UNIVERSAL FOUR-DIMENSIONAL REPRESENTATION OF HIGGS BOSON TO TWO PHOTONS AT TWO LOOPS THROUGH THE LOOP-TREE DUALITY

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OUTLINE

I. The Loop-Tree Duality theorem at one loop

II. The Loop-Tree Duality theorem at two loops

III. Procedure for local renormalisation at two-loop order

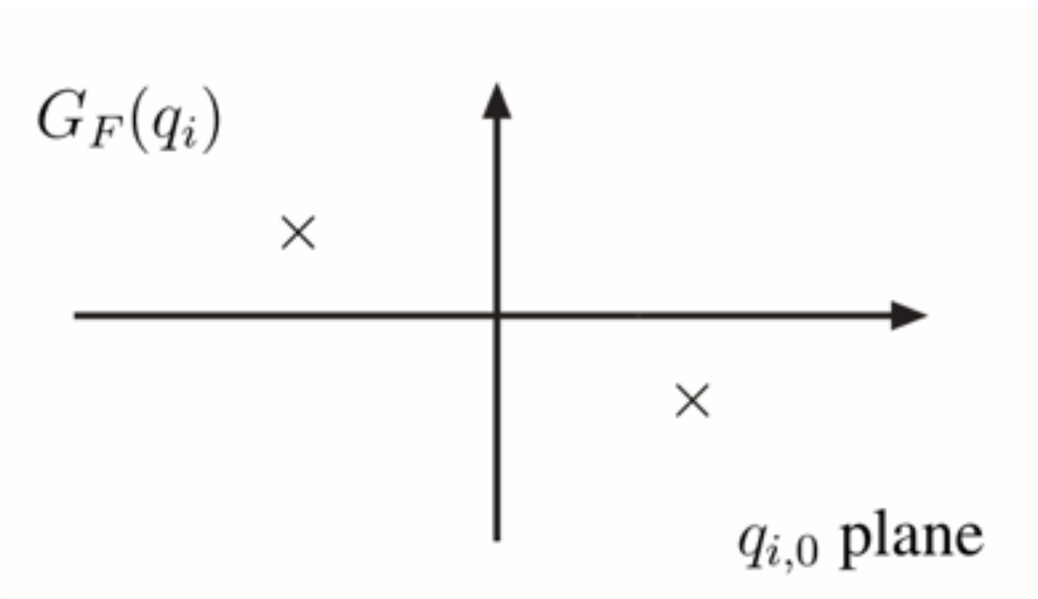
IV. Application to $H \to \gamma\gamma$ at two loops

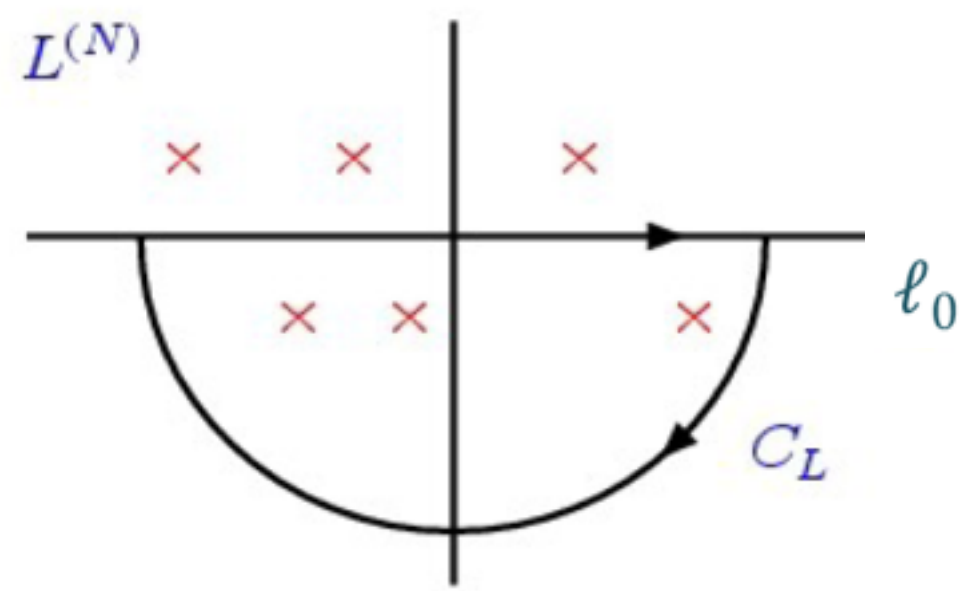
I. The Loop-Tree Duality theorem at one loop

Catani, Gleisberg, Krauss, Rodrigo, Winter, JHEP 09 (2008) 065

THE LOOP-TREE DUALITY THEOREM

Cauchy residue theorem
in the loop energy complex plane





Feynman Propagator +i0:
positive frequencies are propagated forward in time, and negative backward.

selects residues with definite **positive energy and negative imaginary part** (indeed in any coordinate system)

$$G_F(q_i) = \frac{1}{q_i^2 - m_i^2 + \imath 0}$$

$$q_i = \ell + \sum_{k=1}^{i} p_k$$

$$q_{i,0} = \pm\sqrt{\mathbf{q}_i^2 + m_i^2 - \imath 0}$$

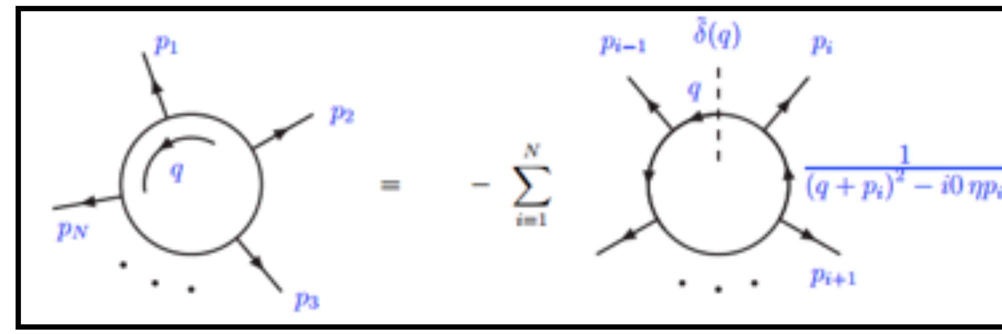
$$\boxed{q_{i,0}^{(+)} = +\sqrt{\mathbf{q}_i^2 + m_i^2 - \imath 0}}$$

Catani, Gleisberg, Krauss, Rodrigo, Winter, JHEP 09 (2008) 065

THE LOOP-TREE DUALITY THEOREM

One-loop integrals (or scattering amplitudes in any relativistic, local and unitary QFT) represented as a linear combination of *N* **single-cut phase-space** integrals

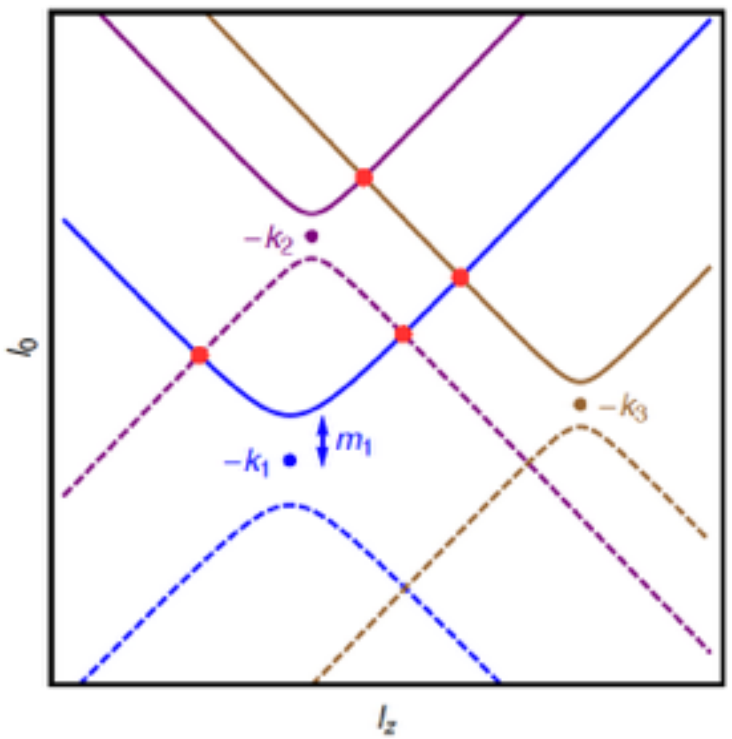
$$\int_\ell \prod_i G_F(q_i) = -\sum_i \int_\ell \tilde{\delta}(q_i) \prod_{j\neq i} G_D(q_i;q_j)$$

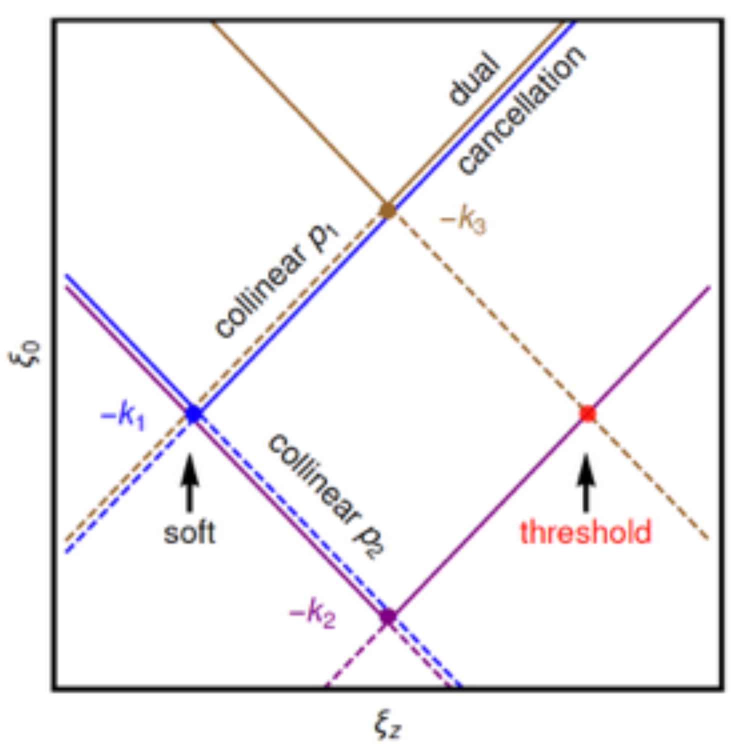


- $\tilde{\delta}(q_i) = i\,2\pi\,\theta(q_{i,0})\,\delta(q_i^2 - m_i^2)$ sets internal line on-shell, positive energy mode
- $G_D(q_i;q_j) = \dfrac{1}{q_j^2 - m_j^2 - i0\,\eta\,k_{ji}}$ **dual propagator,** $k_{ji} = q_j - q_i$
- LTD realized by **modifying the customary +i0 prescription** of the Feynman propagators, it compensates for the absence of multiple-cut contributions that appear in the Feynman Tree Theorem
- Lorentz-covariant dual prescription with η a **future-like** vector; from now on, $\eta^\mu = (1,\mathbf{0})$
- Integration domain now **Euclidean**, with the integration variable being the loop three-momentum

Sborlini, FDM, Hernandez, Rodrigo, JHEP 08 (2016) 160

SINGULARITIES OF THE DUAL INTEGRANDS





- **LTD**: Equivalent to integrating along **forward** on-shell hyperboloids/light-cones (positive energy modes)
- The dual loop integrand becomes singular when **more than one** internal propagators go on-shell while integrating
- **Cancellations** of singularities among dual amplitudes at forward-forward intersections: dual +i0 prescription change signs (proof of consistency)
- IR and threshold singularities illustrated by forward-backward intersections

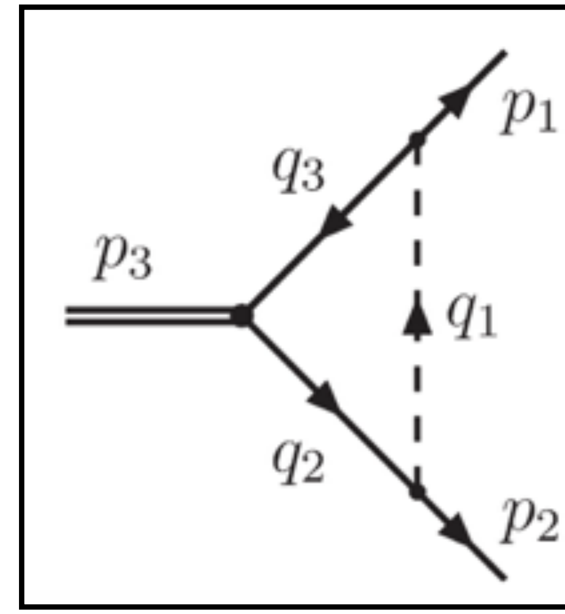
IR and threshold singularities are restricted to a **compact region** of the loop three-momentum

Sborlini, FDM, Hernandez, Rodrigo, JHEP 08 (2016) 160

EXPLICIT EXAMPLE: THE SCALAR THREE-POINT FUNCTION

$$L^{(1)}(p_1, p_2, -p_3) = \int_\ell \prod_{i=3}^{N} G_F(q_i) = \sum_{i=1}^{3} I_i$$

LTD



$$I_1 = \frac{4}{s_{12}} \int \frac{\xi_{1,0}^{-1}\, d[\xi_{1,0}] d[v_1]}{1-(1-2v_1)^2\beta^2}$$

$$I_2 = \frac{2}{s_{12}} \int \frac{\xi_{2,0}^{-1}\, \xi_2^2\, d[\xi_2] d[v_2]}{(1-\xi_{2,0}+\imath 0)(\xi_{2,0}+\beta\xi_2(1-2v_2)-m^2)}$$

$$I_3 = -\frac{2}{s_{12}} \int \frac{\xi_{3,0}^{-1}\, \xi_3^2\, d[\xi_3] d[v_3]}{(1+\xi_{3,0})(\xi_{3,0}-\beta\xi_3(1-2v_3)+m^2)}$$

Modulus of the loop three-momentum

$$d[\xi_i] = \frac{(4\pi)^{\epsilon-2}}{\Gamma(1-\epsilon)} \left(\frac{s_{12}}{\mu^2}\right)^{-\epsilon} \xi_i^{-2\epsilon} d\xi_i$$

$$d[v_i] = v_i(1-v_i)^{-\epsilon} dv_i$$

Polar angle of the loop three-momentum

II. The Loop-Tree Duality theorem at two loops

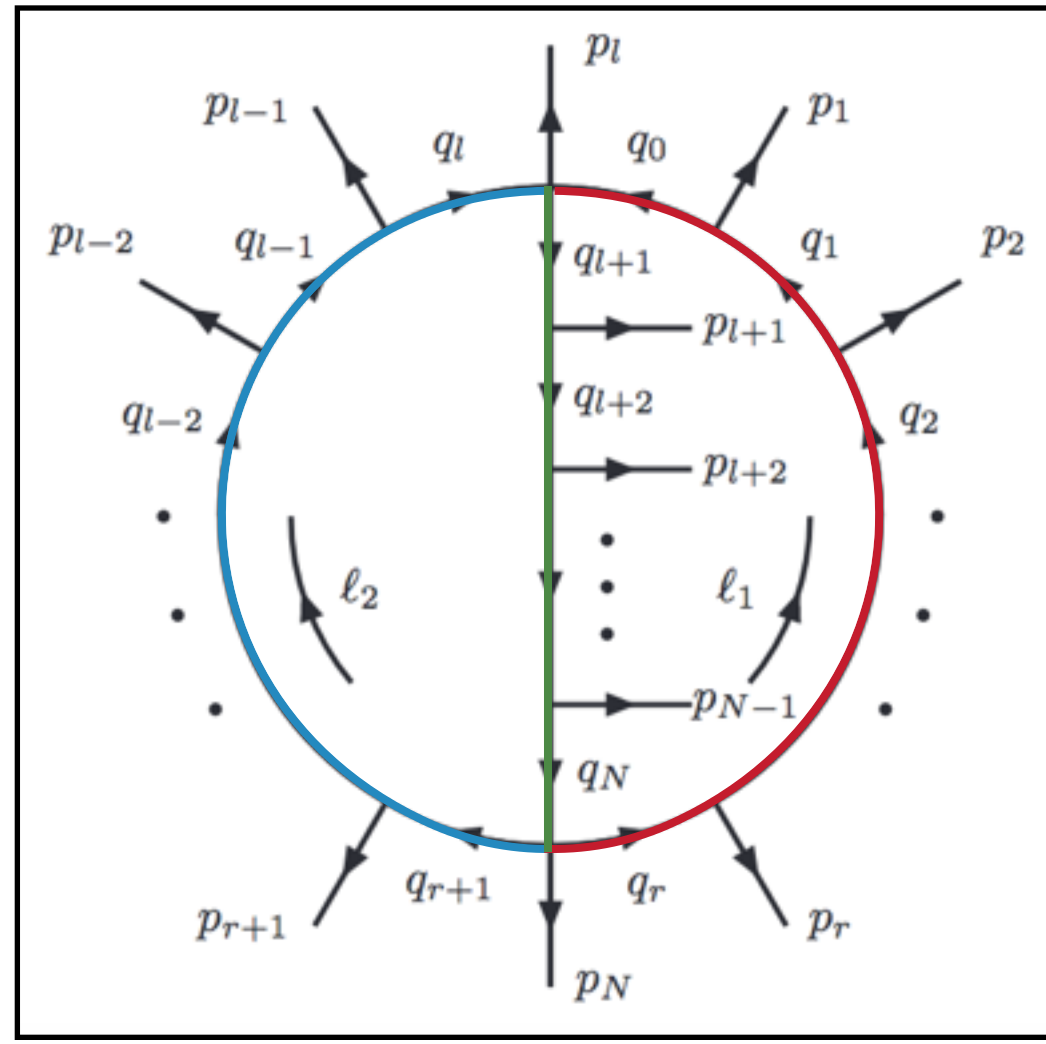
Bierenbaum, Catani, Draggiotis, Rodrigo, JHEP 1010:073,2010

GENERALIZATION OF THE LTD THEOREM AT TWO LOOPS

- Consider three sets of momenta

$$\begin{cases} \alpha_1 = \{\ell_1 + p_i, & i \in \{0,\dots,r\}\} \\ \alpha_2 = \{\ell_2 + p_i, & i \in \{r+1,\dots,l\}\} \\ \alpha_3 = \{\ell_1 + \ell_2 + p_i, & i \in \{l+1,\dots,N\}\} \end{cases}$$

- Two loops means… cutting twice: we need to impose two conditions on the couple $(\ell_1,\ \ell_2)$.
- The idea is therefore to put on shell two particles belonging to two different sets



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GENERALIZATION OF THE LTD THEOREM AT TWO LOOPS

- For a given set α_k, or a union of sets, we introduce

$$G_F(\alpha_k) = \prod_{i\in\alpha_k} G_F(q_i) \ , \quad G_D(\alpha_k) = \sum_{i\in\alpha_k} \tilde{\delta}(q_i) \prod_{\substack{j\in\alpha_k \\ j\neq i}} G_D(q_i;q_j)$$

- It is possible to show that these functions fulfill the following identity…

$$\boxed{G_D(\alpha_i \cup \alpha_j) = G_D(\alpha_i)\, G_D(\alpha_j) + G_D(\alpha_i)\, G_F(\alpha_j) + G_F(\alpha_i)\, G_D(\alpha_j)}$$

- … which allows to iteratively extend LTD to two loops, and even beyond

FDM, Sborlini, Torres, Rodrigo JHEP 02 (2019) 143

GENERALIZATION OF THE LTD THEOREM AT TWO LOOPS

- With these notations, the LTD theorem at one loop can be written

$$\mathcal{A}_N^{(1)} = \int_{\ell_1} \mathcal{N}(\ell_1, \{p_i\}_N)\, G_F(\alpha_1) = -\int_{\ell_1} \mathcal{N}(\ell_1, \{p_i\}_N)\, G_D(\alpha_1)$$

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- Using this, and starting from the Feynman amplitude

$$\mathcal{A}_N^{(2)} = \int_{\ell_1}\int_{\ell_2} \mathcal{N}(\ell_1, \ell_2, \{p_i\}_N) G_F(\alpha_1) \underline{G_F(\alpha_2 \cup \alpha_3)}$$
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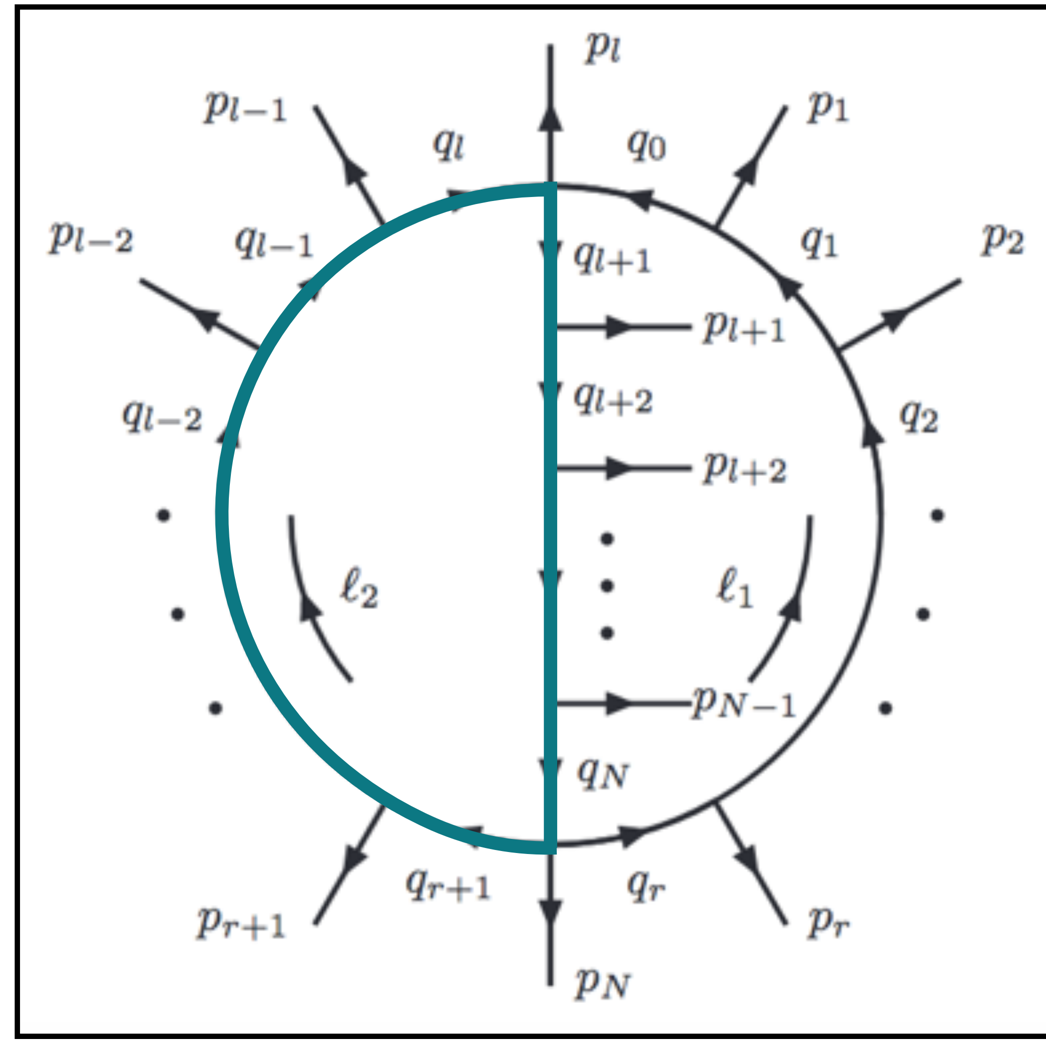

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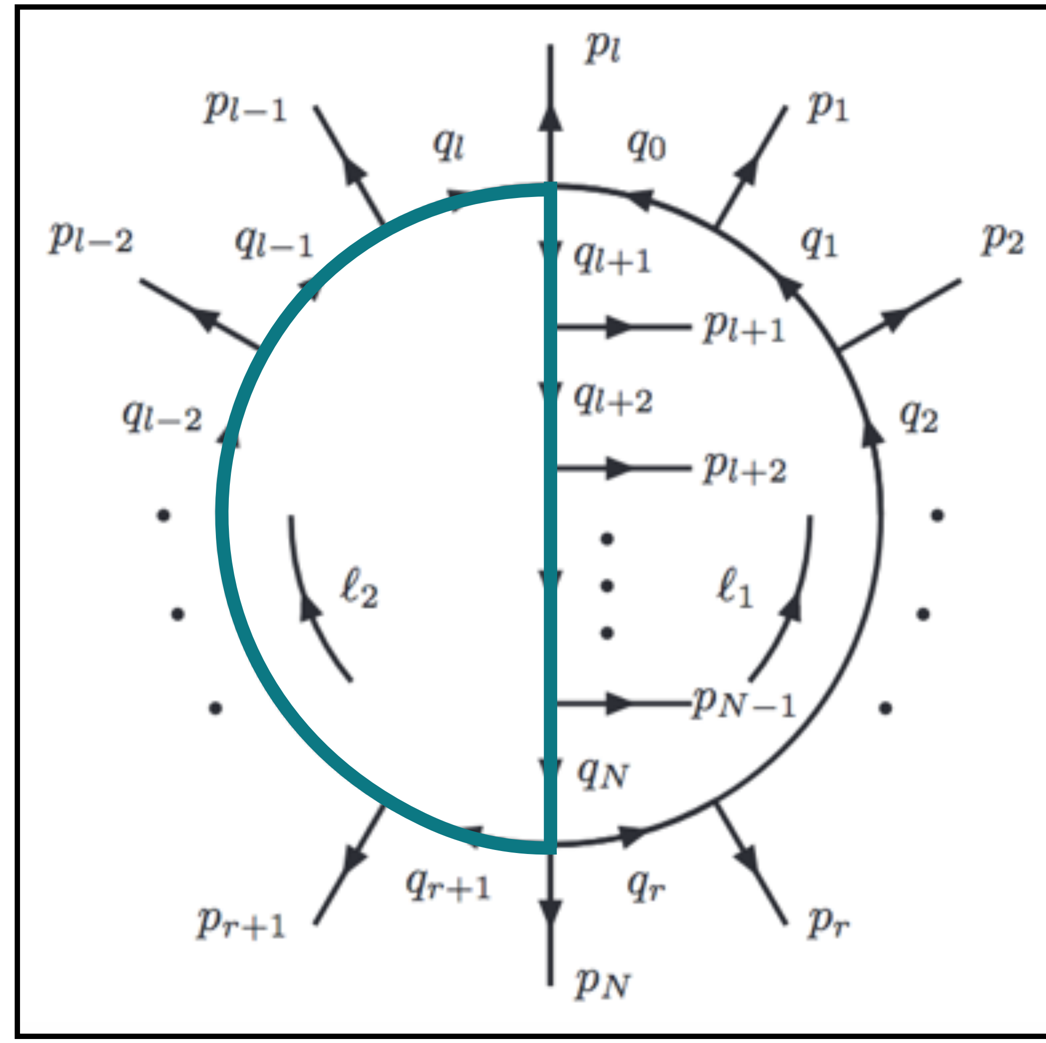

$$\Rightarrow \quad G_F(\alpha_1)G_D(\alpha_2)G_D(\alpha_3) + G_F(\alpha_1)G_F(\alpha_2)G_D(\alpha_3) + G_F(\alpha_1)G_D(\alpha_2)G_F(\alpha_3)$$

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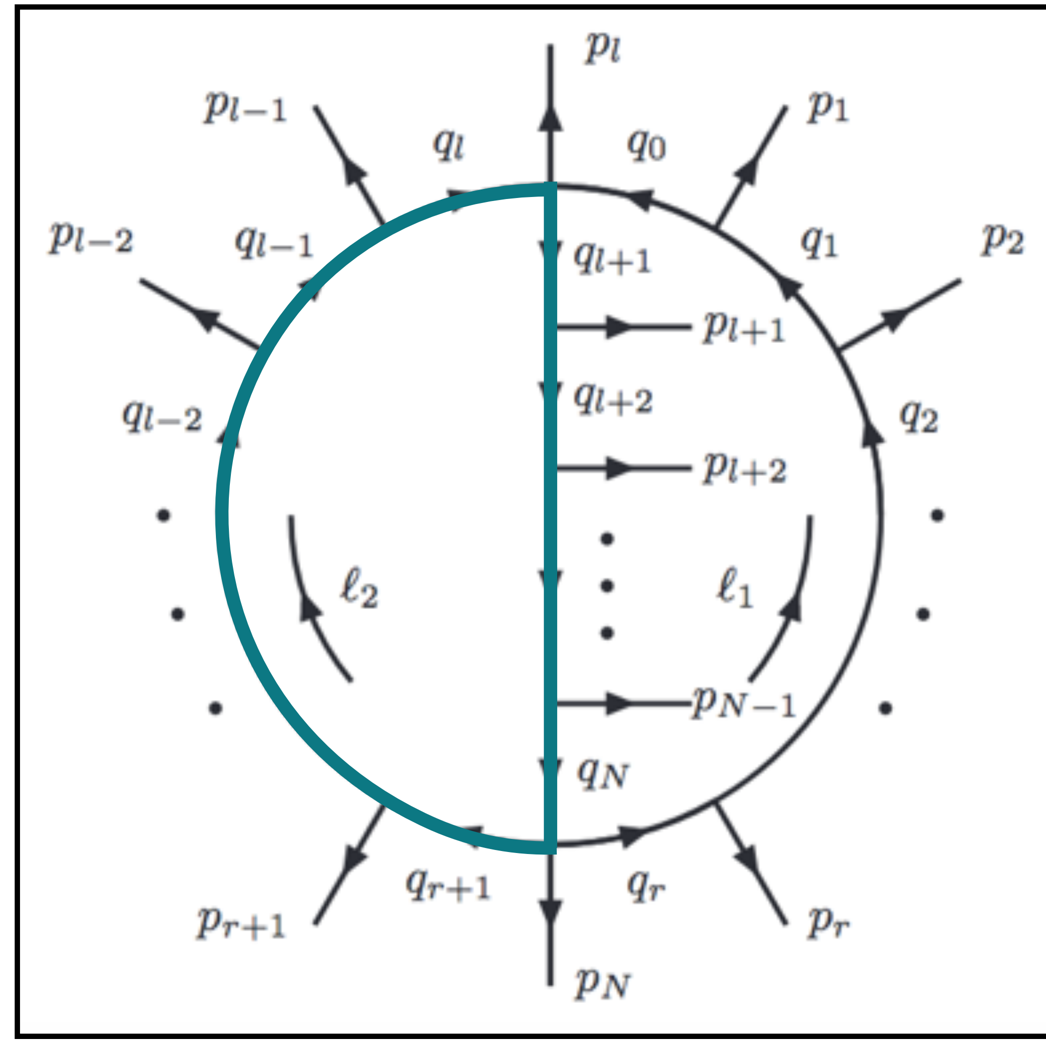

$$\Rightarrow \quad G_F(\alpha_1)G_D(\alpha_2)G_D(\alpha_3) + G_F(\alpha_1)G_F(\alpha_2)G_D(\alpha_3) + G_F(\alpha_1)G_D(\alpha_2)G_F(\alpha_3)$$

Ok

$$\boxed{-G_D(-\alpha_2 \cup \alpha_1)} \qquad \boxed{-G_D(\alpha_1 \cup \alpha_3)}$$

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GENERALIZATION OF THE LTD THEOREM AT TWO LOOPS

- Which leads to the master formula at two loops

$$\boxed{\mathcal{A}_N^{(2)} = \int_{\ell_1}\int_{\ell_2} \mathcal{N}(\ell_1, \ell_2, \{p_i\}_N) \left[G_D(\alpha_2)\, G_D(\alpha_1 \cup \alpha_3) + G_D(-\alpha_2 \cup \alpha_1)\, G_D(\alpha_3) - G_F(\alpha_1)\, G_D(\alpha_2)\, G_D(\alpha_3) \right]}$$

($\alpha_1, \alpha_2, \alpha_3$ completely interchangeable)

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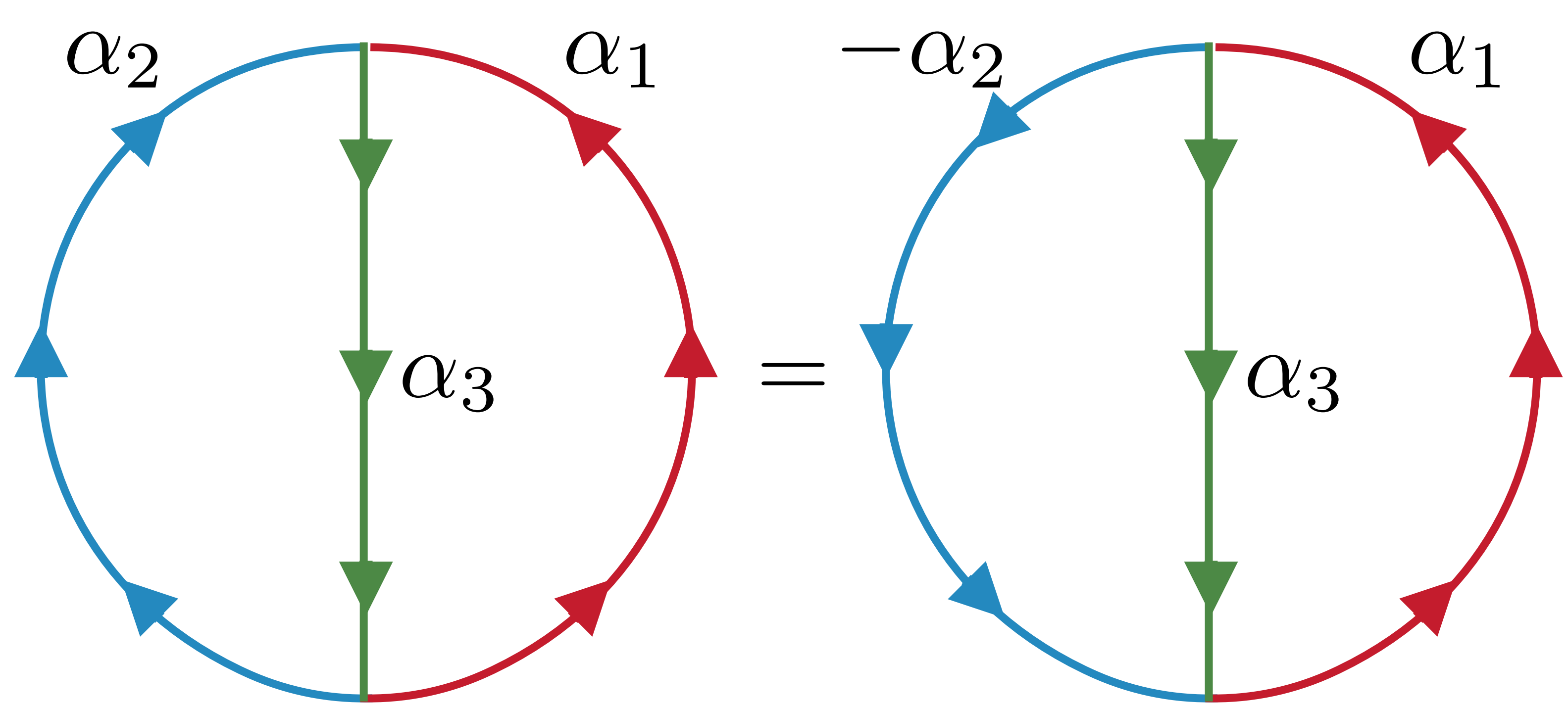
- Notice the minus sign in the second term $-\alpha_k = \{-q,\ q \in \alpha_k\}$



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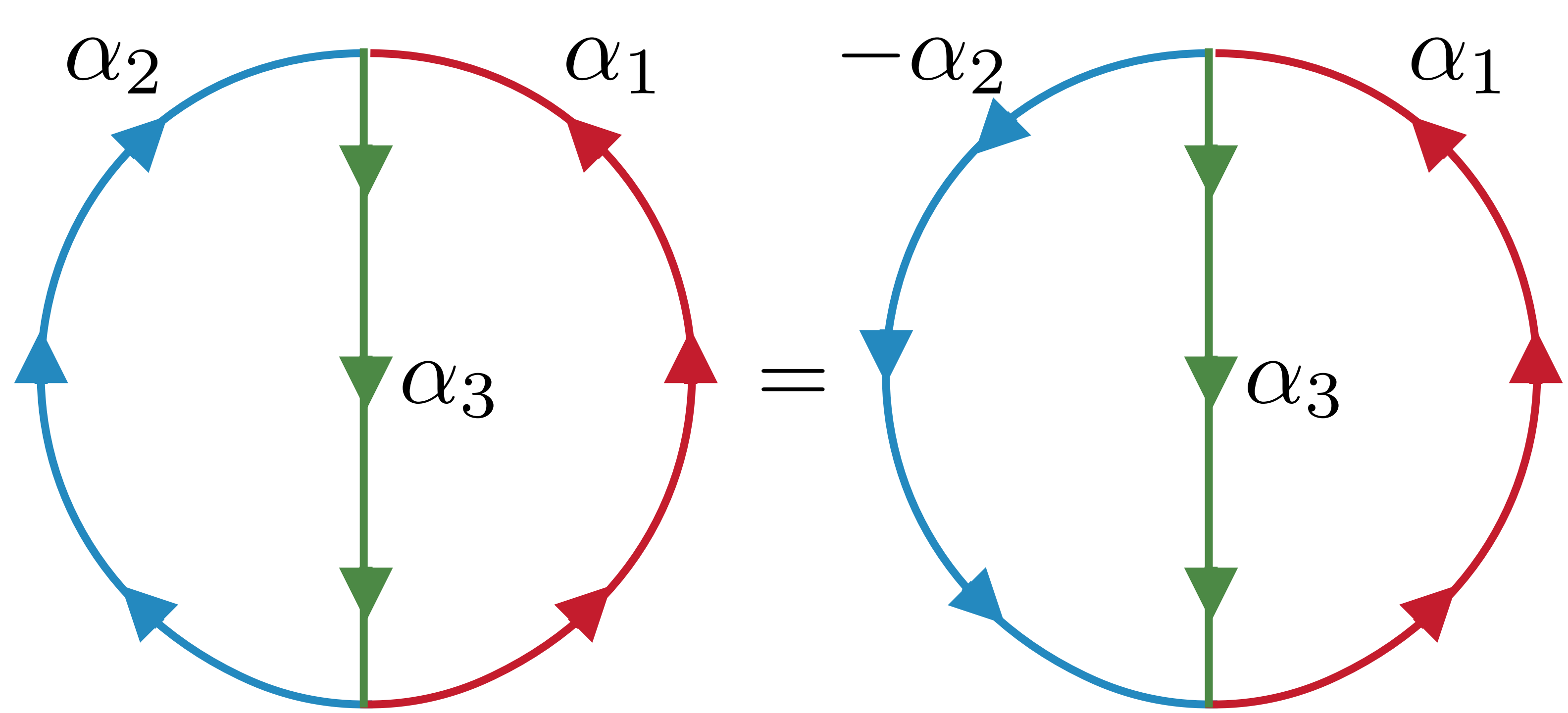
- The on-shell delta is modified accordingly

$$\boxed{\tilde{\delta}(-q_j) = \frac{i\,\pi}{q_{j,0}^{(+)}}\,\delta(q_{j,0}+q_{j,0}^{(+)})}$$



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ALGEBRAIC REDUCTION OF TWO-LOOP AMPLITUDES

- Say we have a **planar** two-loop diagram with **fixed external ordering**, we can write

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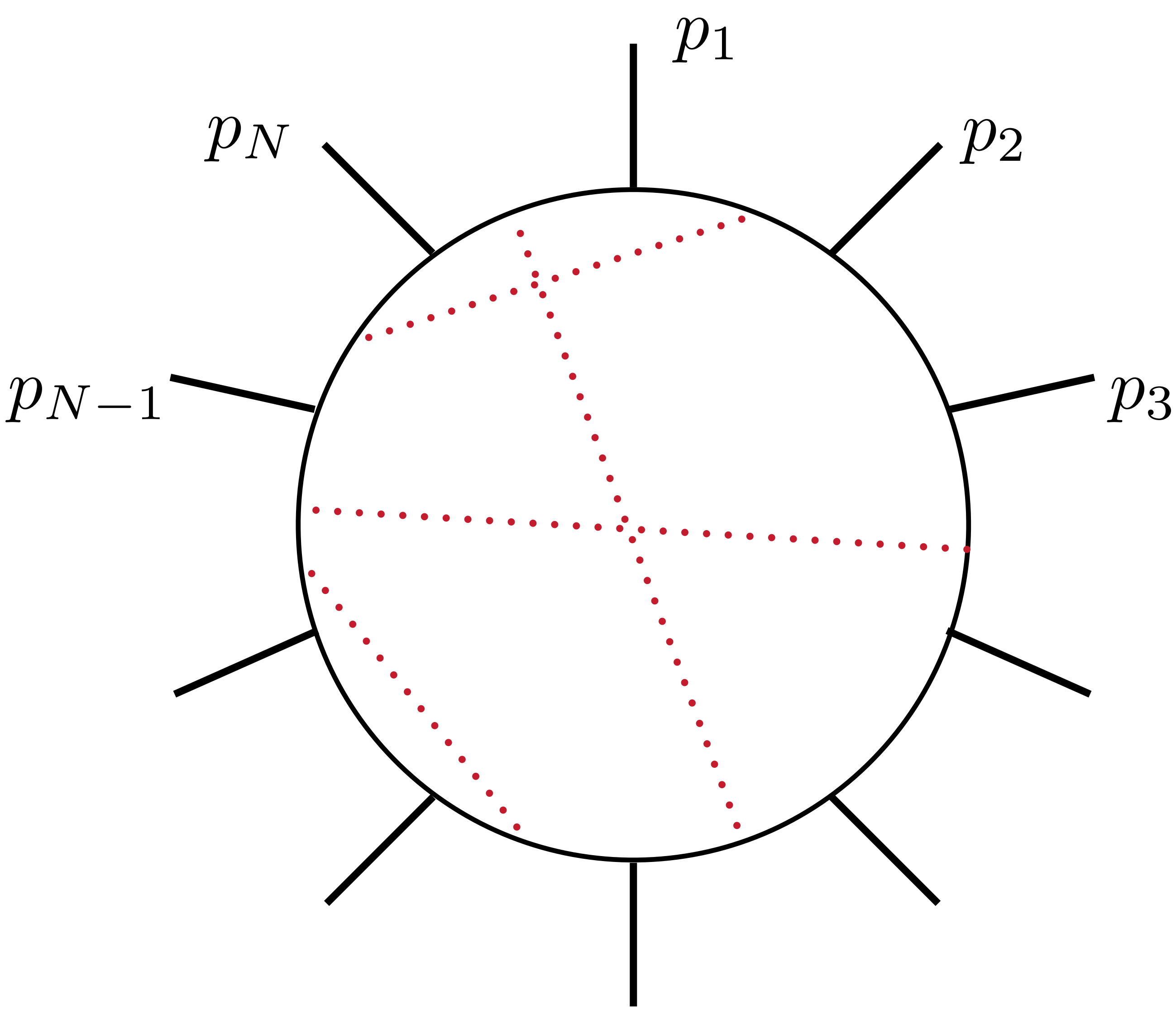

$$\alpha_1 = \{q_1, q_{12}, \ldots, q_{1N}\}$$
$$\alpha_2 = \{q_{N+1}\}$$
$$\alpha_3 = \{q_{\overline{1}}, q_{\overline{12}}, \ldots, q_{\overline{1N}}\}$$

$$q_{1j} = \ell_1 + p_1 + p_2 + \cdots + p_j$$
$$q_{N+1} = \ell_2$$
$$q_{\overline{1j}} = \ell_1 + \ell_2 + p_1 + p_2 + \cdots + p_j$$

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ALGEBRAIC REDUCTION OF TWO-LOOP AMPLITUDES

- This sums up to $\mathrm{N}(\alpha_1+\alpha_2+\alpha_3)=2N+1$ Feynman propagators for the **uncut** integrals… but applying LTD removes two of them, so for a given cut $\tilde{\delta}(q_i,q_j)$, we have in the end $2N-1$ dual propagators

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- The **independent** scalar products we can encounter in the numerator are

$$\{\ell_1 \cdot p_i,\ \ell_2 \cdot p_i,\ \ell_1 \cdot \ell_2 \mid i \in \{1,\ 2, \ldots,\ N-1\}\}$$

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➡ There are as many dual propagators as scalar products

- It is therefore possible to rewrite the numerators in terms of dual propagators, and this **in a unique way**

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ALGEBRAIC REDUCTION OF TWO-LOOP AMPLITUDES

- We can rewrite any planar two-loop integrand $\mathcal{A}_N^{(2)}(\ell_1, \ell_2, \{p_i\}_N)$ as

$$\mathcal{A}_N^{(2)} = \int_{\ell_1} \int_{\ell_2} \mathcal{N}(\ell_1, \ell_2, \{p_i\}_N)\, G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3) + \text{perm.} \qquad \text{(Before cutting)}$$

$$= \int_{\ell_1} \int_{\ell_2} \sum_{j,k} \left[\frac{c_{a_0;a_1,\ldots,a_{2N-1}}(\{p_i\}_N)}{(\kappa_j)^{a_0} (d_{i_1})^{a_1} (d_{i_2})^{a_2} \cdots (d_{i_{2N-1}})^{a_{2N-1}}} \right] \tilde{\delta}(q_j, q_k) + \text{perm.} \qquad \text{(After cutting)}$$

- The idea is to rearrange the expressions of the dual cuts so we have the minimum amount of independent coefficients $c_{a_0;a_1,\ldots,a_{2N-1}}$

III. Procedure for local renormalisation at two-loop order

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ONE-LOOP PROCEDURE

- We consider a Feynman (uncut) integrand $I(\ell, \{p_i\}_N)$, and the replacement

$$S : \begin{cases} \ell^2 \to \lambda^2 \, \ell^2 + (1-\lambda^2)\mu^2 \\ \ell \cdot p_i \to \lambda \, \ell \cdot p_i \end{cases}$$

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- Computing the **local UV counter-term** C of I is done by
  - Applying the replacement S on I
  - Taking the limit $\lambda \to \infty$
  - Selecting the divergent terms, which gives (unfixed) C
  - Fixing the finite part so C integrates to the desired quantity ($\mathcal{O}(\epsilon^0) = 0$ in $\overline{\mathrm{MS}}$)

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- We then obtain a counter-term C and the quantity $I - C$ is **locally UV safe**

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TWO-LOOP PROCEDURE (SINGLE UV)

- This time, we consider a two-loop Feynman integrand $I(\ell_1, \ell_2, \{p_i\}_N)$

- Applying the one-loop procedure to each loop momenta **independently**, using the replacements

$$S_1 : \begin{cases} \ell_1^2 \to \lambda^2\, \ell_1^2 + (1-\lambda^2)\mu^2 \\ \ell_1 \cdot p_i \to \lambda\, \ell_1 \cdot p_i \end{cases} \qquad S_2 : \begin{cases} \ell_2^2 \to \lambda^2\, \ell_2^2 + (1-\lambda^2)\mu^2 \\ \ell_2 \cdot p_i \to \lambda\, \ell_2 \cdot p_i \end{cases}$$

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- We obtain two counter-terms, C_1 and C_2, but $I - C_1 - C_2$ is still **not UV safe**

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- We obtain two counter-terms, C_1 and C_2, but $I - C_1 - C_2$ is still **not UV safe**

$\Rightarrow$ We need to subtract the double UV limit (when both loop momenta go to infinity)

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TWO-LOOP PROCEDURE (DOUBLE UV)

- Computing the double UV behavior is very similar to the one-loop procedure, with some subtleties. We consider the replacement

$$S_{12}: \begin{cases} \ell_i^2 \to \lambda^2\, \ell_i^2 + (1-\lambda^2)\mu^2 \\ \ell_1 \cdot \ell_2 \to \lambda^2\, \ell_1 \cdot \ell_2 - (1-\lambda^2)\mu^2/2 \\ \ell_i \cdot p_k \to \lambda\, \ell_i \cdot p_k \end{cases}$$

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- We then take $I - C_1 - C_2$, and the counter-term is obtained by
  - Applying the replacement S_{12} on $I - C_1 - C_2$
  - Taking the limit $\lambda \to \infty$
  - Selecting the divergent terms, which gives (unfixed) C_{12}
  - Fixing the finite part so C_{12} integrates to the desired quantity ($\mathcal{O}(\epsilon^0) = 0$ in $\overline{\mathrm{MS}}$)

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TWO-LOOP PROCEDURE (DOUBLE UV)

- This iterative way is similar to what is done in DREG, but you **don't need to integrate anything** to compute the actual counter-terms

- In addition to fixing the potential additional singularities introduced by C_1 and C_2 , C_{12} **also removes singularities occurring when** $(\ell_1, \ell_2) \to (\infty, \infty)$

- $I_{ren} = I - C_1 - C_2 - C_{12}$ is therefore completely free of any UV singularity, and, after applying LTD, **can safely be integrated in four dimensions**!

IV. Application to $H \to \gamma\gamma$ at two loops

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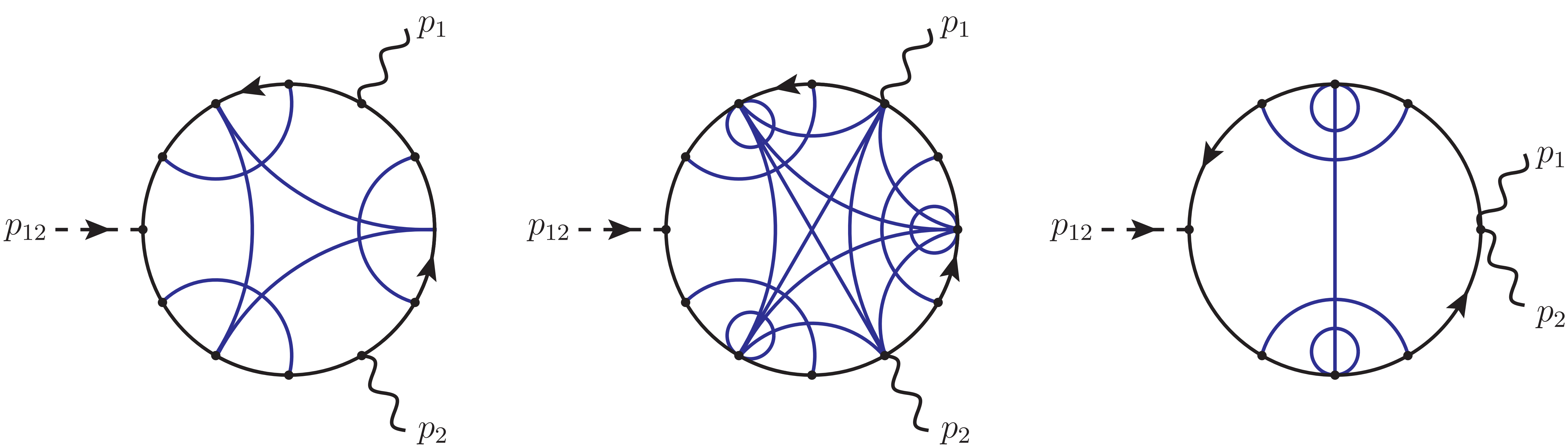
"NON-MIXED" QED CORRECTIONS



12 diagrams with a top as the internal particle

37 diagrams with a charged scalar as the internal particle

(Blue lines are photons)

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SIMPLIFYING THE MASTER FORMULA

- If the Higgs boson is on shell, we are below threshold, i.e. $4M_f^2 > M_H^2$

$\Rightarrow$ No imaginary part $\Rightarrow$ Prescriptions unnecessary

- This simplifies a lot the two-loop representation of LTD

$$\Big[G_D(\alpha_2)\, G_D(\alpha_1 \cup \alpha_3) + G_D(-\alpha_2 \cup \alpha_1)\, G_D(\alpha_3) - G_F(\alpha_1)\, G_D(\alpha_2)\, G_D(\alpha_3) \Big]$$

$$\Downarrow$$

$$\Big[G_D(\alpha_1)\, G_D(\alpha_2)\, G_F(\alpha_3) + G_F(\alpha_1)\, G_D(-\alpha_2)\, G_D(\alpha_3) + G_D(\alpha_1)\, G_F(\alpha_2)\, G_D(\alpha_3) \Big]$$

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$$\Big[G_D(\alpha_2)\,G_D(\alpha_1\cup\alpha_3)+G_D(-\alpha_2\cup\alpha_1)\,G_D(\alpha_3)-G_F(\alpha_1)\,G_D(\alpha_2)\,G_D(\alpha_3)\Big]$$

$$\Downarrow$$

$$\Big[\underbrace{G_D(\alpha_1)\,G_D(\alpha_2)\,G_F(\alpha_3)}_{\text{4 double cuts}}+\underbrace{G_F(\alpha_1)\,G_D(-\alpha_2)\,G_D(\alpha_3)}_{\text{4 double cuts}}+\underbrace{G_D(\alpha_1)\,G_F(\alpha_2)\,G_D(\alpha_3)}_{\text{14 double cuts}}\Big]$$

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- The 22 dual double cuts can be written with 9 generators, for instance

$$
\begin{aligned}
\mathcal{A}_1^{(2,f)}(q_i,q_4) = g_f^{(2)} \int_{\ell_1}\int_{\ell_2} \tilde{\delta}(q_i,q_4) \Bigg\{ & -\frac{r_f\, c_1^{(f)}}{D_3\, D_{12}} \Big(G(D_{\bar{i}},\kappa_i,c_{4,u}^{(f)})\big(1+H(D_3\, D_{12},\kappa_i)\big) + F(D_{\bar{i}},\kappa_4/\kappa_i) \Big) \\
& + \Bigg(c_7^{(f)} \bigg(\frac{1}{D_{\bar{i}}} - \frac{1}{D_{\bar{3}}} \bigg(1 - \frac{D_3}{D_{12}} \bigg(1 - \frac{D_{\overline{12}}}{D_{\bar{i}}} \bigg)\bigg)\bigg)\Bigg) + \frac{1}{D_3} \bigg(c_8^{(f)} \bigg(\frac{1}{D_{\bar{3}}} - \frac{1}{D_{\bar{i}}} \bigg) - \frac{1}{D_{\overline{12}}} \bigg(c_9^{(f)} - c_{10}^{(f)} \frac{D_{\bar{3}}}{D_{\bar{i}}} \bigg)\bigg) \\
& + 2\, r_f \Bigg[\frac{1}{D_3\, D_{12}} \Bigg(c_1^{(f)} \bigg(\frac{1}{D_3\, D_{\bar{3}}} + \frac{1}{D_{\bar{i}}} \bigg(\frac{1}{D_{\bar{3}}} - \frac{1}{D_3} \bigg)\bigg)\Bigg) + \frac{c_{14}^{(f)}}{D_{\bar{3}}} + \frac{c_{20}^{(f)}}{D_{\bar{i}}} - c_{16}^{(f)} \\
& + c_{17}^{(f)} \bigg(\frac{D_{\bar{i}} - D_{\overline{12}}}{D_{\bar{3}}} + \frac{D_{\bar{3}}}{D_{\bar{i}}} \bigg)\bigg) - \frac{1}{D_{\bar{i}}\, D_{\bar{3}}} \Bigg(\frac{c_7^{(f)}}{D_{12}} + c_{18}^{(f)} \Bigg) \Bigg] + \{3 \leftrightarrow 12\} \Bigg) \Bigg\}
\end{aligned}
$$

- The $c_i^{(f)}$ are scalar coefficients and depend only on the reduced mass $r_f = \dfrac{s_{12}}{M_f^2}$ and the dimension d, while the D_i are normalized dual propagators

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$$c_{4,u}^{(t)} = -\frac{d-2}{4}\,, \qquad c_{4,nu}^{(t)} = -\frac{d-2}{4}\,, \qquad c_{7}^{(t)} = -\frac{1}{4}(c_1^{(t)} - r_t)\,,$$

$$c_{8}^{(t)} = c_1^{(t)} + \frac{(d-6)d+10}{2(d-2)}\,r_t\,, \qquad c_{9}^{(t)} = c_1^{(t)} - \frac{(d-8)d+10}{2(d-2)}\,r_t\,, \qquad c_{10}^{(t)} = c_1^{(t)} - \frac{(d-8)d+14}{2(d-2)}\,r_t\,,$$

$$c_{11}^{(t)} = c_1^{(t)} + \frac{(d-8)d+18}{2(d-2)}\,r_t\,, \qquad c_{12}^{(t)} = -\frac{(d-4)(d-5)}{d-2}\,r_t\,, \qquad c_{13}^{(t)} = -\frac{(d-6)d+12}{2(d-2)}\,r_t\,,$$

$$c_{14}^{(t)} = \frac{3}{4}\left(c_1^{(f)} - \frac{d}{3(d-2)}\,r_t\right)\,, \qquad c_{15}^{(t)} = -\frac{1}{2}\left(c_1^{(f)} + \frac{r_t}{2}\right)\,, \qquad c_{16}^{(t)} = \frac{d-8}{4}\,,$$

$$c_{17}^{(t)} = \frac{d-4}{4}\,, \qquad c_{18}^{(t)} = -\frac{(d-4)^2}{4(d-2)}\,, \qquad c_{19}^{(t)} = \frac{1}{2}\left(c_1^{(t)} + \frac{1}{d-2}\,r_t\right)\,,$$

$$c_{20}^{(t)} = \frac{1}{4}(c_1^{(t)} + r_t)\,, \qquad c_{21}^{(t)} = -\frac{2(d-4)}{d-2} + \frac{(d-10)d+18}{4(d-2)}\,r_t\,, \qquad c_{22}^{(t)} = -2 + \frac{(d-4)d}{4(d-2)}\,r_t\,,$$

$$c_{4,u}^{(\phi)} = -\frac{d-2}{4}\,, \qquad c_{4,nu}^{(\phi)} = \frac{1}{4}\,, \qquad c_{7}^{(\phi)} = -\frac{1}{4}\,c_1^{(\phi)}\,,$$

$$c_{8}^{(\phi)} = c_1^{(\phi)}\,, \qquad c_{9}^{(\phi)} = c_1^{(\phi)} - \frac{1}{(d-2)}\,r_\phi, \qquad c_{10}^{(\phi)} = c_1^{(\phi)}\,,$$

$$c_{11}^{(\phi)} = c_1^{(\phi)} + \frac{d-4}{d-2}\,r_\phi\,, \qquad c_{12}^{(\phi)} = -\frac{3(d-4)}{2(d-2)}\,r_\phi\,, \qquad c_{13}^{(\phi)} = \frac{1}{d-2}\,r_\phi\,,$$

$$c_{14}^{(\phi)} = \frac{3}{4}\,c_1^{(\phi)}\,, \qquad c_{15}^{(\phi)} = -\frac{1}{2}\,c_1^{(\phi)}\,, \qquad c_{16}^{(\phi)} = \frac{1}{2}\,,$$

$$c_{17}^{(\phi)} = 0\,, \qquad c_{18}^{(\phi)} = 0\,, \qquad c_{19}^{(\phi)} = \frac{1}{2}\,c_1^{(\phi)}\,,$$

$$c_{20}^{(\phi)} = \frac{1}{4}\,c_1^{(\phi)}\,, \qquad c_{21}^{(\phi)} = -\frac{3}{d-2}\,, \qquad c_{22}^{(\phi)} = -\frac{1}{d-2}\,.$$

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SINGLE UV COUNTER-TERMS

- There are three things to renormalize:
  - The Higgs boson vertex
  - The photon vertices
  - The self-energies

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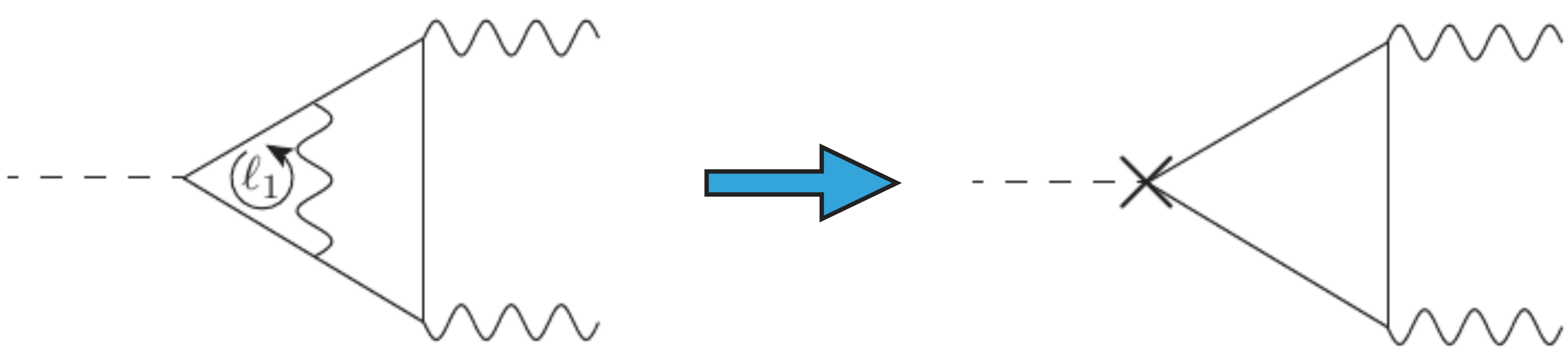
SINGLE UV COUNTER-TERMS

- There are three things to renormalize:
  - The Higgs boson vertex
  - The photon vertices
  - The self-energies
- The single UV counter-terms are built by taking ℓ_1 or $\ell_{12} = \ell_1 + \ell_2$ to infinity in the relevant diagrams
- For instance, for the Higgs boson vertex correction



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HIGGS BOSON VERTEX RENORMALISATION

- There are two contributing diagrams for the top, three for the scalar and the counter-term is computed by taking $\ell_1 \to \infty$ at integrand level
- The Higgs vertex corrections read, for both particles,

$$\begin{aligned}\mathbf{\Gamma}_{H,\mathrm{UV}}^{(1,f)} &= (e\, e_f)^2 \int_{\ell_1} (G_F(q_{1,\mathrm{UV}}))^2 \left(c_{H,\mathrm{UV}}^{(f)} - G_F(q_{1,\mathrm{UV}})\, d_{H,\mathrm{UV}}^{(f)}\, \mu_{\mathrm{UV}}^2 \right) \mathbf{\Gamma}_H^{(0,f)} \\ &= (e\, e_f)^2 \frac{\tilde{S}_\epsilon}{16\pi^2} \left(\frac{\mu_{\mathrm{UV}}^2}{\mu^2} \right)^{-\epsilon} \frac{C_{H,\mathrm{UV}}^{(f)}}{\epsilon} \mathbf{\Gamma}_H^{(0,f)} ,\end{aligned}$$

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- The Higgs vertex corrections read, for both particles,

Depends on what we renormalise (here the Higgs vertex)

Depends on the renormalisation scheme

$$
\begin{aligned}
\mathbf{\Gamma}^{(1,f)}_{H,\mathrm{UV}} &= (e\, e_f)^2 \int_{\ell_1} \left(G_F(q_{1,\mathrm{UV}})\right)^2 \left(c^{(f)}_{H,\mathrm{UV}} - G_F(q_{1,\mathrm{UV}})\, d^{(f)}_{H,\mathrm{UV}}\, \mu^2_{\mathrm{UV}} \right) \mathbf{\Gamma}^{(0,f)}_{H} \\
&= (e\, e_f)^2 \frac{\tilde{S}_\epsilon}{16\pi^2} \left(\frac{\mu^2_{\mathrm{UV}}}{\mu^2} \right)^{-\epsilon} \frac{\boxed{C^{(f)}_{H,\mathrm{UV}}}}{\epsilon} \mathbf{\Gamma}^{(0,f)}_{H} \,,
\end{aligned}
$$

Is a combination of $c^{(f)}_{H,\mathrm{UV}}$ and $d^{(f)}_{H,\mathrm{UV}}$ and is obtained by integrating in d dimensions

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PHOTON VERTEX RENORMALISATION

- The idea is exactly the same (there are more diagrams though), with this time the limit that needs to be considered being $\ell_{12} \to \infty$
- The corresponding counter-term for the top reads

$$\begin{aligned}\boldsymbol{\Gamma}^{(1,t)}_{\gamma,\mathrm{UV}} &= (e\, e_t)^2 \int_{\ell_2} (G_F(q_{12,\mathrm{UV}}))^2 \left(\left(c^{(t)}_{\gamma,\mathrm{UV}} - G_F(q_{12,\mathrm{UV}})\, d^{(t)}_{\gamma,\mathrm{UV}}\, \mu^2_{\mathrm{UV}} \right) \boldsymbol{\Gamma}^{(0,t)}_{\gamma} + c^{(t)}_{\gamma,\mathrm{UV}}\, \boldsymbol{\Delta}^{(1,t)}_{\gamma,\mathrm{UV}} \right) \\ &= (e\, e_t)^2 \frac{\tilde{S}_\epsilon}{16\pi^2} \left(\frac{\mu^2_{\mathrm{UV}}}{\mu^2} \right)^{-\epsilon} \frac{C^{(t)}_{\gamma,\mathrm{UV}}}{\epsilon}\, \boldsymbol{\Gamma}^{(0,t)}_{\gamma}\end{aligned}$$

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$$= (e\, e_t)^2 \frac{\tilde{S}_\epsilon}{16\pi^2} \left(\frac{\mu^2_{\mathrm{UV}}}{\mu^2} \right)^{-\epsilon} \frac{C^{(t)}_{\gamma,\mathrm{UV}}}{\epsilon} \mathbf{\Gamma}^{(0,t)}_{\gamma}$$

- The additional term $\mathbf{\Delta}^{(1,t)}_{\gamma,\mathrm{UV}}$ integrates to 0 in d dimensions but is needed for **local renormalisation**

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DOUBLE UV RENORMALISATION

- Accorded to the replacement S_{12}, the double UV counter-term must have the form

$$\mathcal{A}_{\mathrm{UV}^2}^{(2,f)} = g_f\, s_{12}(e\, e_f)^2 \int_{\ell_1} \int_{\ell_2} \Big[(G_F(q_{1,\mathrm{UV}}))^{n_1} \, (G_F(q_{2,\mathrm{UV}}))^{n_2} \, (G_F(q_{12,\mathrm{UV}}))^{n_{12}} \, \mathcal{N}^{(f)} - 4 \, (G_F(q_{1,\mathrm{UV}}))^3 \, (G_F(q_{12,\mathrm{UV}}))^3 \, d_{\mathrm{UV}^2}^{(f)} \, \mu_{\mathrm{UV}}^4 \Big] \,,$$

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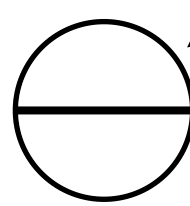
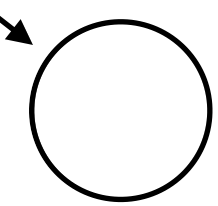
- By using IBP, we can show that $\mathcal{A}^{(2,f)}_{\mathrm{UV}^2} = c^{(f)}_{\ominus} \, I_{\ominus} + c^{(f)}_{\odot} \, I^2_{\odot}$

Sunrise diagram with vanishing external momenta





Massive tadpole

- By replacing the integrals by their values in d dimensions, we can choose $d^{(f)}_{\mathrm{UV}^2}$ to fix the renormalisation scheme

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DOUBLE UV RENORMALISATION

- The total double UV counter-terms for the top and the scalar read

$$\mathcal{A}^{(2,t)}_{\mathrm{UV}^2} = g_f\, s_{12}\, (e\, e_t)^2 \left(\frac{\tilde{S}_\epsilon}{16\pi^2}\right)^2 \left(\frac{\mu^2_{\mathrm{UV}}}{\mu^2}\right)^{-2\epsilon} \left(40 + \frac{16 K_\ominus}{3} + 4(d^{(t)}_{H,\mathrm{UV}} - d^{(t)}_{\gamma,\mathrm{UV}}) - d^{(t)}_{\mathrm{UV}^2} + \mathcal{O}(\epsilon)\right)$$

$$\mathcal{A}^{(2,\phi)}_{\mathrm{UV}^2} = g_f\, s_{12}\, (e\, e_\phi)^2 \left(\frac{\tilde{S}_\epsilon}{16\pi^2}\right)^2 \left(\frac{\mu^2_{\mathrm{UV}}}{\mu^2}\right)^{-2\epsilon} \left(-18 - \frac{8 K_\ominus}{3} - d^{(\phi)}_{\mathrm{UV}^2} + \mathcal{O}(\epsilon)\right) ,$$

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- Even though they do not actually *renormalise* anything, their presence is still necessary to remove **local double UV divergences**

- This is very similar to the one-loop case: it is finite, but still requires the presence of a local counter-term to obtain the correct result

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NUMERICAL INTEGRATION

- We use the following parametrizations for the amplitude

$$\begin{aligned}
\boldsymbol{\ell}_1 &= \frac{\sqrt{s_{12}}}{2}\xi_1\left(\sin(\theta_1), 0, \cos(\theta_1)\right) \\
\boldsymbol{\ell}_{12} = \boldsymbol{\ell}_1 + \boldsymbol{\ell}_2 &= \frac{\sqrt{s_{12}}}{2}\xi_{12}\left(\sin(\theta_{12})\cos(\varphi_{12}), \sin(\theta_{12})\sin(\varphi_{12}), \cos(\theta_{12})\right) \\
\mathbf{p}_1 &= \frac{\sqrt{s_{12}}}{2}(0,0,1) \\
\mathbf{p}_2 &= \frac{\sqrt{s_{12}}}{2}(0,0,-1)
\end{aligned}$$

- And we compactify the integration domain by using the change of variables

$$\xi_i \to \frac{x_i}{1-x_i} \qquad \text{for} \qquad x_i \in [0,1]$$

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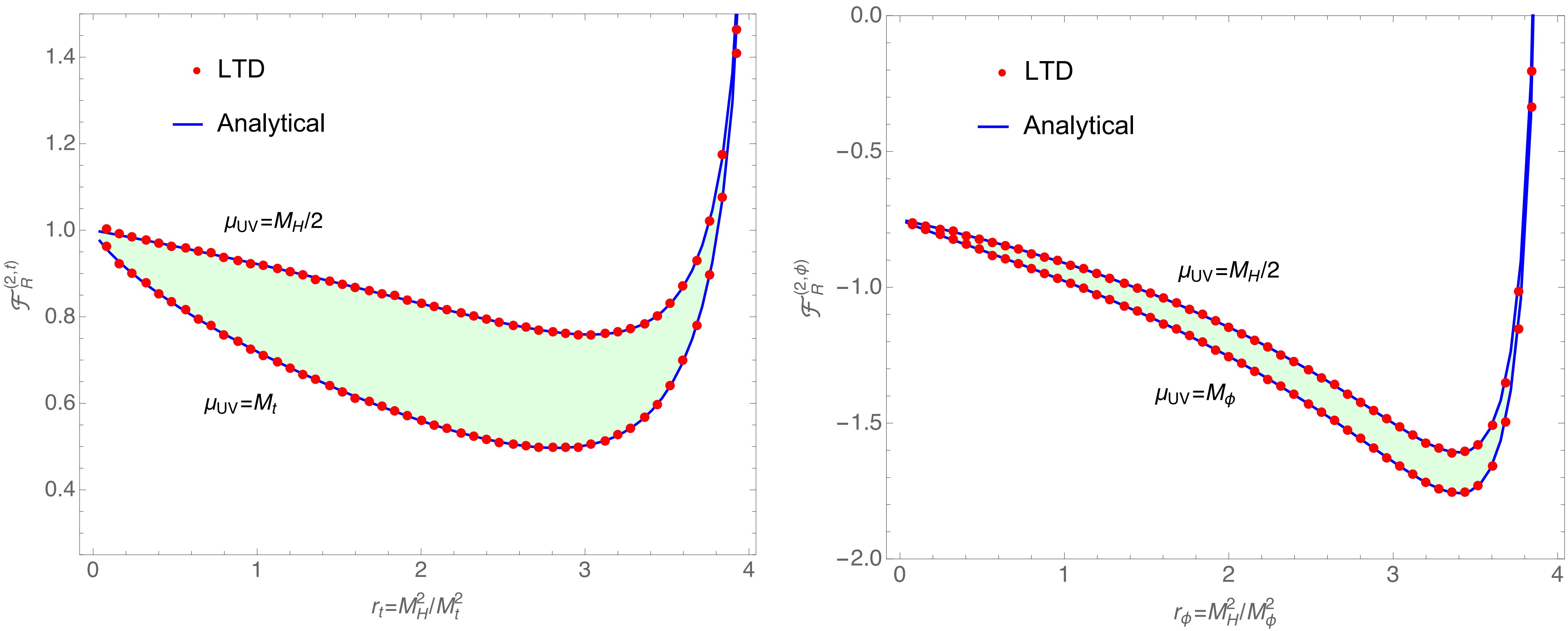
NUMERICAL INTEGRATION



Results in the $\overline{\mathrm{MS}}$ scheme, with two different values of the renormalisation scale

Integration time (with Mathematica on a desktop computer) is $\mathcal{O}(1')$ for each point

SUMMARY & OUTLOOK

What we have achieved…

- The Loop-Tree Duality theorem has been extended to two loops and applied to the $H \to \gamma\gamma$ process at NLO, in a (almost) fully automatized way
- All UV divergences have been dealt with by computing **local counter-terms**, allowing a straightforward numerical integration in four dimensions

SUMMARY & OUTLOOK

What we have achieved…

- The Loop-Tree Duality theorem has been extended to two loops and applied to the $H \to \gamma\gamma$ process at NLO, in a (almost) fully automatized way
- All UV divergences have been dealt with by computing **local counter-terms**, allowing a straightforward numerical integration in four dimensions

What remains to be done…

- Fully functioning automated code at two-loop, from input to plot
- Dealing with potential physical **threshold singularities** (contour deformation) and compute the respective imaginary part
- Dealing with potential **infrared singularities** (i.e. extending FDU at two loops)

Thank you!

Backup slides

Sborlini, FDM, Hernandez, Rodrigo, JHEP 08 (2016) 160

DEALING WITH THE SINGULARITIES

Infrared singularities

$(|\ell| \to 0$ and $\boldsymbol{\ell} \parallel \boldsymbol{p}_i)$

↓

Cancelled by the real contributions

↓

Mapping real kinematics to match the virtual one

Ultraviolet singularities

$(|\ell| \to \infty)$

↓

Dealt with renormalization

↓

Building integrand-level counter-terms to achieve local cancellations

Sborlini, FDM, Hernandez, Rodrigo, JHEP 08 (2016) 160

THE MOMENTUM MAPPING

Defining the mappings requires two steps:

1. Separating the singularities of a same type by splitting the real phase-space into several regions (there cannot be more than one collinear singularity in a given region of the phase-space)

Sborlini, FDM, Hernandez, Rodrigo, JHEP 08 (2016) 160

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2. Implementing an optimized mapping in each region, to allow a fully local cancellation of IR singularities with those present in the dual contributions

In Region i:
$(q_i \,||\, p_i)$

$$\boxed{\begin{array}{ll} p'_r = q_i \ , & p'_i = p_i - q_i + \alpha_i \, p_j \\ p'_j = (1-\alpha_i) p_j \ , & p'_k = p_k \end{array}}$$

Becker, Reuschle, Weinzierl, JHEP 1012:013,2010

BUILD LOCAL UV COUNTER-TERMS

- Expand the **uncut and unintegrated** amplitude around the UV propagator

$$G_F(q_i) = \frac{1}{q_{\mathrm{UV}}^2 - \mu_{\mathrm{UV}}^2 + \imath 0} + \dots \qquad q_{\mathrm{UV}} = \ell + k_{\mathrm{UV}}$$

- By choosing $k_{\mathrm{UV}} = 0$, this is equivalent to applying the following replacement…

$$\begin{cases} \ell^2 \to \lambda^2 q_{\mathrm{UV}}^2 + (1-\lambda^2)\mu_{\mathrm{UV}}^2 \\ \ell \cdot p_i \to \lambda\, q_{\mathrm{UV}} \cdot p_i \end{cases}$$

- … and then expanding around λ and taking only the divergent terms

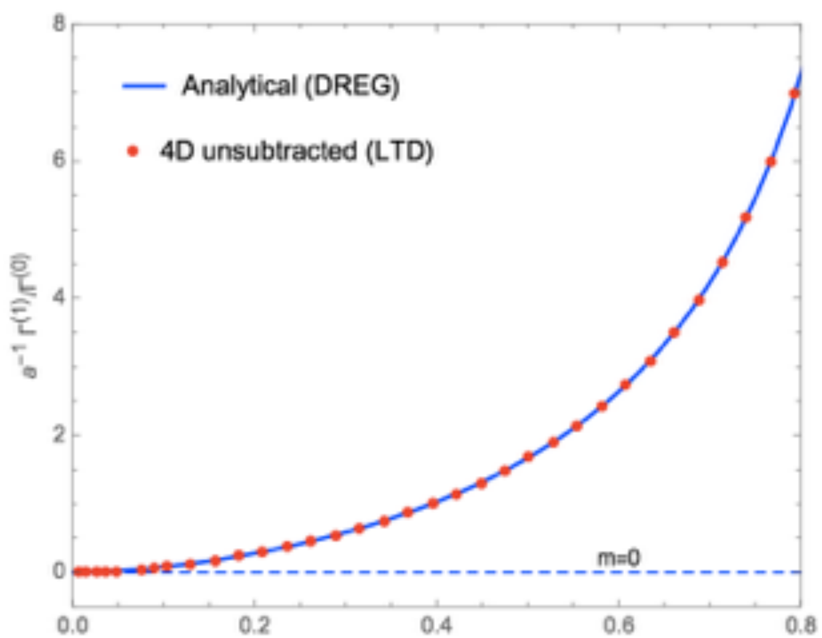
- For the scalar two-point function

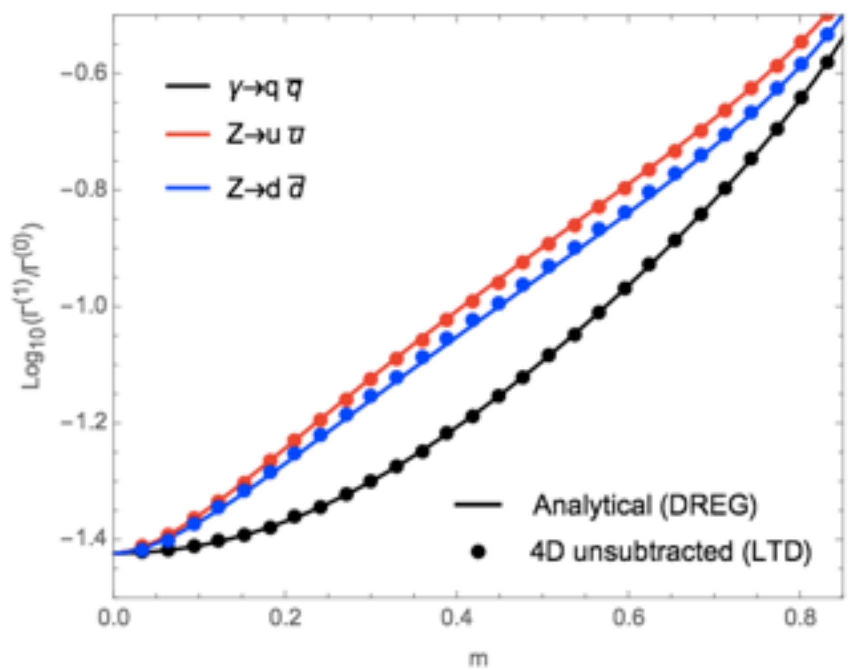
$$I = \int_\ell \frac{1}{(\ell^2 - M^2 + \imath 0)((\ell+p)^2 - M^2 + \imath 0)} \quad \Longrightarrow \quad I_{\mathrm{UV}}^{\mathrm{cnt}} = \int_\ell \frac{1}{(q_{\mathrm{UV}}^2 - \mu_{\mathrm{UV}}^2 + \imath 0)^2}$$

- Apply LTD on this local counter-term, and subtract it from the amplitude

COMPUTATION SAMPLES

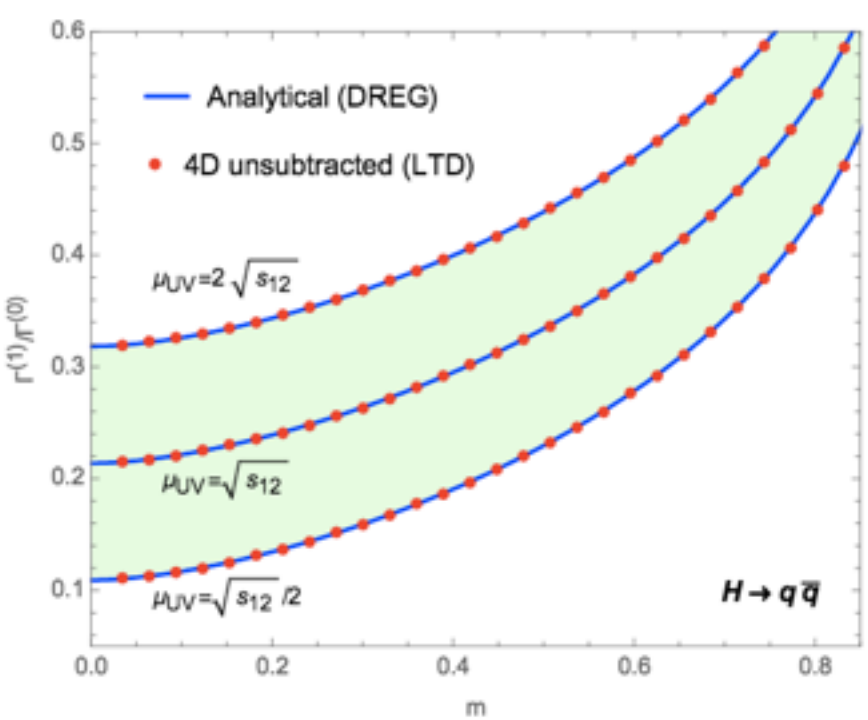
Three-point scalar function

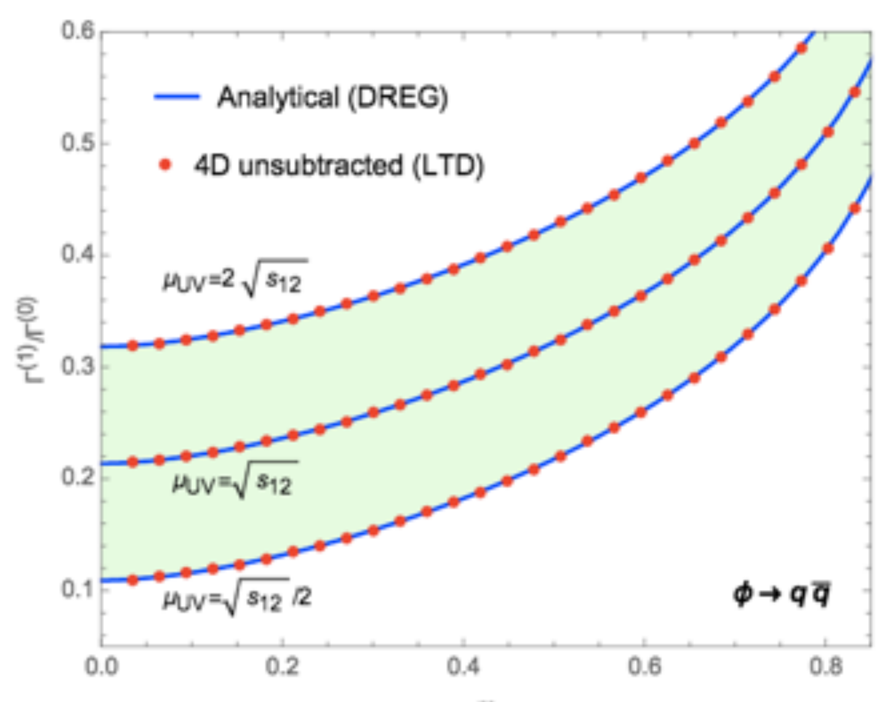




$\gamma/Z \to q\bar{q}$

$H \to q\bar{q}$

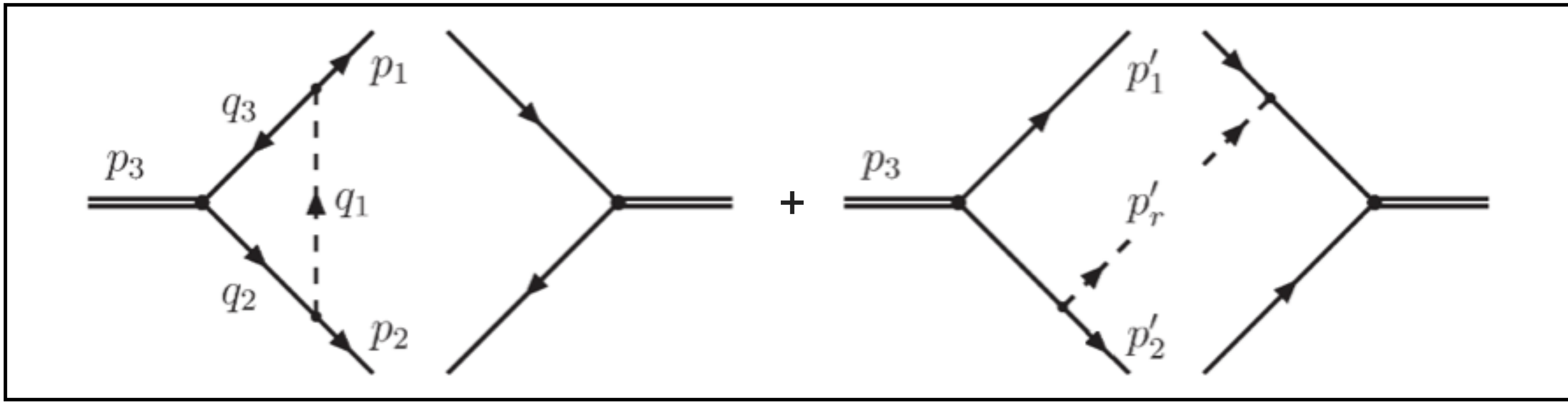




$\phi \to q\bar{q}$

KINOSHITA-LEE-NAUENBERG THEOREM

- The Standard Model is **infrared finite**



- In the traditional approach, the singularities have different signs **after integration**
- Within FDU, cancellations are performed **locally**

THE MOMENTUM MAPPING

Defining the mappings requires two steps:

1. Separating the singularities of a same type by splitting the real phase-space into several regions (there cannot be more than one given type of IR singularity in a given region of the phase-space), for instance

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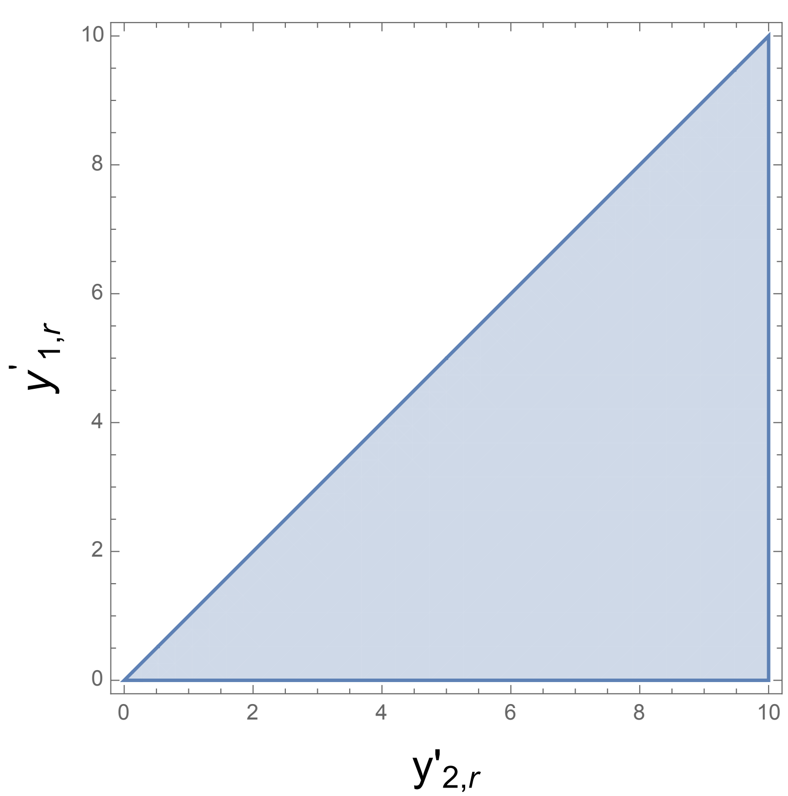
1. Separating the singularities of a same type by splitting the real phase-space into several regions (there cannot be more than one given type of IR singularity in a given region of the phase-space), for instance

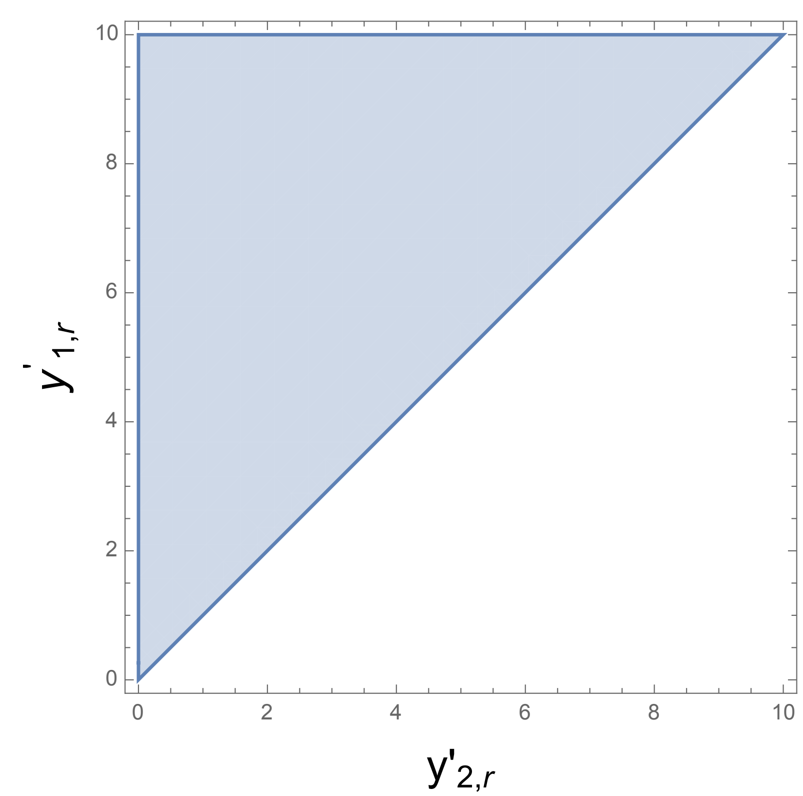
REGION 1

$$y'_{1,r} \leq y'_{2,r}$$



REGION 2

$$y'_{2,r} < y'_{1,r}$$



$$\left(y'_{i,r} = (2p'_i \cdot p'_r)/s_{12}\right)$$

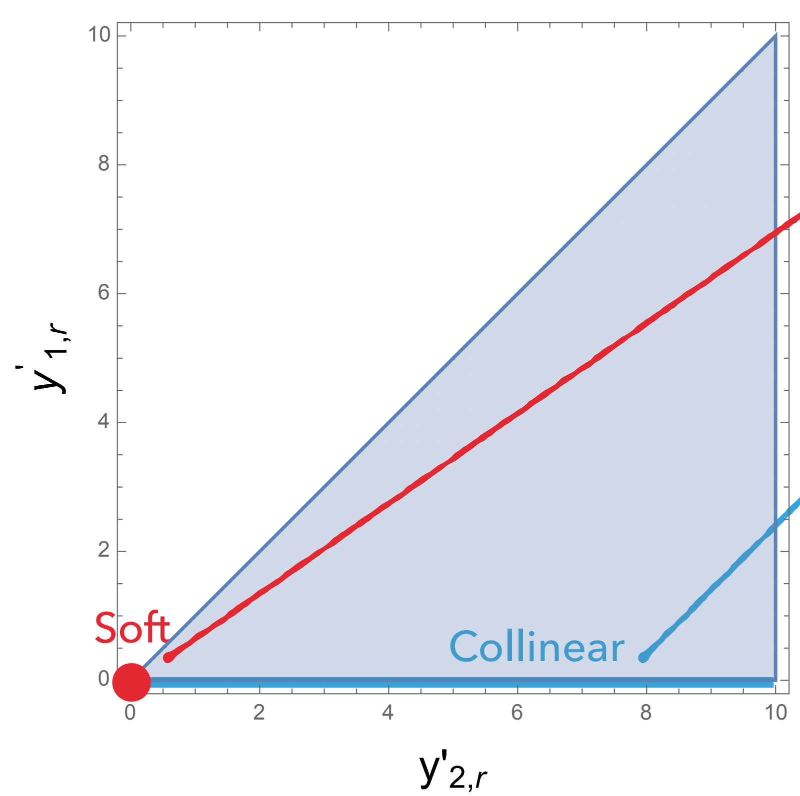
THE MOMENTUM MAPPING

Defining the mappings requires two steps:

1. Separating the singularities of a same type by splitting the real phase-space into several regions (there cannot be more than one given type of IR singularity in a given region of the phase-space), for instance

REGION 1

$$y'_{1,r} \le y'_{2,r}$$



Needed to cancel...

$\boxed{\xi_{1,0} \to 0 \text{ in } I_1}$

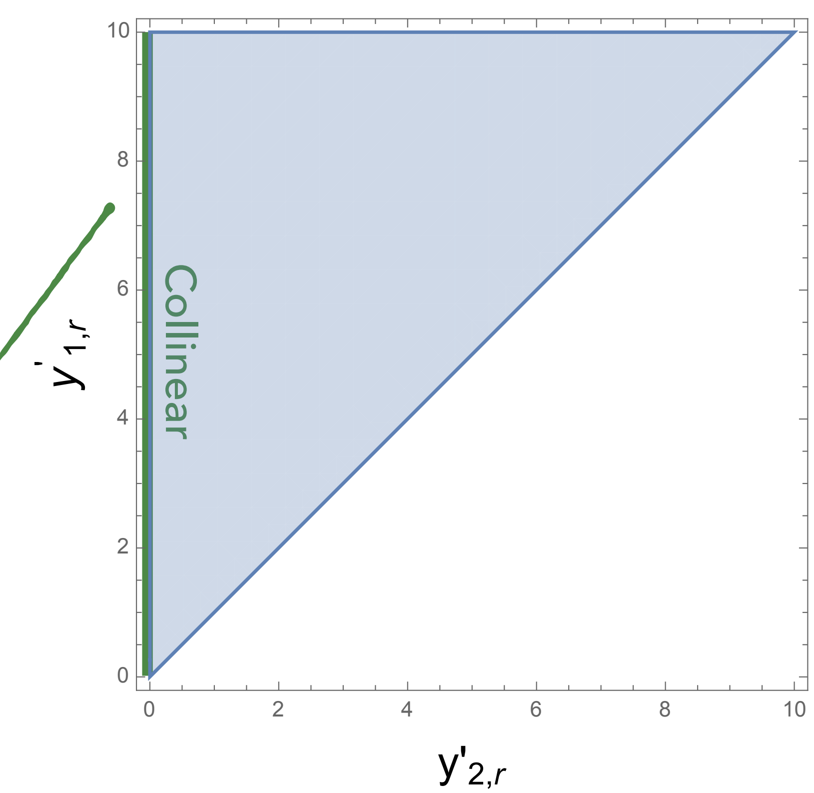
$\boxed{v_1 \to 0 \text{ in } I_1}$

$\boxed{v_2 \to 1 \text{ in } I_2}$

$$(y'_{i,r} = (2p'_i \cdot p'_r)/s_{12})$$

REGION 2

$$y'_{2,r} < y'_{1,r}$$



THE MOMENTUM MAPPING

Defining the mappings requires two steps:

2. Implementing an optimized mapping in each region, to allow a fully local cancellation of IR singularities with those present in the dual contributions

Motivated by QCD factorization properties, we can use

REGION 1:

$$p_r'^{\mu} = q_1^{\mu} \,, \qquad p_1'^{\mu} = p_1^{\mu} - q_1^{\mu} + \alpha_1\, p_2^{\mu} \,,$$
$$p_2'^{\mu} = (1-\alpha_1)\, p_2^{\mu} \,, \qquad \alpha_1 = \frac{q_3^2}{2 q_3 \cdot p_2} \,,$$

REGION 2:

$$p_2'^{\mu} = q_2^{\mu} \,, \qquad p_r'^{\mu} = p_2^{\mu} - q_2^{\mu} + \alpha_2\, p_1^{\mu} \,,$$
$$p_1'^{\mu} = (1-\alpha_2)\, p_1^{\mu} \,, \qquad \alpha_2 = \frac{q_1^2}{2 q_1 \cdot p_1} \,,$$

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$$\Rightarrow \quad y'_{1r} = \frac{v_1\, \xi_{1,0}}{1-(1-v_1)\,\xi_{1,0}} \qquad y'_{12} = 1 - \xi_{1,0}$$
$$y'_{2r} = \frac{(1-v_1)(1-\xi_{1,0})\,\xi_{1,0}}{1-(1-v_1)\,\xi_{1,0}}$$

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$$p_2'^\mu = q_2^\mu \,, \qquad p_r'^\mu = p_2^\mu - q_2^\mu + \alpha_2\, p_1^\mu \,,$$
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$$\Rightarrow \quad y'_{1r} = 1 - \xi_{2,0} \qquad y'_{2r} = \frac{(1-v_2)\,\xi_{2,0}}{1-v_2\,\xi_{2,0}}$$
$$y'_{12} = \frac{v_2\,(1-\xi_{2,0})\,\xi_{2,0}}{1-v_2\,\xi_{2,0}}$$

which we solve using on-shell conditions and momentum conservation.

THE MOMENTUM MAPPING (THE MASSIVE CASE)

- Rewrite the **emitter** and the **spectator** in terms of two massless momenta

$$p_i^\mu = \beta_+ \, \hat{p}_i^\mu + \beta_- \, \hat{p}_j^\mu$$
$$p_j^\mu = (1-\beta_+) \, \hat{p}_i^\mu + (1-\beta_-) \, \hat{p}_j^\mu \qquad \hat{p}_i^\mu + \hat{p}_j^\mu = p_i^\mu + p_j^\mu$$

- Mapping and phase-space partition formally equal to the massless case: determine **mapping parameters from on-shell conditions**

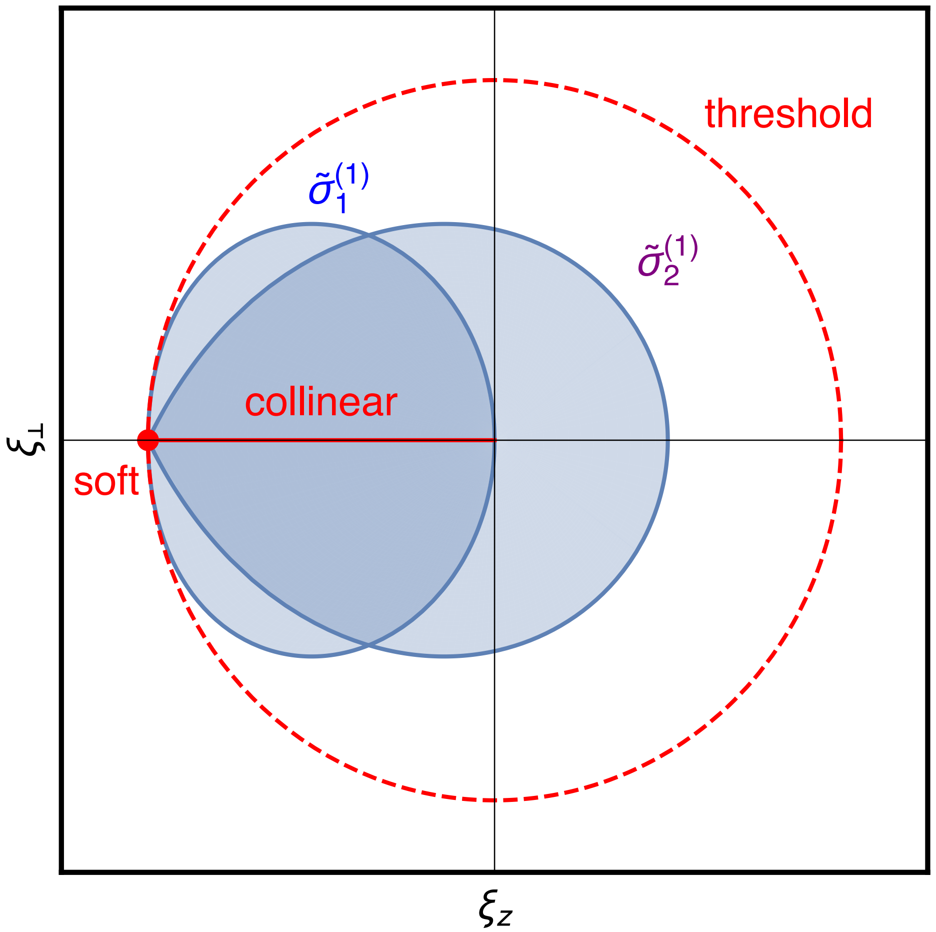
$$p_r'^\mu = q_i^\mu \; ,$$
$$p_i'^\mu = (1-\alpha_i) \, \hat{p}_i^\mu + (1-\gamma_i) \, \hat{p}_j^\mu - q_i^\mu \; ,$$
$$p_j'^\mu = \alpha_i \, \hat{p}_i^\mu + \gamma_i \, \hat{p}_j^\mu \; , \qquad p_k'^\mu = p_k^\mu \; , \quad k \neq i, j$$

- **Quasi-collinear configurations** are conveniently mapped such that the massless limit is smooth

ADDING THE REAL AND THE VIRTUAL CONTRIBUTIONS

$$I_1 = \frac{4}{s_{12}} \int \frac{\xi_{1,0}^{-1}\, d[\xi_{1,0}] d[v_1]}{1-(1-2v_1)^2\beta^2} \Big(\theta(\mathcal{R}_1) + (1-\theta(\mathcal{R}_1))\Big) \qquad I_3$$

$$I_2 = \frac{2}{s_{12}} \int \frac{\xi_{2,0}^{-1}\, \xi_2^2\, d[\xi_{2,0}] d[v_2]}{(1-\xi_{2,0}+\imath 0)(\xi_{2,0}+\beta\xi_2(1-2v_2)-m^2)} \Big(\theta(\mathcal{R}_2) + (1-\theta(\mathcal{R}_2))\Big)$$



Both regions after applying the change of variables

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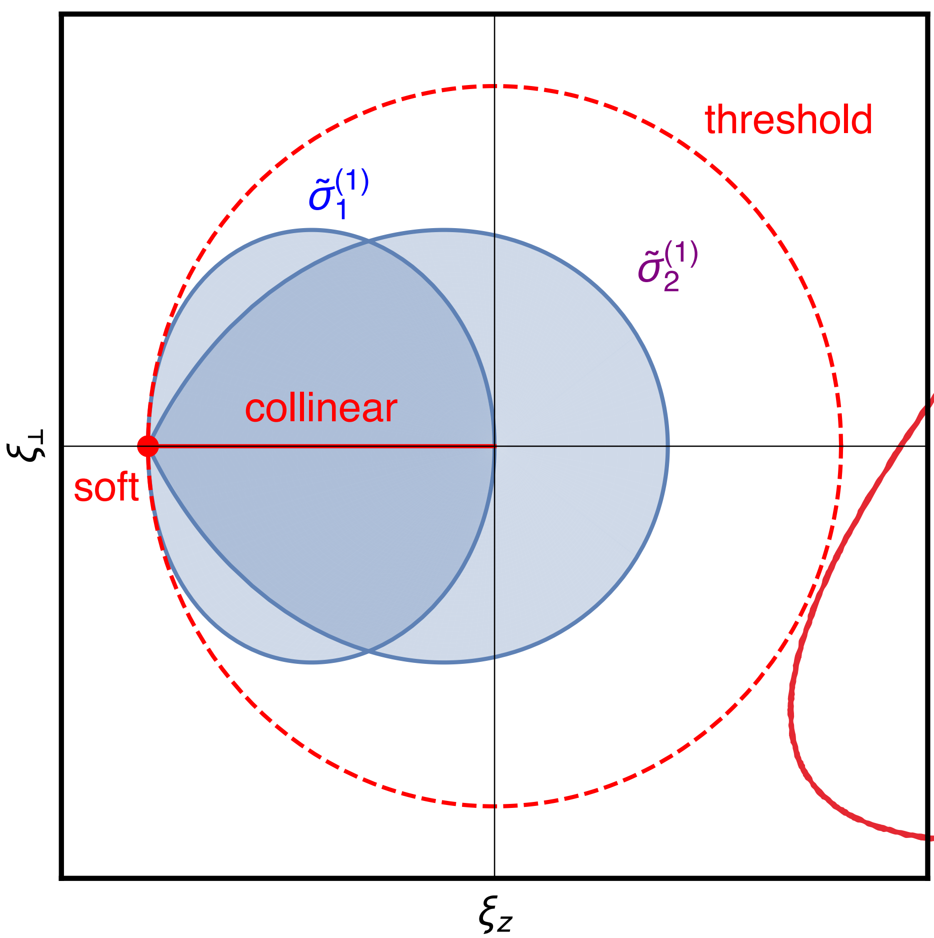


Both regions after applying the change of variables

$$\left.\begin{aligned} \tilde{\sigma}_1^{(1)} &= \tilde{\sigma}_{V,1}^{(1)} + \tilde{\sigma}_{R,1}^{(1)} \\ \tilde{\sigma}_2^{(1)} &= \tilde{\sigma}_{V,2}^{(1)} + \tilde{\sigma}_{R,2}^{(1)} \\ \overline{\sigma}_V^{(1)} &= \overline{\sigma}_{V,1}^{(1)} + \overline{\sigma}_{V,2}^{(1)} + \sigma_{V,3}^{(1)} \end{aligned}\right\} \text{Infrared-safe observables}$$

Ultraviolet singularities remain…

BUILDING A LOCAL COUNTER-TERM

- Expand the dual propagators around a UV propagator…

$$G_F(q_i) = \frac{1}{q_{\mathrm{UV}}^2 - \mu_{\mathrm{UV}}^2 + i0} + \ldots \qquad q_{\mathrm{UV}} = \ell + k_{\mathrm{UV}}$$

- … and adjust the **subleading term** depending on your renormalization scheme (for $\overline{\mathrm{MS}}$, subtract only the pole). For instance for the scalar two-point function:

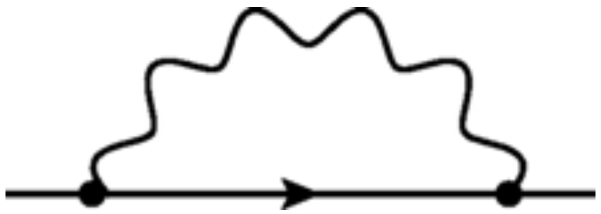
$$I = \int_\ell \frac{1}{(\ell^2 - M^2 + \imath 0)((\ell + p)^2 - M^2 + \imath 0)} \Rightarrow I_{\mathrm{UV}}^{\mathrm{cnt}} = \int_\ell \frac{1}{(q_{\mathrm{UV}}^2 - \mu_{\mathrm{UV}}^2 + \imath 0)^2}$$

- The last step is to subtract the counter-term to the remaining term we had earlier

$$\sigma_V^{(1,\mathrm{R})} = \overline{\sigma}_V^{(1)} - \sigma_V^{(1,\mathrm{UV})}$$

SELF-ENERGY CORRECTIONS

- **Wave function** corrections usually **ignored for massless partons**, but they feature non-trivial IR/UV behavior, required to disentangle both regions, indeed necessary to map the squares of the real amplitudes in the IR



$$= 0 = K \left(\frac{1}{\epsilon_{\mathrm{UV}}} - \frac{1}{\epsilon_{\mathrm{IR}}} \right)$$

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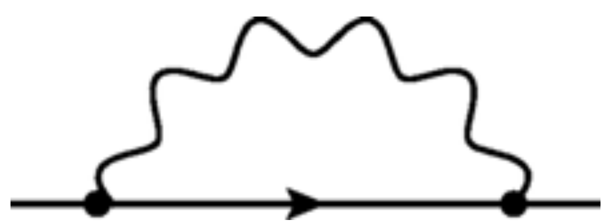


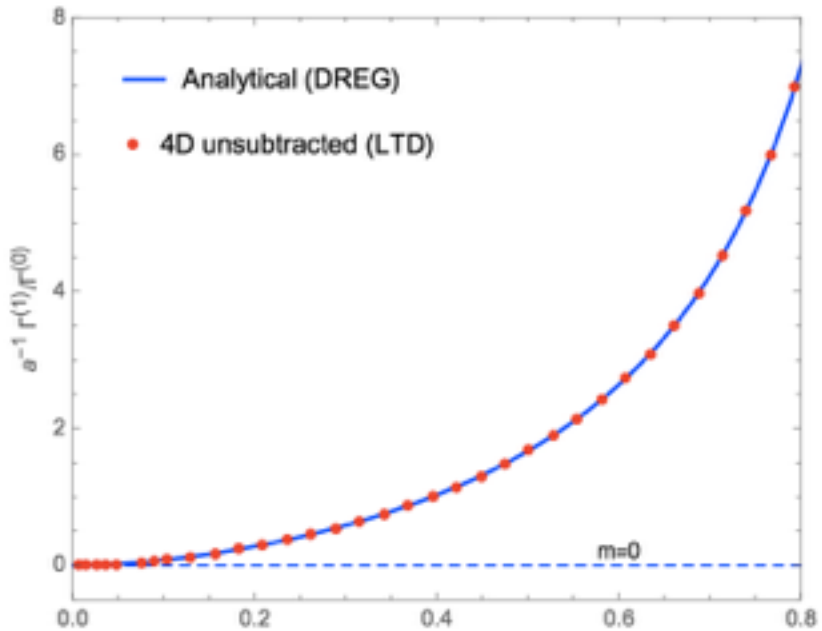
$$= 0 = K\left(\boxed{\frac{1}{\epsilon_{\rm UV}}} - \boxed{\frac{1}{\epsilon_{\rm IR}}}\right)$$

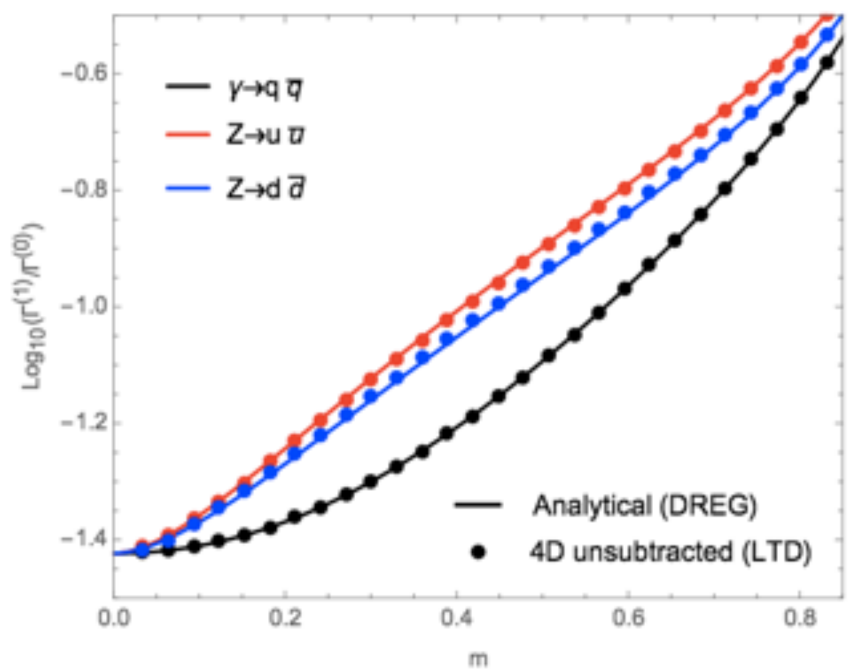
Has to be included in the UV counter-term

Will cancel the square of the virtual contributions

COMPUTATION SAMPLES

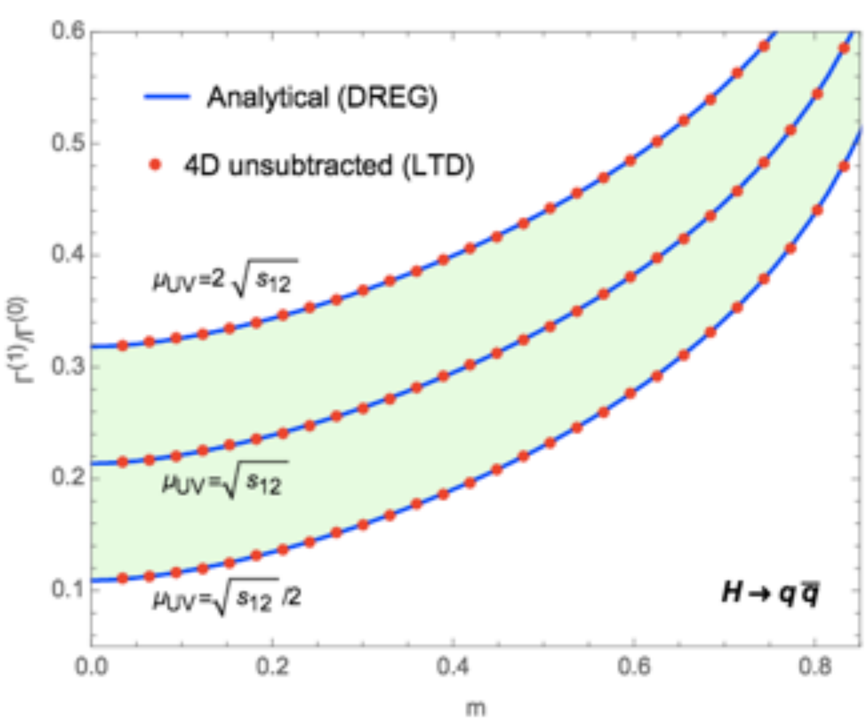
Three-point scalar function

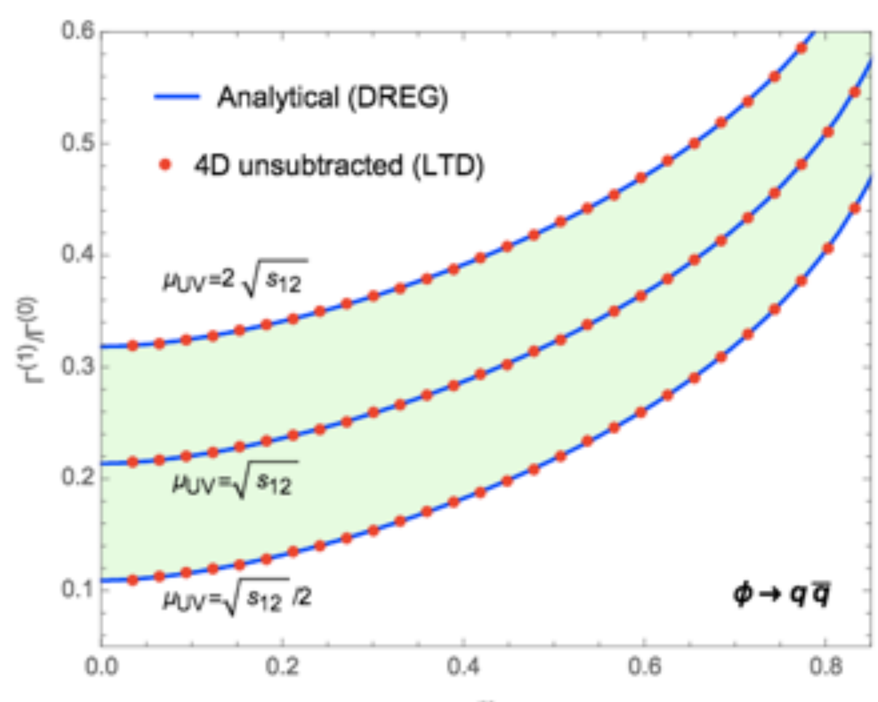




$\gamma/Z \to q\bar{q}$

$H \to q\bar{q}$





$\phi \to q\bar{q}$

COMPARISON WITH DREG

DREG	LTD / FDU
Modify the dimensions of the space-time to $d = 4 - 2\epsilon$	Computations without altering the $d = 4$ **space-time** dimensions
Singularities manifest **after** integration as $\frac{1}{\epsilon}$ **poles:** • **IR** cancelled through suitable **subtraction terms,** which need to be integrated over the unresolved phase-space • **UV** renormalized	Singularities killed **before** integration: • **Unsubtracted** summation over degenerate IR states at integrand level through a suitable **momentum mapping** • **UV** through local counter-terms
Virtual and real contributions are considered **separately:** phase-space with **different number of final-state particles**	Virtual and real contributions are considered **simultaneously:** more efficient Monte Carlo implementation

THE FOUR-DIMENSIONAL UNSUBTRACTION…

- … is a **new algorithm/regularization scheme** for higher-orders in perturbative QFT based on LTD: summation over degenerate soft, final-state collinear singularities and quasi-collinear configurations achieved through a mapping of momenta between real and virtual kinematics

- … allows for fully local cancellations of IR and UV singularities in four dimensions

- … is optimized for **smooth massless limits** due to proper treatment of quasi-collinear configurations

- … allows the **simultaneous generation** of real and virtual corrections, which is advantageous, particularly for multi-leg processes