Precision kaon physics and lattice QCD

Chris Sachrajda

(RBC-UKQCD Collaborations)

Department of Physics and Astronomy
University of Southampton
Southampton SO17 1BJ
UK

Towards the Ultimate Precision in Flavour Physics
IPPP - Durham
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LEVERHULME TRUST

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1. Radiative corrections to decay amplitudes (with colleagues from Rome 1,2,3)
2. Evaluating long-distance contributions (with RBC/UKQCD collaborations)
3. $K \to \pi\pi$ decays (with RBC/UKQCD collaborations)

A full set of references is presented at the end of the slides.

- For many standard quantities, including in kaon physics, lattice results now have remarkable precision, e.g. from $N_f = 2+1+1$ simulations we have:

$$f_K = 155.7(3)\,\text{MeV}\,; \quad \frac{f_K}{f_\pi} = 1.1932(19)\,; \quad f_+(0) = 0.9707(27)\,.$$

2019 FLAG Review, arXiv:1902.08191

  - By “standard", I mean the evaluation of matrix elements of the form

$$\langle 0|O(0)|H\rangle \quad \text{or} \quad \langle H_2|O(0)|H_1\rangle,$$

where H, H_1, H_2 are single hadrons and $O(0)$ is a local composite operator.

- Calculations such as these enable the determination of CKM matrix elements, e.g.

$$\Gamma(K^- \to \ell^- \bar{\nu}_\ell) = \frac{G_F^2\,|V_{us}|^2 f_K^2}{8\pi}\, m_K\, m_\ell^2 \left(1 - \frac{m_\ell^2}{m_K^2}\right)^2 .$$

- Since radiative and isospin-breaking corrections are of

$$O(\alpha_{\text{em}}) \quad \text{and} \quad O\left(\frac{m_d - m_u}{\Lambda_{\text{QCD}}}\right) \sim O(1\%)$$

these effects must be included for further progress in flavour physics phenomenology.

- With colleagues in Rome, we have been developing and exploiting techniques to compute electromagnetic and strong isospin breaking effects in lattice simulations. see CTS in 2018 TUPiFP in Warwick
- Selected results:

$$\Gamma(\pi^\pm \to \mu^\pm \nu_\ell[\gamma]) = (1.0159 \pm 0.0020)\,\Gamma^{(0)}(\pi^\pm \to \mu^\pm \nu_\ell),$$

$$\Gamma(K^\pm \to \mu^\pm \nu_\ell[\gamma]) = (1.0032 \pm 0.0011)\,\Gamma^{(0)}(K^\pm \to \mu^\pm \nu_\ell),$$

  where $\Gamma^{(0)}$ is the result in isosymmetric QCD defined in the GRS scheme which is a particular (and conventionally used) definition.
  - IB corrections are 1.6% for $\pi_{\mu 2}$ decays and 0.3% for $K_{\mu 2}$ decays, in line with expectations.
  - These results can be compared to 1.76(21)% and 0.64(24)% found using ChPT and large N_C (and adopted by the PDG).
    V.Cirigliano and H.Neufeld, arXiv:1102.0563
  - We find $V_{us} = 0.22533(38)$ and $V_{ud}^2 + V_{us}^2 + V_{ub}^2 = 0.99986(44)$.
- For leptonic and semileptonic decays, it would be helpful to have the experimental distributions as a function of the cut-off on the final-state photon's energy (particularly for π_{e2} and K_{e2} decays).

Finite-Volume Effects

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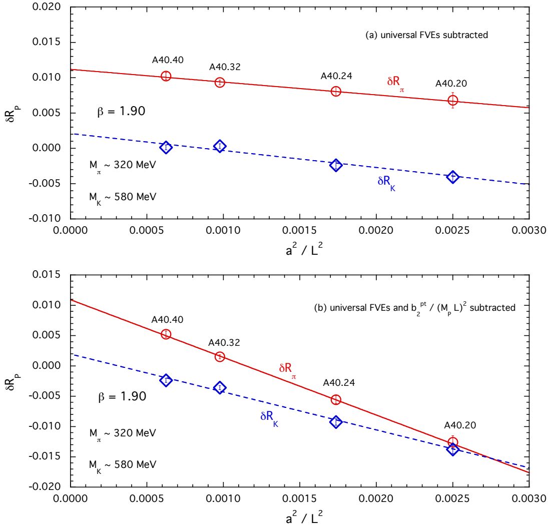


The RBC & UKQCD collaborations

BNL and BNL/RBRC

Yasumichi Aoki (KEK)
Mattia Bruno
Taku Izubuchi
Yong-Chull Jang
Chulwoo Jung
Christoph Lehner
Meifeng Lin
Aaron Meyer
Hiroshi Ohki
Shigemi Ohta (KEK)
Amarjit Soni

UC Boulder

Oliver Witzel

Columbia University

Ziyuan Bai
Norman Christ
Duo Guo
Christopher Kelly
Bob Mawhinney
Masaaki Tomii
Jiqun Tu
Bigeng Wang
Tianle Wang
Evan Wickenden
Yidi Zhao

University of Connecticut

Tom Blum
Dan Hoying (BNL)
Luchang Jin (RBRC)
Cheng Tu

Edinburgh University

Peter Boyle
Guido Cossu
Luigi Del Debbio
Tadeusz Janowski
Richard Kenway
Julia Kettle
Fionn O'haigan
Brian Pendleton
Antonin Portelli
Tobias Tsang
Azusa Yamaguchi

KEK

Julien Frison

University of Liverpool

Nicolas Garron

MIT

David Murphy

Peking University

Xu Feng

University of Southampton

Jonathan Flynn
Vera Guelpers
James Harrison
Andreas Juettner
James Richings
Chris Sachrajda

Stony Brook University

Jun-Sik Yoo
Sergey Syritsyn (RBRC)

York University (Toronto)

Renwick Hudspith

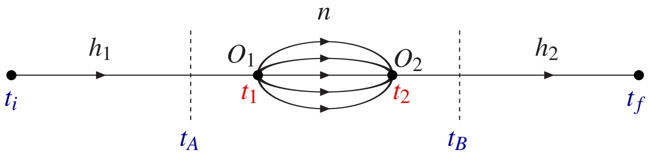
- RBC-UKQCD Collaborations are developing and exploiting techniques to evaluate long-distance contributions to kaon physics, i.e. evaluating matrix elements of the form

$$\int d^4x \, \langle f \,|\, T[O_1(x)\, O_2(0)] \,|\, i\rangle \, .$$

  - Long-distance here means scales $\gtrsim \frac{1}{m_c}$.
  - As well as computing the non-perturbative long-distance contributions from scales of $O(\Lambda_{\rm QCD})$, we aim to avoid the necessity of performing perturbation theory at the scale of m_c. For Δm_K this has proved particularly slowly convergent. J.Brod & M.Gorbahn, arXiv:1108.2036

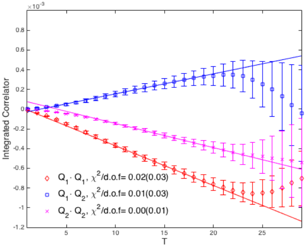
- The techniques are being applied to

  1. $\Delta m_K = m_{K_L} - m_{K_S}$ and ε_K;
  2. Rare kaon decays $K^+ \to \pi^+ \ell^+ \ell^-$ and $K_S \to \pi^0 \ell^+ \ell^-$;
  3. The rare kaon decay $K^+ \to \pi^+ \nu\bar{\nu}$.



1. Fiducial volume: the integration over $t_{1,2}$ is performed in a large, but finite, interval ($t_A \leq t_{1,2} \leq t_B$). This is required to allow sufficiently large intervals $t_A - t_i$ and $t_f - t_B$ to ensure that it is indeed the hadrons $h_{1,2}$ in the initial and final states.
2. Growing exponentials: If there are intermediate states n with lower energies that those of the external states, then unphysical terms of relative size $e^{(E_{i,f}-E_n)T}$ (where $T = t_B - t_A$) are generated.
   - For kaon physics the number of such terms is small and can be handled. For heavy mesons this is much more challenging.
3. Renormalisation: New UV divergences may be generated as $x_1 \to x_2$.
   - For Δm_K and $K \to \pi \ell^+ \ell^-$ decays this doesn't happen with $N_f = 4$.
   - The additional renormalisation, necessary for ε_K and $K \to \pi \nu \bar{\nu}$ decays, has been developed and implemented. N.H.Christ, X.Feng, CTS, A.Portelli, arXiv:1605.04442
4. Finite-volume effects: Power-like FV effects can be calculated by developing an extension of the Lüscher formalism. N.H.Christ, X.Feng, G.Martinelli, CTS, arXiv:1504.01170

- Following the development of the theoretical background and exploratory numerical studies, we presented the first numerical results at physical masses at Lattice 2017 Z.Bai, N.H.Christ, CTS; EPJ Web Conf 175 (2018) 13017
  and updated them at Lattice 2018 B.Wang; arXiv:1812.05302.
- The calculation is performed on a $64^3 \times 128 \times 12$ lattice with Möbius DWF and the Iwasaki gauge action. a^{-1}=2.359(7) GeV, $m_\pi = 135.9(3)$ MeV and $m_K = 496.9(7)$ MeV. T.Blum et al., RBC-UKQCD Collabs., arXiv:1411.7017
  Charm-physics studies with this action $\Rightarrow am_c \simeq 0.32 - 0.33$. We have used $am_c \simeq 0.31$ and studied the dependence on m_c.



all diagrams

- 2018 preliminary result is
$$\Delta m_K = 7.9(1.3)(2.1) \times 10^{-12}\,\mathrm{MeV}\,,$$
  to be compared to the physical value
$$(\Delta m_K)^{\rm phys} = 3.483(6) \times 10^{-12}\,\mathrm{MeV}\,.$$
- The dominant systematic error is due to discretisation effects because $am_c \simeq 0.31$.

- Future project planned on finer lattices at SUMMIT.

RBC-UKQCD, **Z.Bai** et al.

$$\varepsilon_K^{\text{Exp}} = 2.228(11) \times 10^{-3}$$

- There has been no journal publication on the long-distance contribution to ε_K even though the whole theoretical background has been developed.
- A number of conference papers have been presented including:

  "Long distance part of ε_K from lattice QCD"

  Z.Bai, arXiv:1611.06601
- The preliminary results below were obtained from 200 configurations on a $N_f = 2+1$ flavour ensemble using DWF and Iwasaki gauge action on a $24^3 \times 64 \times 16$ lattice with $a^{-1} = 1.78\,$GeV. C.Allton et al, arXiv:0804.0473
- The quark masses are unphysical, $m_\pi = 339\,$MeV, $m_K \simeq 592\,$MeV and $m_c^{\overline{\text{MS}}}(2\,\text{GeV}) = 968\,$MeV.
- Our preliminary result for the LD contribution at these unphysical masses is

$$\varepsilon_K^{\text{LD}} = 0.11(0.08) \times 10^{-3}\,.$$

  - The central (unphysical) value is about 5% of the physical ε_K which is consistent with expectations of the long-distance contribution.
- It is planned to restart the computation of the long-distance contributions to ε_K in the autumn?

G.Isidori, G.Martinelli and P.Turchetti, hep-lat/0506026

- We now turn to the CPC decays $K_S \to \pi^0 \ell^+ \ell^-$ and $K^+ \to \pi^+ \ell^+ \ell^-$ and consider

$$A^i_\mu = \int d^4x\, e^{-iq\cdot x} \langle \pi(p) \,|\, \mathrm{T}\{J^\mu_{\mathrm{em}}(x)\, Q_i(0)\,\} \,|\, K(k)\rangle\,,$$

  where Q_i is an operator from the $\Delta S = 1$ effective weak Hamiltonian.

- EM gauge invariance implies that

$$A^i_\mu = \frac{\omega_i(q^2)}{(4\pi)^2} \left\{ q^2 (p+k)^\mu - (m_K^2 - m_\pi^2)\, q^\mu \right\}.$$

- The theoretical framework has been developed and an exploratory numerical calculation for the K^+ decay has been performed.

N.Christ, X.Feng, A.Portelli and CTS, arXiv1507.03094

N.Christ, X.Feng, A.Jüttner, A.Lawson, A.Portelli and CTS, arXiv1608.07585

- These decays are an important focus for the UK effort on the DiRAC resources it has been allocated last week.

Exploratory numerical study of the $K^+ \to \pi^+ \ell^+ \ell^-$ amplitude

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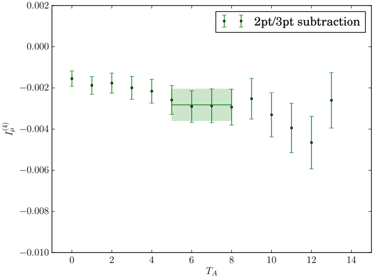
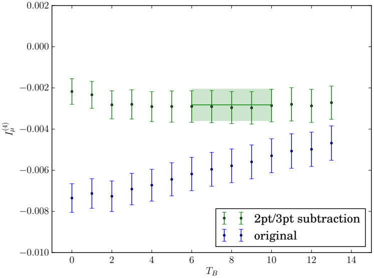
- The exploratory numerical study was performed on a $24^3 \times 64$ DWF+Iwasaki RBC-UKQCD ensembles with $m_\pi \simeq 420$ MeV, $m_K \simeq 625$ MeV), $a^{-1} \simeq 1.78$ fm.

  N.Christ, X.Feng, A.Jüttner, A.Lawson, A.Portelli and CTS, arXiv:1608.07585

  - 128 configurations were used with $\vec{k} = \vec{0}$ and $\vec{p} =$(1,0,0), (1,1,0) and (1,1,1) in units of $2\pi/L$. With this kinematics we are in the unphysical region, $q^2 < 0$ and the charm quark is also lighter than physical $m_c^{\overline{\text{MS}}}(2\,\text{GeV}) \simeq 520$ MeV.



$$\int_{t_J - T_A}^{t_J + 8} \tilde{\Gamma}_0^{(4)}\, dt_H$$



$$\int_{t_J - 6}^{t_J + T_B} \tilde{\Gamma}_0^{(4)}\, dt_H$$

$$A_0^+(q^2) = -0.0028(6).$$

- NA62 ($K^+ \to \pi^+\nu\bar{\nu}$) and KOTO ($K_L \to \pi^0\nu\bar{\nu}$) are beginning their experimental programme to study these decays. These FCNC processes provide ideal probes for the observation of new physics effects.
- The dominant contributions from the top quark $\Rightarrow$ they are also very sensitive to V_{ts} and V_{td}.
- Experimental results and bounds:

$$\mathrm{Br}(K^+ \to \pi^+\nu\bar{\nu})_{\mathrm{exp}} = 1.73^{+1.15}_{-1.05} \times 10^{-10}$$

A.Artamonov et al. (E949), arXiv:0808.2459

$$\mathrm{Br}(K_L \to \pi^0\nu\bar{\nu}) \leq 2.6 \times 10^{-8} \text{ at 90\% confidence level},$$

J.Ahn et al. (E291a), arXiv:0911.4789

- Sample recent theoretical predictions:

$$\mathrm{Br}(K^+ \to \pi^+\nu\bar{\nu})_{\mathrm{SM}} = (9.11 \pm 0.72) \times 10^{-11}$$
$$\mathrm{Br}(K_L \to \pi^0\nu\bar{\nu})_{\mathrm{SM}} = (3.00 \pm 0.30) \times 10^{-11},$$

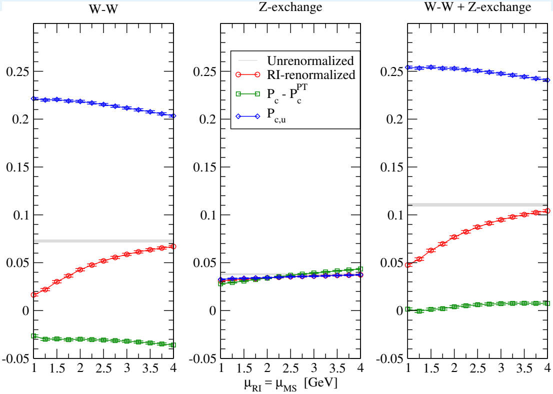
A.Buras, D.Buttazzo, J.Girrbach-Noe, R.Knejgens, arXiv:1503.02693

- To what extent can lattice calculations reduce the theoretical uncertainty?

- To what extent can lattice calculations reduce the theoretical uncertainty?
- $K \to \pi\nu\bar{\nu}$ decays are SD dominated and the hadronic effects can be determined from CC semileptonic decays such as $K^+ \to \pi^0 e^+ \nu$.
- LD contributions, i.e. contributions from distances greater than $1/m_c$ are negligible for K_L decays and are expected to be $O(5\%)$ for K^+ decays.
    - K_L decays are therefore one of the cleanest places to search for the effects of new physics.
    - The aim of our lattice study is to compute the LD effects in K^+ decays. (These provide a significant, if probably still subdominant, contribution to the theoretical uncertainty, which is dominated by the uncertainties in CKM matrix elements.)
- Lattice QCD can provide a first-principles determination of the LD contribution with controlled errors.
- The theoretical framework has been developed and implemented in an exploratory calculation. N.Christ, X.Feng, A.Portelli & CTS, arXiv:1605.04442

  Z.Bai, N.Christ, X.Feng, A,Lawson, A.Portelli & CTS, arXiv:1701.02858 & 1806.11520
- Ongoing work, led by X.Feng, includes a study of the momentum dependence on a 32^3 lattice at a^{-1}=1.37 GeV with $m_\pi \simeq 170$ MeV but lighter m_c as well as preparatory work for a physical simulation.

$K^+ \to \pi^+ \nu\bar{\nu}$ decays - Results from exploratory calculation



- Details of simulation: 800 configs on a $16^3 \times 32$ lattice with $N_f = 2+1$ DWF, $a^{-1} \simeq 1.73\,\mathrm{GeV}$, $m_\pi \simeq 420\,\mathrm{MeV}$, $m_K \simeq 563\,\mathrm{MeV}$ and $m_c^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV}) \simeq 863\,\mathrm{MeV}$.
- For this unphysical kinematics, we find

$$P_c = 0.2529(\pm 13)(\pm 32)(-45) \quad \text{and} \quad \Delta P_c = 0.0040(\pm 13)(\pm 32)(-45)\,.$$

- Large cancellation between WW and Z-exchange contributions.

- In 2015 RBC-UKQCD published our first result for ε'/ε computed at physical quark masses and kinematics, albeit still with large relative errors:

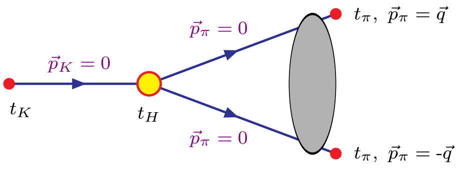
$$\frac{\varepsilon'}{\varepsilon}\Big|_{\text{RBC-UKQCD}} = (1.38 \pm 5.15 \pm 4.59) \times 10^{-4}$$

to be compared with

$$\frac{\varepsilon'}{\varepsilon}\Big|_{\text{Exp}} = (16.6 \pm 2.3) \times 10^{-4}\,.$$

RBC-UKQCD, arXiv:1505.07863

- This is by far the most complicated project that I have ever been involved with.
- This single result hides much important (and much more precise) information which we have determined along the way.
- We are updating the results with about ≥ 6 times the statistics and much improved techniques for reducing the systematic uncertainties.
- It is planned to present updated results within the next few months.



- $K \to \pi\pi$ correlation function is dominated by lightest state, i.e. the state with two-pions at rest. Maiani and Testa, PL B245 (1990) 585

$$C(t_\pi) = A + B_1 e^{-2m_\pi t_\pi} + B_2 e^{-2E_\pi t_\pi} + \cdots$$

  - Solution 1: Study an excited state. Lellouch and Lüscher, hep-lat/0003023
  - Solution 2: Introduce suitable boundary conditions such that the $\pi\pi$ ground state is $|\pi(\vec{q})\pi(-\vec{q})\rangle$. RBC-UKQCD, C.h.Kim hep-lat/0311003

For B-decays, with so many intermediate states below threshold, this is the main obstacle to producing reliable calculations.

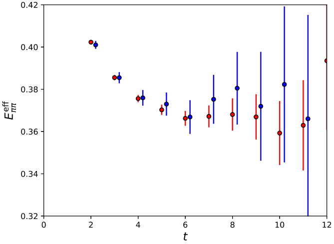
- Elastic $\pi\pi$ phase-shifts are obtained by measuring $E_{\pi\pi} - 2E_\pi$ and using Lüscher's formula.
- Isospin 0, two-pion correlators are noisy (primarily due to the vacuum subtraction) and measuring the ground-state two-pion energy is challenging.
- A puzzle from our 2015 paper was that we found $\delta_S^{I=0}(m_K^2) = (23.8 \pm 4.9 \pm 1.2)^\circ$ to be compared to $\sim 35^\circ$ from dispersive analyses.

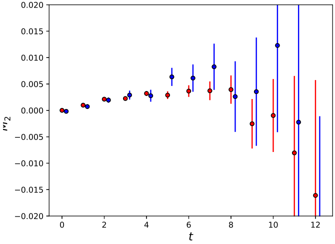
  G.Colangelo, J.Gasser, & H.Leutwyler, hep-ph/0103088
- With increased statistics, and more importantly the use of additional interpolating operators for the two-pion state, we are able to understand and reduce the contamination from excited states and now find $\delta_S^{I=0}(m_K^2) = (30.9 \pm 1.5 \pm 3.0)^\circ$.

  T.Wang and C.Kelly, PoS Lattice 2018 (2018) 276

Puzzle is resolved.

- In order to reduce possible contamination of excited states, we have now included the additional two-pion interpolating operators into the $K \to \pi\pi$ analysis and plan to present the results before the summer.





Q_2

- Blue points - 216 configurations   Red points 1438 configurations
- I stress that more important in the improved analysis is the use of additional two-pion interpolating operators.

- The amplitude A_2 is considerably simpler to evaluate that A_0.
- Our first results for A_2 at physical kinematics were obtained at a single, rather coarse, value of the lattice spacing ($a \simeq 0.14$ fm). Estimated discretization errors at 15%. arXiv:1111.1699, arXiv:1206.5142
- Our latest results were obtained on two new ensembles, 48^3 with $a \simeq 0.11$ fm and 64^3 with $a \simeq 0.084$ fm so that we can make a continuum extrapolation:

$$\begin{aligned} \mathrm{Re}(A_2) &= 1.50(4)_{\mathrm{stat}}(14)_{\mathrm{syst}} \times 10^{-8}\ \mathrm{GeV}. \\ \mathrm{Im}(A_2) &= -6.99(20)_{\mathrm{stat}}(84)_{\mathrm{syst}} \times 10^{-13}\ \mathrm{GeV}. \end{aligned}$$

arXiv:1502.00263

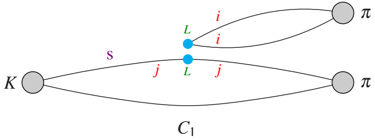
- The experimentally measured value is $\mathrm{Re}(A_2) = 1.479(4) \times 10^{-8}$ GeV.
- Although the precision can still be significantly improved (partly by perturbative calculations), the calculation of A_2 at physical kinematics can now be considered as standard.

RBC-UKQCD Collaboration, arXiv:1212.1474

- Re A_2 is dominated by a simple operator:

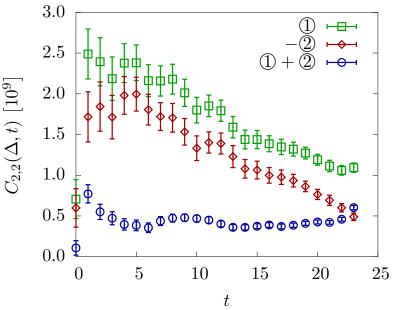
$$O^{3/2}_{(27,1)} = (\bar{s}^i d^i)_L \left\{ (\bar{u}^j u^j)_L - (\bar{d}^j d^j)_L \right\} + (\bar{s}^i u^i)_L\, (\bar{u}^j d^j)_L$$

and two diagrams:



- Re A_2 is proportional to $C_1 + C_2$.
- The contribution to Re A_0 from Q_2 is proportional to $2C_1 - C_2$ and that from Q_1 is proportional to $C_1 - 2C_2$ with the same overall sign.
- Colour counting might suggest that $C_2 \simeq \frac{1}{3} C_1$.
- We find instead that $C_2 \approx -C_1$ so that A_2 is significantly suppressed!
- **We believe that the strong suppression of Re A_2 and the (less-strong) enhancement of Re A_0 is a major factor in the $\Delta I = 1/2$ rule.**

Evidence for the Suppression of Re A_2



Physical Kinematics

- Notation ⓘ $\equiv C_i,\ i = 1,2$.
- Of course before claiming a quantitative understanding of the $\Delta I = 1/2$ rule we needed to compute Re A_0 at physical kinematics and found a results of $\simeq 31 \pm 12$ to be compared to the experimental value of 22.5.
- Much early phenomenology was based on the vacuum insertion approach. although the qualitative picture we find had been suggested by Bardeen, Buras and Gerard in 1987.

Many exciting lattice studies in kaon physics underway.

- For leptonic and semileptonic decays, in order to make further progress in phenomenology it is necessary to compute the isospin-breaking corrections (including radiative correction).
  - This is underway!
- For the evaluation of long-distance contributions to physical quantities in kaon physics, the theoretical framework has been developed and exploratory computations have been performed (at unphysical masses). Physical mass calculations are beginning (for Δm_K they are well advanced) and results will be available in $\sim$ 2 years.
- Our results for ε'/ε will be updated this summer, along with many related results.

1. *QED Corrections to Hadronic Processes in Lattice QCD,*
N.Carrasco, V.Lubicz, G.Martinelli, C.T.Sachrajda, N.Tantalo, C.Tarantino and M.Testa,
Phys. Rev. D **91** (2015) no.7, 07450 [arXiv:1502.00257 [hep-lat]].

2. *Finite-Volume QED Corrections to Decay Amplitudes in Lattice QCD,*
V.Lubicz, G.Martinelli, C.T.Sachrajda, F.Sanfilippo, S.Simula and N.Tantalo,
Phys. Rev. D **95** (2017) no.3, 034504 [arXiv:1611.08497 [hep-lat]].

3. *First Lattice Calculation of the QED Corrections to Leptonic Decay Rates*,
D.Giusti, V.Lubicz, G.Martinelli, C.T.Sachrajda, F.Sanfilippo, S.Simula, N.Tantalo and C.Tarantino,
Phys. Rev. Lett. **120** (2018) 072001 [arXiv:1711:06537]

4. *Radiative corrections to decay amplitudes in lattice QCD,*
D.Giusti, V.Lubicz, G.Martinelli, C.T.Sachrajda, F.Sanfilippo, S.Simula and N.Tantalo,
arXiv:1811.06364 [hep-lat].

5. *Light-meson leptonic decay rates in lattice QCD+QED*
M.Di Carlo, D.Giusti, V.Lubicz, G.Martinelli, C.T.Sachrajda, F.Sanfilippo, S.Simula and N.Tantalo,
in preparation.

1. "Long-distance contribution to the $K_L - K_S$ mass difference,"
   N.H. Christ, T. Izubuchi, CTS, A. Soni and J. Yu,
   Phys.Rev. D88 (2013) 014508 (arXiv:1212.5931)
2. "$K_L - K_S$ mass difference from Lattice QCD,"
   Z. Bai, N.H. Christ, T.Izubuchi, CTS, A.Soni and J. Yu,
   Phys.Rev.Lett. 113 (2014) 112003 (arXiv:1406.0916)
3. "Neutral Kaon Mixing from Lattice QCD"
   Z.Bai, Columbia University Thesis (2017)
4. "The K_L-K_S mass difference"
   Z.Bai, N.H.Christ and CTS, Proceedings of the 2017 International Symposium on Lattice Field Theory.
   EPJ Web Conf 175 (2018) 13017
5. "Results for the mass difference between long- and shot-lived K mesons for physical quark masses"
   B.Wang, Proceedings of the 2018 International Symposium on Lattice Field Theory.
   arXiv:1812.05302

1. "Prospects for a lattice computation of rare kaon decay amplitudes: I, $K \to \pi \ell^+ \ell^-$ decays"
   N.H.Christ, X.Feng, A.Portelli and CTS Phys.Rev. D. **92** (2015) 094512 [arXiv:1507.03094]
2. "Prospects for a lattice computation of rare kaon decay amplitudes: II, $K \to \pi \nu \bar{\nu}$ decays"
   N.H.Christ, X.Feng, A.Portelli and CTS Phys.Rev. D **93** (2016) 114517 [arXiv:1605:04442]
3. "First exploratory calculation of the long distance contributions to the rare kaon decay $K \to \pi \ell^+ \ell^-$"
   N.H.Christ, X.Feng, A.Jüttner, **A.Lawson, A.Portelli** and CTS
   Phys.Rev. D **94** (2016) 114516 [arXiv:1608.07585]
4. "Exploratory lattice QCD study of the rare kaon decay $K^+ \to \pi^+ \nu \bar{\nu}$"
   Z.Bai, N.H.Christ, **X.Feng**, A.Jüttner, A.Lawson, A.Portelli and CTS
   Phys.Rev.Lett. **118** (2017) 252001 [arXiv:1701.02858]
5. "$K^+ \to \pi^+ \nu \bar{\nu}$ decay amplitude from lattice QCD"
   Z.Bai, N.H.Christ, **X.Feng**, A.Jüttner, A.Lawson, A.Portelli and CTS
   Phys.Rev. D **98** (2018) 074509 [arXiv:1806.11520]

1. A_0 and A_2 amplitudes with unphysical quark masses and with the pions at rest.

"K to $\pi\pi$ decay amplitudes from lattice QCD,"
T.Blum, P.A.Boyle, N.H.Christ, N.Garron, E.Goode, T.Izubuchi, C.Lehner, Q.Liu, R.D. Mawhinney, C.T.S, A.Soni, C.Sturm, H.Yin and R. Zhou, Phys. Rev. D **84** (2011) 114503 [arXiv:1106.2714 [hep-lat]].

"Kaon to two pions decay from lattice QCD, $\Delta I = 1/2$ rule and CP violation"
Q.Liu, Ph.D. thesis, Columbia University (2010)

2. A_2 at physical kinematics and a single coarse lattice spacing.

"The $K\to(\pi\pi)_{I=2}$ Decay Amplitude from Lattice QCD,"
T.Blum, P.A.Boyle, N.H.Christ, N.Garron, E.Goode, T.Izubuchi, C.Jung, C.Kelly, C.Lehner, M.Lightman, Q.Liu, A.T.Lytle, R.D.Mawhinney, C.T.S., A.Soni, and C.Sturm
Phys. Rev. Lett. **108** (2012) 141601 [arXiv:1111.1699 [hep-lat]],

"Lattice determination of the $K\to(\pi\pi)_{I=2}$ Decay Amplitude A_2"
Phys. Rev. D **86** (2012) 074513 [arXiv:1206.5142 [hep-lat]]

"Emerging understanding of the $\Delta I = 1/2$ Rule from Lattice QCD,"
P.A. Boyle, N.H. Christ, N. Garron, E.J. Goode, T. Janowski, C. Lehner, Q. Liu, A.T. Lytle, C.T. Sachrajda, A. Soni, and D.Zhang, Phys. Rev. Lett. **110** (2013) 15, 152001 [arXiv:1212.1474 [hep-lat]].

3. A_2 at physical kinematics on two finer lattices $\Rightarrow$ continuum limit taken.
   "$K \to \pi\pi\ \Delta I = 3/2$ decay amplitude in the continuum limit,"
   T.Blum, P.A.Boyle, N.H.Christ, J.Frison, N.Garron, T.Janowski, C.Jung, C.Kelly, C.Lehner, A.Lytle, R.D.Mawhinney, C.T.S., A.Soni, H.Yin, and D.Zhang
   Phys. Rev. D **91** (2015) 7, 074502 [arXiv:1502.00263 [hep-lat]].

4. A_0 at physical kinematics and a single coarse lattice spacing.
   "Standard-model prediction for direct CP violation in $K \to \pi\pi$ decay,"
   Z.Bai, T.Blum, P.A.Boyle, N.H.Christ, J.Frison, N.Garron, T.Izubuchi, C.Jung, C.Kelly, C.Lehner, R.D.Mawhinney, C.T.S, A. Soni, and D. Zhang,
   Phys. Rev. Lett. **115** (2015) 21, 212001 [arXiv:1505.07863 [hep-lat]].

See also: "Calculation of ε'/ε on the lattice" C.Kelly et al. PoS FPCP2016 (2017) 017