

New results on splitting functions in QCD at four- and five loops

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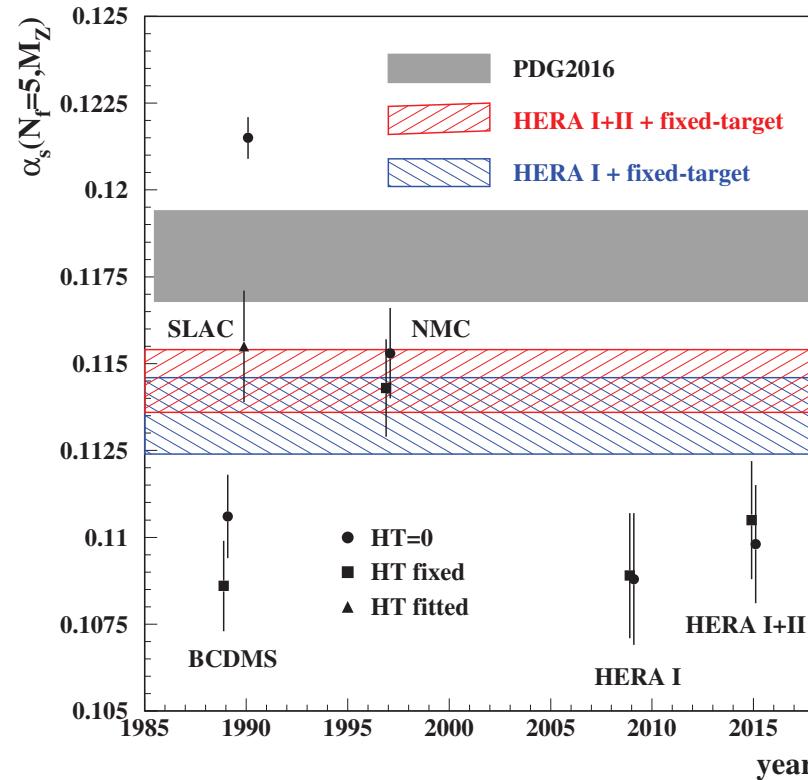
Based on work done in collaboration with:

- *Five-loop contributions to low- N non-singlet anomalous dimensions in QCD*
F. Herzog, S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt
[arXiv:1812.11818](#)
- *On quartic colour factors in splitting functions and the gluon cusp anomalous dimension*
S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt [arXiv:1805.09638](#)
- *Four-Loop Non-Singlet Splitting Functions in the Planar Limit and Beyond*
S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt [arXiv:1707.08315](#)
- Many more papers of MVV and friends ...
[2001 – ...](#)

Motivation

Theory considerations in α_s determinations

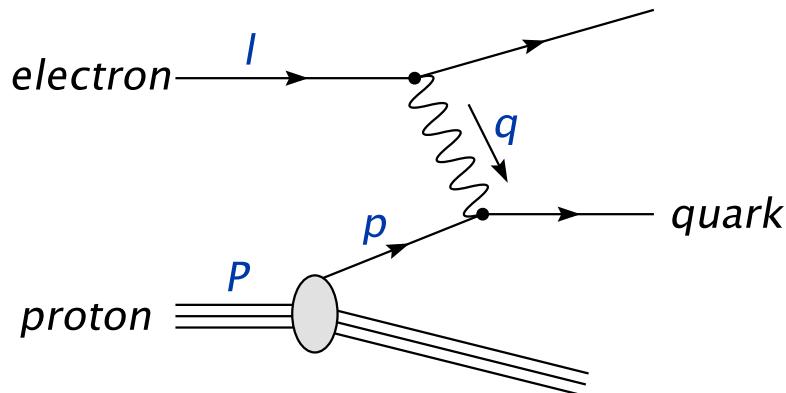
- Correlation of errors among different data DIS sets
- Target mass corrections (powers of nucleon mass M_N^2/Q^2)
- Higher twist $F_2^{\text{ht}} = F_2 + \text{ht}^{(4)}(x)/Q^2 + \dots$
- Variants with no higher twist give larger α_s values Alekhin, Blümlein, S.M. '17



- Theoretical uncertainty of α_s at NNLO from DIS data $\gtrsim \mathcal{O}(1\dots 2)\%$

Theoretical framework

Deep-inelastic scattering



Kinematic variables

- momentum transfer $Q^2 = -q^2$
- Bjorken variable $x = Q^2/(2p \cdot q)$

- Structure functions (up to order $\mathcal{O}(1/Q^2)$)

$$F_a(x, Q^2) = \sum_i [C_{a,i}(\alpha_s(\mu^2), \mu^2/Q^2) \otimes PDF(\mu^2)](x)$$

- Coefficient functions up to N^3LO

$$C_{a,i} = \alpha_s^n \left(c_{a,i}^{(0)} + \alpha_s c_{a,i}^{(1)} + \alpha_s^2 c_{a,i}^{(2)} + \alpha_s^3 c_{a,i}^{(3)} + \dots \right)$$

- Evolution equations up to N^3LO

- non-singlet ($2n_f - 1$ scalar) and singlet (2×2 matrix) equations

$$\frac{d}{d \ln \mu^2} PDF(x, \mu^2) = [P(\alpha_s(\mu^2)) \otimes PDF(\mu^2)](x)$$

- splitting functions $P_{ij} = \alpha_s P_{ij}^{(0)} + \alpha_s^2 P_{ij}^{(1)} + \alpha_s^3 P_{ij}^{(2)} + \alpha_s^4 P_{ij}^{(3)} + \dots$

Evolution equations

- Parton distribution functions $q_i(x, \mu^2)$, $\bar{q}_i(x, \mu^2)$ and $g(x, \mu^2)$ for quarks, antiquarks of flavour i and gluons

- Flavor non-singlet combinations

$$q_{ns,ik}^\pm = (q_i \pm \bar{q}_i) - (q_k \pm \bar{q}_k) \text{ and } q_{ns}^v = \sum_{i=1}^{n_f} (q_i - \bar{q}_i)$$

- splitting functions P_{ns}^\pm and $P_{ns}^v = P_{ns}^- + P_{ns}^s$

- Flavor singlet evolution

$$\frac{d}{d \ln \mu^2} \begin{pmatrix} q_s \\ g \end{pmatrix} = \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} q_s \\ g \end{pmatrix} \text{ and } q_s = \sum_{i=1}^{n_f} (q_i + \bar{q}_i)$$

- quark-quark splitting function $P_{qq} = P_{ns}^+ + P_{ps}$

- Mellin transformation relates to anomalous dimensions $\gamma_{ik}(N)$ of twist-two operators

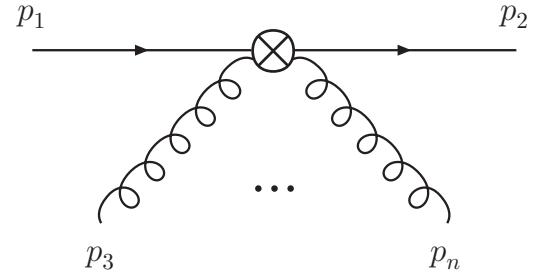
$$\gamma_{ik}^{(n)}(N, \alpha_s) = - \int_0^1 dx \ x^{N-1} P_{ik}^{(n)}(x, \alpha_s)$$

Non-singlet

Operator matrix elements

- Non-singlet operator of spin- N and twist two

$$O_{\{\mu_1, \dots, \mu_N\}}^{\text{ns}} = \bar{\psi} \lambda^\alpha \gamma_{\{\mu_1} D_{\mu_2} \dots D_{\mu_N\}} \psi ,$$
$$\alpha = 3, 8, \dots, (n_f^2 - 1)$$



Calculation

- Anomalous dimensions $\gamma(N)$ from ultraviolet divergence of loop corrections to operator in (anti-)quark two-point function
- Feynman diagrams for operator matrix elements generated up to four loops with **Qgraf** Nogueira '91
- Parametric reduction of four-loop massless propagator diagrams with **Forcer** Ruijl, Ueda, Vermaseren '17
- Symbolic manipulations with **Form** Vermaseren '00; Kuipers, Ueda, Vermaseren, Vollinga '12 and multi-threaded version **TForm** Tentyukov, Vermaseren '07
- Diagrams of same topology and color factor combined to meta diagrams
 - 1 one-, 7 two-, 53 three- and 650 four-loop meta diagrams for γ_{ns}^\pm
 - 1 three- and 29 four-loop meta diagrams for γ_{ns}^s

Anomalous dimensions

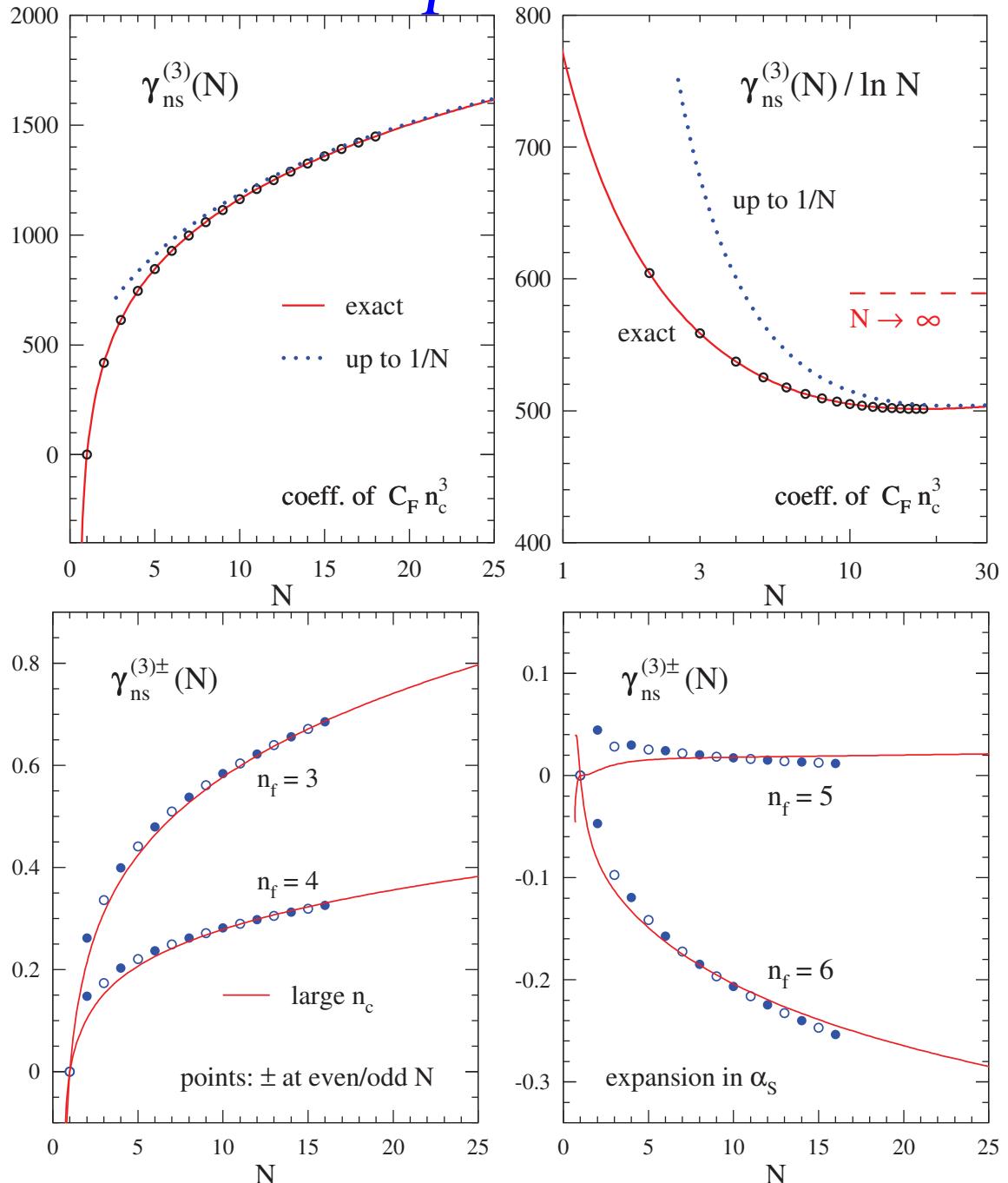
- Anomalous dimensions $\gamma(N)$ of leading twist non-singlet local operators
 - expressible in harmonic sums up to weight 7
$$S_{\pm m_1, \dots, m_k}(N) = \sum_{i=1}^N \frac{(\pm 1)^i}{i^{m_1}} S_{m_2, \dots, m_k}(i)$$
 - $2 \cdot 3^{w-1}$ sums at weight w
- Reciprocity relation $\gamma(N) = \gamma_u(N + \gamma(N) - \beta(a))$ reduces number of 2^{w-1} sums at weight w for γ_u
 - additional denominators with powers $1/(N+1)$ give $2^{w+1} - 1$ objects (255 at weight 7)
- Constraints at large- x /small- x ($N \rightarrow \infty/N \rightarrow 0$) give additional 46 conditions

Upshot

- Computation of Mellin moments up to $N = 18$ for anomalous dimensions feasible
- Reconstruction of analytic all- N expressions in large- n_c limit from solution of Diophantine equations

Mellin moments at four loops

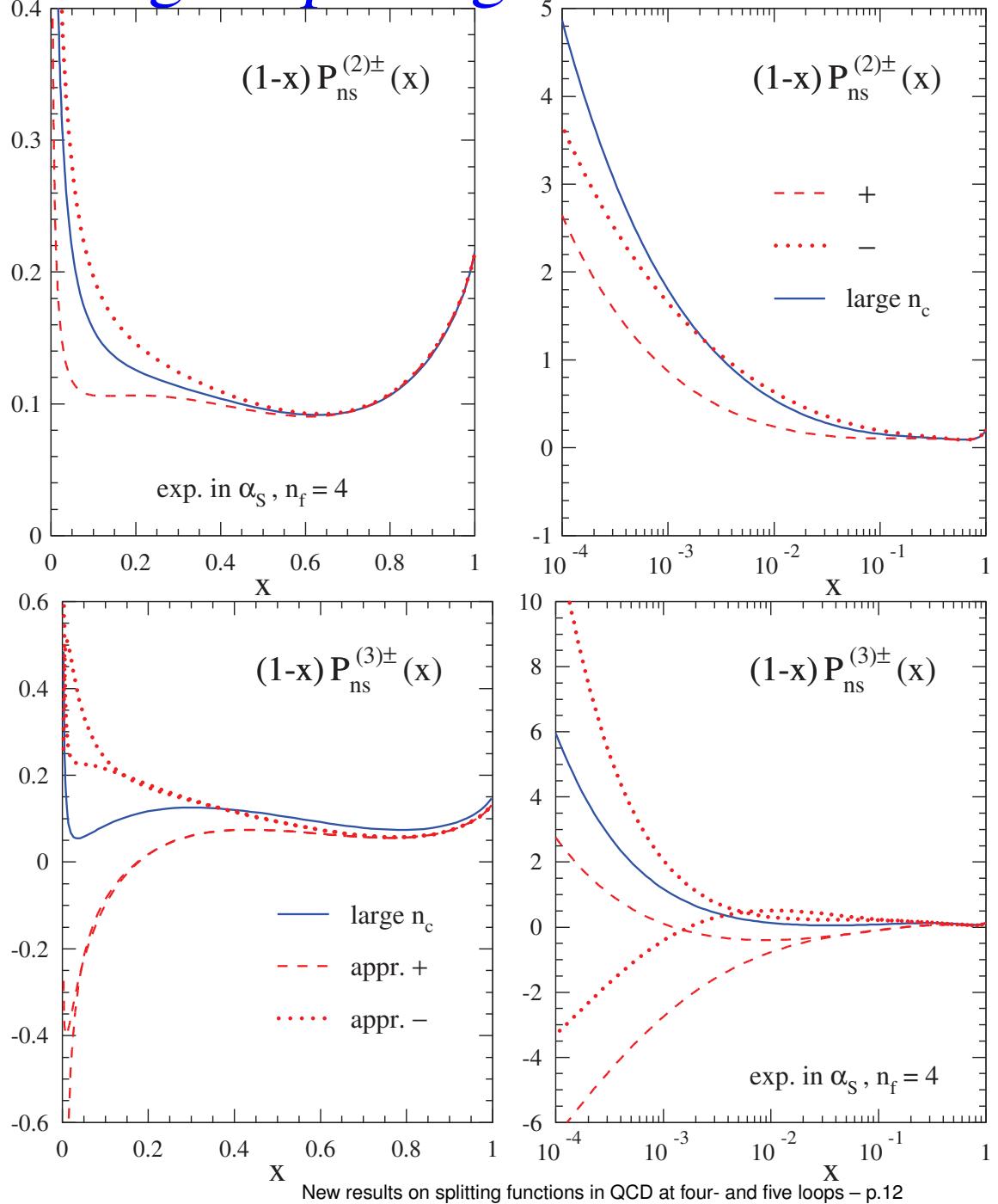
- Top:
 n_f^0 part of anomalous dimensions $\gamma_{ns}^{(3)\pm}(N)$ in large- n_c limit and large- N expansion



- Bottom: results for even- N ($\gamma_{ns}^{(3)+}(N)$) and odd- N ($\gamma_{ns}^{(3)-}(N)$) in large- n_c limit for $n_f = 3, \dots, 6$

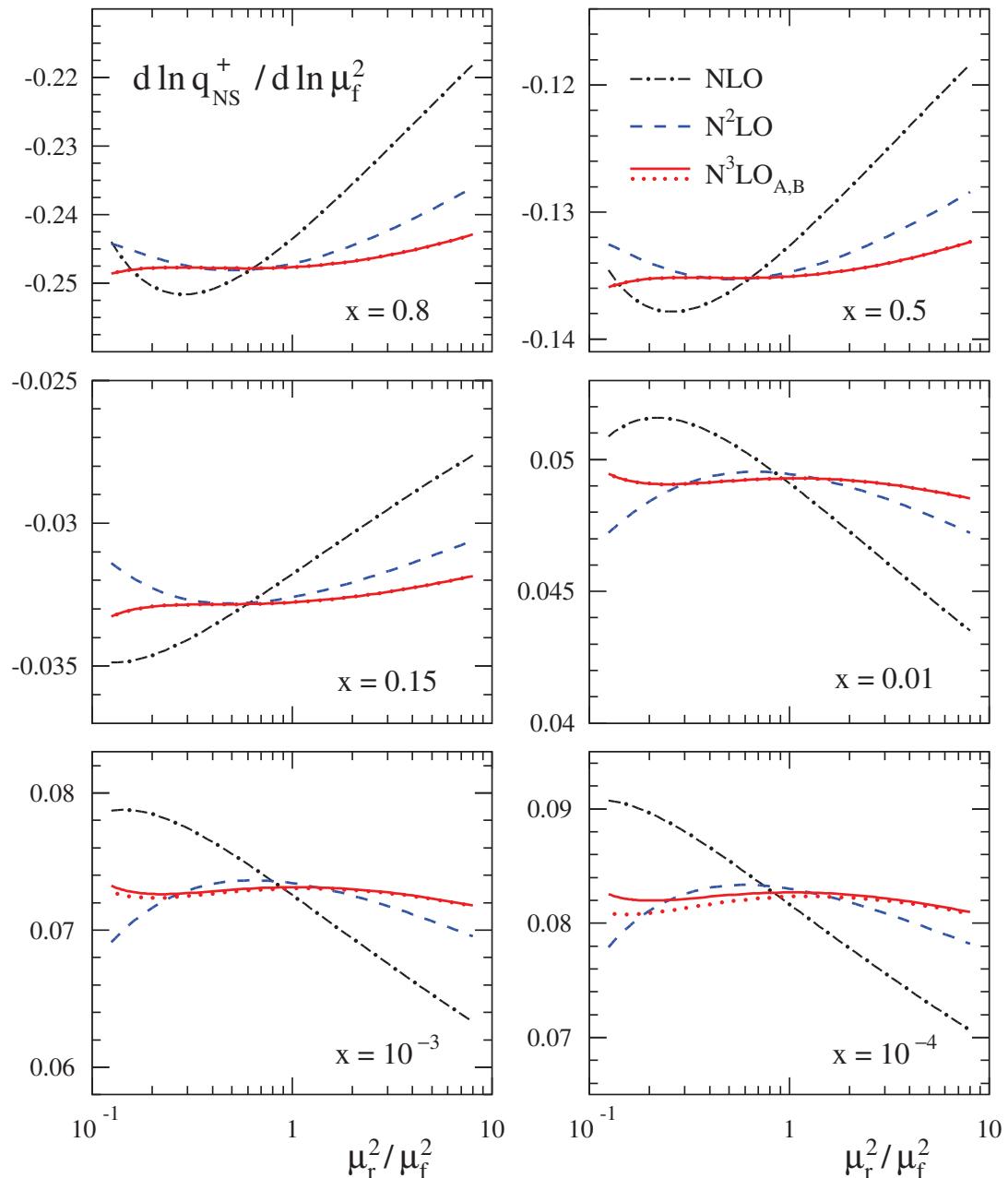
Four-loop non-singlet splitting functions

- Top:
three-loop $P_{\text{ns}}^{(2)\pm}(x)$
and large- n_c limit
with $n_f = 4$
- Bottom:
four-loop $P_{\text{ns}}^{(3)\pm}(x)$
and uncertainty bands
beyond large- n_c limit
with $n_f = 4$



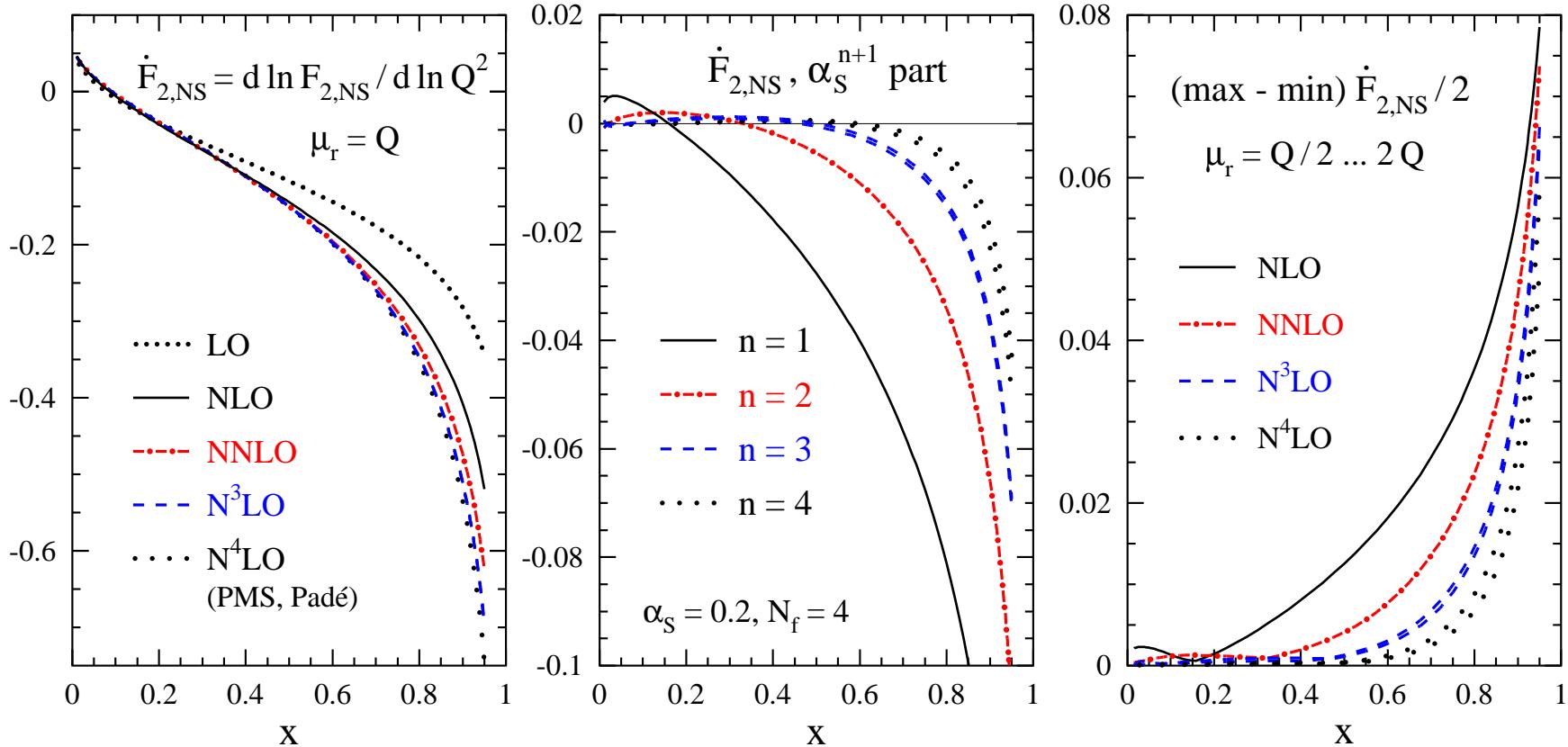
Scale stability of evolution

- Renormalization scale dependence of evolution kernel $d \ln q_{\text{ns}}^+ / d \ln \mu_f^2$
 - non-singlet shape
 $xq_{\text{ns}}^+(x, \mu_0^2) = x^{0.5}(1-x)^3$
- NLO, NNLO and N³LO predictions
 - remaining uncertainty of four-loop splitting function $P_{\text{ns}}^{(3)+}$ almost invisible



The structure function F_2 (non-singlet)

- Large- x convergence of perturbative series
 - approx. N³LO structure functions S.M., Vermaseren, Vogt '05



- Potential for ‘gold-plated’ determination of α_s
 - theory uncertainty $\Delta_{\text{pert.}} \alpha_s < 1\%$

Mellin moments at five loops

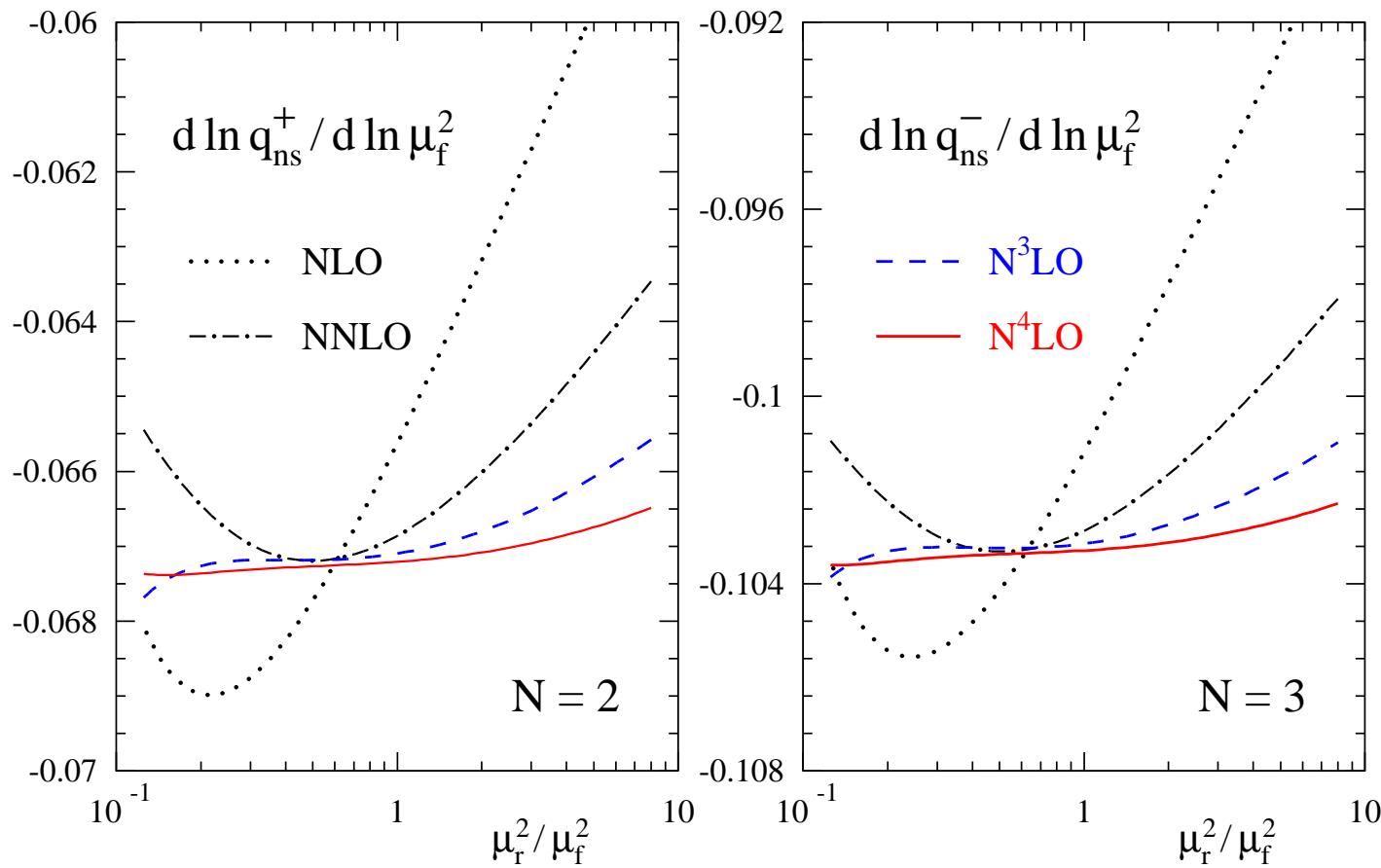
- Moments $N = 2$ and $N = 3$ for nonsinglet anomalous dimensions γ_{ns}^{\pm}
- implementation by Herzog, Ruijl '17 of local R^* operation Chetyrkin, Tkachov '82; Chetyrkin, Smirnov '84 for reduction of five-loop self-energy diagrams to four-loop ones computed with **Forcer** Ruijl, Ueda, Vermaseren '17

$$\begin{aligned} \gamma_{\text{ns}}^{(4)+}(N=2) = & C_F^5 \left[\frac{9306376}{19683} - \frac{802784}{729} \zeta_3 - \frac{557440}{81} \zeta_5 + \frac{12544}{9} \zeta_3^2 + 8512 \zeta_7 \right] \\ & - C_A C_F^4 \left[\frac{81862744}{19683} - \frac{1600592}{243} \zeta_3 + \frac{59840}{81} \zeta_4 - \frac{142240}{27} \zeta_5 + 3072 \zeta_3^2 - \frac{35200}{9} \zeta_6 + 19936 \zeta_7 \right] \\ & + C_A^2 C_F^3 \left[\frac{63340406}{6561} - \frac{1003192}{243} \zeta_3 - \frac{229472}{81} \zeta_4 + \frac{6196}{27} \zeta_5 + \frac{30976}{9} \zeta_3^2 - \frac{35200}{9} \zeta_6 + 15680 \zeta_7 \right] \\ & - C_A^3 C_F^2 \left[\frac{220224724}{19683} + \frac{4115536}{729} \zeta_3 - \frac{170968}{27} \zeta_4 - \frac{3640624}{243} \zeta_5 + \frac{70400}{27} \zeta_3^2 + \frac{123200}{27} \zeta_6 + \frac{331856}{27} \zeta_7 \right] \\ & + C_A^4 C_F \left[\frac{266532611}{39366} + \frac{2588144}{729} \zeta_3 - \frac{221920}{81} \zeta_4 - \frac{3102208}{243} \zeta_5 + \frac{74912}{81} \zeta_3^2 + \frac{334400}{81} \zeta_6 + \frac{178976}{27} \zeta_7 \right] \\ & - \frac{d_{AA}^{(4)}}{N_F} C_F \left[\frac{15344}{81} - \frac{12064}{27} \zeta_3 - \frac{704}{3} \zeta_4 + \frac{58400}{27} \zeta_5 - \frac{6016}{3} \zeta_3^2 - \frac{19040}{9} \zeta_7 \right] \\ & + \frac{d_{FA}^{(4)}}{N_F} C_F \left[\frac{23968}{81} - \frac{733504}{81} \zeta_3 + \frac{176320}{81} \zeta_5 + \frac{6400}{3} \zeta_3^2 + \frac{77056}{9} \zeta_7 \right] \\ & - \frac{d_{FA}^{(4)}}{N_F} C_A \left[\frac{82768}{81} - \frac{555520}{81} \zeta_3 + \frac{10912}{9} \zeta_4 - \frac{1292960}{81} \zeta_5 + \frac{84352}{27} \zeta_3^2 + \frac{140800}{27} \zeta_6 + 12768 \zeta_7 \right] \\ & + n_f C_F \left[\frac{182496}{19683} - \frac{463520}{243} \zeta_3 + \frac{21248}{81} \zeta_4 - \frac{16480}{9} \zeta_5 + \frac{6656}{3} \zeta_3^2 - \frac{6400}{9} \zeta_6 + \frac{8960}{3} \zeta_7 \right] \\ & - n_f C_A C_F^3 \left[\frac{3375082}{6561} - \frac{420068}{243} \zeta_3 - \frac{48256}{81} \zeta_4 + \frac{458032}{81} \zeta_5 + \frac{3968}{3} \zeta_3^2 - \frac{8000}{3} \zeta_6 + \frac{4480}{9} \zeta_7 \right] \\ & + n_f C_A C_F^2 \left[\frac{15291499}{13122} + \frac{1561600}{243} \zeta_3 - \frac{114536}{27} \zeta_4 - \frac{252544}{243} \zeta_5 + \frac{24896}{27} \zeta_3^2 + \frac{13600}{27} \zeta_6 + \frac{11200}{27} \zeta_7 \right] \\ & - n_f C_A^2 C_F \left[\frac{48846580}{19683} + \frac{4314308}{729} \zeta_3 - \frac{274768}{81} \zeta_4 - \frac{1389080}{81} \zeta_5 + \frac{27080}{81} \zeta_3^2 + \frac{184000}{81} \zeta_6 + \frac{39088}{27} \zeta_7 \right] \\ & + n_f \frac{d_{FA}^{(4)}}{N_F} \left[\frac{22096}{27} + \frac{43712}{81} \zeta_3 - \frac{512}{9} \zeta_4 - \frac{217280}{81} \zeta_5 - \frac{25088}{27} \zeta_3^2 + \frac{25600}{9} \zeta_6 - 2464 \zeta_7 \right] \\ & - n_f C_F \frac{d_{FA}^{(4)}}{N_F} \left[\frac{170752}{81} - \frac{328832}{81} \zeta_3 + \frac{650240}{81} \zeta_5 - \frac{8192}{9} \zeta_3^2 - \frac{35840}{9} \zeta_7 \right] \\ & + n_f C_A \frac{d_{AA}^{(4)}}{N_F} \left[\frac{207824}{81} + \frac{251392}{81} \zeta_3 - \frac{5632}{9} \zeta_4 - \frac{522880}{81} \zeta_5 + \frac{15872}{27} \zeta_3^2 + \frac{70400}{27} \zeta_6 - \frac{29120}{9} \zeta_7 \right] \\ & + n_f^2 C_F^3 \left[\frac{1082297}{6561} - \frac{145792}{243} \zeta_3 + \frac{1072}{81} \zeta_4 + \frac{55552}{81} \zeta_5 + \frac{1792}{9} \zeta_3^2 - \frac{3200}{9} \zeta_6 \right] \\ & + n_f^2 C_A C_F^2 \left[\frac{332254}{2187} + \frac{85016}{243} \zeta_3 + \frac{20752}{27} \zeta_4 - \frac{28544}{81} \zeta_5 - \frac{13952}{27} \zeta_3^2 + \frac{1600}{27} \zeta_6 \right] \\ & + n_f^2 C_A^2 C_F \left[\frac{631400}{6561} + \frac{214268}{243} \zeta_3 - \frac{784}{81} \zeta_4 - \frac{53344}{81} \zeta_5 + \frac{25472}{81} \zeta_3^2 + \frac{22400}{81} \zeta_6 \right] \\ & - n_f^2 \frac{d_{FF}^{(4)}}{N_F} \left[\frac{43744}{81} - \frac{35648}{81} \zeta_3 - \frac{1792}{9} \zeta_4 - \frac{52480}{81} \zeta_5 + \frac{2048}{27} \zeta_3^2 + \frac{12800}{27} \zeta_6 \right] \\ & + n_f^3 C_F^2 \left[\frac{166510}{19683} + \frac{11872}{729} \zeta_3 - \frac{2752}{3} \zeta_4 + \frac{512}{27} \zeta_5 \right] \\ & + n_f^3 C_A C_F \left[\frac{168677}{19683} + \frac{11872}{729} \zeta_3 + \frac{2752}{81} \zeta_4 - \frac{4096}{81} \zeta_5 \right] - n_f^4 C_F \left[\frac{5504}{19683} + \frac{1024}{729} \zeta_3 - \frac{128}{81} \zeta_4 \right] \end{aligned}$$

$$\begin{aligned} \gamma_{\text{ns}}^{(4)-}(N=3) = & C_F^5 \left[\frac{99382175}{80621568} + \frac{23328}{80621568} \zeta_3 - \frac{3395975}{162} \zeta_5 - \frac{9650}{9} \zeta_3^2 + \frac{34685}{2} \zeta_7 \right] \\ & - C_A C_F^4 \left[\frac{286028134219}{80621568} - \frac{23916529}{776} \zeta_3 - \frac{4490}{81} \zeta_5 + \frac{134090}{81} \zeta_4 - \frac{2468075}{108} \zeta_6 - \frac{55000}{9} \zeta_3 + \frac{155155}{4} \zeta_7 \right] \\ & + C_A^2 C_F^3 \left[\frac{20173099267}{3359232} - \frac{154041281}{864} \zeta_3 + \frac{732787}{1296} \zeta_4 + \frac{1972075}{216} \zeta_5 - \frac{63830}{9} \zeta_3^2 - \frac{79750}{9} \zeta_6 + \frac{139895}{4} \zeta_7 \right] \\ & - C_A^3 C_F^2 \left[\frac{166662991819}{20155392} - \frac{36397493}{2916} \zeta_3 - \frac{103763}{54} \zeta_4 + \frac{30994565}{3888} \zeta_5 - \frac{133990}{27} \zeta_3^2 - \frac{72875}{54} \zeta_6 + \frac{2127335}{108} \zeta_7 \right] \\ & + C_A^4 C_F \left[\frac{75932799665}{1007769} - \frac{27693563}{23328} \zeta_3 - \frac{1791229}{1296} \zeta_4 - \frac{9417425}{1944} \zeta_5 - \frac{96700}{81} \zeta_3^2 + \frac{163625}{81} \zeta_6 + \frac{199640}{27} \zeta_7 \right] \\ & - \frac{d_{AA}^{(4)}}{N_F} C_F \left[\frac{81725}{162} - \frac{33505}{18} \zeta_3 - \frac{1100}{3} \zeta_4 + \frac{52025}{18} \zeta_5 - \frac{7000}{3} \zeta_3^2 - \frac{48125}{36} \zeta_7 \right] \\ & - \frac{d_{FA}^{(4)}}{N_F} C_F \left[\frac{231575}{36} - \frac{6351445}{324} \zeta_3 - \frac{2927225}{162} \zeta_5 + \frac{23210}{3} \zeta_3^2 - \frac{200410}{9} \zeta_7 \right] \\ & + \frac{d_{FA}^{(4)}}{N_F} C_A \left[\frac{165871}{54} + \frac{1816625}{162} \zeta_3 - \frac{41800}{9} \zeta_4 - \frac{4456145}{162} \zeta_5 + \frac{196880}{27} \zeta_3^2 + \frac{200750}{27} \zeta_6 - \frac{7525}{4} \zeta_7 \right] \\ & + n_f C_F \left[\frac{40310784}{40310784} - \frac{1776521549}{486} - \frac{1332919}{81} \zeta_3 + \frac{5000}{9} \zeta_5 + \frac{33290}{81} \zeta_4 - \frac{30325}{81} \zeta_6 - \frac{10000}{9} \zeta_5 + \frac{14000}{3} \zeta_7 \right] \\ & - n_f C_A C_F^3 \left[\frac{3737356319}{3359232} - \frac{2237111}{432} \zeta_3 + \frac{1280}{3} \zeta_5 + \frac{262069}{648} \zeta_4 + \frac{1693715}{162} \zeta_5 - \frac{14000}{3} \zeta_6 + \frac{7000}{3} \zeta_7 \right] \\ & + n_f C_A^2 C_F^2 \left[\frac{5637513931}{2711207} - \frac{2711207}{27} \zeta_3 - \frac{5020}{27} \zeta_4 - \frac{47499}{108} \zeta_5 + \frac{508820}{243} \zeta_6 - \frac{20375}{27} \zeta_6 + \frac{50155}{108} \zeta_7 \right] \\ & - n_f C_A^3 C_F \left[\frac{8766012215}{2519424} + \frac{45697231}{5832} \zeta_3 + \frac{1195}{81} \zeta_5 - \frac{2848403}{648} \zeta_4 - \frac{1808870}{243} \zeta_6 + \frac{222250}{81} \zeta_6 + \frac{250915}{108} \zeta_7 \right] \\ & - n_f C_F \frac{d_{FA}^{(4)}}{N_F} \left[\frac{24385}{27} - \frac{334010}{81} \zeta_3 - \frac{8480}{9} \zeta_5 + \frac{1622600}{81} \zeta_5 - \frac{135380}{81} \zeta_7 \right] \\ & + n_f^2 C_F \frac{d_{FF}^{(4)}}{N_F} \left[\frac{297889}{162} + \frac{154970}{81} \zeta_3 - \frac{62600}{27} \zeta_5 + \frac{3700}{9} \zeta_4 - \frac{122780}{81} \zeta_5 - \frac{36500}{27} \zeta_6 - \frac{910}{27} \zeta_7 \right] \\ & + n_f^2 C_A \frac{d_{AA}^{(4)}}{N_F} \left[\frac{241835}{162} + \frac{33487}{81} \zeta_3 + \frac{30560}{27} \zeta_5 - \frac{10780}{9} \zeta_4 - \frac{316900}{81} \zeta_5 + \frac{110000}{27} \zeta_6 - \frac{71960}{9} \zeta_7 \right] \\ & + n_f^2 C_F^3 \left[\frac{512848319}{1679616} - \frac{57109}{54} \zeta_3 + \frac{2800}{9} \zeta_5^2 + \frac{9118}{81} \zeta_4 + \frac{86440}{81} \zeta_5 - \frac{5000}{9} \zeta_6 \right] \\ & + n_f^2 C_A C_F^2 \left[\frac{1080083}{5832} - \frac{296729}{972} \zeta_3 - \frac{21800}{27} \zeta_5^2 + \frac{56327}{54} \zeta_4 - \frac{42860}{81} \zeta_5 + \frac{2500}{27} \zeta_6 \right] \\ & + n_f^2 C_A^2 C_F \left[\frac{61747877}{419904} + \frac{2496811}{1944} \zeta_3 + \frac{39800}{81} \zeta_5^2 - \frac{3503}{3} \zeta_4 - \frac{88990}{243} \zeta_5 + \frac{35000}{81} \zeta_6 \right] \\ & - n_f^2 \frac{d_{FF}^{(4)}}{N_F} \left[\frac{19435}{27} - \frac{53366}{81} \zeta_3 + \frac{3200}{27} \zeta_5^2 - \frac{3160}{9} \zeta_4 - \frac{70000}{81} \zeta_5 + \frac{20000}{27} \zeta_6 \right] \\ & + n_f^3 C_F^2 \left[\frac{28758139}{1259712} + \frac{21673}{729} \zeta_3 - \frac{610}{9} \zeta_4 + \frac{800}{27} \zeta_5 \right] \\ & + n_f^3 C_A C_F \left[\frac{13729181}{1259712} + \frac{14947}{729} \zeta_3 + \frac{4390}{81} \zeta_4 - \frac{6400}{81} \zeta_5 \right] - n_f^4 C_F \left[\frac{259993}{629856} + \frac{1660}{729} \zeta_3 - \frac{200}{81} \zeta_4 \right] \end{aligned}$$

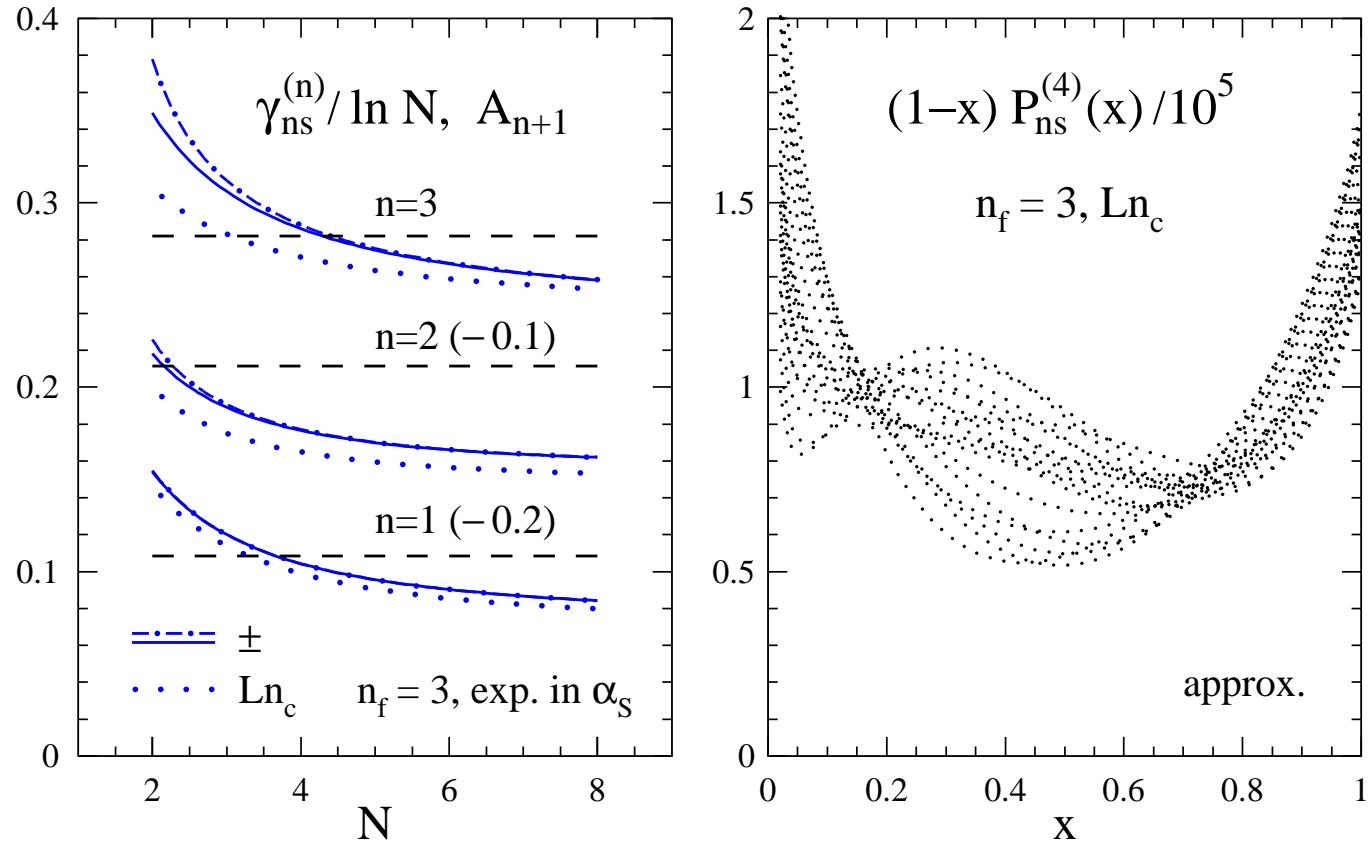
$$\begin{aligned} \gamma_{\text{ns}}^{(4)\nu}(N=3) = & \gamma_{\text{ns}}^{(4)-}(N=3) \\ & + n_f \frac{d_{abc} d^{abc}}{N_F} \left[C_F^2 \left[\frac{79906955}{46656} + \frac{246955}{54} \zeta_3 - \frac{504550}{81} \zeta_5 \right] \right. \\ & \left. - C_A C_F \left[\frac{797321}{3888} - \frac{475655}{54} \zeta_3 + \frac{17600}{9} \zeta_4 + \frac{516950}{81} \zeta_5 - \frac{500}{9} \zeta_3^2 + \frac{2800}{9} \zeta_7 \right] \right] \\ & + C_A^2 \left[\frac{166535}{486} - \frac{1783913}{324} \zeta_3 + \frac{5555}{9} \zeta_4 + \frac{507515}{81} \zeta_5 - \frac{2035}{27} \zeta_3^2 - \frac{5500}{27} \zeta_6 - \frac{2765}{18} \zeta_7 \right] \\ & + n_f C_A \left[\frac{285985}{3888} + \frac{41954}{81} \zeta_3 + \frac{160}{27} \zeta_4 - \frac{1010}{81} \zeta_5 - \frac{56480}{81} \zeta_6 + \frac{1000}{27} \zeta_7 \right] \\ & + n_f C_F \left[\frac{1098323}{3888} - \frac{49720}{81} \zeta_3 + \frac{3200}{9} \zeta_4 \right] - n_f^2 \left[\frac{21823}{1944} \right] \end{aligned}$$

Scale stability of evolution



- Renormalization-scale dependence of $d \ln q_{ns}^\pm / d \ln \mu_f^2$ at $N = 2$ and $N = 3$ using NLO, NNLO, N³LO and N⁴LO predictions with $\alpha_s(\mu_f) = 0.2$ and $n_f = 4$

Five-loop splitting function at large- x



- Left: Non-singlet anomalous dimensions $\gamma_{\text{ns}}^{(n)\pm}(N) / \ln N$ for non-even/odd $2 \leq N \leq 8$ for $n_f = 3$ compared to their limits for large- n_c and for $N \rightarrow \infty$ (straight lines)
- Right: 20 trial functions approximating $P_{\text{ns}}^{(4)\pm}(N)$ in large- n_c limit for $n_f = 3$ with uncertainty band for five-loop cusp anomalous dimension A_5

Singlet

Color factors of $SU(n_c)$

- Quadratic Casimir factors $C_r \delta^{ab} \equiv \text{Tr} (T_r^a T_r^b)$
 - fundamental representation $C_F = (n_c^2 - 1)/(2n_c)$;
adjoint representation $C_A = n_c$
- Quartic Casimir invariants occur for the first time at four loops
 - $d_{xy}^{(4)} \equiv d_x^{abcd} d_y^{abcd}$ for representations labels x, y with generators T_r^a

$$d_r^{abcd} = \frac{1}{6} \text{Tr} (T_r^a T_r^b T_r^c T_r^d + \text{five } bcd \text{ permutations})$$
- $SU(n_c)$ with fermions in fundamental representation

$$d_{AA}^{(4)}/n_A = \frac{1}{24} n_c^2 (n_c^2 + 36) ,$$

$$d_{FA}^{(4)}/n_A = \frac{1}{48} n_c (n_c^2 + 6) ,$$

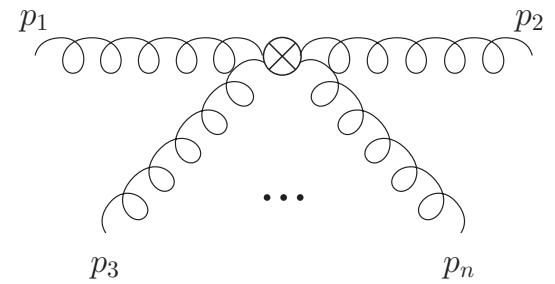
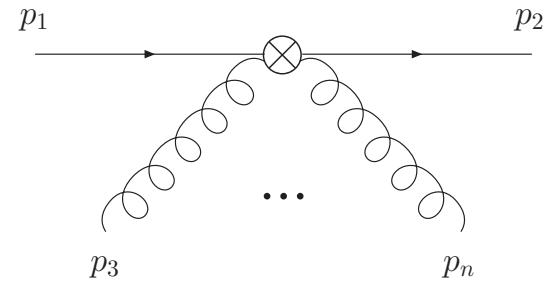
$$d_{FF}^{(4)}/n_A = \frac{1}{96} (n_c^2 - 6 + 18n_c^{-2})$$
 - trace-normalized with $T_F = \frac{1}{2}$
 - dimensions of representations $n_F = n_c$ and $n_A = (n_c^2 - 1)$

Operator matrix elements

- Singlet operators of spin- N and twist two

$$O_{\{\mu_1, \dots, \mu_N\}}^q = \bar{\psi} \gamma_{\{\mu_1} D_{\mu_2} \dots D_{\mu_N\}} \psi ,$$

$$O_{\{\mu_1, \dots, \mu_N\}}^g = F_{\nu \{\mu_1} D_{\mu_2} \dots D_{\mu_{N-1}} F_{\mu_N\}}^\nu$$



- Quartic Casimir terms at four loops
are effectively ‘leading-order’

- anomalous dimensions fulfil relation for $\mathcal{N} = 1$ supersymmetry

$$\gamma_{qq}^{(3)}(N) + \gamma_{gq}^{(3)}(N) - \gamma_{qg}^{(3)}(N) - \gamma_{gg}^{(3)}(N) \stackrel{Q}{=} 0$$

- color-factor substitutions for $n_f = 1$ Majorana fermions (factor $2n_f$)

$$(2n_f)^2 \frac{d_{FF}^{(4)}}{n_A} = 2n_f \frac{d_{FA}^{(4)}}{n_A} = 2n_f \frac{d_{FF}^{(4)}}{n_F} = \frac{d_{FA}^{(4)}}{n_F} = \frac{d_{AA}^{(4)}}{n_A}$$

- Eigenvalues of singlet splitting functions (conjectured to be) composed of reciprocity-respecting sums

- quartic Casimir terms fulfil stronger condition Belitsky, Müller, Schäfer ‘99

$$\gamma_{qg}^{(0)}(N) \gamma_{gq}^{(3)}(N) \stackrel{Q}{=} \gamma_{gq}^{(0)}(N) \gamma_{qg}^{(3)}(N)$$

Calculation and results

- Computation of Mellin moments up to $N = 16$ for anomalous dimensions feasible
- Numerical approximaton x -space expressions
 - used for large- x limit

$$P_{ii}^{(n-1)}(x) = \frac{A_{n,i}}{(1-x)_+} + B_{n,i} \delta(1-x) + C_{n,i} \ln(1-x) + D_{n,i}$$

- Cusp anomalous dimensions related by Casimir scaling up to three loops

$$A_{n,g} = \frac{C_A}{C_F} A_{n,q} \text{ for } n \leq 3$$

- Casimir scaling at four loops broken due to new color factors

- $A_{n,q}$ contains $\frac{d_F^{abcd} d_A^{abcd}}{n_F}$ and $\frac{d_F^{abcd} d_F^{abcd}}{n_F}$
- $A_{n,g}$ contains $\frac{d_A^{abcd} d_A^{abcd}}{n_A}$, $\frac{d_F^{abcd} d_A^{abcd}}{n_A}$ and $\frac{d_F^{abcd} d_F^{abcd}}{n_A}$

- Large n_c -limit at four loops restores Casimir scaling

Dixon '17

$$A_{4,g}|_{\text{large-}n_c} = \frac{C_A}{C_F} A_{4,q}|_{\text{large-}n_c}$$

Quark and gluon cusp anomalous dimensions

- Large- n_c limit of quark cusp anomalous dimension (agrees with Henn, Lee, Smirnov, Smirnov, Steinhauser '16)

$$\begin{aligned} A_{4,q}|_{\text{large-}n_c} = & C_F n_c^3 \left(\frac{84278}{81} - \frac{88832}{81} \zeta_2 + \frac{20992}{27} \zeta_3 + 1804 \zeta_4 - \frac{352}{3} \zeta_2 \zeta_3 - 352 \zeta_5 \right. \\ & \left. - 32 \zeta_3^2 - 876 \zeta_6 \right) \\ & - C_F n_c^2 n_f \left(\frac{39883}{81} - \frac{26692}{81} \zeta_2 + \frac{16252}{27} \zeta_3 + \frac{440}{3} \zeta_4 - \frac{256}{3} \zeta_2 \zeta_3 - 224 \zeta_5 \right) \\ & + C_F n_c n_f^2 \left(\frac{2119}{81} - \frac{608}{81} \zeta_2 + \frac{1280}{27} \zeta_3 - \frac{64}{3} \zeta_4 \right) - C_F n_f^3 \left(\frac{32}{81} - \frac{64}{27} \zeta_3 \right) \end{aligned}$$

- Result includes non-vanishing coefficients of quartic Casimir

contributions $\frac{d_F^{abcd} d_A^{abcd}}{n_F}$ and $\frac{d_F^{abcd} d_F^{abcd}}{n_F}$

Generalized ‘Casimir scaling’

quark	gluon	$A_{4,q}$	$A_{4,g}$
C_F^4	—	0	—
$C_F^3 C_A$	—	0	—
$C_F^2 C_A^2$	—	0	—
$C_F C_A^3$	C_A^4	610.25 ± 0.1	
$d_{FA}^{(4)}/n_F$	$d_{AA}^{(4)}/n_A$	-507.0 ± 2.0	-507.0 ± 5.0
<hr/>			
$n_f C_F^3$	$n_f C_F^2 C_A$	-31.05543	
$n_f C_F^2 C_A$	$n_f C_F C_A^2$	38.79538	
$n_f C_F C_A^2$	$n_f C_A^3$	-440.6670	
$n_f d_{FF}^{(4)}/n_F$	$n_f d_{FA}^{(4)}/n_A$	-123.8949	-124.0 ± 0.6
<hr/>			
$n_f^2 C_F^2$	$n_f^2 C_F C_A$	-21.31439	
$n_f^2 C_F C_A$	$n_f^2 C_A^2$	58.36737	
—	$n_f^2 d_{FF}^{(4)}/n_A$	—	0.0 ± 0.1
$n_f^3 C_F$	$n_f^3 C_A$	2.454258	2.454258

- Exact results (rounded to seven digits) for $n_f C_F^3$ by Grozin ‘18 and for $n_f C_F^2 C_A$, $n_f C_F C_A^2$, $n_f d_{FF}^{(4)}/n_F$ by Henn, Peraro, Stahlhofen, Wasser ‘19 all in agreement with approximations from S.M., Ruijl, Ueda, Vermaseren, Vogt ‘17

Numerical implications

- Numerical results (expansion in powers of $\alpha_s/(4\pi)$)

$$A_{4,q} = 20702(2) - 5171.916 n_f + 195.5772 n_f^2 + 3.272344 n_f^3,$$

$$A_{4,g} = 40880(30) - 11714.25 n_f + 440.0488 n_f^2 + 7.362774 n_f^3$$

- Casimir scaling between $A_{4,g}$ and $A_{4,q}$ broken by almost 15% in n_f^0
- non-leading large- n_c part of quartic-Casimir term
(factors ‘36’ and ‘6’ in $A_{4,g}$ and $A_{4,q}$)
- Perturbative expansion very benign for quark

$$A_q(\alpha_s, n_f=3) = 0.42441 \alpha_s [1 + 0.72657 \alpha_s + 0.73405 \alpha_s^2 + 0.6647(2) \alpha_s^3 + \dots]$$

$$A_q(\alpha_s, n_f=4) = 0.42441 \alpha_s [1 + 0.63815 \alpha_s + 0.50998 \alpha_s^2 + 0.3168(2) \alpha_s^3 + \dots]$$

$$A_q(\alpha_s, n_f=5) = 0.42441 \alpha_s [1 + 0.54973 \alpha_s + 0.28403 \alpha_s^2 + 0.0133(3) \alpha_s^3 + \dots]$$

and gluon

$$A_g(\alpha_s, n_f=3) = 0.95493 \alpha_s [1 + 0.72657 \alpha_s + 0.73405 \alpha_s^2 + 0.415(2) \alpha_s^3 + \dots]$$

$$A_g(\alpha_s, n_f=4) = 0.95493 \alpha_s [1 + 0.63815 \alpha_s + 0.50998 \alpha_s^2 + 0.064(2) \alpha_s^3 + \dots]$$

$$A_g(\alpha_s, n_f=5) = 0.95493 \alpha_s [1 + 0.54973 \alpha_s + 0.28403 \alpha_s^2 - 0.243(2) \alpha_s^3 + \dots]$$

Analytic results

- Reconstruction of analytic all- N expressions for ζ_5 terms from solution of Diophantine equations

- example for $\gamma_{gg}^{(3)}$ with $\eta = \frac{1}{N} - \frac{1}{N+1}$ and $\nu = \frac{1}{N-1} - \frac{1}{N+2}$

$$\left. \gamma_{gg}^{(3)}(N) \right|_{\zeta_5 d_{AA}^{(4)}/n_A} = \frac{64}{3} \left(30 (12\eta^2 - 4\nu^2 - S_1(4S_1 + 8\eta - 8\nu - 11) - 7\nu) + 188\eta - \frac{751}{3} - \frac{1}{6} N(N+1) \right)$$

- Recall large- N limit of anomalous dimensions

$$\left. \gamma_{ii}^{(k)}(N) \right|_{N \rightarrow \infty} = A_{n,i} \ln(N) + \mathcal{O}(\text{const}_N)$$

- Terms $S_1(N)^2 \sim \ln(N)^2$ and $N(N+1)$ proportional to ζ_5 must be compensated in large- N limit

Summary

- Determination of strong coupling α_s at 1% precision requires QCD radiative corrections to evolution equations at $N^3\text{LO}$
- Matrix elements of local operators of twist two
 - non-singlet anomalous dimensions $\gamma_{ns}^{(3),\pm,v}(N)$ (fixed Mellin moments and exact results for large- n_c) at $N^3\text{LO}$
 - quartic Casimir contributions to singlet anomalous dimension $\gamma_{ij}^{(3)}(N)$ (fixed Mellin moments and exact results for ζ_5 terms) at $N^3\text{LO}$
- Quark and gluon cusp anomalous dimensions
 - generalization of the lower-order ‘Casimir scaling’
- Structural similarities of QCD to $N = 4$ Super Yang-Mills theory