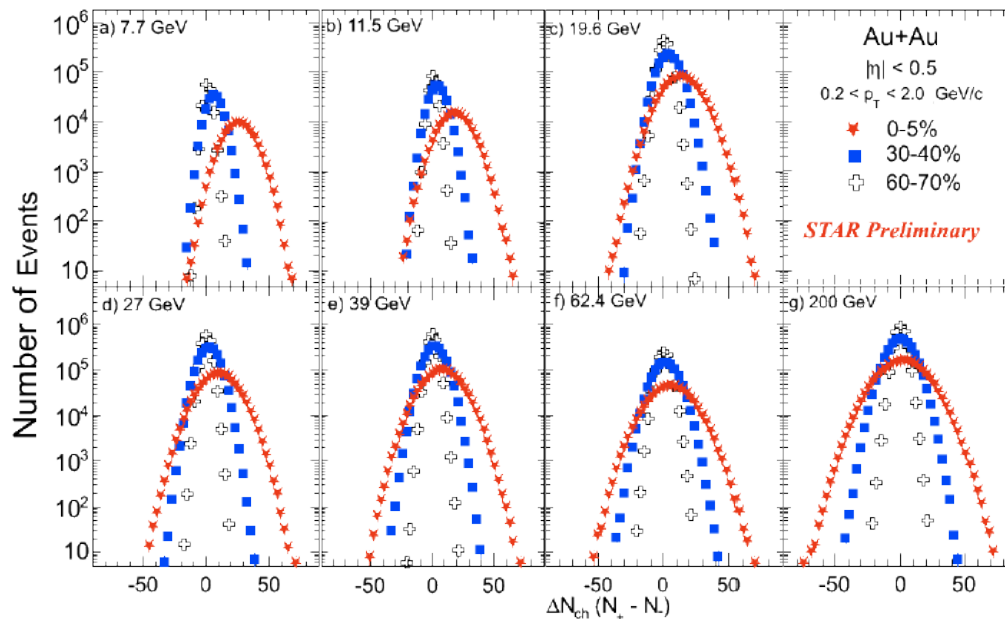


- **Chemical freeze-out:** inelastic reactions cease: the chemical composition of the system is fixed (particle yields and fluctuations)
- **Kinetic freeze-out:** elastic reactions cease: spectra and correlations are frozen (free streaming of hadrons)
- Hadrons reach the detector

# Distribution of conserved charges



- Consider the number of electrically charged particles  $N_Q$
- Its average value over the whole ensemble of events is  $\langle N_Q \rangle$
- In experiments it is possible to measure its **event-by-event distribution**



STAR Collab.: PRL (2014)

# Fluctuations on the lattice



- **Fluctuations** of conserved charges are the **cumulants** of their event-by-event distribution

- Definition: 
$$\chi_{lmn}^{BSQ} = \frac{\partial^{l+m+n} p/T^4}{\partial(\mu_B/T)^l \partial(\mu_S/T)^m \partial(\mu_Q/T)^n}$$

- They can be calculated on the lattice and compared to experiment

- variance:  $\sigma^2 = \chi_2$       Skewness:  $S = \chi_3 / (\chi_2)^{3/2}$       Kurtosis:  $\kappa = \chi_4 / (\chi_2)^2$

$$S\sigma = \chi_3 / \chi_2$$

$$\kappa\sigma^2 = \chi_4 / \chi_2$$

$$M/\sigma^2 = \chi_1 / \chi_2$$

$$S\sigma^3 / M = \chi_3 / \chi_1$$

# Things to keep in mind



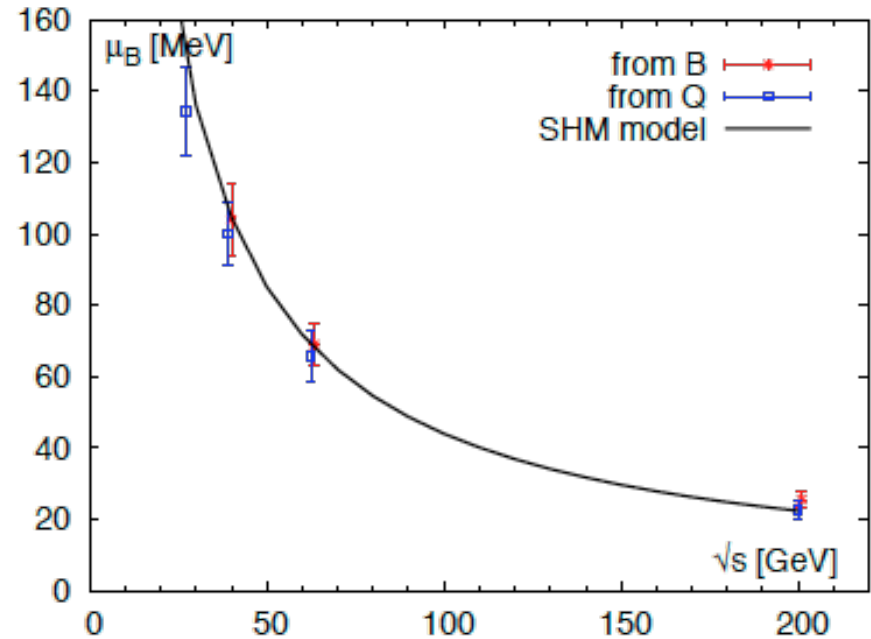
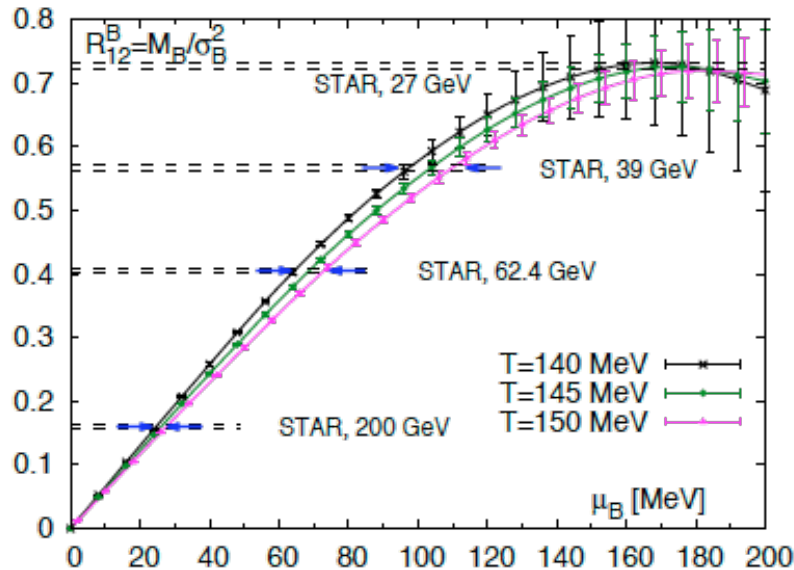
- Effects due to volume variation because of finite centrality bin width
  - Experimentally corrected by centrality-bin-width correction method  
V. Skokov et al., PRC (2013), P. Braun-Munzinger et al., NPA (2017),
- Finite reconstruction efficiency
  - Experimentally corrected based on binomial distribution  
V. Begun and M. Mackowiak-Pawłowska (2017)
- Spallation protons
  - Experimentally removed with proper cuts in  $p_T$   
A. Bzdak, V. Koch, PRC (2012)
- Canonical vs Grand Canonical ensemble
  - Experimental cuts in the kinematics and acceptance  
V. Koch, S. Jeon, PRL (2000)
- Baryon number conservation
  - Experimental data need to be corrected for this effect  
P. Braun-Munzinger et al., NPA (2017)
- Proton multiplicity distributions vs baryon number fluctuations
  - Recipes for treating proton fluctuations  
M. Asakawa and M. Kitazawa, PRC(2012), M. Nahrgang et al., 1402.1238
- Final-state interactions in the hadronic phase
  - Consistency between different charges = fundamental test  
J. Steinheimer et al., PRL (2013)



# Consistency between freeze-out of B and Q



- Independent fit of  $R_{12}^Q$  and  $R_{12}^B$ : consistency between freeze-out chemical potentials



WB: PRL (2014)  
 STAR collaboration, PRL (2014)

$\sqrt{s}$ [GeV]	$\mu_B^f$ [MeV] (from B)	$\mu_B^f$ [MeV] (from Q)
200	$25.8 \pm 2.7$	$22.8 \pm 2.6$
62.4	$69.7 \pm 6.4$	$66.6 \pm 7.9$
39	$105 \pm 11$	$101 \pm 10$
27	-	$136 \pm 13.8$

# Freeze-out line from first principles

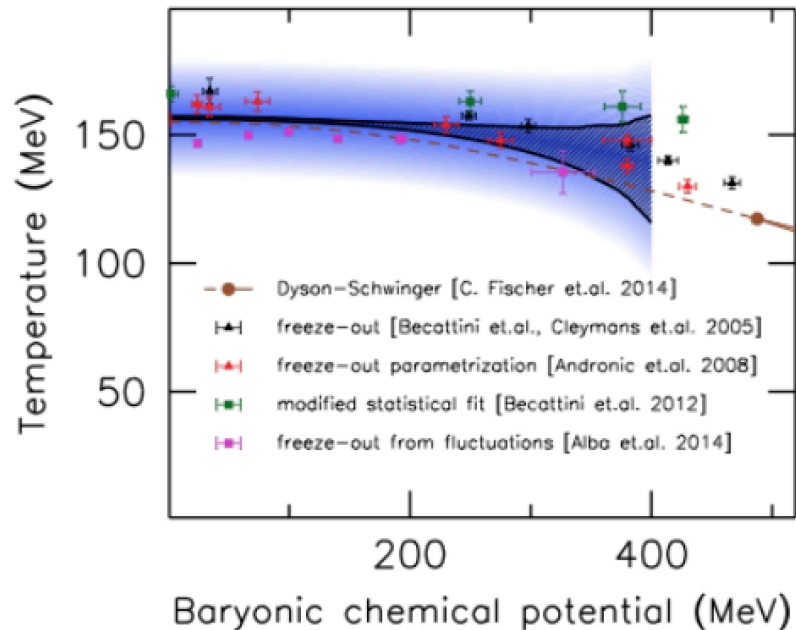
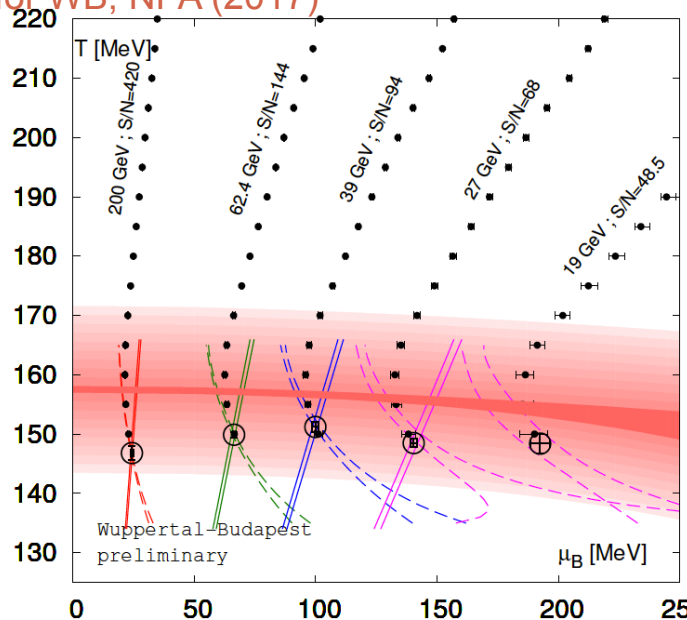


- Use  $T$ - and  $\mu_B$ -dependence of  $R_{12}^Q$  and  $R_{12}^B$  for a combined fit:

$$R_{12}^Q(T, \mu_B) = \frac{\chi_1^Q(T, \mu_B)}{\chi_2^Q(T, \mu_B)} = \frac{\chi_{11}^{QB}(T, 0) + \chi_2^Q(T, 0)q_1(T) + \chi_{11}^{QS}(T, 0)s_1(T)}{\chi_2^Q(T, 0)} \frac{\mu_B}{T} + \mathcal{O}(\mu_B^3).$$

$$R_{12}^B(T, \mu_B) = \frac{\chi_1^B(T, \mu_B)}{\chi_2^B(T, \mu_B)} = \frac{\chi_2^B(T, 0) + \chi_{11}^{BQ}(T, 0)q_1(T) + \chi_{11}^{BS}(T, 0)s_1(T)}{\chi_2^B(T, 0)} \frac{\mu_B}{T} + \mathcal{O}(\mu_B^3)$$

C. Ratti for WB, NPA (2017)



# Kaon fluctuations on the lattice

J. Noronha-Hostler, C.R. et al., 1607.02527

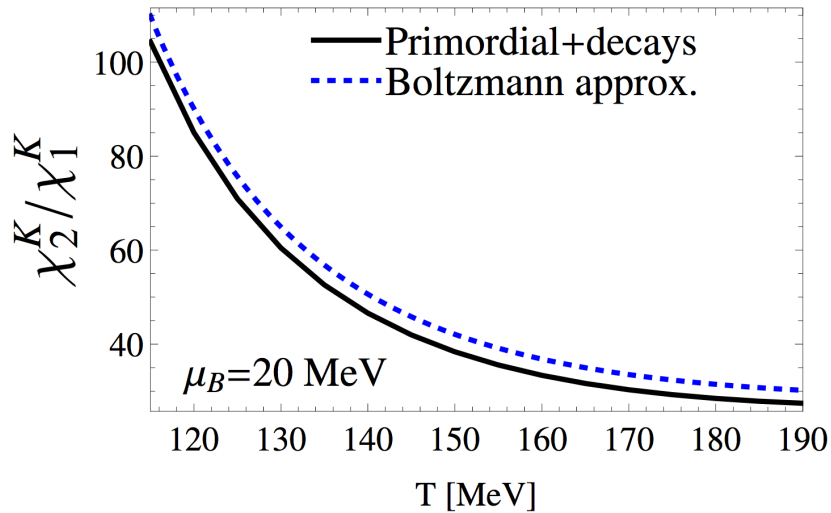
- Lattice QCD works in terms of conserved charges
- Challenge: isolate the fluctuations of a given particle species
- Assuming an HRG model in the Boltzmann approximation, it is possible to write the pressure as:

$$P(\hat{\mu}_B, \hat{\mu}_S) = P_{00}^{BS} + P_{10}^{BS} \cosh(\hat{\mu}_B) + P_{01}^{BS} \cosh(\hat{\mu}_S) + P_{11}^{BS} \cosh(\hat{\mu}_B - \hat{\mu}_S) + P_{12}^{BS} \cosh(\hat{\mu}_B - 2\hat{\mu}_S) + P_{13}^{BS} \cosh(\hat{\mu}_B - 3\hat{\mu}_S)$$

- Kaons in heavy ion collisions: primordial + decays
- Idea: calculate  $\chi_2^K/\chi_1^K$  in the HRG model for the two cases: only primordial kaons in the Boltzmann approximation vs primordial + resonance decay kaons

# Kaon fluctuations on the lattice

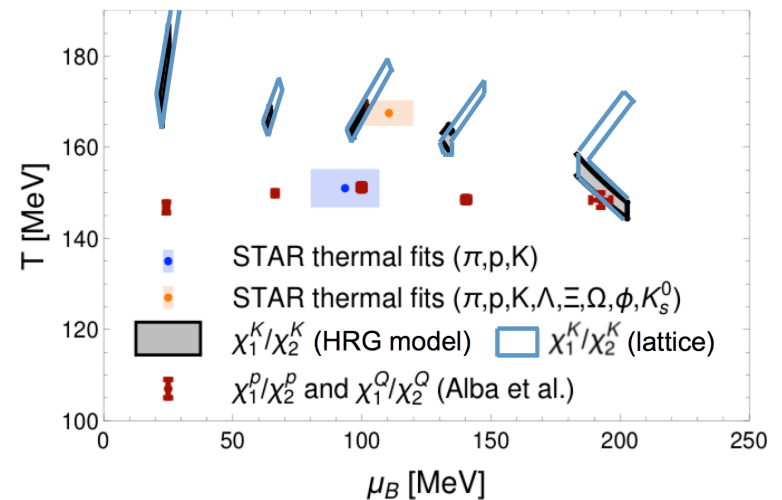
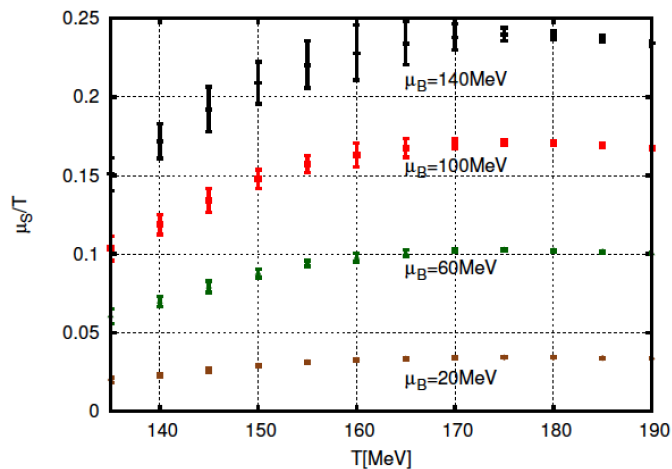
J. Noronha-Hostler, C.R. et al., forthcoming



- Boltzmann approximation works well for lower order kaon fluctuations

$$\frac{\chi_2^K}{\chi_1^K} = \frac{\cosh(\hat{\mu}_S + \hat{\mu}_Q)}{\sinh(\hat{\mu}_S + \hat{\mu}_Q)}$$

- $\chi_2^K/\chi_1^K$  from primordial kaons + decays is very close to the Boltzmann approximation
- $\mu_S$  and  $\mu_Q$  are functions of  $T$  and  $\mu_B$  to match the experimental constraints:



# Fluctuations at the critical point

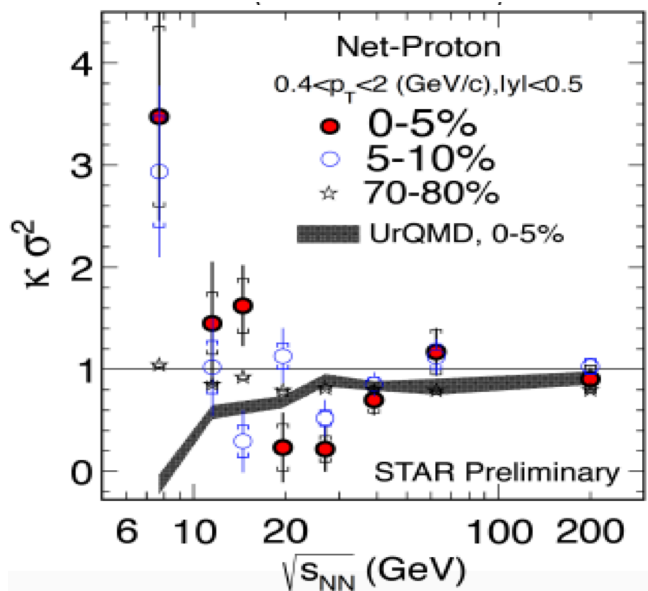
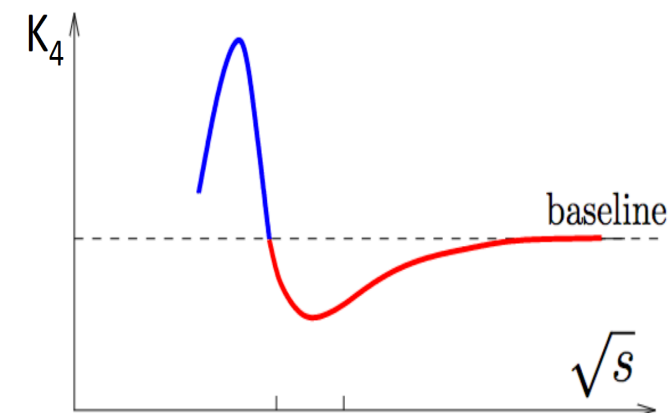
M. Stephanov, PRL (2009).

- **Correlation length** near the critical point  
 $\xi \sim |T - T_c|^{-\nu}$  where  $\nu > 0$

$$\chi_2 = VT\xi^2$$

$$\chi_3 = 2VT^{3/2}\hat{\lambda}_3\xi^{9/2}$$

$$\chi_4 = 6VT^2[2\hat{\lambda}_3^2 - \hat{\lambda}_4]\xi^7$$



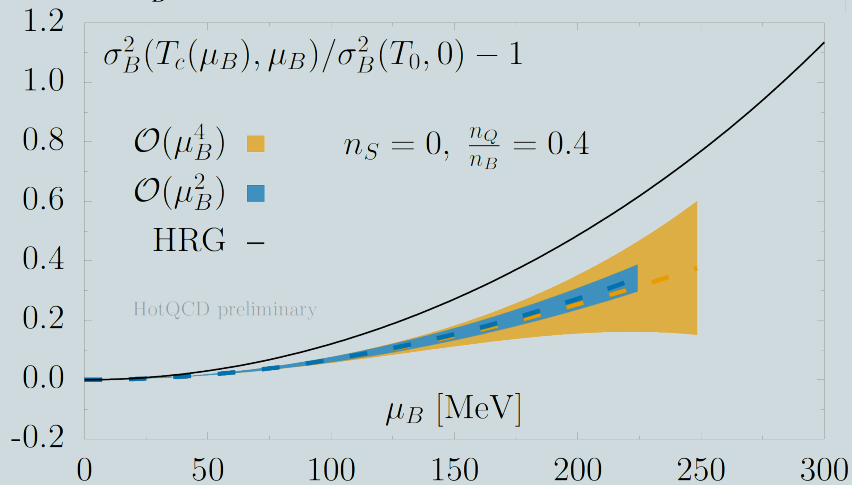
- Fluctuations are expected to diverge at the critical point
- Fourth-order fluctuations should have a non-monotonic behavior
- Preliminary STAR data seem to confirm this
- Can we describe this trend with lattice QCD?

# Fluctuations along the QCD crossover

P. Steinbrecher for HotQCD, 1807.05607

## Net-baryon variance

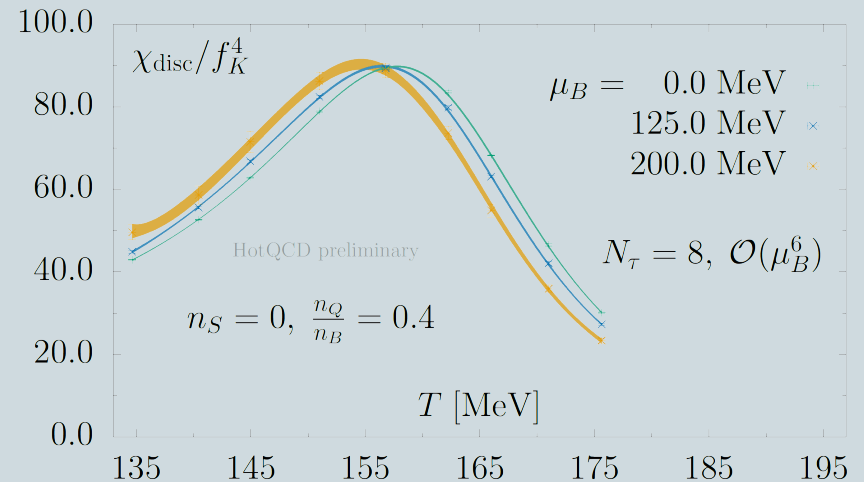
$$\frac{\sigma_B^2(T_c(\mu_B), \mu_B) - \sigma_B^2(T_0, 0)}{\sigma_B^2(T_0, 0)} = \lambda_2 \left(\frac{\mu_B}{T_0}\right)^2 + \lambda_4 \left(\frac{\mu_B}{T_0}\right)^4 + \mathcal{O}(\mu_B^6)$$



- Expected to be larger than HRG model result near the CP
- No sign of criticality

## Disconnected chiral susceptibility

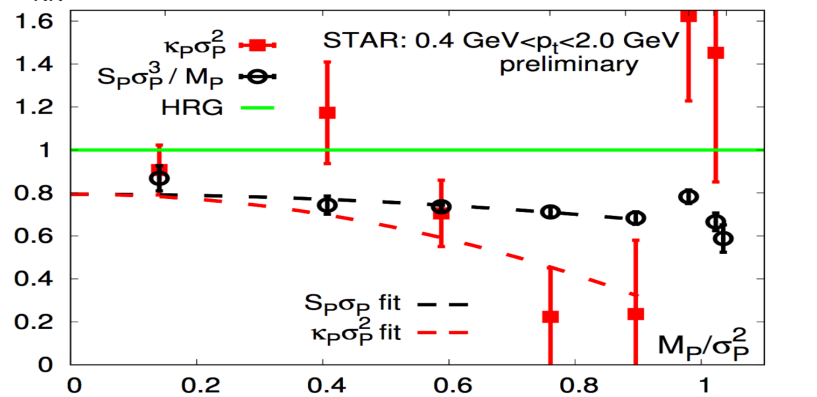
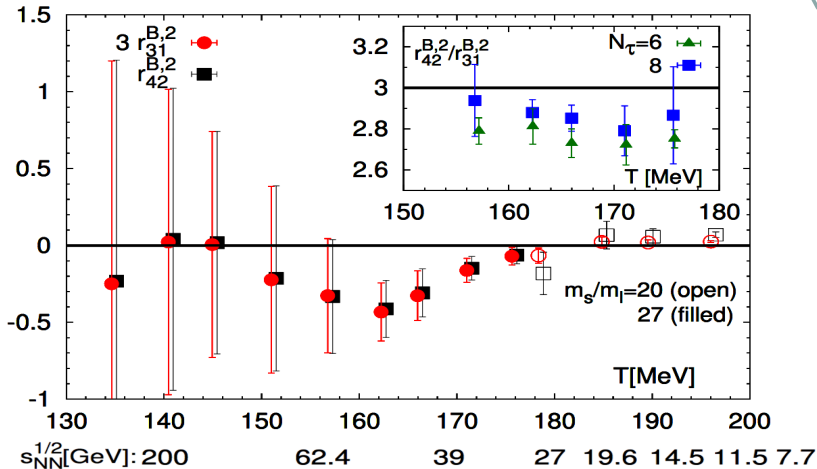
$$\chi_{\text{sub}} \equiv \frac{T}{V} m_s \left( \frac{\partial}{\partial m_u} + \frac{\partial}{\partial m_d} \right) \left[ m_s (\Sigma_u + \Sigma_d) - (m_u + m_d) \Sigma_s \right]$$



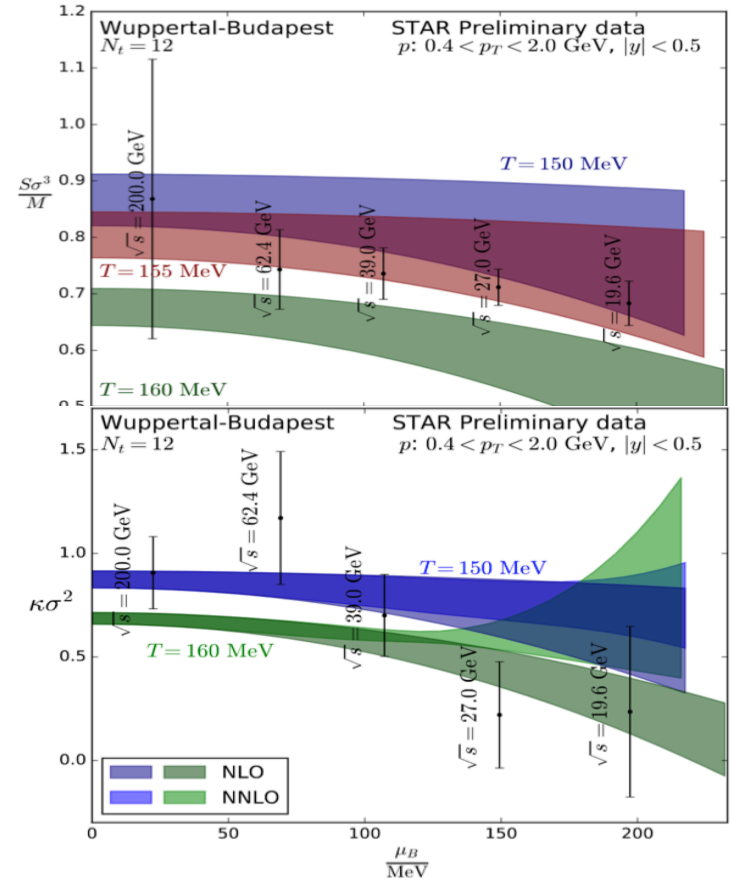
- Peak height expected to increase near the CP
- No sign of criticality

# Higher order fluctuations

HotQCD, PRD (2017)



WB, 1805.04445 (2018)



$$\frac{S_B \sigma_B^3}{M_B} = \frac{\chi_3^B(T, \mu_B)}{\chi_1^B(T, \mu_B)} = \frac{\chi_4^B + s_1 \chi_{31}^{BS} + q_1 \chi_{31}^{BQ}}{\chi_2^B + s_1 \chi_{11}^{BS} + q_1 \chi_{11}^{BQ}} + \mathcal{O}(\mu_B^2) \equiv r_{31}^{B,0} + r_{31}^{B,2} \hat{\mu}_B^2 + \mathcal{O}(\mu_B^4)$$

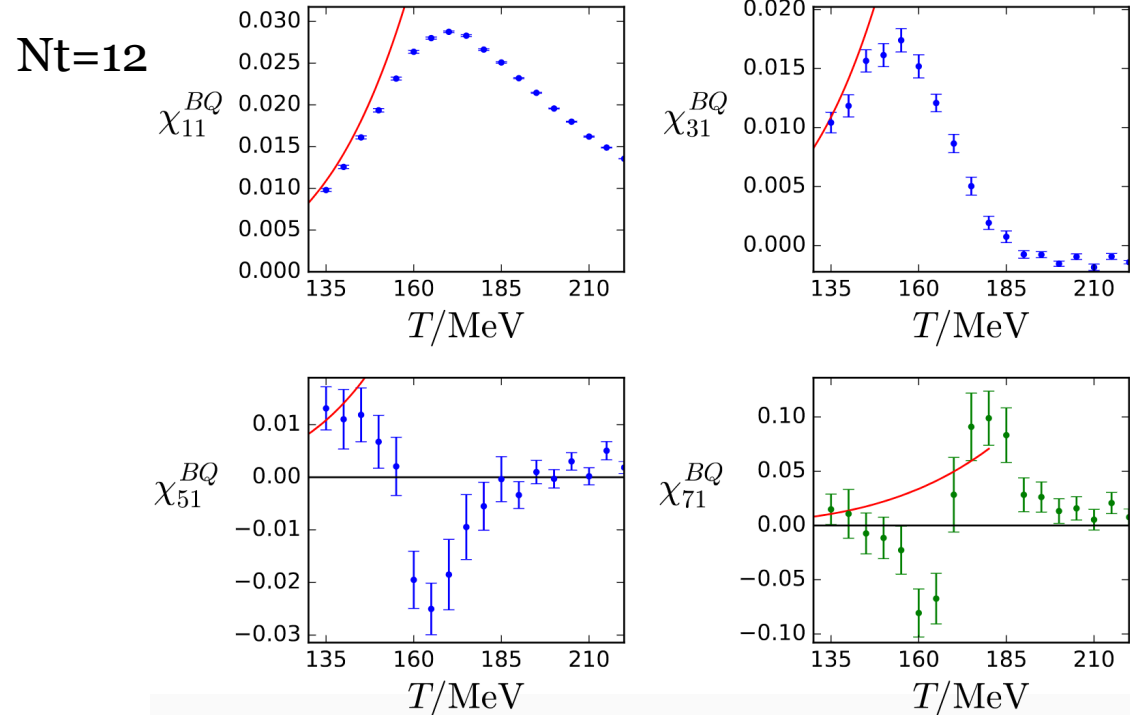
$$\kappa_B \sigma_B^2 = \frac{\chi_4^B(T, \mu_B)}{\chi_2^B(T, \mu_B)} = \frac{\chi_4^B}{\chi_2^B} + \mathcal{O}(\mu_B^2) \equiv r_{42}^{B,0} + r_{42}^{B,2} \hat{\mu}_B^2 + \mathcal{O}(\mu_B^4),$$

A. Rustamov  
@QM2018

Alternative explanation:  
canonical suppression

WB, 1805.04445 (2018)

- Simulation of the lower order correlators at imaginary  $\mu_B$
- Fit to extract higher order terms
- Results exist also for BS, QS and BQS correlators



$$\chi_{11}^{BS}(\hat{\mu}_B) = \chi_{11}^{BS} + \frac{1}{2!} \chi_{31}^{BS} \hat{\mu}_B^2 + \frac{1}{4!} \chi_{51}^{BS} \hat{\mu}_B^4 + \frac{1}{6!} \chi_{71}^{BS} \hat{\mu}_B^6 + \frac{1}{8!} \chi_{91}^{BS} \hat{\mu}_B^8$$

$$\chi_{21}^{BS}(\hat{\mu}_B) = \chi_{31}^{BS} \hat{\mu}_B + \frac{1}{3!} \chi_{51}^{BS} \hat{\mu}_B^3 + \frac{1}{5!} \chi_{71}^{BS} \hat{\mu}_B^5 + \frac{1}{7!} \chi_{91}^{BS} \hat{\mu}_B^7$$

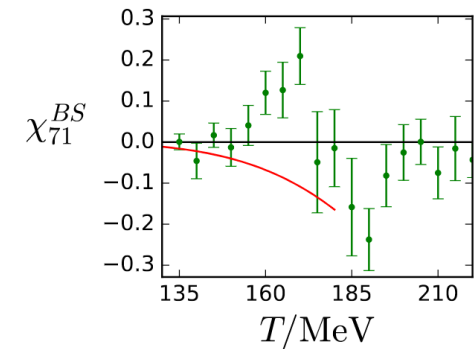
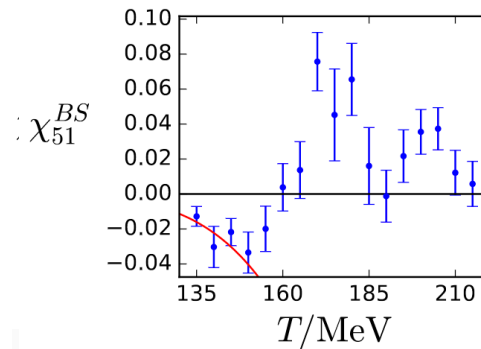
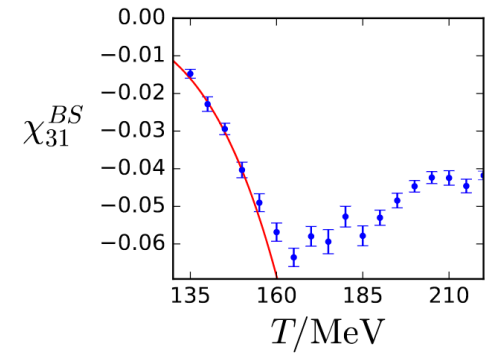
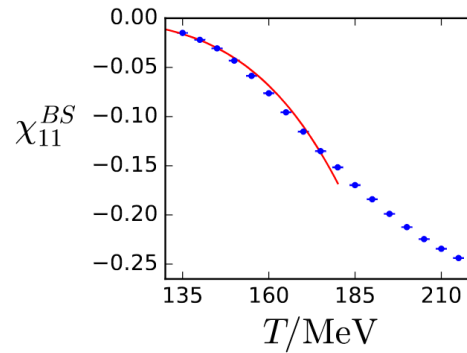
$$\chi_{31}^{BS}(\hat{\mu}_B) = \chi_{31}^{BS} + \frac{1}{2!} \chi_{51}^{BS} \hat{\mu}_B^2 + \frac{1}{4!} \chi_{71}^{BS} \hat{\mu}_B^4 + \frac{1}{6!} \chi_{91}^{BS} \hat{\mu}_B^6$$



WB, 1805.04445 (2018)

- Simulation of the lower order correlators at imaginary  $\mu_B$
- Fit to extract higher order terms
- Results exist also for BS, QS and BQS correlators

Nt=12



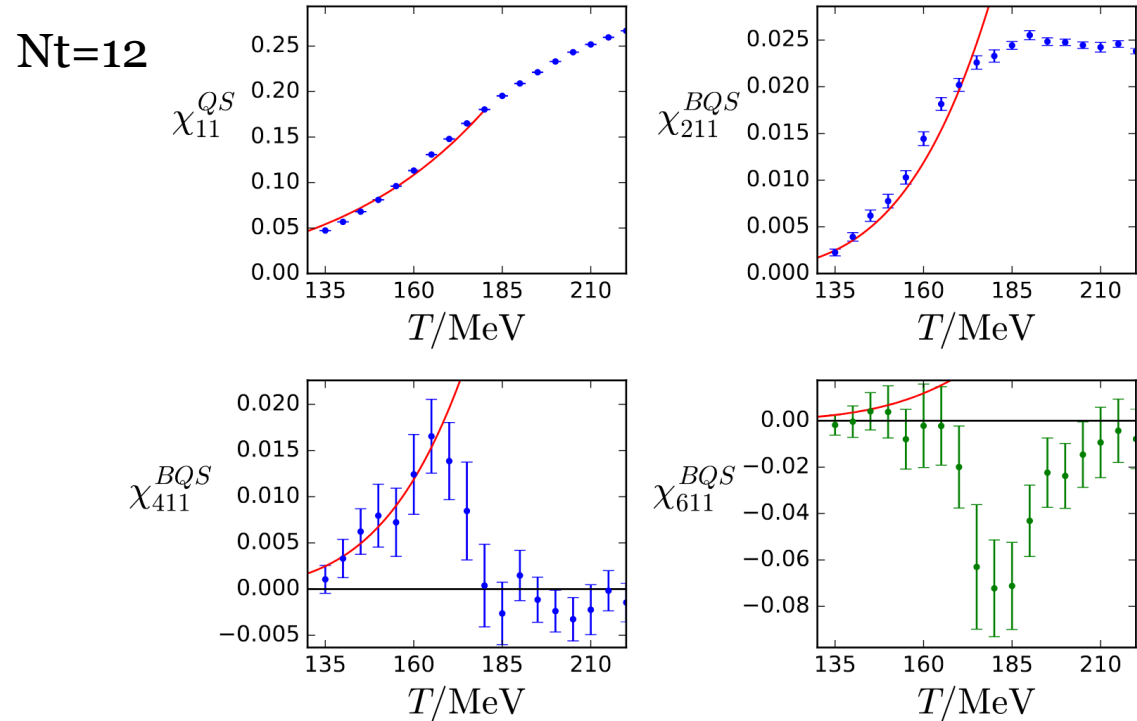
$$\chi_{11}^{BS}(\hat{\mu}_B) = \chi_{11}^{BS} + \frac{1}{2!} \chi_{31}^{BS} \hat{\mu}_B^2 + \frac{1}{4!} \chi_{51}^{BS} \hat{\mu}_B^4 + \frac{1}{6!} \chi_{71}^{BS} \hat{\mu}_B^6 + \frac{1}{8!} \chi_{91}^{BS} \hat{\mu}_B^8$$

$$\chi_{21}^{BS}(\hat{\mu}_B) = \chi_{31}^{BS} \hat{\mu}_B + \frac{1}{3!} \chi_{51}^{BS} \hat{\mu}_B^3 + \frac{1}{5!} \chi_{71}^{BS} \hat{\mu}_B^5 + \frac{1}{7!} \chi_{91}^{BS} \hat{\mu}_B^7$$

$$\chi_{31}^{BS}(\hat{\mu}_B) = \chi_{31}^{BS} + \frac{1}{2!} \chi_{51}^{BS} \hat{\mu}_B^2 + \frac{1}{4!} \chi_{71}^{BS} \hat{\mu}_B^4 + \frac{1}{6!} \chi_{91}^{BS} \hat{\mu}_B^6$$

WB, 1805.04445 (2018)

- Simulation of the lower order correlators at imaginary  $\mu_B$
- Fit to extract higher order terms
- Results exist also for BS, QS and BQS correlators



$$\chi_{11}^{BS}(\hat{\mu}_B) = \chi_{11}^{BS} + \frac{1}{2!} \chi_{31}^{BS} \hat{\mu}_B^2 + \frac{1}{4!} \chi_{51}^{BS} \hat{\mu}_B^4 + \frac{1}{6!} \chi_{71}^{BS} \hat{\mu}_B^6 + \frac{1}{8!} \chi_{91}^{BS} \hat{\mu}_B^8$$

$$\chi_{21}^{BS}(\hat{\mu}_B) = \chi_{31}^{BS} \hat{\mu}_B + \frac{1}{3!} \chi_{51}^{BS} \hat{\mu}_B^3 + \frac{1}{5!} \chi_{71}^{BS} \hat{\mu}_B^5 + \frac{1}{7!} \chi_{91}^{BS} \hat{\mu}_B^7$$

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# Other approaches I did not have time to address



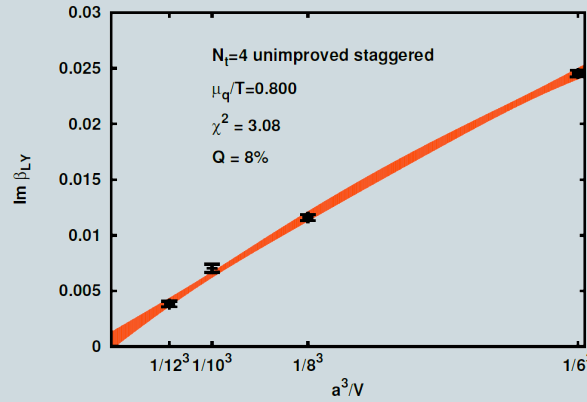
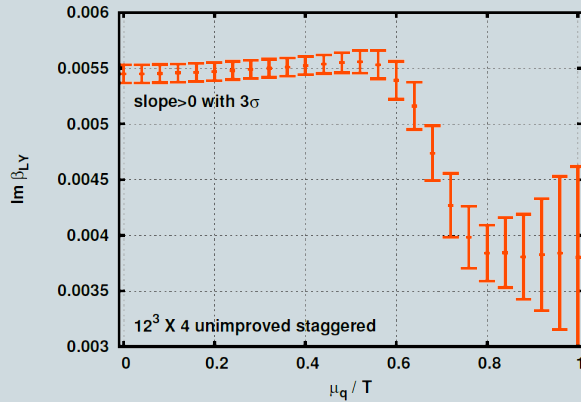
- Reweighting techniques (Fodor & Katz)
- Canonical ensemble (Alexandru et al., Kratochvila, de Forcrand, Ejiri, Bornyakov, Goy, Lombardo, Nakamura)
- Density of state methods (Fodor, Katz & Schmidt, Alexandru et al.)
- Two-color QCD (ITEP Moscow lattice group, Kogut et al., S. Hands et al., von Smekal et al.)
- Scalar field theories with complex actions (See talk by M. Ogilvie on Tuesday)
- Complex Langevin (see talks by D. Sinclair, S. Tsutsui, F. Attanasio, Y. Ito, A. Joseph on Monday)
- Lefshetz Thimble (see talks by K. Zambello, S. Lawrence, N. Warrington, H. Lamm on Monday)
- Phase unwrapping (see talks by G. Kanwar and M. Wagman on Friday)

# Radius of convergence of Taylor series

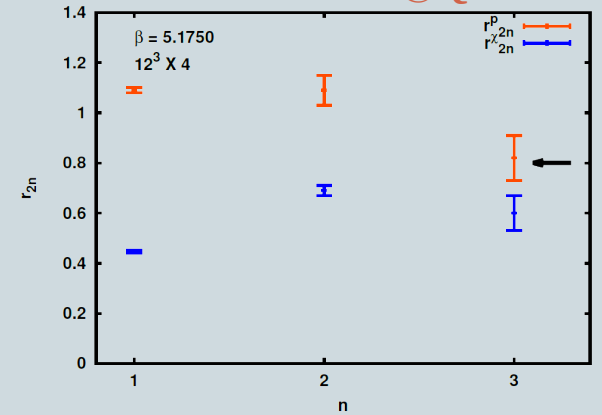


Plenary talk by Sayantan Sharma on Tuesday

- For a genuine phase transition, we expect the  $\infty$ -volume limit of the Lee-Yang zero to be real

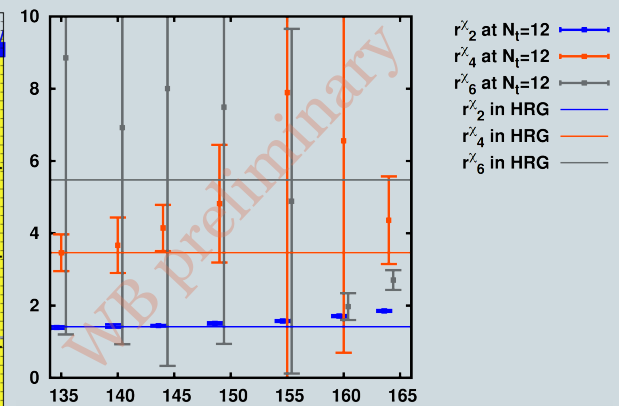
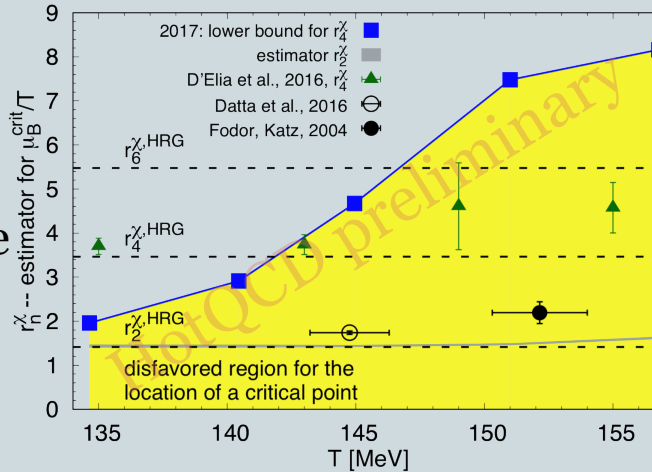


A. Pasztor for WB @QM2018



$$r_{2n}^X = \left| \frac{2n(2n-1)\chi_{2n}^B}{\chi_{2n+2}^B} \right|^{1/2}$$

- It grows as  $\sim n$  in the HRG model



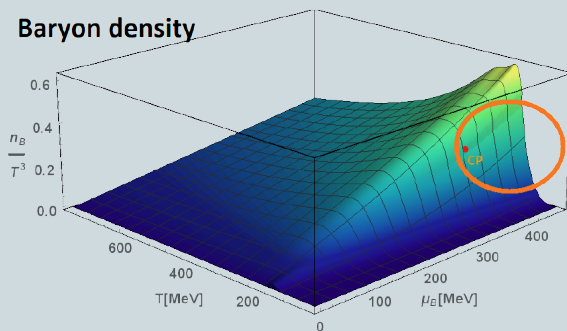
# Alternative EoS at large densities

## EoS for QCD with a 3D-Ising critical point

$$T^4 c_n^{\text{LAT}}(T) = T^4 c_n^{\text{Non-Ising}}(T) + T_c^4 c_n^{\text{Ising}}(T)$$

P. Parotto et al., 1805.05249 (2018)

- Implement scaling behavior of 3D-Ising model EoS
- Define map from 3D-Ising model to QCD
- Estimate contribution to Taylor coefficients from 3D-Ising model critical point
- Reconstruct full pressure



- Density discontinuous at  $\mu_B > \mu_{BC}$

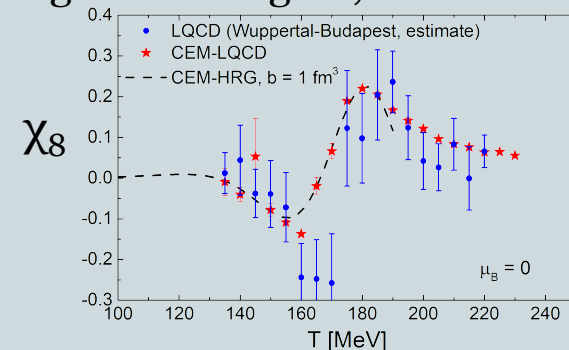
## Cluster expansion model

Vovchenko, Steinheimer, Philipsen, Stoecker, 1711.01261

- HRG-motivated fugacity expansion for  $\rho_B$
- $$\frac{\rho_B(T, \mu_B)}{T^3} = \chi_1^B(T, \mu_B) = \sum_{k=1}^{\infty} b_k(T) \sinh(k\mu_B/T)$$
- $b_1(T)$  and  $b_2(T)$  are model inputs
  - All higher order coefficients predicted:

$$b_k(T) = \alpha_k^{SB} \frac{[b_2(T)]^{k-1}}{[b_1(T)]^{k-2}}$$

- Physical picture: HRG with repulsion at moderate  $T$ , “weakly” interacting quarks and gluons at high  $T$ , no CP



- Plan: integrate  $\rho_B$  and get  $p(T, \mu_B)$

# How can lattice QCD support the experiments?



- Equation of state
  - Needed for **hydrodynamic** description of the QGP
- QCD phase diagram
  - Transition line at finite density
  - Constraints on the location of the critical point
- Fluctuations of conserved charges
  - Can be **simulated** on the lattice and **measured** in experiments
  - Can give information on the **evolution** of heavy-ion collisions
  - Can give information on the **critical point**

# Hadron Resonance Gas model

Dashen, Ma, Bernstein; Prakash, Venugopalan; Karsch, Tawfik, Redlich

- **Interacting** hadronic matter in the **ground state** can be well approximated by a **non-interacting** resonance gas
- The pressure can be written as:

$$p^{HRG}/T^4 = \frac{1}{VT^3} \sum_{i \in \text{mesons}} \ln \mathcal{Z}_{m_i}^M(T, V, \mu_{X^a}) + \frac{1}{VT^3} \sum_{i \in \text{baryons}} \ln \mathcal{Z}_{m_i}^B(T, V, \mu_{X^a})$$

where

$$\ln \mathcal{Z}_{m_i}^{M/B} = \mp \frac{V d_i}{2\pi^2} \int_0^\infty dk k^2 \ln(1 \mp z_i e^{-\varepsilon_i/T}) ,$$

with energies  $\varepsilon_i = \sqrt{k^2 + m_i^2}$ , degeneracy factors  $d_i$  and fugacities

$$z_i = \exp \left( \left( \sum_a X_i^a \mu_{X^a} \right) / T \right) .$$

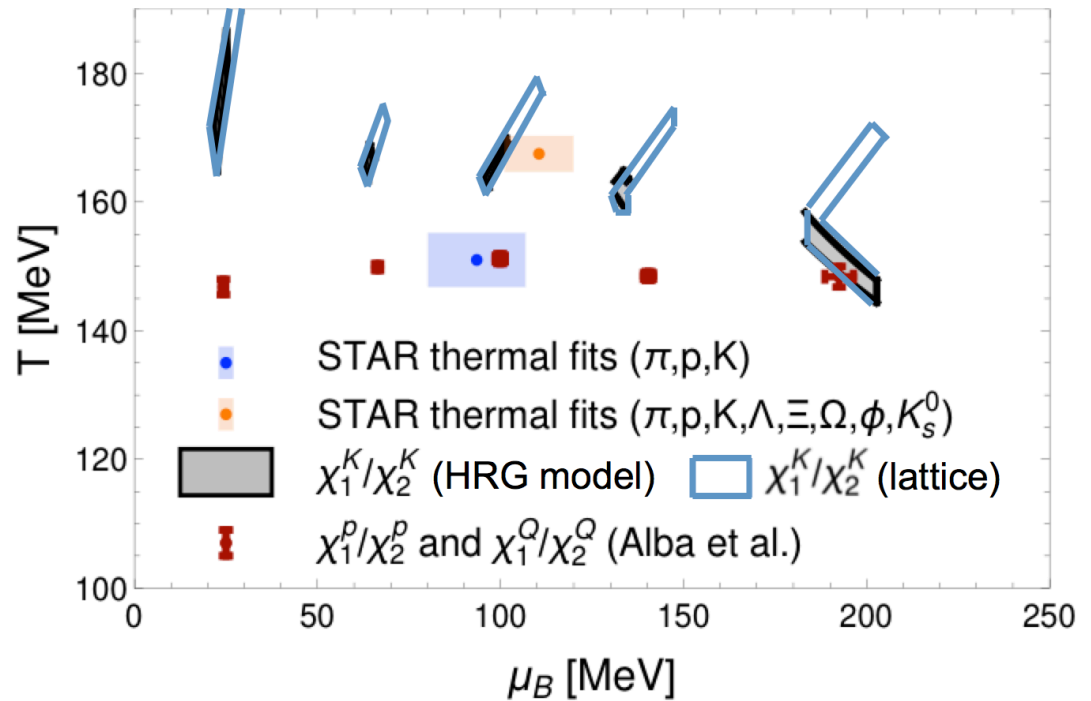
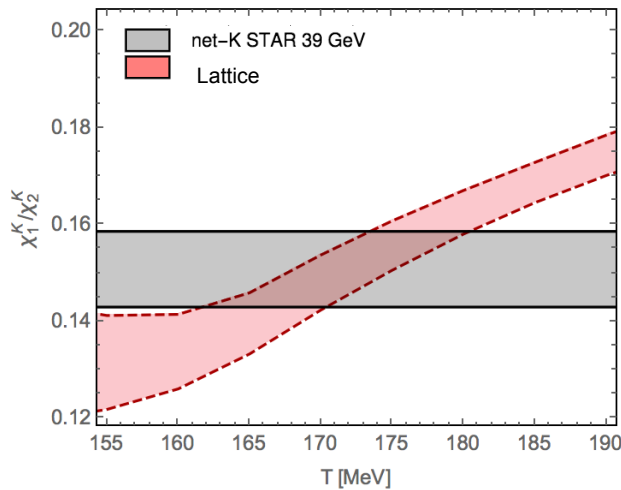
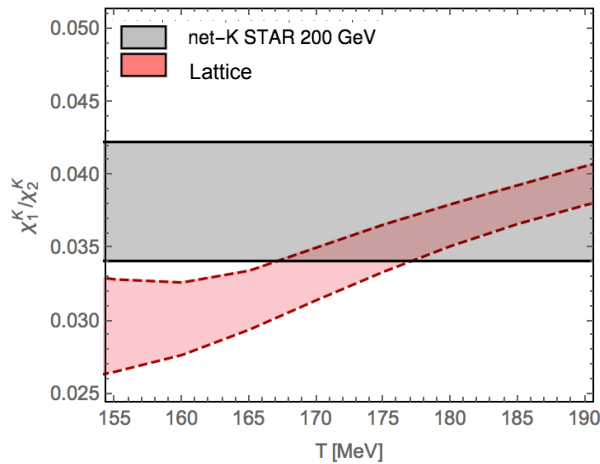
$X^a$ : all possible conserved charges, including the baryon number  $B$ , electric charge  $Q$ , strangeness  $S$ .

- Fugacity expansion for  $\mu_S = \mu_Q = 0$ :  $\frac{p_B}{T^4} = \sum_{i \in B} \frac{d_i}{\pi^2} \left( \frac{m_i}{T} \right)^2 \sum_{N=1}^{\infty} (-1)^{N+1} N^{-2} K_2 \left( N \frac{m_i}{T} \right) \cosh \left[ N \frac{\mu_B}{T} \right]$

**Boltzmann approximation:  $N=1$**

# Kaon fluctuations on the lattice

J. Noronha-Hostler, C.R. et al. forthcoming



- Lattice QCD temperatures have a large uncertainty but they are above the light flavor ones

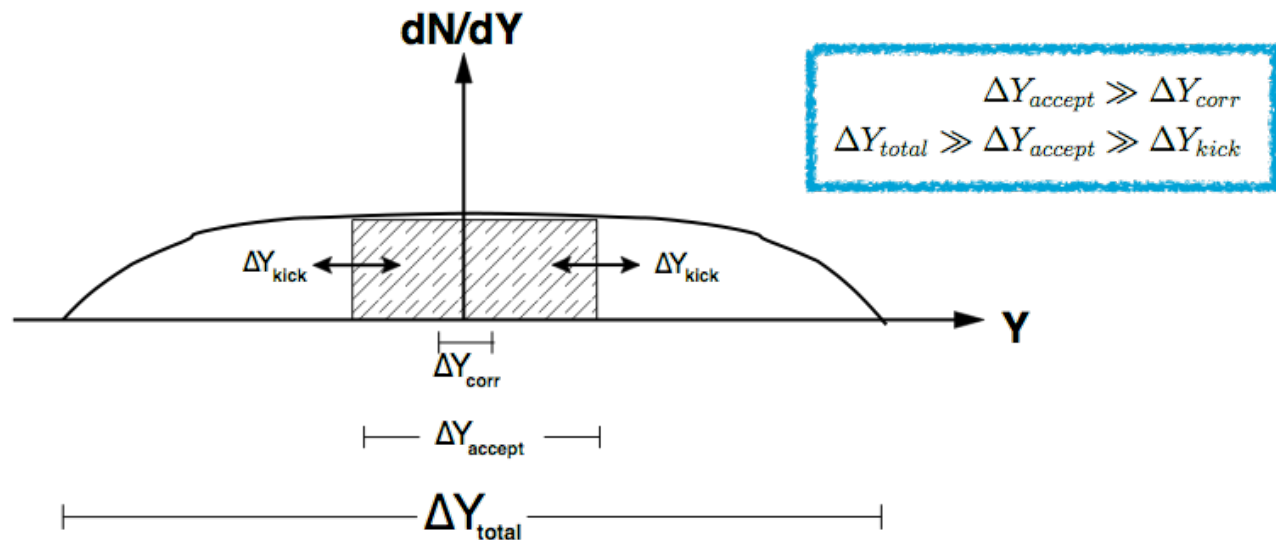


# Fluctuations of conserved charges?



\* If we look at the **entire system**, **none of the conserved charges will fluctuate**

\* By studying a sufficiently **small subsystem**, the fluctuations of conserved quantities become meaningful



- $\Delta Y_{total}$ : range for total charge multiplicity distribution
- $\Delta Y_{accept}$ : interval for the accepted charged particles
- $\Delta Y_{kick}$ : rapidity shift that charges receive during and after hadronization

# QCD matter under extreme conditions

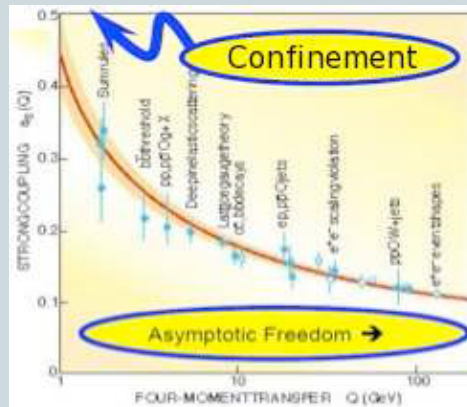


To address these questions we need fundamental theory and experiment

## Theory: Quantum Chromodynamics

- QCD is the fundamental theory of strong interactions
- It describes interactions among quarks and gluons

$$L_{QCD} = \sum_{i=1}^{n_f} \bar{\Psi}_i \gamma_\mu \left( i\partial^\mu - g A_a^\mu \frac{\lambda_a}{2} \right) \Psi_i - m_i \bar{\Psi}_i \Psi_i - \frac{1}{4} \sum_a F_a^{\mu\nu} F_a^{\mu\nu}$$



## Experiment: heavy-ion collisions



- Quark-gluon plasma (QGP) discovery at RHIC and the LHC
- QGP is a strongly interacting (almost) perfect fluid

# Cumulants of multiplicity distribution



- Deviation of  $N_Q$  from its mean in a single event:  $\delta N_Q = N_Q - \langle N_Q \rangle$
- The **cumulants** of the event-by-event distribution of  $N_Q$  are:

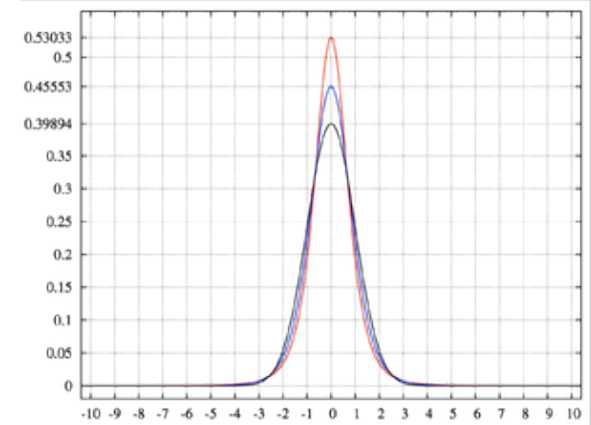
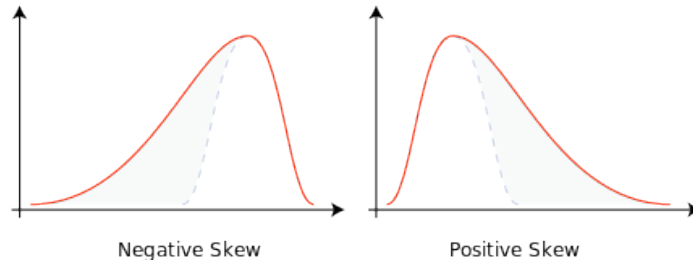
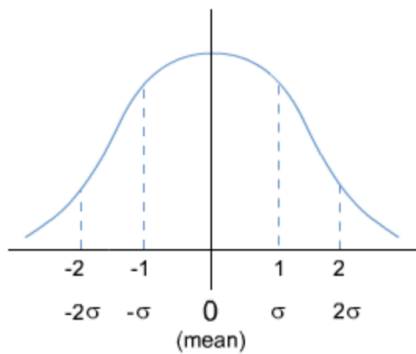
$$\chi_2 = \langle (\delta N_Q)^2 \rangle \quad \chi_3 = \langle (\delta N_Q)^3 \rangle \quad \chi_4 = \langle (\delta N_Q)^4 \rangle - 3 \langle (\delta N_Q)^2 \rangle^2$$

- The cumulants are related to the central moments of the distribution by:

variance:  $\sigma^2 = \chi_2$

Skewness:  $S = \chi_3 / (\chi_2)^{3/2}$

Kurtosis:  $\kappa = \chi_4 / (\chi_2)^2$

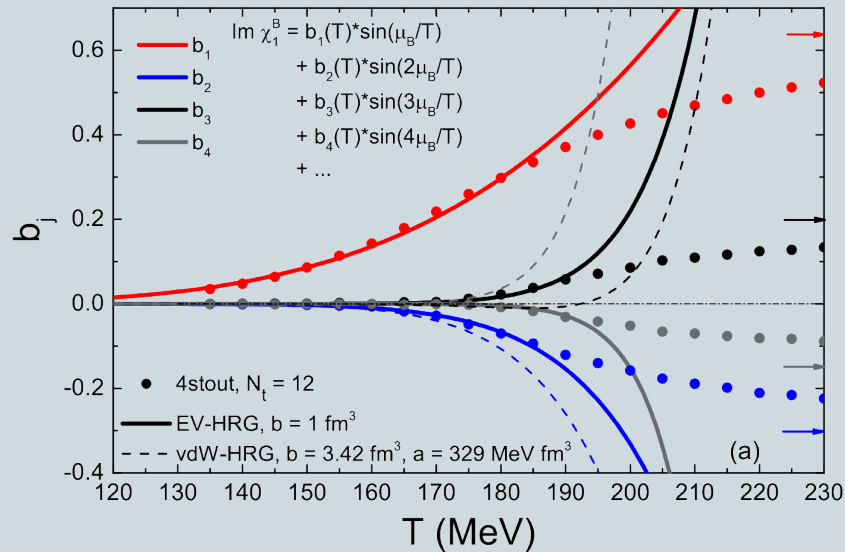


# Fluctuations and hadrochemistry

$$\chi_1^B(T, \mu_B) = \frac{\rho_B(T, \mu_B)}{T^3} = \sum_{k=1}^{\infty} b_k(T) \sinh(k \mu_B/T)$$

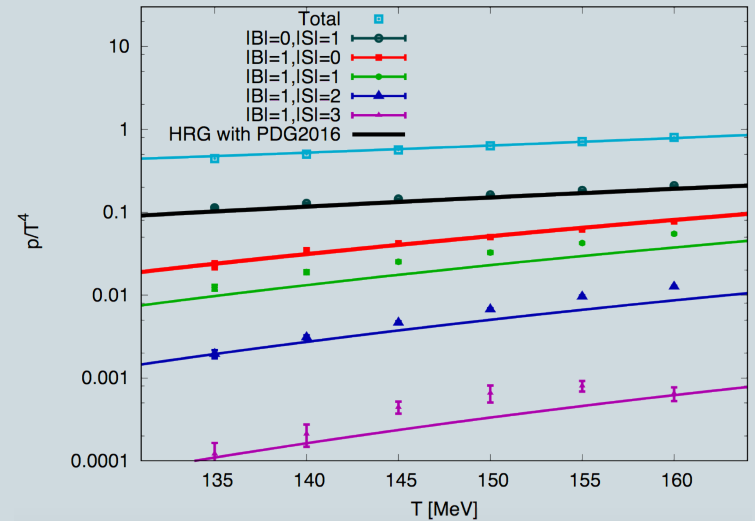
$$P(\hat{\mu}_B, \hat{\mu}_S) = P_{00}^{BS} + P_{10}^{BS} \cosh(\hat{\mu}_B) + P_{01}^{BS} \cosh(\hat{\mu}_S) + P_{11}^{BS} \cosh(\hat{\mu}_B - \hat{\mu}_S) + P_{12}^{BS} \cosh(\hat{\mu}_B - 2\hat{\mu}_S) + P_{13}^{BS} \cosh(\hat{\mu}_B - 3\hat{\mu}_S)$$

V. Vovchenko et al., PLB (2017)



- Consistent with HRG at low temperatures
- Consistent with approach to ideal gas limit
- $b_2$  departs from zero at  $T \sim 160$  MeV
- Deviation from ideal HRG

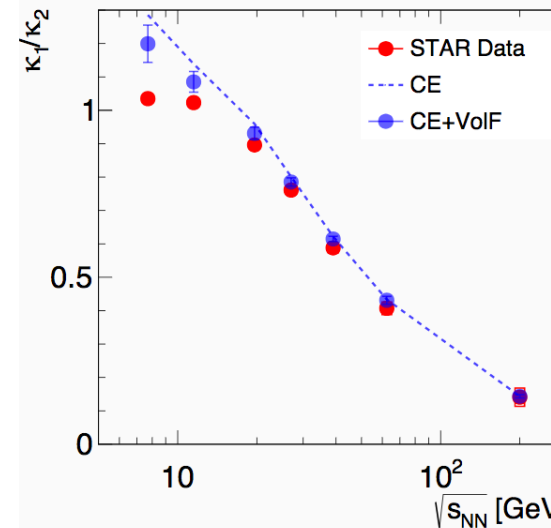
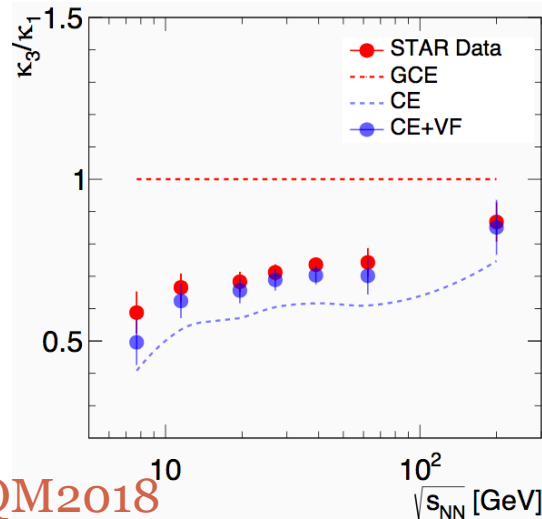
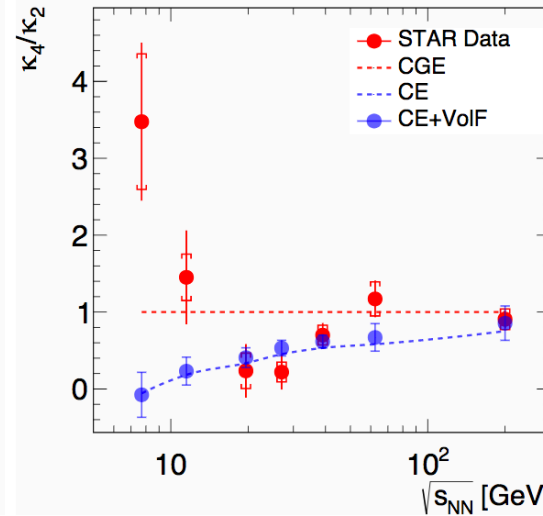
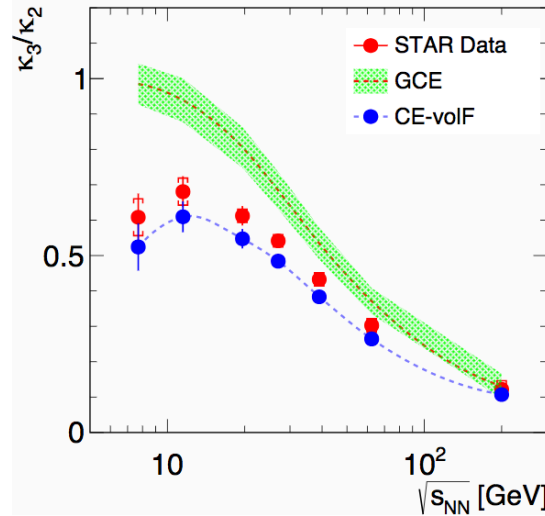
P. Alba et al., PRD (2017)



- Need of additional strange hadrons, predicted by the Quark Model but not yet detected
- First pointed out in Bazavov et al., PRL(2014)

(see talk by J. Glesaaen on Friday)

# Canonical suppression

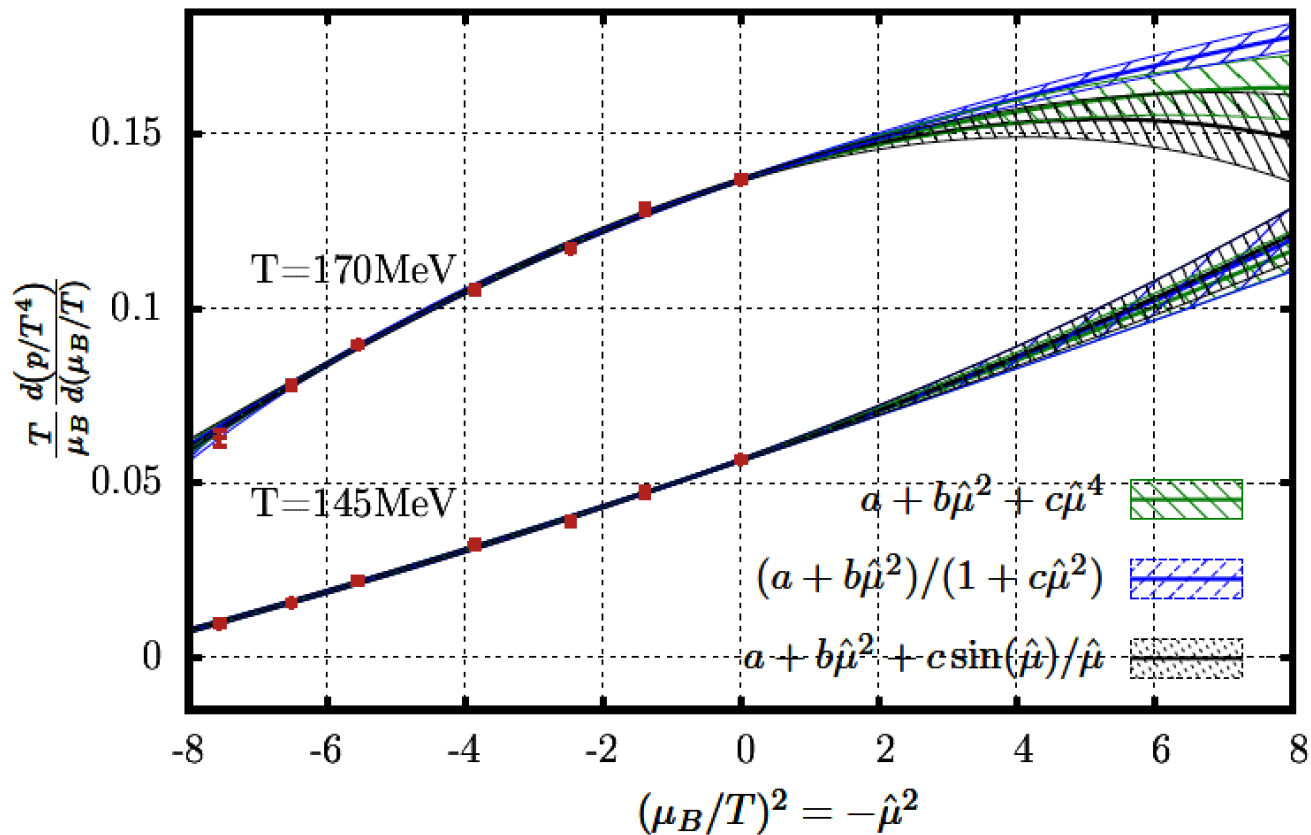


A. Rustamov @QM2018

above 11.5 GeV CE suppression accounts for measured deviations from GCE

# Analytical continuation – illustration of systematics

Analytical continuation on  $N_t = 12$  raw data



# Analytical continuation – illustration of systematics

$$\text{Condition: } \chi_8 \lesssim \chi_4 \longrightarrow f(\hat{\mu}_B) = a + b\hat{\mu}_B^2 + c\hat{\mu}_B^4 + \frac{b\epsilon}{840}\hat{\mu}_B^6$$

Analytical continuation on  $N_t = 12$  raw data

