The Diagrammatic Coaction

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References:
Phys.Rev.Lett. 119 (2017) no.5, 051601,
JHEP 1706 (2017) 114,
JHEP 1712 (2017) 090
and work to appear soon

The Diagrammatic Coaction: Motivation

- Many Feynman diagrams evaluate to Multiple polylogarithms (MPLs).
  MPLs admit a coaction, which maps complicated objects into simpler ones.
  The coaction has many applications including simplifying expressions and taking discontinuities.

- **Can Feynman integrals themselves be endowed with a coaction?**

  The idea that there should be a Hopf Algebra on Feynman graphs / integrals that reproduces the one on MPLs was an inspiration for research over many years [Kreimer, Brown].

- **At one loop we now have an explicit construction in dimensional regularisation with a simple graphical interpretation involving contractions and cuts.**

- This algebraic structure **encodes the analytic properties** of the integrals.
  It facilitates e.g. performing analytic continuation and deriving differential equations.
  It also provides a new perspective on cuts and Master Integrals.

- **Can it be extended to higher loops?**

Coaction on Multiple Polylogs

- Multiple Polylogarithms (MPLs) are **iterated integrals** of the form:

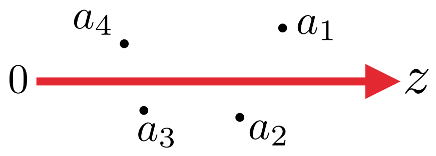
$$G(\underbrace{a_1, a_2, \ldots, a_n}_{\text{weight } n}; z) = \int_0^z \frac{dt}{t-a_1} G(\underbrace{a_2, \ldots, a_n}_{\text{weight } n-1}; t)$$

- They form an algebra with the action $A \times A \to A$:

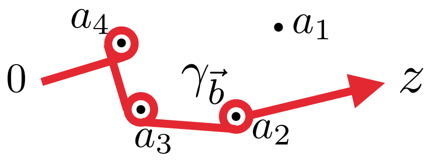
$$G(\vec{a}; z)\, G(\vec{b}; z) = \sum_{\vec{c} \in \vec{a}\, \text{Ш}\, \vec{b}} G(\vec{c}; z)\,,$$

- Define a **coaction**: $A \to A \otimes H$

$$\Delta(G(\vec{c}; z)) = \sum_{\vec{b} \subseteq \vec{c}} \underbrace{G(\vec{b}; z)}_{\text{weight } |\vec{b}|} \otimes \underbrace{G_{\vec{b}}(\vec{c}; z)}_{\text{weight } |\vec{c}|-|\vec{b}|}\,,$$



left: modify the integrand retaining only the poles in $\vec{b}$ (keeping the original contour).



right: modify the contour to encircle the poles in $\vec{b}$ (keeping the original integrand)

- This allows to derive functional identities by simple algebra.

Goncharov, Brown, Goncharov-Spradlin-Vergu-Volovich, Duhr, …

The Master Formula

- Inspired by the coaction on MPLs we define a coaction on integrals:

$$\Delta\left(\int_{\gamma}\omega\right)=\sum_i\int_{\gamma}\omega_i\otimes\int_{\gamma_i}\omega\,,$$

- We have inserted a complete set of master integrands ω_i and corresponding master contours γ_i — these basis elements are paired according to:

$$P_{ss}\left(\int_{\gamma_i}\omega_j\right)=\delta_{ij}\,,$$

- P_{ss} is a semi-simple projector: it retains semi-simple objects $\Delta(x)=x\otimes 1$ including π and rational functions (and complete elliptic integrals) but eliminates logs and polylogs.

Properties of the coaction

- Given a basis of paired integrand ω_i and contour γ_i the coaction on integrals is defined by

$$\Delta\left(\int_{\gamma}\omega\right) \equiv \sum_i \int_{\gamma}\omega_i \otimes \int_{\gamma_i}\omega$$

reproducing the properties of the coaction on MPLs.

- Discontinuities act on the first entry

$$\Delta\left(\mathrm{Disc}\left[\int_{\gamma}\omega\right]\right) = \sum_i \mathrm{Disc}\left[\int_{\gamma}\omega_i\right] \otimes \int_{\gamma_i}\omega\,,$$

— this is useful for taking discontinuities.

- Differentiation acts on the second entry

$$\Delta\left(\partial\left[\int_{\gamma}\omega\right]\right) = \sum_i \int_{\gamma}\omega_i \otimes \partial\left[\int_{\gamma_i}\omega\right]\,,$$

— useful for deriving differential equations.

Diagrammatic Coaction at one loop

- To translate the coaction $\Delta\left(\int_{\gamma} \omega\right) \equiv \sum_i \int_{\gamma} \omega_i \otimes \int_{\gamma_i} \omega$

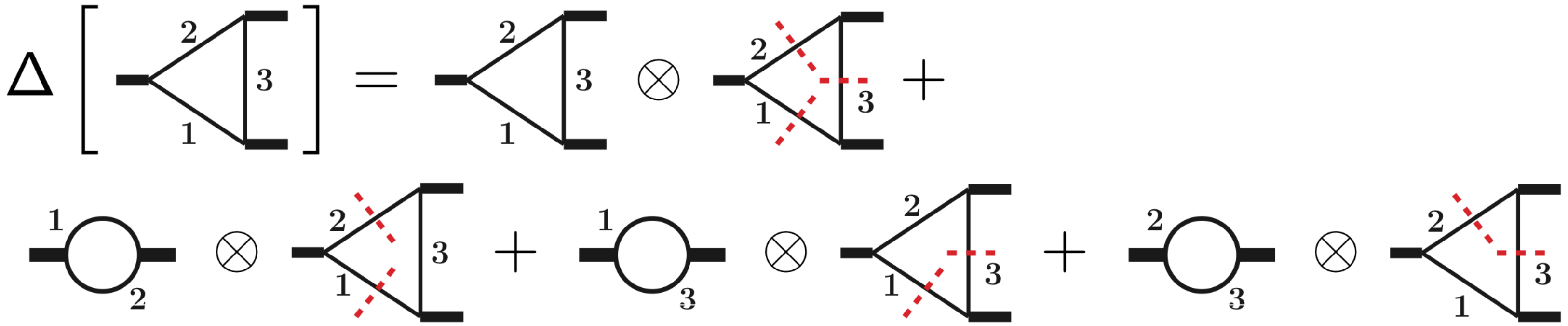
  to Feynman diagrams we need to fix a basis.

- At one loop, a convenient basis of uniform weight Feynman integrals is:

$$\widetilde{J}_n = \mathrm{e}^{\gamma_E \epsilon} \int \frac{d^D k}{i\pi^{D/2}} \prod_{j=0}^{n-1} \frac{1}{(k-q_j)^2 - m_j^2}\,, \qquad D = 2\left\lceil \frac{n}{2} \right\rceil - 2\epsilon$$

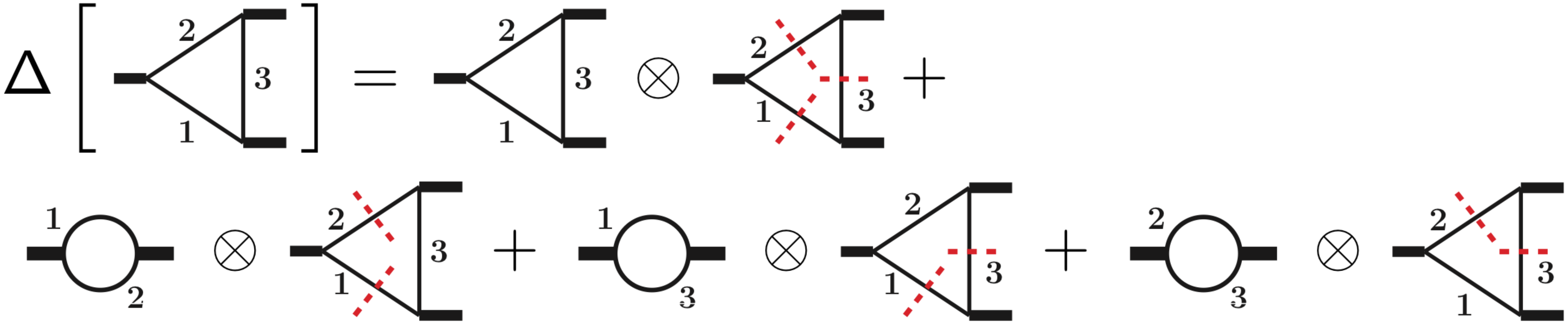
- This fixes the basis for the first entries in the coaction: for a given integral $\widetilde{J}_n$ these are **the diagram itself and all its possible contractions.**

- Next we **need to identify the corresponding contours encircling poles, defining the second entries. These are cuts!**
  First guess: cut all propagators that aren't contracted in the first entry.



Diagrammatic Coaction at one loop — example: the three-mass triangle

- The diagrammatic coaction for a three-mass triangle with massless propagators (to all orders in dimensional regularisation):



- Can be verified, order-by-order in ϵ to reproduce the coaction on MPLs

- Defining $z\,\bar{z} = p_2^2/p_1^2$ and $(1-z)\,(1-\bar{z}) = p_3^2/p_1^2$

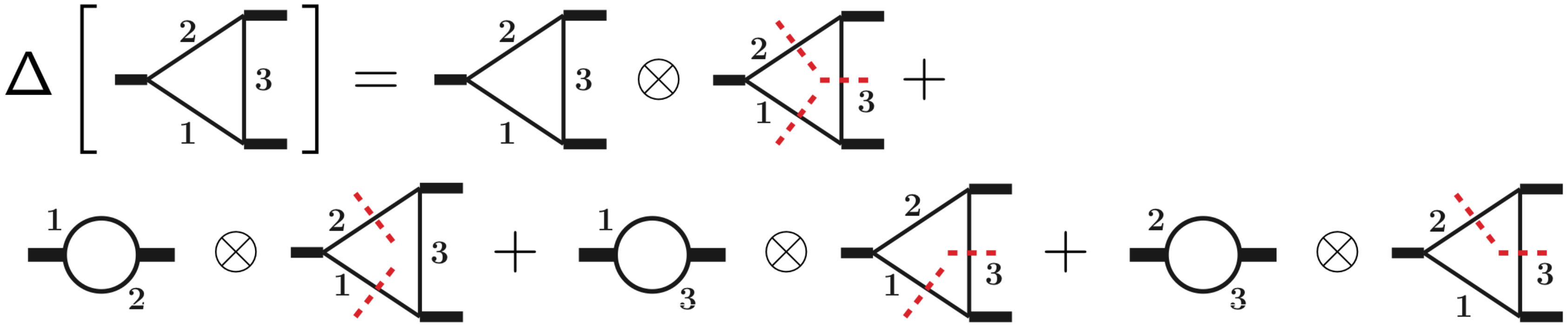
<u>At leading order in ϵ</u> : $\mathcal{T}(z,\bar{z}) = -2\mathrm{Li}_2(z) + 2\mathrm{Li}_2(\bar{z}) - \ln(z\bar{z})\ln\left(\frac{1-z}{1-\bar{z}}\right)$

$$\Delta\left[\mathcal{T}(z,\bar{z})\right] = \mathcal{T}(z,\bar{z})\otimes 1 + 1\otimes\mathcal{T}(z,\bar{z}) + \ln\left(-p_2^2\right)\otimes\ln\frac{1-\bar{z}}{1-z} + \ln\left(-p_3^2\right)\otimes\ln\frac{z}{\bar{z}} + \ln(-p_1^2)\otimes\ln\frac{\bar{z}(1-z)}{z(1-\bar{z})}$$

- We see how the 3 channel discontinuities are captured by the 3 bubbles.

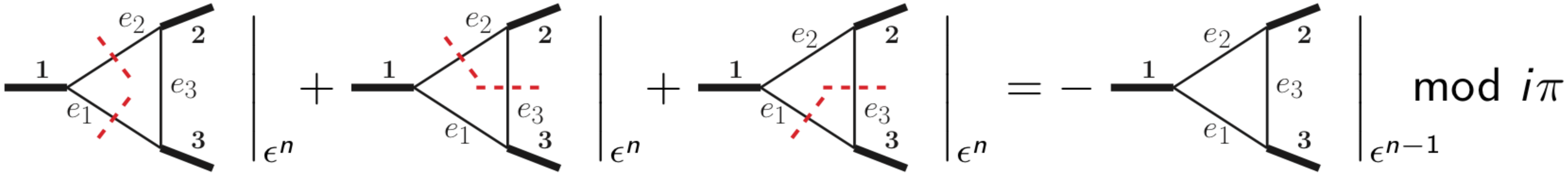
Diagrammatic Coaction at one loop — example: the three-mass triangle

- The diagrammatic coaction for a three-mass triangle:



$$\Delta\left[\mathcal{T}(z,\bar{z})\right] = \mathcal{T}(z,\bar{z})\otimes 1 + 1\otimes\mathcal{T}(z,\bar{z}) + \ln\left(-p_2^2\right)\otimes\ln\frac{1-\bar{z}}{1-z} + \ln\left(-p_3^2\right)\otimes\ln\frac{z}{\bar{z}} + \ln(-p_1^2)\otimes\ln\frac{\bar{z}(1-z)}{z(1-\bar{z})}$$

- The trivial term with the full triangle on the second entry is recovered (to all orders) though the “pole-cancellation identity”:



which in general takes the form: $\displaystyle\sum_{i\in[n]} \mathcal{C}_i \widetilde{J}_n + \sum_{i<j\in[n]} C_{i,j}\widetilde{J}_n = -\epsilon\widetilde{J}_n \quad \mathrm{mod}\ i\pi$

The Diagrammatic Coaction at one loop: the general case

- Translating the coaction on integrals

$$\Delta\left(\int_{\gamma}\omega\right) \equiv \sum_i \int_{\gamma}\omega_i \otimes \int_{\gamma_i}\omega \qquad\qquad P_{ss}\left(\int_{\gamma_i}\omega_j\right) = \delta_{ij}\,,$$

to Feynman diagrams

$$\Delta(J_G) = \sum_{\emptyset\neq C\subseteq E_G} \underset{\overset{\nearrow}{J_{G_C}}}{\int_{\gamma}\omega_C} \otimes \underset{\overset{\nearrow}{\mathbf{?}}}{\boxed{\int_{\gamma_C}\omega_{E_G}}}$$

- The first entry in the diagrammatic coaction are J_{G_C}: graphs whose edges that are not in the subset C are contracted.

- **What is the corresponding contour γ_C defining the second entry?**

- The general solution:

$$\gamma_C = \begin{cases} \Gamma_{C\infty} & \text{for |C| odd} \\ \Gamma_C & \text{for |C| even} \end{cases}$$

where Γ_C encircles the poles in C while $\Gamma_{C\infty}$ encircles in addition the branch point at infinite momentum (it’s a second type Landau singularity).

Coaction for one loop Feynman diagrams

- It turns out contours that encircle the branch point at infinite momentum at one loop can be recast in terms of ordinary cuts!

- This is owing to the (homology) relations:

For even |C|:

$$\mathcal{C}_{C\infty}\widetilde{J}_n = \sum_{i\in[n]\setminus C} \mathcal{C}_{Ci}\widetilde{J}_n + \sum_{i<j\in[n]\setminus C} C_{Cij}\widetilde{J}_n \quad \mod i\pi$$

For odd |C|:

$$\mathcal{C}_{C\infty}\widetilde{J}_n = -2C_C\widetilde{J}_n - \sum_{i\in[n]\setminus C} \mathcal{C}_{Ci}\widetilde{J}_n \quad \mod i\pi$$

D. Fotiadi and F. Pham (1966); Abreu et al. 1702.03163

- The diagrammatic coaction at one loop:

$$\Delta(J_G) = \sum_{\emptyset\neq C\subseteq E_G} \int_\gamma \omega_C \otimes \int_{\gamma_C} \omega_{E_G}$$

➡

$$\Delta(J_G) = \sum_{\emptyset\neq C\subseteq E_G} J_{G_C} \otimes \left(\mathcal{C}_C J_G + a_C \sum_{e\in E_G\setminus C} \mathcal{C}_{Ce} J_G \right)$$

with

$$a_C = \begin{cases} \frac{1}{2} & \text{if } |C| \text{ is odd} \\ 0 & \text{if } |C| \text{ is even} \end{cases}$$

Coaction for one loop Feynman diagrams

- The diagrammatic coaction at one loop, expressed in terms of cuts:

$$\Delta(J_G) = \sum_{\emptyset \neq C \subseteq E_G} \int_\gamma \omega_C \otimes \int_{\gamma_C} \omega_{E_G}$$

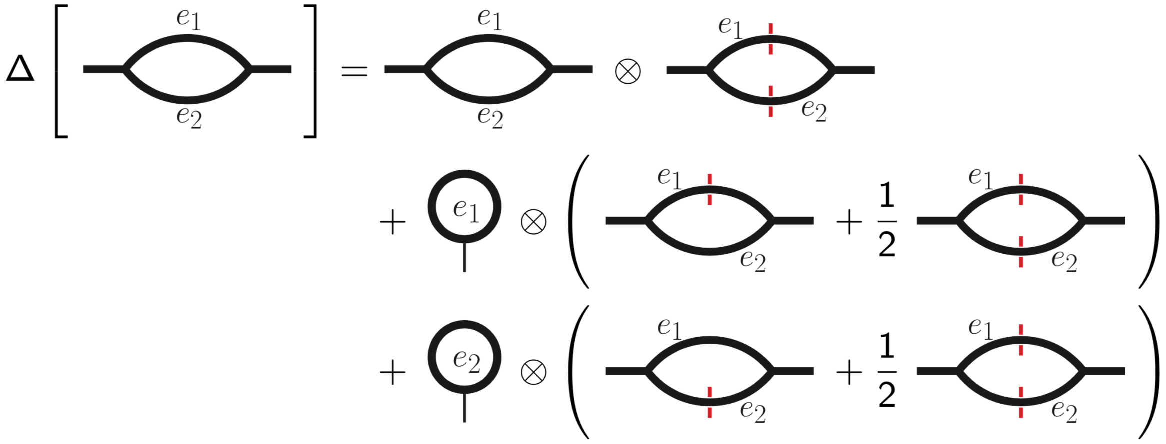
$$\boxed{\Delta(J_G) = \sum_{\emptyset \neq C \subseteq E_G} J_{G_C} \otimes \left(\mathcal{C}_C J_G + a_C \sum_{e \in E_G \setminus C} \mathcal{C}_{Ce} J_G \right)}$$

with
$$a_C = \begin{cases} \frac{1}{2} & \text{if } |C| \text{ is odd} \\ 0 & \text{if } |C| \text{ is even} \end{cases}$$

- <u>Example</u>: bubble with two massive propagators

$$\begin{aligned} \Delta\left(\int_{\Gamma_\emptyset} \omega_{12}\right) &= \int_{\Gamma_\emptyset} \omega_{12} \otimes \int_{\Gamma_{12}} \omega_{12} \\ &+ \int_{\Gamma_\emptyset} \omega_1 \otimes \left(\int_{\Gamma_1} + \frac{1}{2}\int_{\Gamma_{12}}\right) \omega_{12} \\ &+ \int_{\Gamma_\emptyset} \omega_2 \otimes \left(\int_{\Gamma_2} + \frac{1}{2}\int_{\Gamma_{12}}\right) \omega_{12} \end{aligned}$$

$\Leftrightarrow$



ω_i	γ_i
ω_{12}	Γ_{12}
ω_1	$\Gamma_1 + \frac{1}{2}\Gamma_{12} = \Gamma_{\infty 1}$
ω_2	$\Gamma_2 + \frac{1}{2}\Gamma_{12} = \Gamma_{\infty 2}$

satisfy

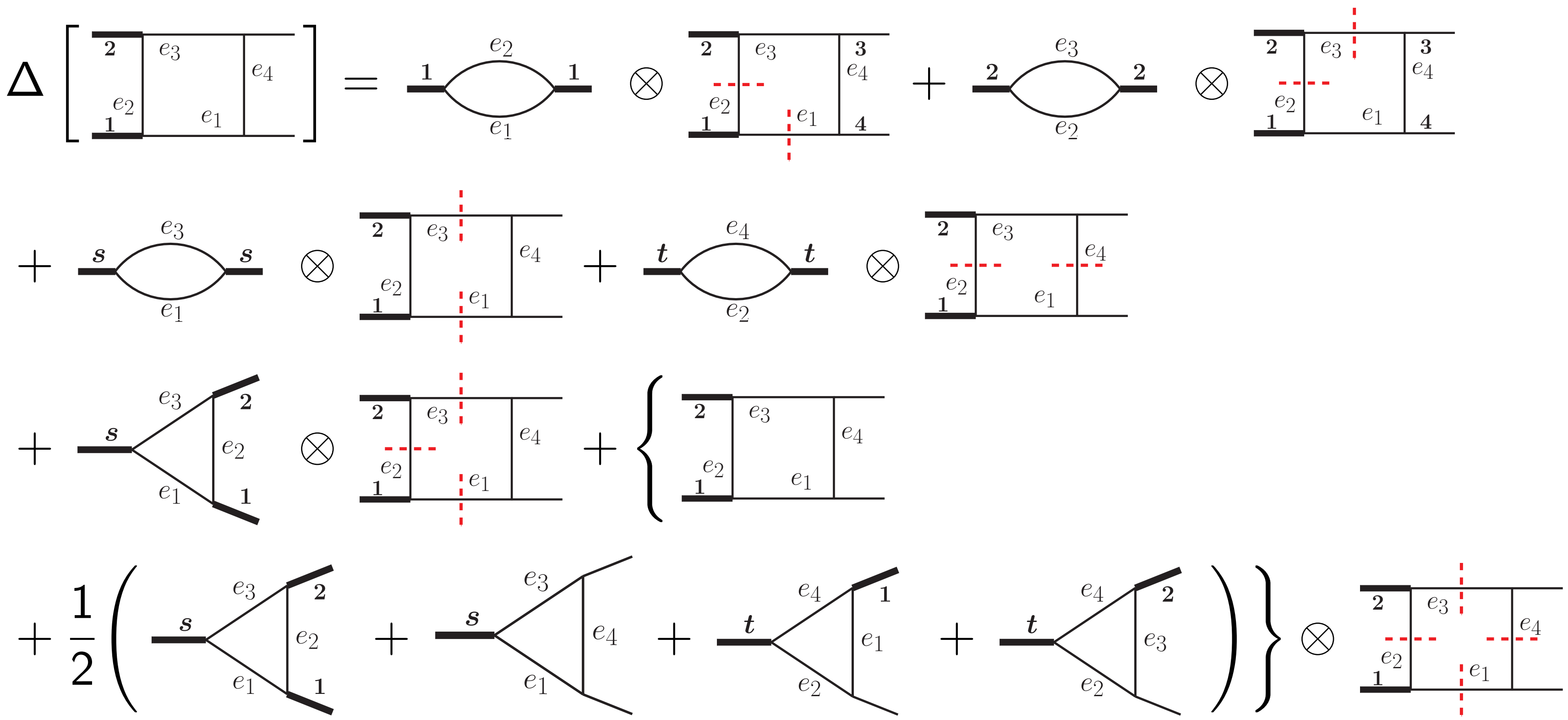
$$P_{ss}\left(\int_{\gamma_i} \omega_j\right) = \delta_{ij}$$

Coaction for one loop Feynman diagrams: the two-mass hard box

$$\Delta(J_G) = \sum_{\emptyset\neq C\subseteq E_G} J_{G_C} \otimes \left(\mathcal{C}_C J_G + a_C \sum_{e\in E_G\setminus C} \mathcal{C}_{Ce} J_G\right)$$

with
$$a_C = \begin{cases} \frac{1}{2} & \text{if } |C| \text{ is odd} \\ 0 & \text{if } |C| \text{ is even} \end{cases}$$

- The diagrammatic coaction for the two-mass hard box:



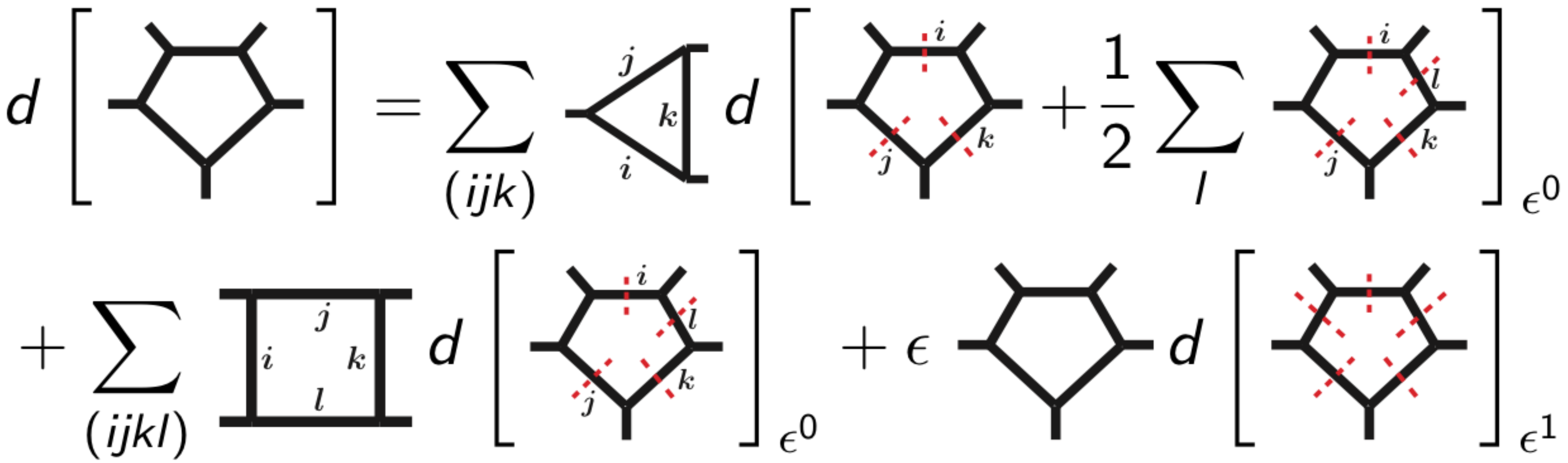
Differential equations from the coaction

- Given that differentiation only acts on the second entries:

$$\Delta\left(\partial\left[\int_{\gamma} \omega\right]\right) = \sum_i \int_{\gamma} \omega_i \otimes \partial\left[\int_{\gamma_i} \omega\right] ,$$

we can derive differential equations from the diagrammatic coaction by focusing on the weight 1 component on the second entry.

- For example, for a pentagon we get:



Coaction on hypergeometric functions

- We have seen that the diagrammatic coaction holds order-by-order in ϵ, recovering the coaction on MPLs.
- **Can it be established directly for the Hypergeometric functions? Yes!**

(Ruth Britto’s talk last week)

Example: Gauss Hypergeometric function 2F1

$$\omega_1 = x^{a\epsilon}(1-x)^{-1+b\epsilon}(1-zx)^{c\epsilon}dx \qquad \Leftrightarrow \qquad \gamma_1 = [0,1]$$

$$\omega_2 = x^{a\epsilon}(1-x)^{b\epsilon}(1-zx)^{-1+c\epsilon}dx \qquad \Leftrightarrow \qquad \gamma_2 = [0,1/z]$$

The period matrix is readily diagonal, so suitable normalization yields: $P_{ss}\left(\int_{\gamma_i}\omega_j\right) = \delta_{ij}$

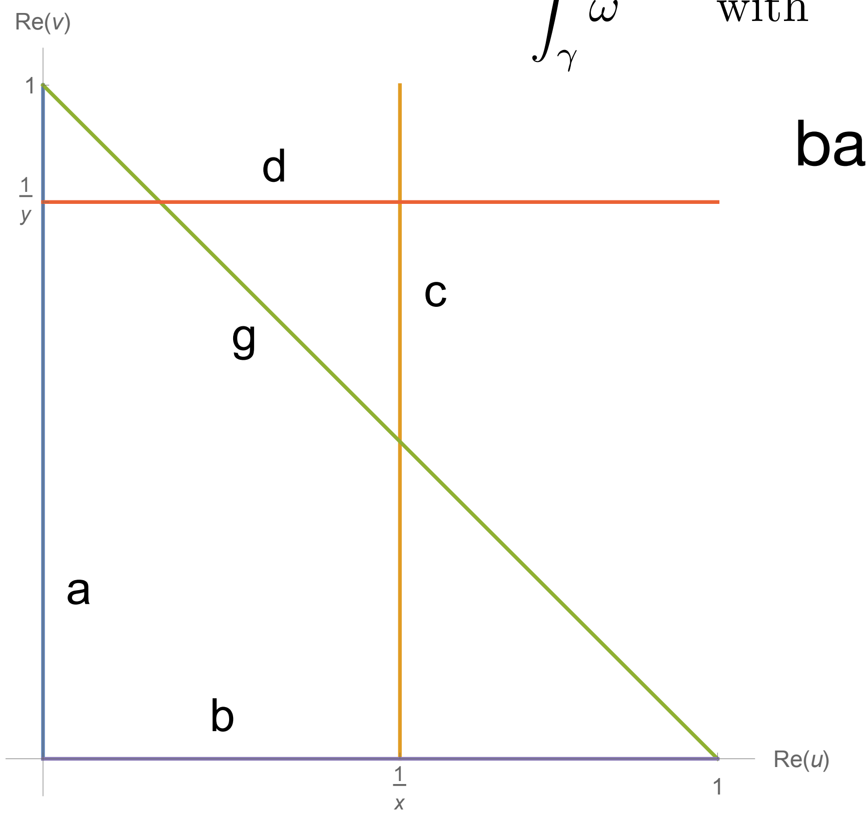
$$\Delta \int_{\gamma_1}\omega = \int_{\gamma_1}\omega_1 \otimes \int_{\gamma_1}\omega + \int_{\gamma_1}\omega_2 \otimes \int_{\gamma_2}\omega$$

Coaction on Appell functions

- A similar construction can be applied to generalised Hypergeometrics. Let's consider Appell F3, depending on two variables.

$$F_3(\alpha,\alpha',\beta,\beta',\gamma;x,y) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')} \times \int_0^1 \int_0^{1-v} u^{\beta-1} v^{\beta'-1} (1-u-v)^{\gamma-\beta-\beta'-1} (1-xu)^{-\alpha} (1-yv)^{-\alpha'}\, du\, dv$$

- The coaction can be obtained from $\Delta\left(\int_{\gamma} \omega\right) \equiv \sum_i \int_{\gamma} \omega_i \otimes \int_{\gamma_i} \omega$



$$\int_{\gamma} \omega \qquad \text{with} \qquad \omega = \Phi\varphi \qquad \Phi = u^{a\epsilon} v^{b\epsilon} (1-xu)^{c\epsilon} (1-yv)^{d\epsilon} (1-u-v)^{g\epsilon}$$

basis of 2-forms:

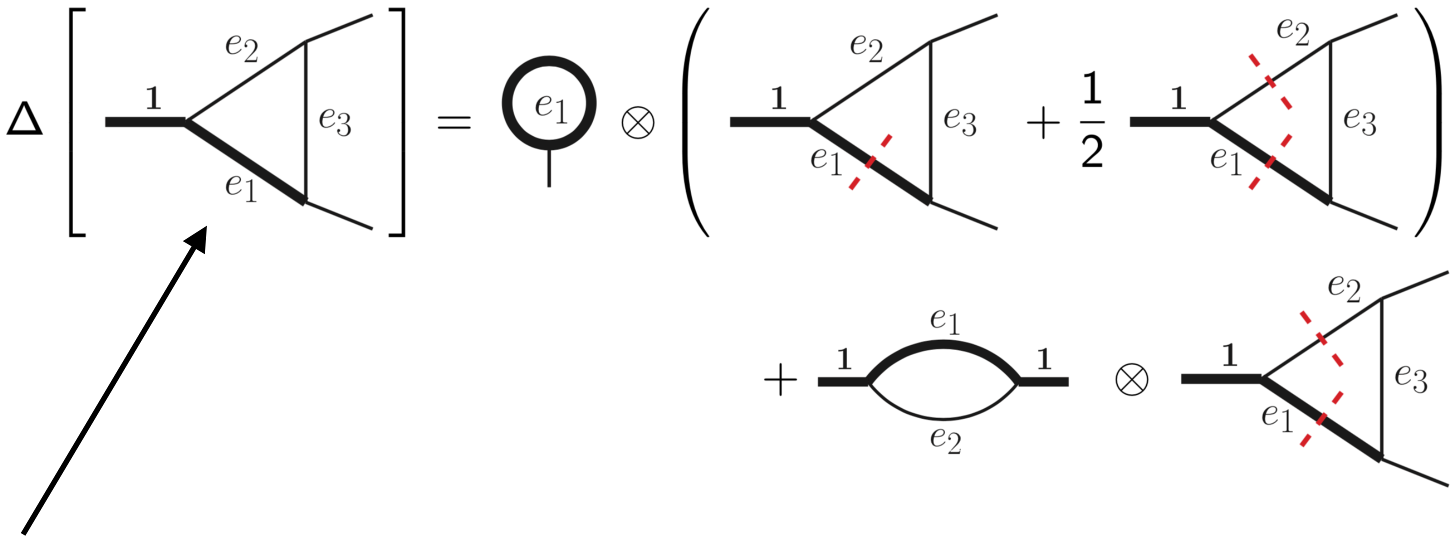
$$\begin{aligned}
\varphi_{ab} &= d\log(u) \wedge d\log(v) &&= \frac{du \wedge dv}{uv} \\
\varphi_{bc} &= d\log(1-xu) \wedge d\log(v) &&= -\frac{x\, du \wedge dv}{(1-xu)v} \\
\varphi_{cd} &= d\log(1-xu) \wedge d\log(1-yv) &&= \frac{xy\, du \wedge dv}{(1-xu)(1-yv)} \\
\varphi_{ad} &= d\log(u) \wedge d\log(1-yv) &&= -\frac{y\, du \wedge dv}{u(1-yv)}
\end{aligned}$$

period matrix:

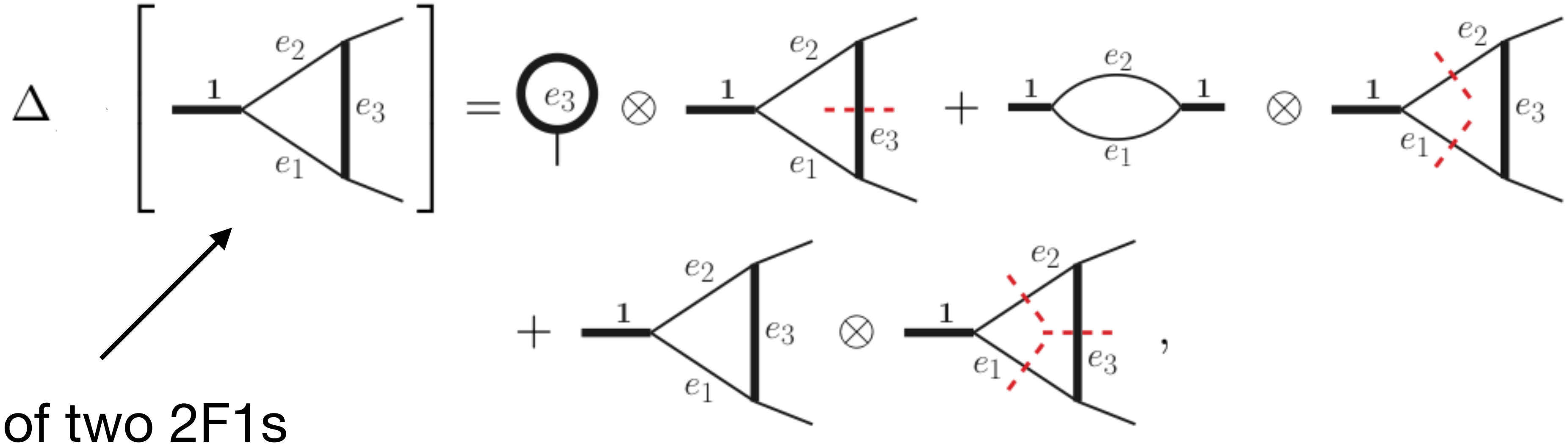
$P_{ss} \int_{\gamma_S} \Phi\varphi_T$	ab	bc	cd	ad
abg	$\frac{1}{ab\epsilon^2}$	0	0	0
bcg	0	$\frac{1}{bc\epsilon^2}$	0	0
cdg	0	0	$\frac{1}{cd\epsilon^2}$	0
adg	0	0	0	$\frac{1}{ad\epsilon^2}$

- The result for the coaction can be fully expressed in terms of Appell F3.

Coaction on hypergeometric functions — application to one loop diagrams



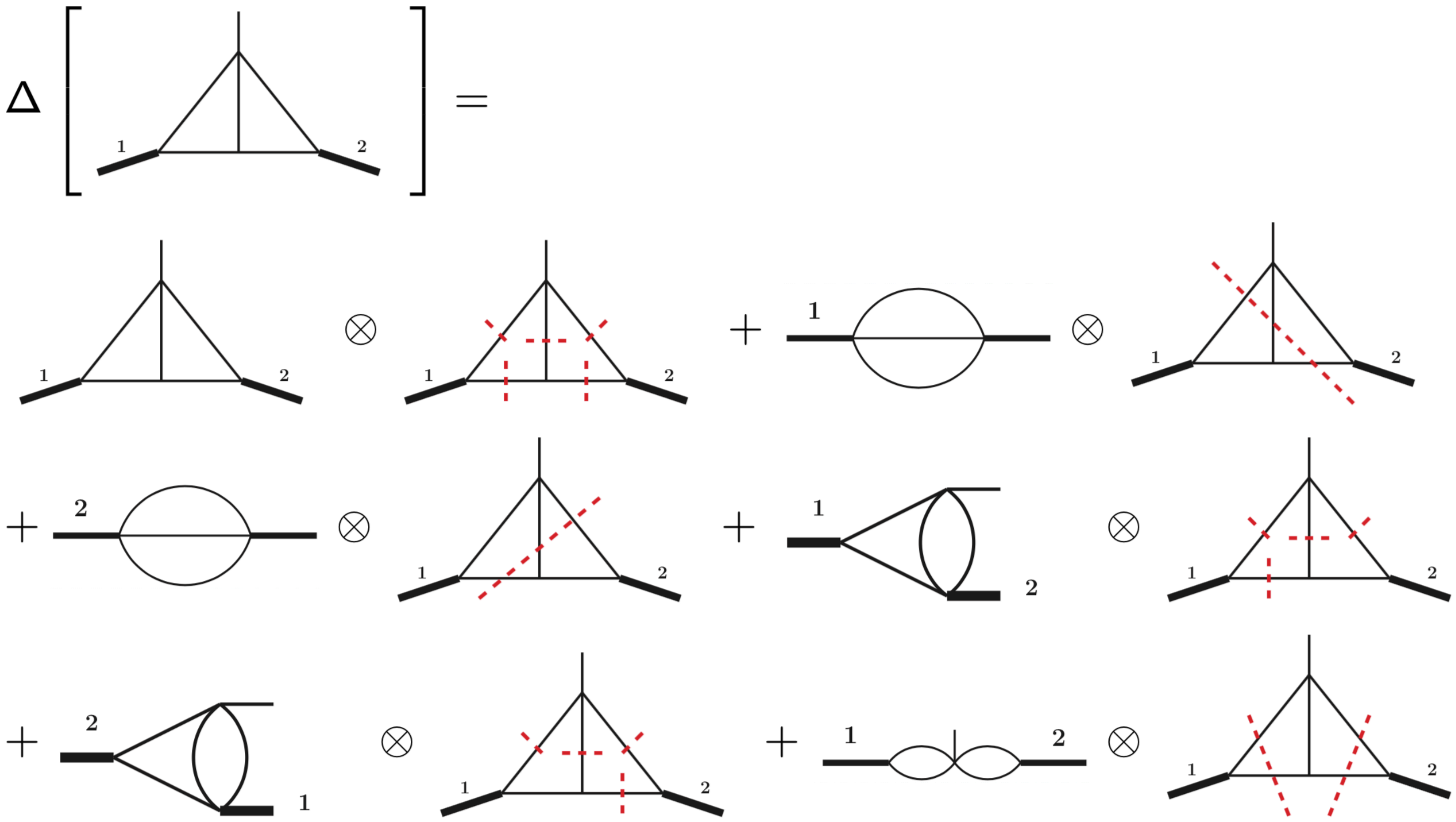
$$\Delta\left({}_2F_1(1,1+\epsilon,2-\epsilon,x)\right) = {}_2F_1(1,\epsilon,1-\epsilon,x) \otimes {}_2F_1(1,1+\epsilon,2-\epsilon,x) + {}_2F_1(1,1+\epsilon,2-\epsilon,x) \otimes {}_2F_1\left(1,\epsilon,1-\epsilon,\frac{1}{x}\right)$$



a sum of two 2F1s

Diagrammatic Coaction at two loops

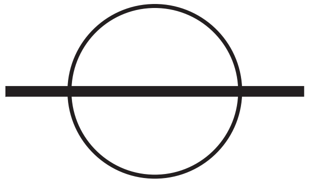
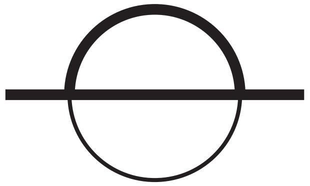
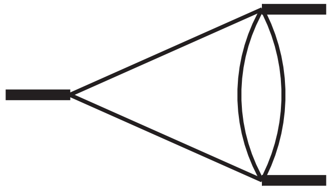
- The double triangle can be written in terms of two 3F2 and a 2F1
- This system has 6 master integrands and contours, out of which only one is of the top topology.



- This coaction was verified directly in terms of hypergeometric functions!
- The correspondence between 1st and 2nd entries is just as at one loop. All 1st entries are two loop diagrams — in 2nd entries cuts affect both loops.

Diagrammatic Coaction at two loops

- In contrast to one loop, where the maximal cut is unique, many two-loop diagrams have multiple maximal cuts (multiple master integrals at the top topology).
- We verified in several non-trivial examples that the second entries in the coaction can all be expressed in terms of cuts of the original graph.

	Uncut integral	**Maximal cut**
	${}_2F_1$ (2 masters)	${}_2F_1$ (2 masters)
	F_4 (4 masters)	F_1 (3 masters)
	F_4 (4 masters)	${}_2F_1$ (2 masters)

Conclusions

- We construct a coaction on integrals based on pairing between master integrands and master contours. This reproduces the coaction on MPLs upon expansion.
- It also extends naturally to hypergeometric functions including all Appell functions. In each case, left entries span the space of functions related by integer shifts, while right entries span the space of solutions to the differential equation.
- It also translates into a coaction of Feynman integrals for any one loop diagram, with any mass configuration. Relations between cuts (homology) are essential to establish it.
- The diagrammatic coaction leads to straightforward derivation of discontinuities and differential equations.
- Recently we used the coaction on hypergeometric functions to find the coaction of several non-trivial two-loop diagrams.
  The general diagrammatic coaction at two loops is not yet known.
- New features at two loops include multiple Master Integrals at the top topology (several maximal cuts) and elliptic polylogs.
  Both these features are present in hypergeometric functions.