Reduction for Special Negative Sectors of Planar Two-Loop Integrals

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Introduction:

The community is gearing towards non-precedented complexity in NNLO QCD computations:

More and more frameworks and schemes are introduced for tackling real radiations, like:

- Antenna: Gehrmann-De Ridder, Gehrmann and Ritzmann
- CoLorFuLNNLO: Del Duca, Somogyi and Trocsanyi
- Sector Improved Residue Subtraction: Czakon and Heymes
- Nested Soft-Collinear Subtraction: Caola, Melnikov and Röntsch
- Local Analytic Sector Subtraction: Magnea, Maina, Pelliccioli, Signorile-Signorile, Torrielli and Uccirati
- Projection to Born: Cacciari, Dreyer, Karlberg, Salam and Zanderighi
- q_T Subtraction: Catani and Grazzini
- N-Jetiness: Boughezal, Focke, Liu and Petriello & Gaunt, Stahlhofen, Tackmann, Walsh

Introduction:

Multi-scale/loop/leg amplitudes are also needed!

The recent years witness an ever-increasing interest and progress in multi-loop calculations:

Achievements are made in two-loop QCD amplitude calculations:

(Sample) Five gluon two-loop amplitude calculations:

Badger, Frellesvig and Zhang, JHEP 12 (2013) 045

Badger, Mogull, Ochirov, and O'Connell, JHEP 10 (2015) 064

Gehrmann, Henn, and Lo Presti, Phys. Rev. Lett. 116 (2016), no. 6

Chawdhry, Lim, and Mitov, arXiv:1805.09182

Badger, Bronnum-Hansen, Hartanto, and Peraro, JHEP 1901 (2019) 186

Introduction:

It is found that numerator structure is very complex and can have very large negative powers (up to -5).

While in the denominator the maximum power is 2 (at least in Feynman-gauge).

$\Longrightarrow$ the numerator seems the technical bottleneck

We have fantastic programs for reduction using Laporta's algorithm:

Reduze (Studerus and Studerus & von Manteuffel)

FIRE (AV Smirnov), FIRE6 is just came out! (arXiv:1901.07808, Smirnov, Chukharev)

KIRA (Maierhofer, Usovitsch and Uwer) and very recently KIRA1.2 (arXiv:1812.01491, Maierhofer and Usovitsch)

Introduction:

Beside the canonical representation the Baikov form can also be used to set up IBP equations.

Using the Baikov rep. the resulting IBP equations seem to be more complicated

Algebraic geometry comes as a rescue:

It is possible to formulate efficient reduction based on the Baikov representation: Bohm, Georgoudis, Larsen, Schonemann and Zhang: JHEP 1809 (2018) 024

A question naturally arises:

Can we do anything without all the fancy math?

The Baikov Representation

The Representation of Baikov:

The canonical representation of an L-loop integral:

$$I^{(L)}_{\alpha_1\ldots\alpha_N} = \int \left(\prod_{i=1}^{L} \frac{\mathrm{d}^d \ell_i}{i\pi^{d/2}} \right) \frac{1}{D_1^{\alpha_1} \cdots D_N^{\alpha_N}}$$

with inverse propagators having the form of:

$$D_a = \sum_{i,j=1}^{L} A_a^{ij} (\ell_i \cdot \ell_j) + \sum_{i=1}^{L} \sum_{j=1}^{E} A_a^{i(j+L)} (\ell_i \cdot p_j) + f_a \,, \quad a \in 1, \ldots, N$$

Note:N is the number of different propagators in the integral family (N=L(L+1)/2 + L E)

L: Number of loops

E: Number of independent external legs

The Representation of Baikov:

In the Baikov representation the integral can be written as:

$$I^{(L)}_{\alpha_1\ldots\alpha_N} = \mathcal{N} \int \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_N}{x_1^{\alpha_1} \cdots x_N^{\alpha_N}} \left(\mathcal{P}^L_N(x_1 - f_1, \ldots, x_N - f_N)\right)^{\frac{d-L-E-1}{2}}$$

$\mathcal{N}$: prefactor, containing space-time dimension, 2's, π's and Γ's

$\mathcal{P}^N_L$: Baikov polynomial

$$\mathcal{P}^L_N(x_1, \ldots, x_N) = G(\ell_1, \ldots, \ell_L, p_1, \ldots, p_E)\big|_{s_{ij}=\sum_{a=1}^N A^{ij}_a x_a}$$

with G being the Gram determinant in L+E momenta and $s_{ij} = q_i \cdot q_j$

$$G(\ell_1, \ldots, \ell_L, p_1, \ldots, p_E) = \sum_{\sigma \in S_{L+E}} \left(\mathrm{sgn}(\sigma) \prod_{i=1}^{L+E} q_i \cdot q_{\sigma_i} \right)$$

Exploiting the Representation at L=2

The L=2 Case

What happens when we focus on L=2?

$$\mathcal{P}_N^L(\ldots) \to \mathcal{P}_N^2(x_1, \ldots, x_N) = G(\ell_1, \ell_2, p_1, \ldots, p_E)\big|_{s_{ij}=\sum_{a=1}^N A_a^{ij} x_a}$$

$$G(\ldots) \to G(\ell_1, \ell_2, p_1, \ldots, p_E) = \sum_{\sigma \in S_{(2+E)}} \left(\mathrm{sgn}(\sigma) \prod_{i=1}^{E+2} q_i \cdot q_{\sigma_i} \right)$$

$$G(\ell_1, \ell_2, p_1, \ldots, p_E) = \begin{vmatrix} \ell_1 \cdot \ell_1 & \ell_1 \cdot \ell_2 & \cdots & \ell_1 \cdot p_E \\ \ell_1 \cdot \ell_2 & \ell_2 \cdot \ell_2 & \cdots & \ell_2 \cdot p_E \\ \vdots & \vdots & \ddots & \vdots \\ \ell_1 \cdot p_E & \ell_2 \cdot p_E & \cdots & p_E \cdot p_E \end{vmatrix}$$

Dot products are replaced by linear combinations of x's

The L=2 Case

The Baikov polynomial is at most quadratic in all x's by construction

Integration boundaries are determined by:

$$\mathcal{P}_N^L(x_1, \ldots, x_N) = 0$$

$\Longrightarrow$ The integrations over x_i's happen between the roots of the corresponding quadratic equation.

$$I^{(L)}_{\alpha_1 \ldots \alpha_N} = \mathcal{N} \int \frac{\mathrm{d}x_1}{x_1^{\alpha_1}} \cdots \int_{x_i^-}^{x_i^+} \frac{\mathrm{d}x_i}{x_i^{\alpha_i}} \cdots \int \frac{\mathrm{d}x_N}{x_N^{\alpha_N}} \left(\ldots (x_i^+ - x_i)(x_i - x_i^-) \right)^{\frac{d-L-E-1}{2}}$$

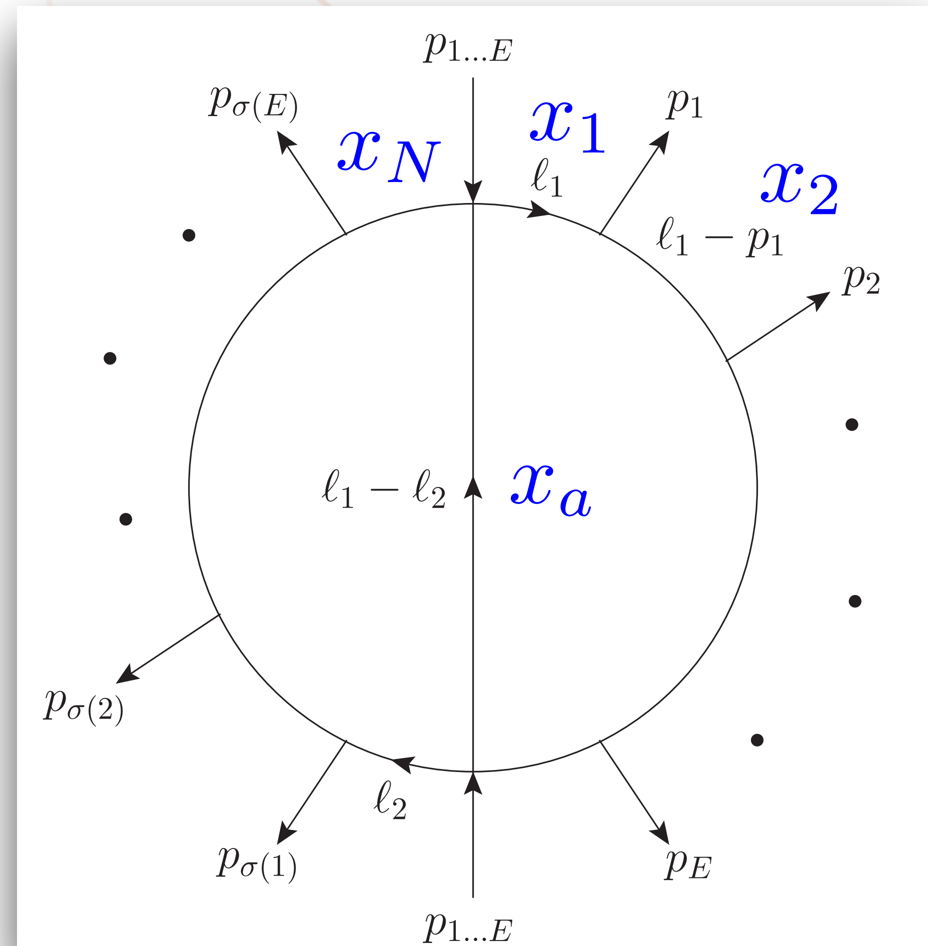
This becomes handy when devising IBP reductions!

In general the structure of the Baikov polynomial is very complicated

We focus on the case of negative sectors: $\alpha_i<0$ (the inverse propagator is in the numerator)

The L=2 Case

Observation: in a planar two-loop integral it can always be achieved that only one propagator contains both loop momenta.



This is the most general two-loop planar topology. Note that the ordering of external momenta does not have to be the same in the two halves!

Can happen when the 1loop x 1loop interference is considered as a genuine two-loop topology

$\Longrightarrow$Monomials quadratic in this Baikov x only depend on external kinematics:

$$\mathcal{P}_N^2(x_1, \dots, x_a, \dots, x_N) = C(\{p_i \cdot p_j\}, \{m_i\}) x_a^2 + \dots$$

The L=2 Case

An alternate reduction strategy can be applied to the special negative sector, i.e., the one having both loop momenta

Note that this topology can also be considered the product of two one-loop tensor integrals coupled in the numerator through a $(\ell_1 \cdot \ell_2)^{-\alpha_a}$

In principle the problem can be attacked by Passarino Veltman reduction. This method is yet another way to attack the same problem in a more modern way

Schematically these integrals in Baikov rep. can be written as:

$$I^{(2)}_{\alpha_1 \dots \alpha_N} = \mathcal{N} \int \frac{\prod \mathrm{d}x_i}{\prod_{i \neq a} x_i^{\alpha_i}} x_a^{-\alpha_a} \mathcal{P}^n$$

with α_a being negative!

Note the short-hands $\mathcal{P} = \mathcal{P}_N^L, \quad n = \dfrac{d - L - E - 1}{2}$

New Reduction for the Special Negative Sector

Reduction when α_a=-1

Considering the case of α_a=-1:

We have an x_a in the numerator, but note that:

$$\partial_{x_a}\mathcal{P} = \partial_a\mathcal{P} = \mathcal{C}\{p_i \cdot p_j\}, \{m_i\})x_a + \ldots$$

the ellipsis stand for further terms independent of x_a

Thus:

$$\mathcal{N}\int \frac{\prod \mathrm{d}x_i}{\prod_{i\neq a} x_i^{\alpha_i}}(\partial_a\mathcal{P})\mathcal{P}^n = \mathcal{C}_{\alpha_1\ldots\alpha_N} I^{(2)}_{\alpha_1\ldots\alpha_N} + \sum_{\substack{\{\beta\}\\ \beta_a=0}} \mathcal{C}_{\beta_1\ldots\beta_N} I^{(2)}_{\beta_1\ldots\beta_N} =$$

$$= \mathcal{C}_{\alpha_1\ldots\alpha_N} I^{(2)}_{\alpha_1\ldots\alpha_N} + \sum_{\substack{\{\beta\}\\ \beta_a=0}} \mathcal{C}_{\beta_1\ldots\beta_N} \left(I^{(1)} \otimes I^{(1)}\right)_{\beta_1\ldots\beta_N}$$

The first term on RHS is our two-loop integral with some prefactors

Reduction when α_a=-1

This can be turned into an IBP relation noting that:

$$\int \frac{\prod \mathrm{d}x_i}{\prod_{i\neq a} x_i^{\alpha_i}} (\partial_a \mathcal{P})\mathcal{P}^n = \frac{1}{n+1}\int \frac{\prod \mathrm{d}x_i}{\prod_{i\neq a} x_i^{\alpha_i}} \partial_a \left(\mathcal{P}^{n+1}\right) =$$

$$= \widetilde{\mathcal{C}}_{\alpha_1\dots\alpha_N} I^{(2)}_{\alpha_1\dots\alpha_N} + \sum_{\substack{\{\beta\}\\ \beta_a=0}} \widetilde{\mathcal{C}}_{\beta_1\dots\beta_N} \left(I^{(1)} \otimes I^{(1)}\right)_{\beta_1\dots\beta_N} = 0$$

Since the integrand is a total derivative in x_a it integrates to zero due to the form of the Baikov polynomial!

$\Longrightarrow$Our two-loop integral is expressible with a sum of products of one-loop integrals.

We also dropped the non-essential $\mathcal{N}$ prefactor hence the change in normalization

Reduction with General α_a

Original integral: $I^{(2)}_{\alpha_1\ldots\alpha_N} = \mathcal{N}\int \frac{\prod \mathrm{d}x_i}{\prod_{i\neq a} x_i^{\alpha_i}} x_a^{-\alpha_a}\mathcal{P}^n$

With a general negative α_a we can exploit the same properties of the representation:

$$\int \frac{\prod \mathrm{d}x_i}{\prod_{i\neq a} x_i^{\alpha_i}} x_a^{-\alpha_a-1}(\partial_a\mathcal{P})\mathcal{P}^n = \widetilde{\mathcal{C}}_{\alpha_1\ldots\alpha_N} I^{(2)}_{\alpha_1\ldots\alpha_N} + \sum_{\substack{\{\beta\}\\ \alpha_a<\beta_a}} \widetilde{\mathcal{C}}_{\beta_1\ldots\beta_N} I^{(2)}_{\beta_1\ldots\beta_N}$$

This time not only our two-loop integral appears on RHS but further ones too having lower rank in the special sector

After some algebra:

$$0 = \int \frac{\prod \mathrm{d}x_i}{\prod_{i\neq a} x_i^{\alpha_i}} x_a^{-\alpha_a-1}(\partial_a\mathcal{P})\mathcal{P}^n - \frac{1+\alpha_a}{n+1}\int \frac{\prod \mathrm{d}x_i}{\prod_{i\neq a} x_i^{\alpha_i}} x_a^{-\alpha_a-2}\mathcal{P}\,\mathcal{P}^n$$

Note that the α_a=-1 choice gives back the previously derived special case!

Reduction with General α_a

The expression and the reduction can be made more compact using the syzygy decomposition of the Baikov polynomial:

$$\mathcal{P} = \sum_{j=1}^{N} g_j \frac{\partial \mathcal{P}}{\partial x_j} + b$$

the coeff.'s g_j and b depend on the Baikov x's!

$$0 = \int \frac{\prod \mathrm{d}x_i}{\prod_{i\neq a} x_i^{\alpha_i}} x_a^{-\alpha_a-1} (\partial_a \mathcal{P}) \mathcal{P}^n - \frac{1+\alpha_a}{n+1} \left\{ \sum_{j=1}^{N} \int \frac{\prod \mathrm{d}x_i}{\prod_{i\neq a} x_i^{\alpha_i}} x_a^{-\alpha_a-2} g_j (\partial_j \mathcal{P}) \mathcal{P}^n + \right.$$
$$\left. + \int \frac{\prod \mathrm{d}x_i}{\prod_{i\neq a} x_i^{\alpha_i}} x_a^{-\alpha_a-2} b\, \mathcal{P}^n \right\}$$

Beside of our original two-loop integral the others appearing have the special sector at a lower rank. $\Longrightarrow$ a straightforward top-down approach can be utilized to get the reduction done!

Checks and Tests

To test the approach several integral families were considered, like the massive and massless double-box and the massless pentabox

High ranks for considered up to 4

The proof-of-concept implementation was in $\mathtt{Mathematica}$ without any optimization

The reductions were able to get done on a laptop and the longest took ~1 hour

For checking purposes we used KIRA1.1 and FIRE5

The traditional programs needed 48 cores and up to a day to do the same reduction

In all cases we found complete agreement

Conclusions

- Alternate reduction approach is present for the mixed negative sectors of two-loop planar integrals
- The new strategy shows a straightforward top-down approach free from a Laporta-style reduction
- The method eliminates the mixed factor in the numerator converting the two-loop integral into the product of two one-loop tensor integrals
- When applied significant speed-up can occur
- Tested and checked on several, very complicated two-loop integral families being in spotlight these days

Thank you for your attention!