Scattering in the High Energy Limit and SVMPLs

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Introduction

Understand the mathematical structure of scattering amplitudes.

(Planar) $\mathcal{N} = 4$ Super Yang-Mills theory is a perfect laboratory.

$\mathcal{N} = 4$ Super Yang-Mills

Maximally supersymmetric $SU(N_c)$ Yang-Mills in 4D.

Many special properties:

- Conformal symmetry (massless, $\beta(g) = 0$ to all orders).
- Maximal transcendentality $\to$ functions appearing in an ℓ loop amplitude are always of 'transcendentality'/weight 2ℓ:

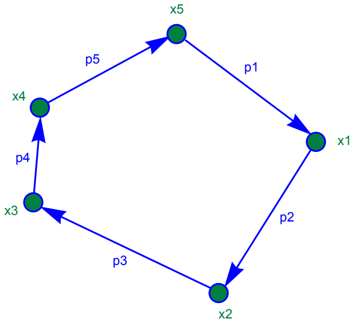
@ 1-Loop : $\text{Li}_2(x), \log^2(x), \zeta_2$

@ ℓ-Loop : Generalised Polylogarithms:

$$G(\underbrace{a_1, ..., a_{2\ell}}_{\text{weight } 2\ell}; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, ..., a_{2\ell}; t).$$

Planar $\mathcal{N} = 4$ Super Yang-Mills

In the limit $N_c \to \infty$ with $g^2 N_c \sim \mathcal{O}(1)$, a hidden symmetry appears



Define new coordinates x_i s.t.

$$x_i - x_{i-1} = p_i.$$

A seperate conformal group acts on x-space.
$\rightarrow$ Dual Conformal Symmetry

[Drummond, Henn, Korchemsky, Sokatchev]

Planar $\mathcal{N} = 4$ Super Yang-Mills

Dual Conformal Symmetry fixes 4 & 5 point amplitudes completely. [Anastasiou, Bern, Dixon, Kosower; Bern, Dixon, Smirnov]

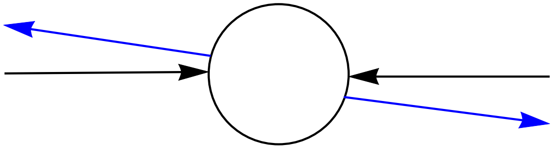
For 6 and 7 points, impressive high-loop results. [Caron-Huot, Del Duca, Dixon, Duhr, Drummond, Golden, Goncharov, Harrington, Henn, Kosower, McLeod, Papathanasiou, Pennington, Smirnov, Spradlin, Vergu, Volovich, Von Hippel, ...]:

8 points and beyond. $\rightarrow$ Special kinematic limit.

The High-Energy Limit

The High-Energy (Regge) Limit

Example: $2 \to 2$ gluon scattering



in the kinematic (Regge) limit

$$s = (p_1 + p_2)^2 \gg -t = \frac{s}{2}\big(1 - \cos\theta\big),$$

$\to$ very forward scattering.

The Regge Limit

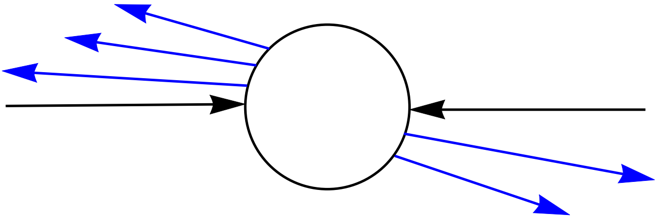
[Balitsky, Fadin, Kuraev, Lipatov]: high-energy limit of QCD
$\rightarrow$ resum large logarithms.

At ℓ loops we have

- $\sim \alpha_S^\ell \log^{\ell-1}\left(\frac{s}{-t}\right)$ is leading logarithmic order
  $\sum$ leading logarithmic terms = Leading Logarithmic Approximation (LLA)
- $\sim \alpha_S^\ell \log^{\ell-2}\left(\frac{s}{-t}\right)$ is next-to-leading logarithmic order
  $\sum$ next-to-leading logarithmic terms = Next-to-Leading Logarithmic Approximation (NLLA)
- ...

The Multi-Regge Limit

Move on to n-gluon scattering.



Strong ordering in angle wrt beam axis ($p_1^+ = p_2^- = \mathbf{p}_1 = \mathbf{p}_2 = 0$)

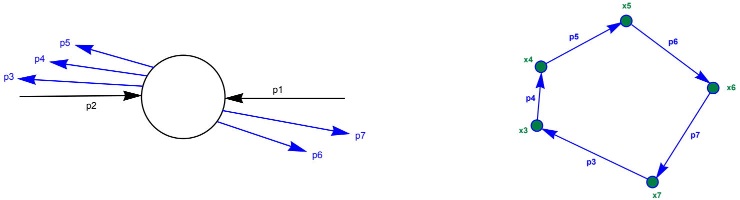
$$p_3^+ \gg p_4^+ \gg \dots p_{N-1}^+ \gg p_N^+ \,.$$

No hierarchy in transverse plane

$$|\mathbf{p}_3| \simeq \dots \simeq |\mathbf{p}_N| \,.$$

The Multi-Regge Limit

All nontrivial kinematics are in the transverse plane.



Dual conformal invariance restricts our variables to

$$z_i = \frac{(\mathbf{x}_1 - \mathbf{x}_{i+3})(\mathbf{x}_{i+2} - \mathbf{x}_{i+1})}{(\mathbf{x}_1 - \mathbf{x}_{i+1})(\mathbf{x}_{i+2} - \mathbf{x}_{i+3})}.$$

For N-point scattering: $N-5$ dual conformal cross ratios.

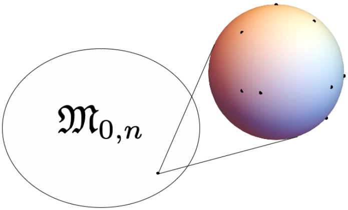
Geometrical Structure

The Moduli Space $\mathfrak{M}_{0,n}$

Kinematics are determined by $n = N - 2$ points in $\mathbb{CP}^1$.

$\mathfrak{M}_{0,n}$= space of configurations for n points on the Riemann sphere.

$\rightarrow$ the phase space for MRK.

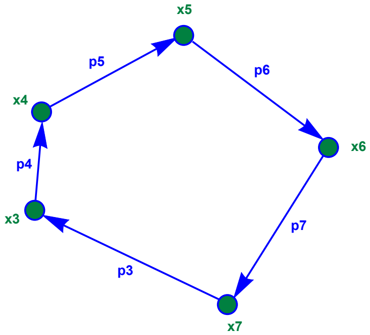


$$\text{dim}_{\mathbb{C}}\mathfrak{M}_{0,N-2} = N - 5$$

$$\downarrow$$

$N - 5$ dual conformal cross ratios $\{z_i\}$.

Iterated Integrals on $\mathfrak{M}_{0,n}$



Degenerate configurations occur when points coincide.

$$\mathbf{x}_i = \mathbf{x}_j$$

Which corresponds to soft limits.

The singularity structure of functions on $\mathfrak{M}_{0,n}$ should respect this.

Iterated Integrals on $\mathfrak{M}_{0,n}$

All iterated integrals on $\mathfrak{M}_{0,n}$ can be written in terms of generalised polylogarithms [Brown]

$$G(a_1, ..., a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, ..., a_n; t).$$

Example:

$$G(a; z) = \log\left(1 - \frac{z}{a}\right)$$
$$G(0, 1; z) = \mathsf{Li}_2(z).$$

Amplitudes in MRK are made up of polylogarithms.

Single-Valued Polylogarithms

Branch cuts occur when

$$(x_i - x_j)^2 = 0.$$

Thus logarithms that appear in the first factor must obey [Gaiotto, Maldacena, Sever, Viera]

$$\Delta(\mathcal{A}) \sim \log(x_i - x_j)^2 \otimes ...$$

For MRK we are restricted to the transverse plane, thus

$$\Delta(\mathcal{A}^{MRK}) \sim \log \underbrace{|\mathbf{x}_i - \mathbf{x}_j|^2}_{\geq 0} \otimes ...$$

and amplitudes in the Multi-Regge limit are single-valued.

[Dixon, Duhr, Pennington]

Single-Valued Polylogarithms

One can combine polylogarithms and their complex conjugates such that all branch cuts cancel.

Associate to each $G(a, b, ...; z)$ a single-valued function $\mathcal{G}(a, b, ...; z)$ such that

$$\partial_z \mathcal{G}(a, b, ...; z) = \frac{1}{z-a} \mathcal{G}(b, ...; z).$$

For example:

$$\mathcal{G}(a; z) = G(a; z) + G(\bar{a}; \bar{z}) = \log \left| 1 + \frac{z}{a} \right|^2$$

$$\begin{aligned} \mathcal{G}(a, b; z) = & G(a, z) G\left(\bar{b}, \bar{z}\right) + G(b, a) G\left(\bar{a}, \bar{z}\right) + G\left(\bar{b}, \bar{a}\right) G\left(\bar{a}, \bar{z}\right) \\ & - G(a, b) G\left(\bar{b}, \bar{z}\right) - G\left(\bar{a}, \bar{b}\right) G\left(\bar{b}, \bar{z}\right) \\ & + G\left(\bar{b}, \bar{a}, \bar{z}\right) + G(a, b, z). \end{aligned}$$

Scattering Amplitudes in MRK at LLA

The MRK Ratio

We are interested in the perturbative expansion of

$$\mathcal{R}_{h_4\dots h_{N-1}} \sim \left.\frac{\mathcal{A}_N(-,+,h_4,\dots,h_{N-1},+,-)}{\mathcal{A}_N^{\mathsf{BDS}}(-,+,h_4,\dots,h_{N-1},+,-)}\right|_{\mathsf{MRK}}$$

at leading logarithmic accuracy. At LLA up to ℓ loops we write for six, seven and a generic number of particles respectively

$$\mathcal{R}^{(\ell)}_{h_4 h_5} = 2\pi i a^\ell \left(\frac{\log^{\ell-1}\tau_1}{(\ell-1)!}\right) g^{(\ell-1)}_{h_4,h_5}(z_1)$$

$$\mathcal{R}^{(\ell)}_{h_4 h_5 h_6} = 2\pi i a^\ell \sum_{i_1+i_2=\ell-1} \left(\frac{\log^{i_1}\tau_1}{i_1!}\frac{\log^{i_2}\tau_2}{i_2!}\right) g^{(i_1,i_2)}_{h_4 h_5 h_6}(z_1,z_2)$$

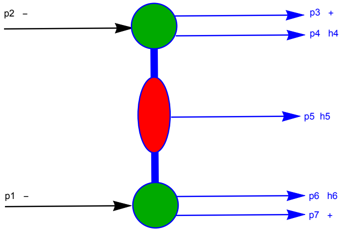
$$\mathcal{R}^{(\ell)}_{h_4\dots h_{N-1}} = 2\pi i a^\ell \sum_{\sum i_k=\ell-1} \left(\prod_{k=1}^{N-5}\frac{\log^{i_k}\tau_k}{i_k!}\right) g^{(i_1,\dots,i_{N-5})}_{h_4\dots h_{N-1}}(\{z_i\}).$$

where a is the coupling constant and $\log(\tau_i)$ are the large logs.

Factorisation

In MRK the amplitude factorises.

[Bartels, Lipatov, Sabio-Vera; Bartels, Kormilitzin, Lipatov, Prygarin]



For example, at 7 points LLA

$$\mathcal{R}_{h_4 h_5 h_6} \sim a\mathcal{F}\left[\chi_0^{h_4}\tau_1^{aE_{\nu 1,n1}} C_0^{h_5}\tau_2^{aE_{\nu 2,n2}}\chi_0^{-h_6}\right]$$

where

$$\mathcal{F}[f] = \sum_{n=-\infty}^{\infty}\int_{\mathbb{R}}\frac{d\nu}{2\pi}\left(\frac{z}{\bar{z}}\right)^{\frac{n}{2}}|z|^{2i\nu}f(\nu,n)$$

denotes the Fourier-Mellin transform.

Factorisation

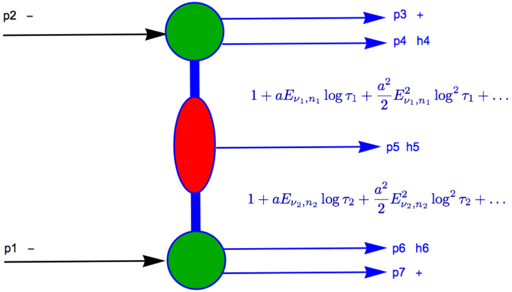
The building blocks are given by

$$\chi_0^{\pm}(\nu, n) = \frac{1}{i\nu \pm \frac{n}{2}} \,,$$

$$E_{\nu n} = -\frac{1}{2}\frac{|n|}{\nu^2 + \frac{n^2}{2}} + \psi\left(1 + i\nu + \frac{|n|}{2}\right) + \psi\left(1 - i\nu + \frac{|n|}{2}\right) - 2\psi(1)\,,$$

$$C_0^{+}(\nu_1, n_1, \nu_2, n_2) = \frac{\Gamma\left(1 - i\nu_1 - \frac{n_1}{2}\right)\Gamma\left(i\nu_2 + \frac{n_2}{2}\right)}{\Gamma\left(1 + i\nu_1 - \frac{n_1}{2}\right)\Gamma\left(-i\nu_2 + \frac{n_2}{2}\right)} \times \frac{\Gamma\left(i\nu_1 - i\nu_2 - \frac{n_1}{2} + \frac{n_2}{2}\right)}{\Gamma\left(1 - i\nu_1 + i\nu_2 - \frac{n_1}{2} + \frac{n_2}{2}\right)}.$$

Convolution Structure



$$\mathcal{F}\left[\chi_0^{h_4}\tau_1^{aE_{\nu 1,n1}}C_0^{h_5}\tau_2^{aE_{\nu 2,n2}}\chi_0^{-h_6}\right] = \sum_{ij}\frac{a^{i+j}}{i!j!}\log^i\tau_1\log^j\tau_2\, g^{(i,j)}_{h_4h_5h_6}$$

$$g^{(i,j)}_{h_4h_5h_6} = \mathcal{F}\left[\chi_0^{h_4}E^i_{\nu_1,n_1}C_0^{h_5}E^j_{\nu_2,n_2}\chi_0^{-h_6}\right]$$

Fourier-Mellin Convolutions

The Fourier-Mellin transform maps products into convolutions

$$\mathcal{F}[f \cdot g] = \mathcal{F}[f] * \mathcal{F}[g] = \int \frac{d^2\omega}{|\omega|^2} \mathcal{F}[f](\omega) \mathcal{F}[g] \left(\frac{z}{\omega} \right).$$

This means that we can compute

$$g^{(i,j)}_{h_4 h_5 h_6} = \mathcal{F} \left[\chi_0^{h_4} E^i_{\nu_1, n_1} C_0^{h_5} E^j_{\nu_2, n_2} \chi_0^{-h_6} \right]$$

by performing convolution integrals over a finite set of building blocks

$$\{\mathcal{F}[\chi], \mathcal{F}[C], \mathcal{F}[E]\}.$$

Stokes' Theorem

We are dealing with single-valued functions with isolated singularities on the Riemann sphere.

$$\downarrow$$

One can use Stokes' theorem to compute convolutions [Schnetz]

$$\int \frac{d^2 z}{\pi} f(z) = \mathrm{Res}_{z=\infty} F(z) - \sum_i \mathrm{Res}_{z=a_i} F(z)$$

where

$$\bar{\partial}_z F = f.$$

Convolutions reduce to residue computations.

Convolution Structure

$$
\begin{aligned}
g^{(i,j)}_{h_4 h_5 h_6} &= \mathcal{F}\left[\chi_0^{h_4} E^{i}_{\nu_1,n_1} C_0^{h_5} E^{j}_{\nu_2,n_2} \chi_0^{-h_6}\right] \\
&= \mathcal{F}\left[E_{\nu_1,n_1}\right] * \mathcal{F}\left[\chi_0^{h_4} E^{i-1}_{\nu_1,n_1} C_0^{h_5} E^{j}_{\nu_2,n_2} \chi_0^{-h_6}\right] \\
&= \mathcal{F}\left[E_{\nu_1,n_1}\right] * g^{(i-1,j)}_{h_4 h_5 h_6}
\end{aligned}
$$

Note that $\mathcal{F}\left[E_{\nu_i,n_i}\right] = -\frac{1}{2}(z_i + \bar{z}_i)/|1-z_i|^2$ and so

$$
g^{(i,j)}_{h_4 h_5 h_6}(z_1, z_2) = \int \frac{d^2\omega}{|\omega|^2} \frac{-(\omega+\bar{\omega})}{2|1-\omega|^2} \underbrace{g^{(i-1,j)}_{h_4 h_5 h_6}\left(\frac{z_1}{\omega}, z_2\right)}_{\text{SV-POLYLOGS}}
$$

$\rightarrow$ Obtain higher-order coefficients by convoluting with $\mathcal{F}\left[E_{\nu,n}\right]$ from low-loop result.

[Del Duca, Druc, Drummond, Duhr ,Dulat, Marzucca, Papathanasiou, BV]

Convolution Structure

$$g^{(i,j)}_{-++} = \mathcal{F}\left[\chi_0^- E^i_{\nu_1,n_1} C_0^+ E^j_{\nu_2,n_2} \chi_0^-\right]$$
$$= \mathcal{F}\left[\chi_0^- / \chi_0^+\right] * \mathcal{F}\left[\chi_0^+ E^i_{\nu_1,n_1} C_0^+ E^j_{\nu_2,n_2} \chi_0^-\right]$$
$$= \mathcal{F}\left[\chi_0^- / \chi_0^+\right] * g^{(i,j)}_{+++}$$

Note that $\mathcal{F}\left[\chi_0^- / \chi_0^+\right] = -z_i/(1-z_i)^2$ and so

$$g^{(i,j)}_{-++} = \int \frac{d^2\omega}{|\omega|^2} \frac{-\omega}{(1-\omega)^2} \underbrace{g^{(i,j)}_{+++}\left(\frac{z_1}{\omega}, z_2\right)}_{\text{SV-POLYLOGS}}$$

$\rightarrow$ Obtain any helicity configuration by performing a convolution on the MHV coefficients.

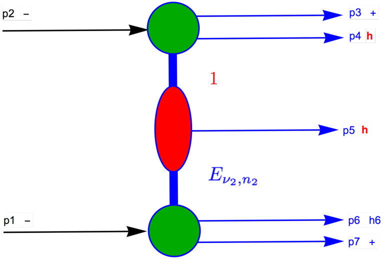
[Del Duca, Druc, Drummond, Duhr ,Dulat, Marzucca, Papathanasiou, BV]

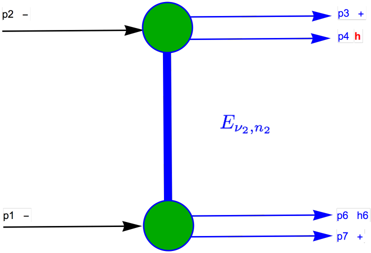
Factorisation of Perturbative Coefficients

When expressed in terms of $\{\mathbf{x}_i\}$ we see that

$$g^{(0,1)}_{\mathbf{hh}h_6}(\mathbf{x}_1, \mathbf{x}_2) = g^{(1)}_{\mathbf{h}h_6}(\mathbf{x}_2).$$

Known at 2 loops [Bartels, Kormilitzin, Lipatov, Prygarin]





In general: propagators with no E where the neighbouring helicities are equal can be dropped.

[Del Duca, Druc, Drummond, Duhr ,Dulat, Marzucca, Papathanasiou, BV]

Factorisation of Perturbative Coefficients

This allows us to determine all n-point LLA MHV scattering amplitudes up to a certain loop order.

$$g_{+\dots+}^{(0,\dots,0,i_{a_1},0,\dots,0,i_{a_2},0,\dots,0,i_{a_k},0,\dots,0)}(\mathbf{x}_1,\dots,\mathbf{x}_{N-5}) = g_{+\dots+}^{(i_{a_1},i_{a_2},\dots,i_{a_k})}(\mathbf{x}_{a_1},\dots,\mathbf{x}_{a_k})$$

[Del Duca, Druc, Drummond, Duhr ,Dulat, Marzucca, Papathanasiou, BV]

Factorisation of Perturbative Coefficients

For example

$$\mathcal{R}^{(2)}_{+\cdots+} = \sum_i \log \tau_i \, g^{(1)}_{++}(\mathbf{x}_i)$$

[Bartels, Kormilitzin, Lipatov, Prygarin]

$$\mathcal{R}^{(3)}_{+\cdots+} = \sum_i \log^2 \tau_i \, g^{(2)}_{++}(\mathbf{x}_i) + \frac{1}{2} \sum_{i \neq j} \log \tau_i \log \tau_j \, g^{(1,1)}_{+++}(\mathbf{x}_i, \mathbf{x}_j).$$

The MHV LLA amplitude at ℓ loops is completely fixed by contributions up to $\ell + 4$ particles.

[Del Duca, Druc, Drummond, Duhr ,Dulat, Marzucca, Papathanasiou, BV]

Factorisation of Perturbative Coefficients

Beyond MHV, we encounter unfactorizable contributions

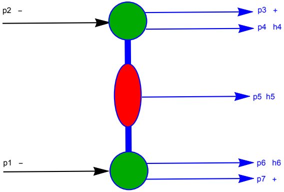
$$\mathcal{R}^{(2)}_{-+\ldots} = \log \tau_1\, g^{(1)}_{-+}(\mathbf{x}_1) + \sum_{j=2}^{N-5} \log \tau_j\, g^{(0,1)}_{-++}(\mathbf{x}_1, \mathbf{x}_j),$$

$$\mathcal{R}^{(2)}_{+-+\ldots} = \log \tau_1\, g^{(1,0)}_{+-+}(\mathbf{x}_1, \mathbf{x}_2) + \log \tau_2\, g^{(0,1)}_{+-+}(\mathbf{x}_1, \mathbf{x}_2) + \sum_{j=3}^{N-5} \log \tau_j\, g^{(0,0,1)}_{+-++}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_j).$$

Scattering Amplitudes in MRK at NLLA

NLO building blocks

At NLLA we get contributions from NLO building blocks



For example, at 7 points:

$$f_{h_4 h_5 h_6} = a\mathcal{F}\left[\chi^{h_4}\tau_1^{-\omega(\nu_1,n_1)} C^{h_5}\tau_2^{-\omega(\nu_2,n_2)}\chi^{-h_6}\right]$$

where

$$\chi^{h_i} = \chi_0^{h_i}(1 + a\kappa_1^{h_i} + \dots)$$

$$C^{h_i} = C_0^{h_i}(1 + ac_1^{h_i} + \dots)$$

$$\omega(\nu_i, n_i) = -a(E_{\nu_i,n_i} + aE^{(1)}_{\nu_i,n_i} + \dots)$$

NLO building blocks

$$\chi^{h_i} = \chi_0^{h_i}(1 + a\kappa_1^{h_i} + \dots)$$
$$C^{h_i} = C_0^{h_i}(1 + ac_1^{h_i} + \dots)$$
$$\omega(\nu_i, n_i) = -a(E_{\nu_i,n_i} + aE^{(1)}_{\nu_i,n_i} + \dots)$$

The building blocks χ^{h_i} and $\omega(\nu_i, n_i)$ are known to all orders. [Basso, Caron-Huot, Sever]

The NLO contribution of the CEB $c_1^{h_i}$ was extracted from the 2-loop NLLA symbol in MRK [Bargheer, Papathanasiou, Schomerus; Del Duca, Druc, Drummond, Duhr ,Dulat, Marzucca, Papathanasiou, BV]

NLO building blocks

$$\kappa_1^+(\nu, n) = \frac{1}{4}\left(E^2 + \frac{3}{4}N^2 - NV + \frac{\pi^2}{3}\right)$$

$$E_{\nu,n}^{(1)} = -\frac{1}{4}\left(D^2 E - 2VDE + 4\zeta_2 E + 12\zeta_3\right)$$

$$V(\nu, n) \equiv \frac{i\nu}{\nu^2 + \frac{n^2}{4}}\,, \qquad N(\nu, n) = \frac{n}{\nu^2 + \frac{n^2}{4}}\,, \qquad D_\nu = -i\partial/\partial\nu\,,$$

NLO building blocks

$$
\begin{aligned}
c_1^{+}(\nu_1, n_1, \nu_2, n_2) = & \frac{1}{2}\Big[DE_1 - DE_2 + E_1E_2 + \tfrac{1}{4}(N_1+N_2)^2 + V_1V_2 \\
& + (V_1 - V_2)\big(M - E_1 - E_2\big) + 2\zeta_2 \\
& + i\pi(V_2 - V_1 - E_1 - E_2)\Big] \\
& - \tfrac{1}{4}(E_1^2 + E_2^2 + N_1V_1 - N_2V_2) \\
& - \tfrac{3}{16}(N_1^2 + N_2^2) - \zeta_2
\end{aligned}
$$

$$
V(\nu, n) \equiv \frac{i\nu}{\nu^2 + \frac{n^2}{4}}\,, \qquad N(\nu, n) = \frac{n}{\nu^2 + \frac{n^2}{4}}\,, \qquad D_\nu = -i\partial/\partial\nu\,,
$$

$$
\begin{aligned}
M(\nu_1, n_1, \nu_2, n_2) = & \psi\big(i(\nu_1 - \nu_2) - \tfrac{n_1 - n_2}{2}\big) \\
& + \psi\big(1 - i(\nu_1 - \nu_2) - \tfrac{n_1 - n_2}{2}\big) - 2\psi(1)\,.
\end{aligned}
$$

Convolutions at NLLA

$$\chi^{h_i} = \chi_0^{h_i}(1 + a\kappa_1^{h_i} + \dots)$$
$$C^{h_i} = C_0^{h_i}(1 + ac_1^{h_i} + \dots)$$
$$\omega(\nu_i, n_i) = -a(E_{\nu_i,n_i} + aE^{(1)}_{\nu_i,n_i} + \dots)$$

At NLLA every term can at most have one NLO contribution. If we write

$$\varpi_7 = \chi_0^+ C_0^+ \chi_0^-$$

we get for example at two loops

$$\begin{aligned} f^{(2),\text{NLLA}}_{h_4h_5h_6} &= a\mathcal{F}\left[\chi^{h_4}\tau_1^{-\omega(\nu_1,n_1)} C^{h_5}\tau_2^{-\omega(\nu_2,n_2)}\chi^{-h_6}\right]\Big|_{\text{2-loop NLLA}} \\ &= a^2\left(\mathcal{F}\left[\kappa_1^{h_4}\varpi_7\right] + \mathcal{F}\left[c_1^{h_5}\varpi_7\right] + \mathcal{F}\left[\kappa_1^{-h_6}\varpi_7\right]\right) \\ &= a^2\mathcal{F}\left[(\kappa_1^{h_4} + c_1^{h_5} + \kappa_1^{-h_6})\varpi_7\right]. \end{aligned}$$

Convolutions at NLLA

The three-loop contribution would give us

$$f^{(3),\text{NLLA}}_{h_4 h_5 h_6} = a^3 \sum_i \left(\mathcal{F}\left[E_{\nu_i,n_i} (\kappa_1^{h_4} + c_1^{h_5} + \kappa_1^{-h_6}) \varpi_7 \right] + \mathcal{F}\left[E^{(1)}_{\nu_i,n_i} \varpi_7 \right] \right).$$

Which we can write as

$$a^3 \sum_i \left(\mathcal{F}\left[E_{\nu_i,n_i} \right] * \mathcal{F}\left[(\kappa_1^{h_4} + c_1^{h_5} + \kappa_1^{-h_6}) \varpi_7 \right] + \mathcal{F}\left[E^{(1)}_{\nu_i,n_i} \varpi_7 \right] \right).$$

So if we know

$$\left\{ \mathcal{F}\left[\kappa_1^{h_i} \varpi_7 \right], \mathcal{F}\left[c_1^{h_i} \varpi_7 \right], \mathcal{F}\left[E^{(1)}_{\nu_i,n_i} \varpi_7 \right] \right\}$$

we can build up NLLA amplitudes at arbitrary loop orders through convolutions with $\mathcal{F}\left[E_{\nu_i,n_i} \right]$.

Convolutions at NLLA

The NLLA MHV 2-loop amplitude is known for any number of particles, which gives us $\mathcal{F}\left[(\kappa_1^{h_4} + c_1^{h_5} + \kappa_1^{-h_6})\varpi_7\right]$. [Bargheer, Papathanasiou, Schomerus; Del Duca, Druc, Drummond, Duhr ,Dulat, Marzucca, Papathanasiou, BV]

The term $\mathcal{F}\left[E^{(1)}_{\nu_i,n_i}\varpi_7\right]$ can be extracted from the 3-loop 6 point NLLA MHV amplitude [Dixon, Duhr, Pennington] through NLLA factorisation of perturbative coefficients. [Del Duca, Druc, Drummond, Duhr ,Dulat, Marzucca, Papathanasiou, BV]

NLLA Helicity Flips

To move beyond MHV

$$f_{-++} = \mathcal{F}\left[\chi^- \tau_1^{-\omega(\nu_1,n_1)} C^+ \tau_2^{-\omega(\nu_2,n_2)} \chi^-\right]$$
$$= \mathcal{F}\left[\frac{\chi^-}{\chi^+}\right] * \mathcal{F}\left[\chi^+ \tau_1^{-\omega(\nu_1,n_1)} C^+ \tau_2^{-\omega(\nu_2,n_2)} \chi^-\right]$$
$$= \mathcal{F}\left[\frac{\chi^-}{\chi^+}\right] * f_{+++}.$$

We may expand

$$\mathcal{F}\left[\frac{\chi^-}{\chi^+}\right] = \mathcal{F}\left[\frac{\chi_0^-}{\chi_0^+}\right] + a\mathcal{H}_1 + \ldots$$

and note that

$$f_{-++}^{(\ell),\text{NLLA}} = \mathcal{F}\left[\frac{\chi_0^-}{\chi_0^+}\right] * f_{+++}^{(\ell),\text{NLLA}} + a\,\mathcal{H}_1 * f_{+++}^{(\ell-1),\text{LLA}}.$$

An NLLA Perturbative Coefficient

Example: The NLLA 7-particle 2-loop NMHV (-++)

$$\tilde{g}^{(0,0)}_{-++}(\rho_1,\rho_2) = \mathfrak{ga}^{(2;0,0)}_{-++}(\rho_1,\rho_2) + \frac{\rho_1(\rho_2-1)}{(\rho_1-1)\rho_2}\mathfrak{gb}^{(2;0,0)}_{1,-++}(\rho_1,\rho_2) + \frac{\rho_1}{\rho_1-1}\mathfrak{gb}^{(2;0,0)}_{2,-++}(\rho_1,\rho_2)$$

An NLLA Perturbative Coefficient

$$
\begin{aligned}
\mathfrak{ga}^{(2;0,0)}_{-++}(\rho_1,\rho_2) =& \frac{1}{4}\mathcal{G}_{0,0,1}(\rho_2) + \frac{1}{8}\mathcal{G}_{0,1,0}(\rho_1) - \frac{1}{4}\mathcal{G}_{0,1,0}(\rho_2) - \frac{1}{4}\mathcal{G}_{0,1,1}(\rho_2) \\
& - \frac{1}{4}\mathcal{G}_{1,0,0}(\rho_1) + \frac{1}{4}\mathcal{G}_{1,0,0}(\rho_2) - \frac{1}{4}\mathcal{G}_{1,0,1}(\rho_2) + \frac{1}{8}\mathcal{G}_{1,0,\rho_2}(\rho_1) \\
& - \frac{1}{4}\mathcal{G}_{1,1,0}(\rho_1) - \frac{1}{4}\mathcal{G}_{1,1,0}(\rho_2) + \frac{1}{2}\mathcal{G}_{1,1,1}(\rho_2) - \frac{1}{4}\mathcal{G}_{1,1,\rho_2}(\rho_1) \\
& + \frac{1}{8}\mathcal{G}_{1,\rho_2,0}(\rho_1) - \frac{1}{4}\mathcal{G}_{1,\rho_2,\rho_2}(\rho_1) - \frac{1}{4}\mathcal{G}_{\rho_2,0,1}(\rho_1) - \frac{1}{8}\mathcal{G}_{\rho_2,1,0}(\rho_1) \\
& - \frac{1}{4}\mathcal{G}_{\rho_2,1,1}(\rho_1) + \frac{1}{4}\mathcal{G}_{\rho_2,1,\rho_2}(\rho_1) - \frac{1}{4}\zeta_2\mathcal{G}_0(\rho_1) - \frac{1}{4}\zeta_2\mathcal{G}_1(\rho_1) + \frac{1}{4}\zeta_2\mathcal{G}_1(\rho_2) \\
& - \frac{1}{4}\mathcal{G}_1(\rho_1)\mathcal{G}_{0,0}(\rho_2) + \frac{1}{4}\mathcal{G}_1(\rho_1)\mathcal{G}_{0,1}(\rho_2) + \frac{1}{8}\mathcal{G}_0(\rho_2)\mathcal{G}_{1,0}(\rho_1) \\
& - \frac{1}{8}\mathcal{G}_1(\rho_2)\mathcal{G}_{1,0}(\rho_1) + \frac{1}{2}\mathcal{G}_1(\rho_1)\mathcal{G}_{1,0}(\rho_2) - \frac{1}{4}\mathcal{G}_{\rho_2}(\rho_1)\mathcal{G}_{1,0}(\rho_2) \\
& - \frac{1}{4}\mathcal{G}_0(\rho_2)\mathcal{G}_{1,1}(\rho_1) + \frac{1}{4}\mathcal{G}_1(\rho_2)\mathcal{G}_{1,1}(\rho_1) - \frac{1}{2}\mathcal{G}_1(\rho_1)\mathcal{G}_{1,1}(\rho_2) \\
& - \frac{1}{4}\mathcal{G}_0(\rho_2)\mathcal{G}_{1,\rho_2}(\rho_1) + \frac{1}{4}\mathcal{G}_1(\rho_2)\mathcal{G}_{1,\rho_2}(\rho_1) + \frac{1}{4}\mathcal{G}_1(\rho_2)\mathcal{G}_{\rho_2,0}(\rho_1) \\
& + \frac{1}{4}\mathcal{G}_0(\rho_2)\mathcal{G}_{\rho_2,1}(\rho_1)
\end{aligned}
$$

An NLLA Perturbative Coefficient

$$
\begin{aligned}
\mathfrak{gb}^{(2;0,0)}_{1,-++}(\rho_1,\rho_2) = &-\frac{1}{4}\mathcal{G}_{0,0,1}(\rho_2)+\frac{1}{8}\mathcal{G}_{0,1,0}(\rho_2)-\frac{1}{4}\mathcal{G}_{0,1,1}(\rho_1)+\frac{3}{8}\mathcal{G}_{0,1,1}(\rho_2)\\
&+\frac{1}{8}\mathcal{G}_{0,1,\rho_2}(\rho_1)-\frac{1}{8}\mathcal{G}_{0,\rho_2,1}(\rho_1)-\frac{1}{8}\mathcal{G}_{1,0,1}(\rho_1)+\frac{1}{8}\mathcal{G}_{1,0,1}(\rho_2)\\
&-\frac{1}{8}\mathcal{G}_{1,1,0}(\rho_2)-\frac{1}{4}\mathcal{G}_{1,1,1}(\rho_2)+\frac{1}{8}\mathcal{G}_{1,\rho_2,1}(\rho_1)+\frac{1}{8}\mathcal{G}_{\rho_2,0,1}(\rho_1)\\
&+\frac{1}{4}\mathcal{G}_{\rho_2,1,1}(\rho_1)-\frac{1}{8}\mathcal{G}_{\rho_2,1,\rho_2}(\rho_1)-\frac{1}{8}\mathcal{G}_{1}(\rho_2)\,\mathcal{G}_{\rho_2,1}(\rho_1)\\
&+\frac{1}{8}\mathcal{G}_{0}(\rho_2)\,\mathcal{G}_{0,1}(\rho_1)+\frac{1}{8}\mathcal{G}_{1}(\rho_2)\,\mathcal{G}_{0,1}(\rho_1)+\frac{1}{8}\mathcal{G}_{0}(\rho_1)\,\mathcal{G}_{0,1}(\rho_2)\\
&-\frac{1}{8}\mathcal{G}_{1}(\rho_1)\,\mathcal{G}_{0,1}(\rho_2)+\frac{1}{8}\mathcal{G}_{1}(\rho_2)\,\mathcal{G}_{0,\rho_2}(\rho_1)+\frac{1}{8}\mathcal{G}_{1}(\rho_2)\,\mathcal{G}_{1,0}(\rho_1)\\
&-\frac{1}{8}\mathcal{G}_{0}(\rho_1)\,\mathcal{G}_{1,0}(\rho_2)+\frac{1}{8}\mathcal{G}_{\rho_2}(\rho_1)\,\mathcal{G}_{1,0}(\rho_2)-\frac{1}{4}\mathcal{G}_{0}(\rho_1)\,\mathcal{G}_{1,1}(\rho_2)\\
&+\frac{1}{8}\mathcal{G}_{1}(\rho_1)\,\mathcal{G}_{1,1}(\rho_2)+\frac{1}{8}\mathcal{G}_{\rho_2}(\rho_1)\,\mathcal{G}_{1,1}(\rho_2)-\frac{1}{8}\mathcal{G}_{1}(\rho_2)\,\mathcal{G}_{1,\rho_2}(\rho_1)\\
&-\frac{1}{8}\mathcal{G}_{1}(\rho_2)\,\mathcal{G}_{\rho_2,0}(\rho_1)-\frac{1}{8}\mathcal{G}_{0}(\rho_2)\,\mathcal{G}_{\rho_2,1}(\rho_1)
\end{aligned}
$$

An NLLA Perturbative Coefficient

$$
\begin{aligned}
\mathfrak{gb}^{(2;0,0)}_{2,-++}(\rho_1,\rho_2) =& \frac{3}{8}\mathcal{G}_{0,0,0}(\rho_1) - \frac{3}{8}\mathcal{G}_{0,0,0}(\rho_2) + \frac{1}{4}\mathcal{G}_{0,0,1}(\rho_2) - \frac{1}{8}\mathcal{G}_{0,0,\rho_2}(\rho_1) \\
& - \frac{1}{4}\mathcal{G}_{0,1,0}(\rho_1) + \frac{3}{8}\mathcal{G}_{0,1,0}(\rho_2) + \frac{1}{4}\mathcal{G}_{0,1,1}(\rho_1) - \frac{1}{4}\mathcal{G}_{0,1,1}(\rho_2) \\
& - \frac{1}{8}\mathcal{G}_{0,\rho_2,0}(\rho_1) + \frac{1}{4}\mathcal{G}_{0,\rho_2,\rho_2}(\rho_1) - \frac{1}{4}\mathcal{G}_{1,0,0}(\rho_1) + \frac{1}{4}\mathcal{G}_{1,0,0}(\rho_2) \\
& + \frac{1}{8}\mathcal{G}_{1,0,1}(\rho_1) - \frac{1}{4}\mathcal{G}_{1,0,1}(\rho_2) + \frac{1}{8}\mathcal{G}_{1,0,\rho_2}(\rho_1) + \frac{1}{2}\mathcal{G}_{1,1,0}(\rho_1) \\
& + \frac{1}{8}\mathcal{G}_{\rho_2,0,0}(\rho_1) + \frac{1}{8}\mathcal{G}_{\rho_2,0,1}(\rho_1) - \frac{1}{8}\mathcal{G}_{\rho_2,0,\rho_2}(\rho_1) + \frac{1}{2}\zeta_2\mathcal{G}_0(\rho_1) - \zeta_3 \\
& - \frac{1}{8}\mathcal{G}_0(\rho_2)\mathcal{G}_{0,0}(\rho_1) + \frac{1}{8}\mathcal{G}_1(\rho_2)\mathcal{G}_{0,0}(\rho_1) + \frac{1}{4}\mathcal{G}_0(\rho_1)\mathcal{G}_{0,0}(\rho_2) \\
& - \frac{1}{8}\mathcal{G}_1(\rho_1)\mathcal{G}_{0,0}(\rho_2) + \frac{1}{8}\mathcal{G}_{\rho_2}(\rho_1)\mathcal{G}_{0,0}(\rho_2) - \frac{1}{4}\mathcal{G}_1(\rho_2)\mathcal{G}_{0,1}(\rho_1) \\
& - \frac{1}{4}\mathcal{G}_0(\rho_1)\mathcal{G}_{0,1}(\rho_2) + \frac{1}{8}\mathcal{G}_1(\rho_1)\mathcal{G}_{0,1}(\rho_2) - \frac{1}{8}\mathcal{G}_{\rho_2}(\rho_1)\mathcal{G}_{0,1}(\rho_2) \\
& + \frac{1}{4}\mathcal{G}_0(\rho_2)\mathcal{G}_{0,\rho_2}(\rho_1) - \frac{1}{4}\mathcal{G}_1(\rho_2)\mathcal{G}_{0,\rho_2}(\rho_1) + \frac{1}{8}\mathcal{G}_0(\rho_2)\mathcal{G}_{1,0}(\rho_1) \\
& - \frac{1}{4}\mathcal{G}_1(\rho_2)\mathcal{G}_{1,0}(\rho_1) - \frac{1}{4}\mathcal{G}_0(\rho_1)\mathcal{G}_{1,0}(\rho_2) + \frac{1}{2}\mathcal{G}_0(\rho_1)\mathcal{G}_{1,1}(\rho_2) \\
& - \frac{1}{8}\mathcal{G}_0(\rho_2)\mathcal{G}_{\rho_2,0}(\rho_1)
\end{aligned}
$$

Conclusion

- Higher loop and leg amplitudes at LLA in MRK in Planar $\mathcal{N} = 4$ SYM can be computed easily using convolutions of simple building blocks.
- Beyond LLA the formalism still holds. It has been applied as follows:
  - All MHV 5-loop amplitudes at LLA and 8-point LLA amplitudes for any helicity configuration up to 4 loops [1606.08807].
  - 7-point NLLA MHV up to 4 and NMHV up to 3 loops [1801.10605].
  - 8-point NLLA up to 3-loops for any helicity configuration [Marzucca, BV, (In Progress)].
- The understanding of the function space of MRK leads to great computational simplifications.

Thank you