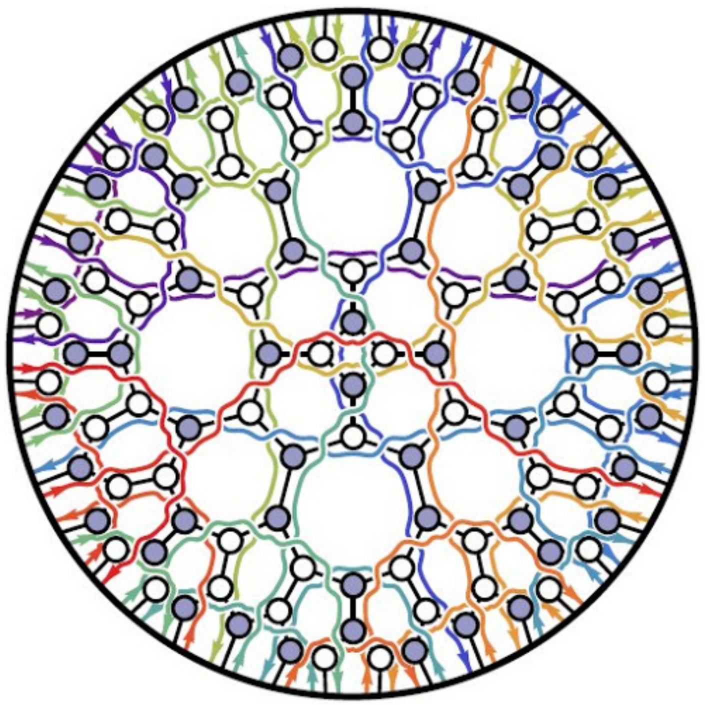
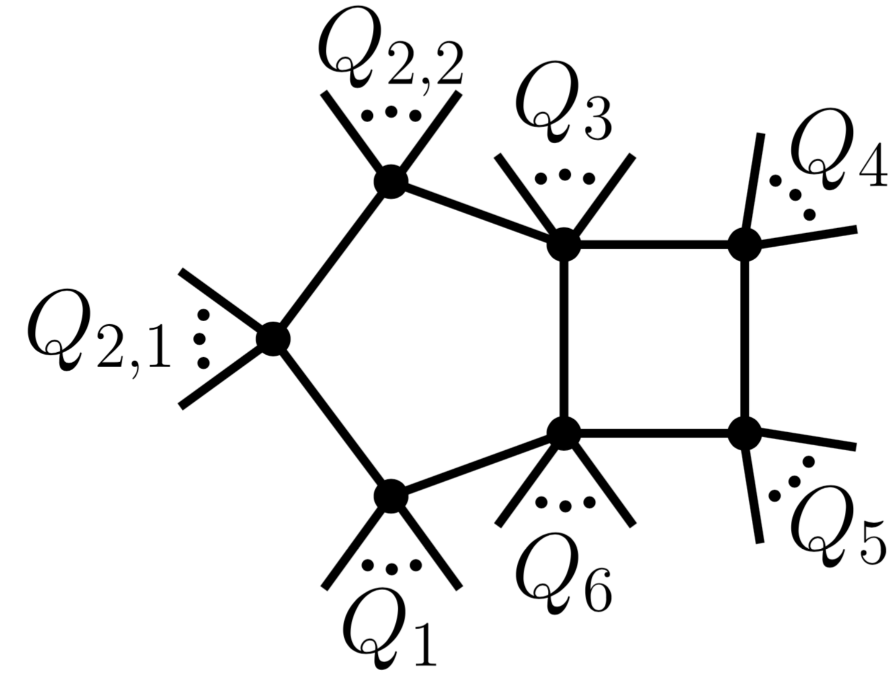




Enrico Herrmann

SLAC NATIONAL ACCELERATOR LABORATORY

In collaboration with: Jaroslav Trnka

+ work in progress + Alex Edison, Cameron Langer, Julio Parra-Martinez

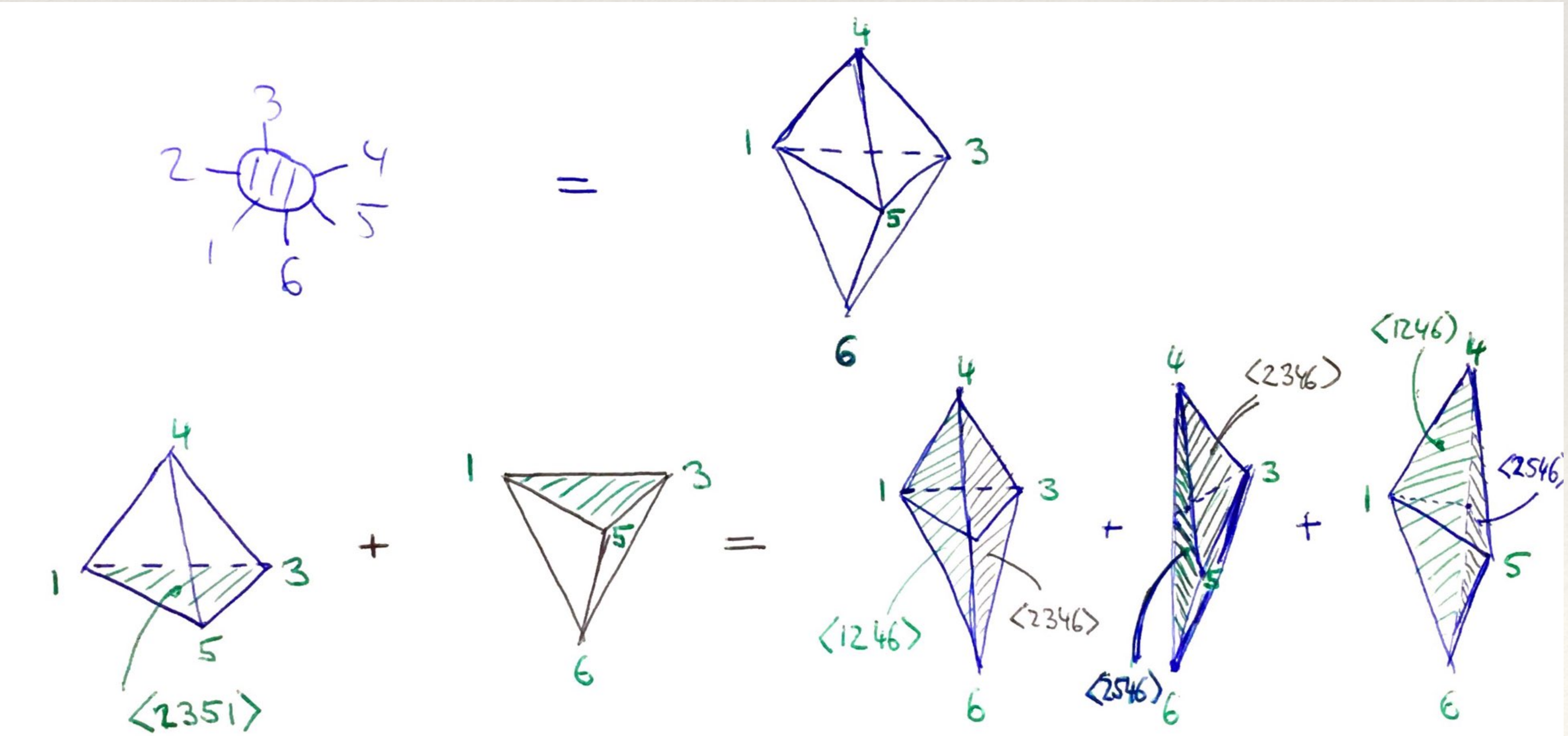
Loop integrands in N=4 sYM and N=8 sugra

The Galileo Galilei Institute
For Theoretical Physics
10/31/2018

(0) Motivation

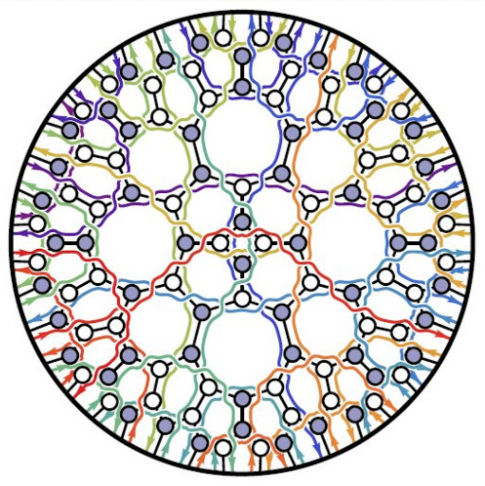
- grand idea: reformulate QFT: replace unitarity & locality by new mathematical principles
- 1^{st} hint Hodges: 6pt tree-amp = volume of polyhedron in $\mathbb{P}^3$

$$\frac{\langle 1345\rangle^3}{\langle 1234\rangle\langle 1245\rangle\langle 2345\rangle\langle 2351\rangle}+\frac{\langle 1356\rangle^3}{\langle 1235\rangle\langle 1256\rangle\langle 2356\rangle\langle 2361\rangle}=\frac{\langle 1346\rangle^3}{\langle 1234\rangle\langle 1236\rangle\langle 1246\rangle\langle 2346\rangle}+\frac{\langle 3456\rangle^3}{\langle 2345\rangle\langle 2356\rangle\langle 2346\rangle\langle 2546\rangle}+\frac{\langle 5146\rangle^3}{\langle 1245\rangle\langle 1256\rangle\langle 1246\rangle\langle 2546\rangle}$$



(0) Motivation

- fascinating interplay between physics & geometry in scattering amplitudes
- novel geometric structures primarily in planar N=4 sYM:
  - Grassmannian [space of k-planes in n-dim]
    [Arkani-Hamed,Bourjaily,Cachazo,Goncharov,Postnikov,Trnka]
  - Amplituhedron
    [Arkani-Hamed,Trnka]
- What about other theories?
  - φ^3-theory: Associahedron [Arkani-Hamed,Bai,He,Yan]
  - nonplanar YM? [Bern,Litsey,Stankowicz,EH,Trnka]
  - gravity? [EH,Trnka] + work in progress [Edison,EH,Langer,Parra-Martinez,Trnka]
  - N<4 sYM? work in progress [EH,Langer,Trnka]

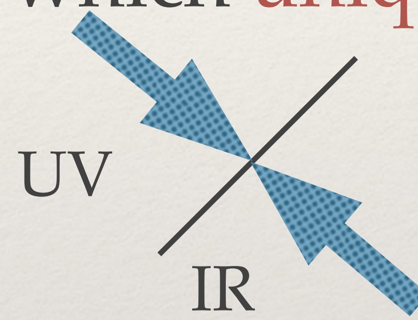


(0) Motivation

- comparison planar N=4 sYM, nonplanar sYM, gravity

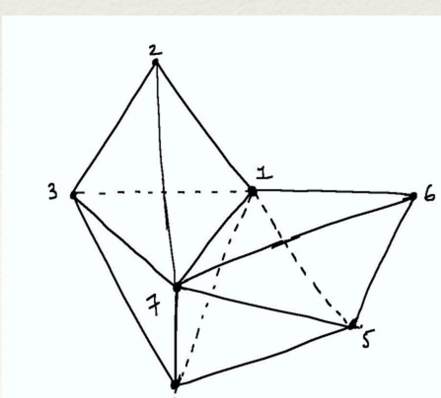
planar N=4 sYM	nonplanar N=4 sYM	gravity

- identify homogeneous properties which uniquely fix amplitude
  - constrain UV & IR



dlog-forms; no poles at infinity		What are the gravity properties?

- reformulate constraints as inequalities that define geometry

	?	?

(1) Outline

- i) setting the stage:
  amplitudes, integrands, cuts and on-shell diagrams
- ii) properties of on-shell (OS) diagrams
- iii) from OS-diagrams to properties of amplitudes
- iv) Gravity
  - IR - properties [EH,Trnka '16]
  - UV - properties [EH,Trnka '18] <— focus on this part
  - Fixing the amplitude in progress [Edison,EH,Langer,Parra-Martinez,Trnka]
- v) Conclusions

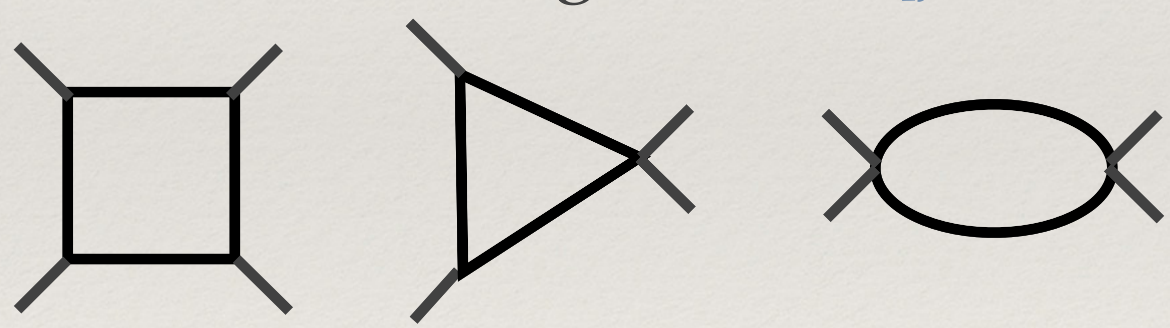
i) loop-amplitudes

- loop-amplitudes in 4d:

kinematic coefficients

$$\mathscr{A}^{(L)} = \sum_k c_k \int \mathscr{I}_k \, d^4\ell_1 \cdots d^4\ell_L$$

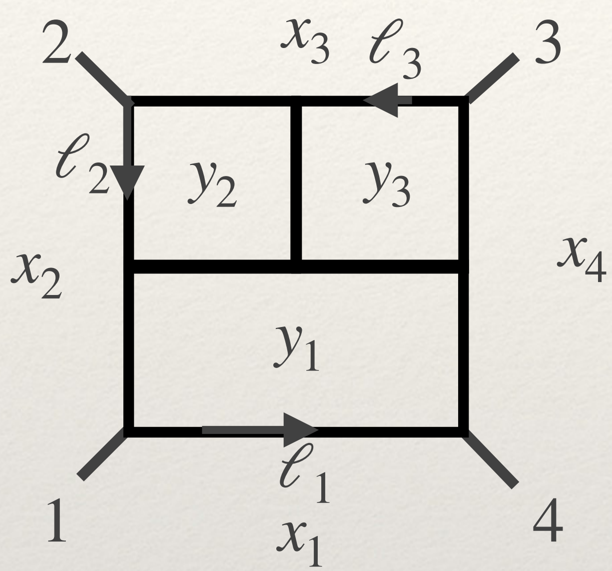
basis integrands [Jake's talk]



- generalized unitarity: match amplitude on cuts —> fix c's

i) planar integrand

- planar integrand $\Leftrightarrow$ unambiguous labels!



$$p_i^\mu = (x_{i+1}^\mu - x_i^\mu)$$

$$\ell_i^\mu = (y_i^\mu - x_i^\mu)$$

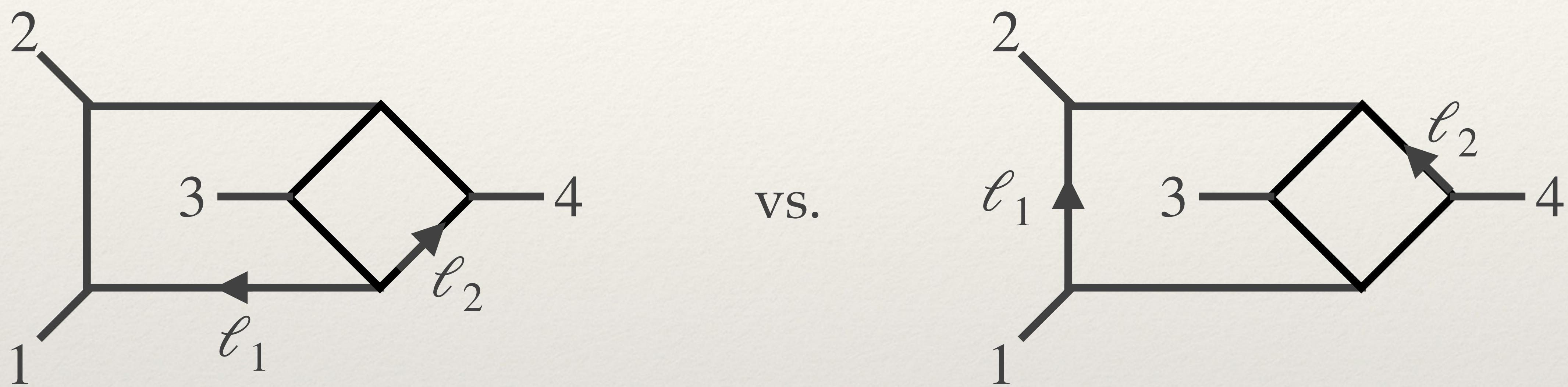
dual-variables

$$\mathscr{A}^{(L)} = \sum_k c_k \int \mathcal{I}_k \, d^4\ell_1 \cdots d^4\ell_L = \int \mathcal{I} \, d^4y_1 \cdots d^4y_L$$

- well-defined notion of an integrand
  - rational function
  - properties of integrated answer encoded in $\mathcal{I}$

i) ambiguity in non-planar integrands

- no global loop-variables in nonplanar diagrams:



- no global definition of an integrand —> stick with diagrams

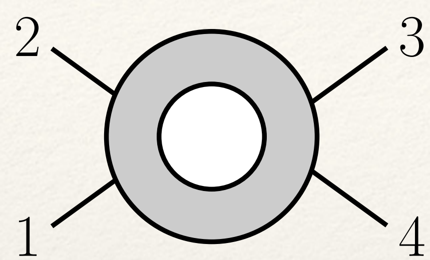
$$\mathscr{A}^{(L)} = \sum_k c_k \int \mathscr{I}_k \, d^4\ell_1 \cdots d^4\ell_L$$

- expansion objects for:
    - non-planar YM
    - gravity

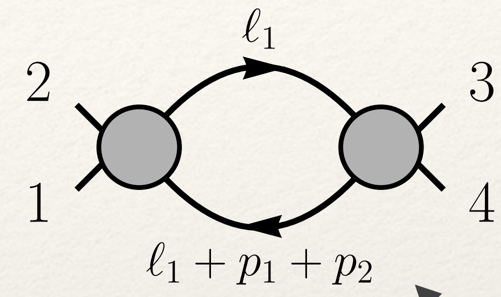
Is there a way out?

i) cuts of loop-integrands

- unitarity cut:



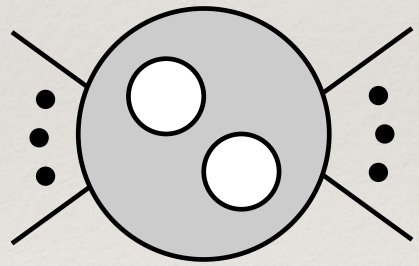
$$\ell_1^2 = (\ell_1 + p_1 + p_2)^2 = 0$$



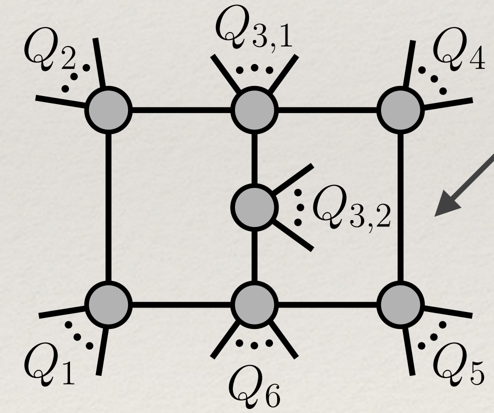
$$\operatorname*{Res}_{\ell_1^2=0=(\ell_1+1+2)^2} \mathscr{A}^{(1)}(1234) = \sum_{\text{states}} \mathscr{A}_L^{(0)} \times \mathscr{A}_R^{(0)}$$

on-shell functions

- generalized unitarity:



$$\ell_1^2 = \cdots = \ell_8^2 = 0$$

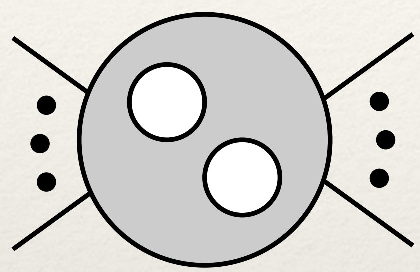


$$\operatorname*{Res}_{\ell_i^2=0} \mathscr{A}^{(2)} = \sum_{\text{states}} \mathscr{A}_1^{(0)} \times \cdots \times \mathscr{A}_7^{(0)}$$

- well-define loop-variables on cuts!

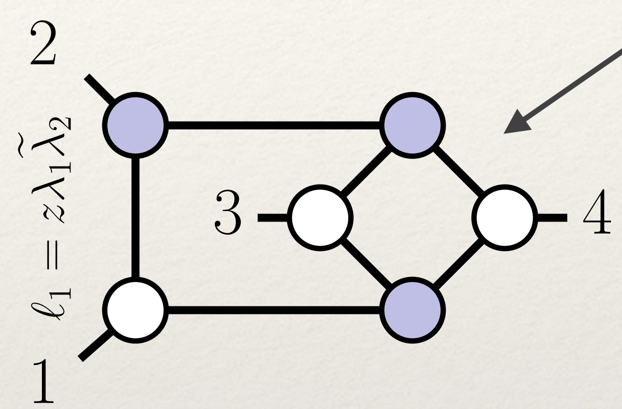
i) on-shell diagrams

- generalized unitarity:



$$\ell_1^2 = \cdots = \ell_7^2 = 0$$

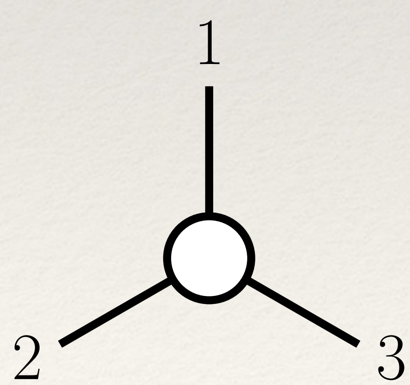
on-shell diagram



$$\underset{\ell_i^2=0}{\mathrm{Res}}\ \mathscr{A}^{(2)}(1234) = \sum_{\text{states}} \mathscr{A}_1^{(0)} \times \cdots \times \mathscr{A}_6^{(0)} = f(z; \lambda_i, \tilde{\lambda}_i)$$

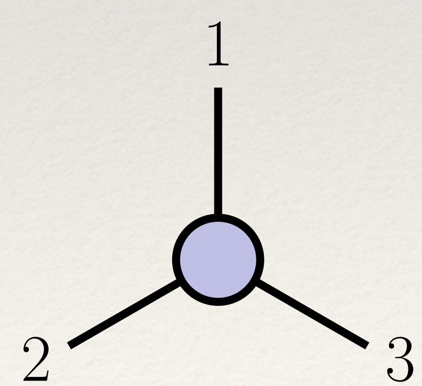
- elementary building blocks:

$\mathscr{A}_3^{\overline{\mathrm{MHV}}}$:



$\lambda_1 \sim \lambda_2 \sim \lambda_3$

$\mathscr{A}_3^{\mathrm{MHV}}$:



$\tilde{\lambda}_1 \sim \tilde{\lambda}_2 \sim \tilde{\lambda}_3$

ii) Grassmannian and on-shell diagrams

- fascinating connection between physics and mathematics

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High Energy Physics – Theory

Scattering Amplitudes and the Positive Grassmannian

Nima Arkani-Hamed, Jacob L. Bourjaily, Freddy Cachazo, Alexander B. Goncharov, Alexander Postnikov, Jaroslav Trnka

(Submitted on 21 Dec 2012 (v1), last revised 17 Mar 2014 (this version, v2))

We establish a direct connection between scattering amplitudes in planar four-dimensional theories and a remarkable mathematical structure known as the positive Grassmannian. The central physical idea is to focus on on-shell diagrams as objects of fundamental importance to scattering amplitudes. We show that the all-loop integrand in N=4 SYM is naturally represented in this way. On-shell diagrams in this theory are intimately tied to a variety of mathematical objects, ranging from a new graphical representation of permutations to a beautiful stratification of the Grassmannian G(k,n) which generalizes the notion of a simplex in projective space. All physically important operations involving on-shell diagrams map to canonical operations on permutations; in particular, BCFW deformations correspond to adjacent transpositions. Each cell of the positive Grassmannian is naturally endowed with positive coordinates and an invariant measure which determines the on-shell function associated with the diagram. This understanding allows us to classify and compute all on-shell diagrams, and give a geometric understanding for all the non-trivial relations among them. Yangian invariance of scattering amplitudes is transparently represented by diffeomorphisms of G(k,n) which preserve the positive structure. Scattering amplitudes in (1+1)-dimensional integrable systems and the ABJM theory in (2+1) dimensions can both be understood as special cases of these ideas. On-shell diagrams in theories with less (or no) supersymmetry are associated with exactly the same structures in the Grassmannian, but with a measure deformed by a factor encoding ultraviolet singularities. The Grassmannian representation of on-shell processes also gives a new understanding of the all-loop integrand for scattering amplitudes, presenting all integrands in a novel dLog form which directly reflects the underlying positive structure.

Comments: a handful of minor corrections and citations added/updated; 158 pages, 264 figures
Subjects: **High Energy Physics – Theory (hep-th)**; Algebraic Geometry (math.AG); Combinatorics (math.CO)
Cite as: **arXiv:1212.5605 [hep-th]**
(or **arXiv:1212.5605v2 [hep-th]** for this version)

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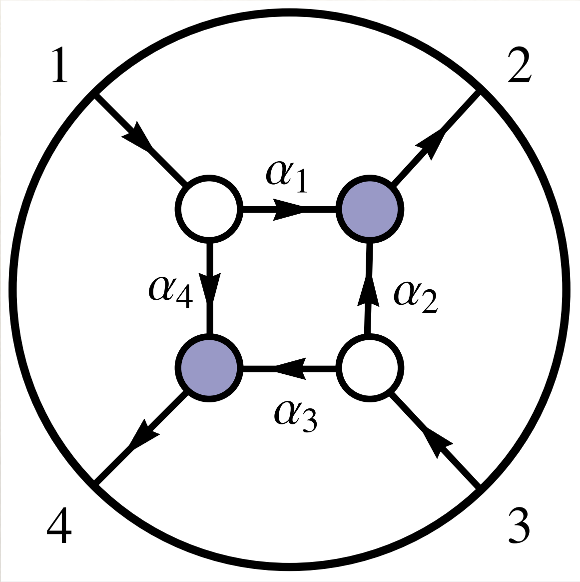
- connection to algebraic geometry, combinatorics, …

ii) Grassmannian and on-shell diagrams

- planar diagrams in mathematics: building matrices with positive minors

$$\mathrm{Gr}_{\geq}(k,n) \simeq \{[(k\times n)\text{ matrices}]/\mathrm{GL}(k)\,|\,\text{ordered }(k\times k)\text{ minors}\geq 0\}$$



$$\leftrightarrow \qquad C = \begin{pmatrix} 1 & \alpha_1 & 0 & -\alpha_4 \\ 0 & \alpha_2 & 1 & \alpha_3 \end{pmatrix}, \qquad \alpha_i > 0$$

k : helicity-sector / R-charge

n : # external legs

- connection to physics: value of N=4 sYM OS-diag is

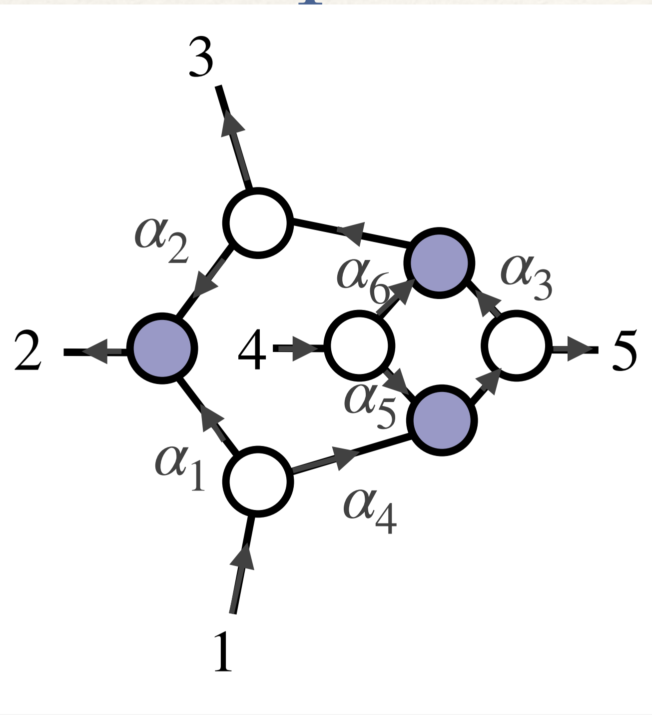
[Arkani-Hamed,Bourjaily,Cachazo,Goncharov,Postnikov,Trnka]

$$\Omega^{\mathcal{N}=4\mathrm{sYM}} = \frac{d\alpha_1}{\alpha_1}\cdots\frac{d\alpha_r}{\alpha_r}\,\delta(C\cdot\mathcal{Z})$$

all external kinematics

ii) Grassmannian and on-shell diagrams

- non-planar diagrams —> give up positivity



$$Gr(k,n) \simeq \{[(k \times n)\text{ matrices}]/GL(k)\}$$

$$\leftrightarrow \quad C = \begin{pmatrix} 1 & \alpha_1 + \alpha_2\alpha_3\alpha_4 & \alpha_3\alpha_4 & 0 & \alpha_4 \\ 0 & \alpha_2(\alpha_6 + \alpha_3\alpha_5) & (\alpha_6 + \alpha_3\alpha_5) & 1 & \alpha_5 \end{pmatrix}$$

k : helicity-sector / R-charge

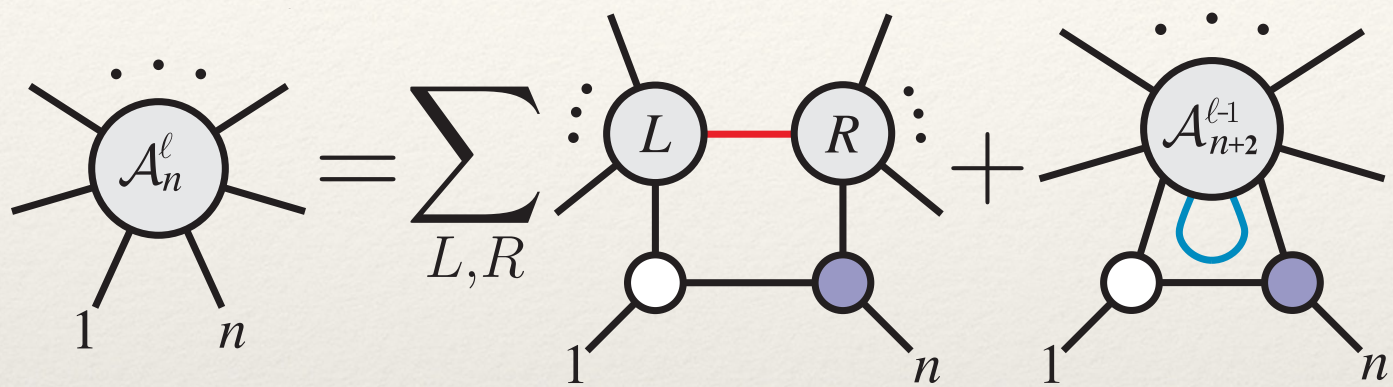
n : # external legs

- connection to physics: value of N=8 sugra OS-diag is [EH, Trnka]

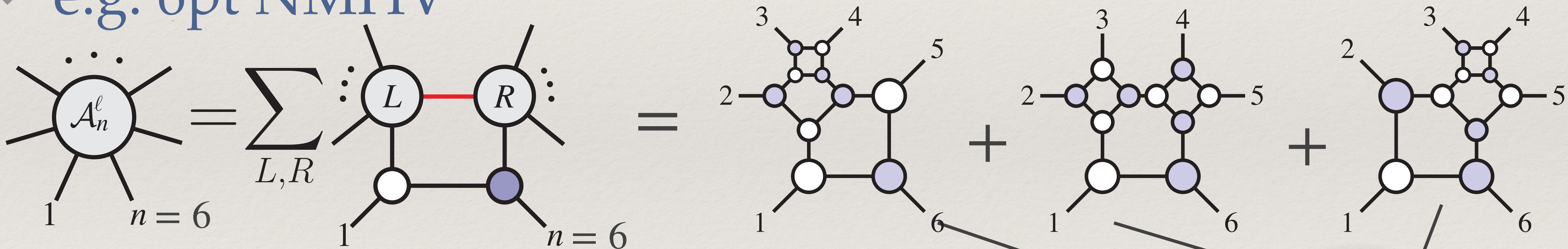
$$\Omega^{\mathcal{N}=8 \text{ sugra}} = \left[\frac{d\alpha_1}{\alpha_1^3} \cdots \frac{d\alpha_r}{\alpha_r^3} \prod_v \Delta_v\right] \delta(C \cdot \mathcal{Z})$$

iii) from OS-diags to amplitudes

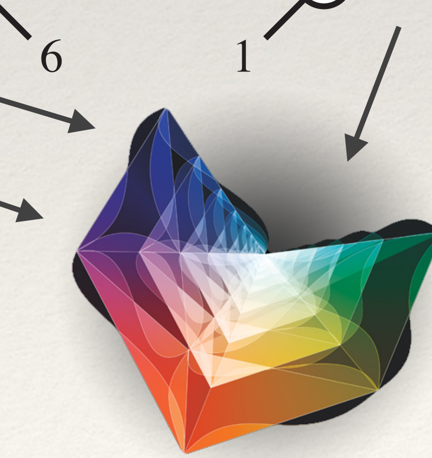
- planar N=4 sYM —> BCFW loop-recursion relations



- e.g. 6pt NMHV



- amplitudes inherit properties of OS-diags!



- theories where BCFW-loop recursion unknown:
  OS-diags <—> cuts of loop integrands: encode properties of amplitude

iii-1) from OS-diags to amplitudes: YM

- N=4 sYM (planar & non-planar)
- IR-property: logarithmic singularities!

$$\Omega^{\mathcal{N}=4\text{sYM}} = \frac{d\alpha_1}{\alpha_1} \cdots \frac{d\alpha_r}{\alpha_r}\ \delta(C \cdot \mathcal{Z})$$

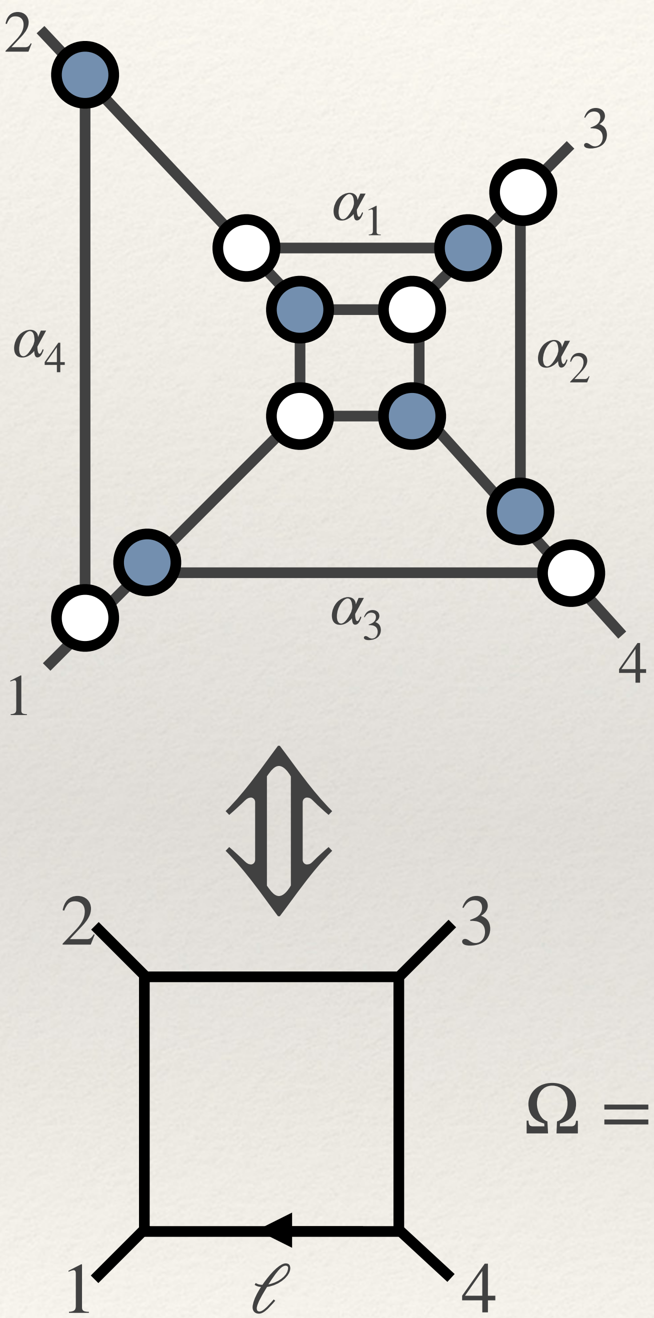
all external kinematics

- IR-condition on analytic properties of amplitudes:

$$\mathcal{A} \sim \frac{dx}{x-a} R(x, \ldots) \qquad , \text{ as } x \to a \text{ (singular point)}$$

- nontrivial constraints on possible local integrand basis elements!

interlude: Feynman integrals in dlog-form



$$\Omega = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4} \times \mathscr{A}_4^{\text{tree}} \times \delta(C\cdot\mathscr{Z})$$

logarithmic form in Grassmannian variables!

can identify and solve for Feynman loop variables ℓ^μ

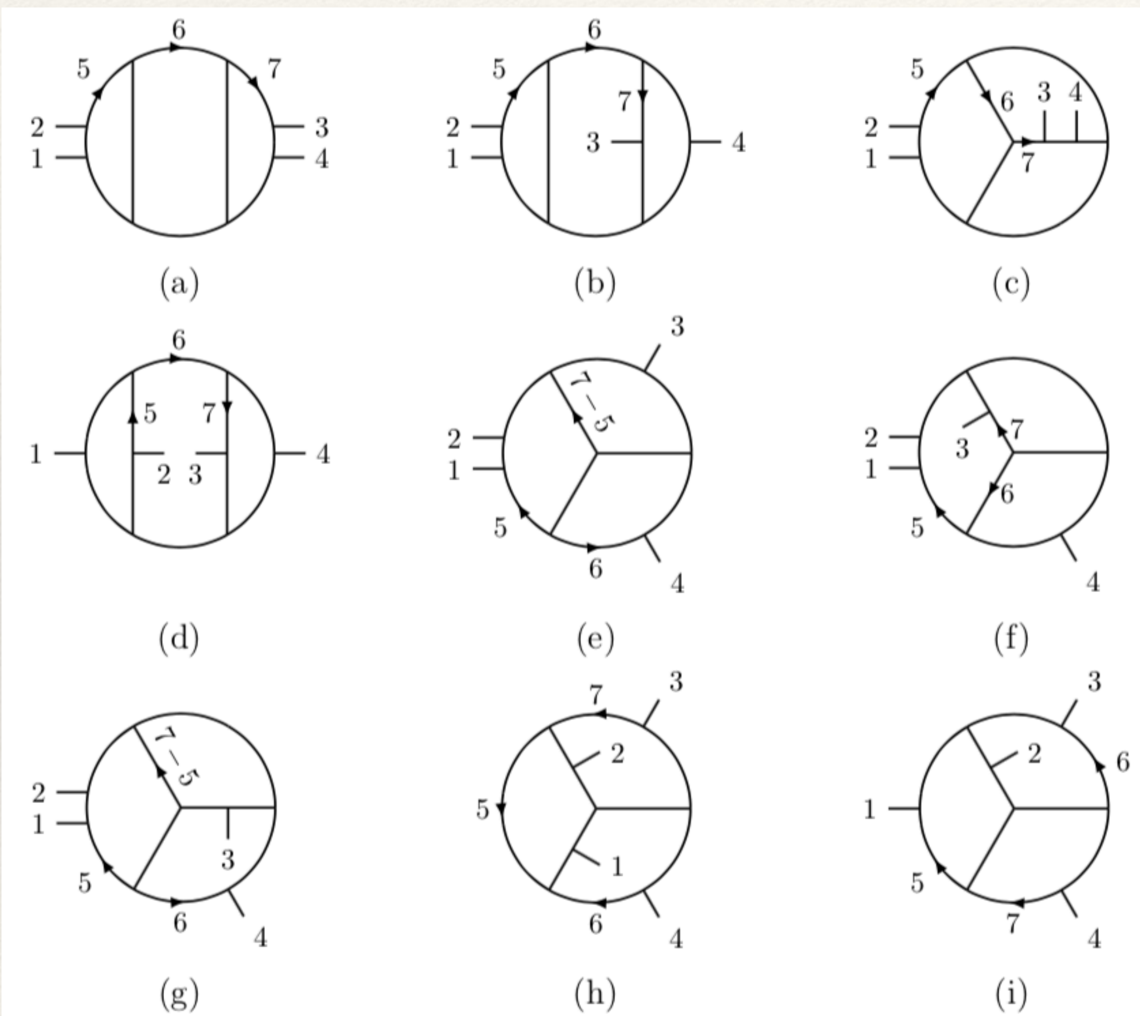
Arkani-Hamed,Cachazo,Goncharov,Postnikov,Trnka: 1212.5605

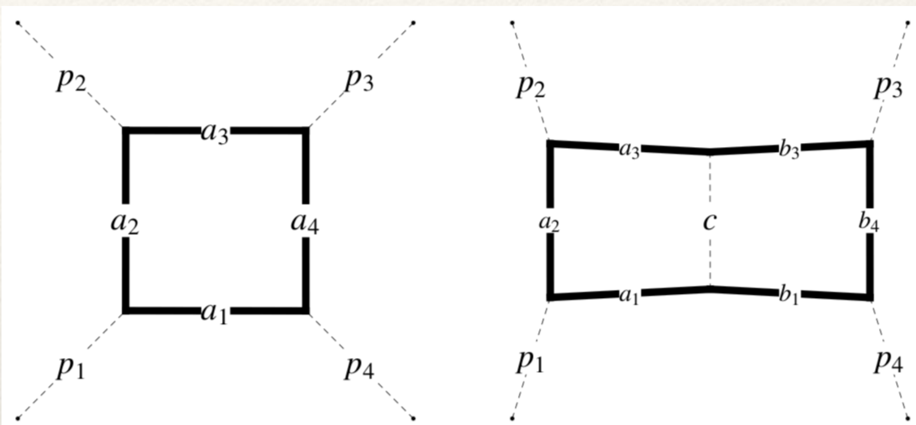
$$\Omega = d\log\frac{\ell^2}{(\ell-\ell^*)^2}\,d\log\frac{(\ell-p_1)^2}{(\ell-\ell^*)^2}\,d\log\frac{(\ell-p_1-p_2)^2}{(\ell-\ell^*)^2}\,d\log\frac{(\ell+p_4)^2}{(\ell-\ell^*)^2}$$

new representation of Feynman integrals

dlog-representation exists for more general FI

Bern,EH,Litsey,Stankowicz,Trnka: 1412.8584, 1512.08591



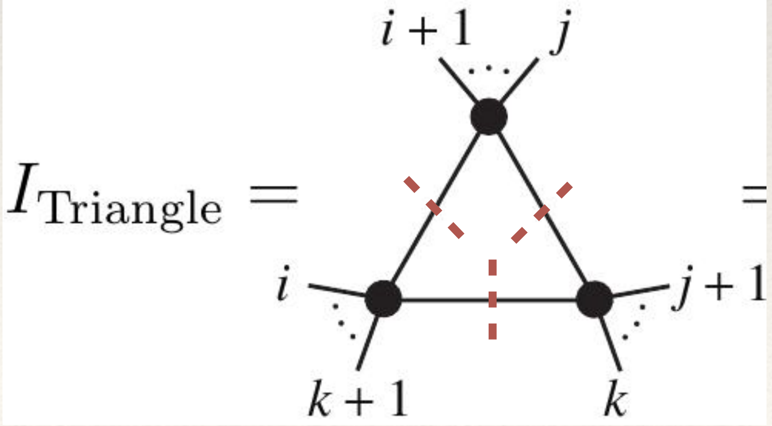


dlog forms exist for special integrals

- related to UT conjecture of $\mathcal{N} = 4$ sYM
- basis of integrals for Henn diff. eqs.
- new symmetries of nonplanar theories?
- potential geometric interpretation?

iii-2) from OS-diags to amplitudes: YM

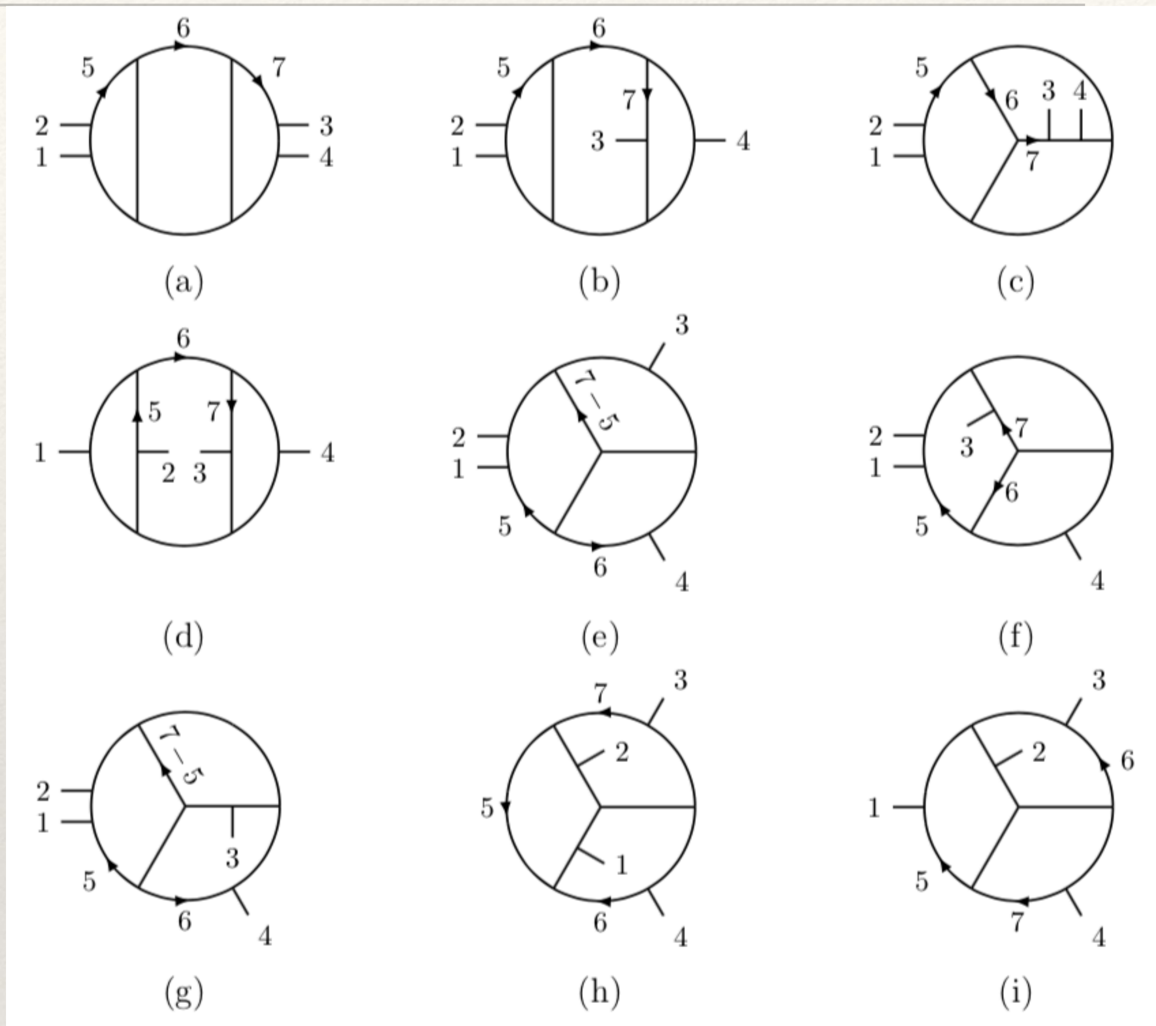
- N=4 sYM (planar & non-planar)
- UV-property: no poles at infinity!
  - planar: manifest in terms of mom. twistors
  - non-planar: need to check in local expansion, term-by-term analysis
- stronger than UV-finiteness, e.g. triangle integral



$$= \sim \frac{dz}{z}, \quad \ell^\mu(z) \sim z, \text{ has Res @ } \ell \to \infty$$

iii-3) uniqueness of YM

- non-planar N=4 sYM
- Combine IR- & UV-properties
  - term-by-term analysis
  - dlog-forms
  - no poles at infinity



- new non-planar symmetry? [Bern,Enciso,Ita,Shen,Zeng; Chicherin,Henn,Sokatchev]

$$\mathscr{A}^{(L)} = \sum_k c_k \int \mathscr{I}_k \, d^4\ell_1 \cdots d^4\ell_L$$

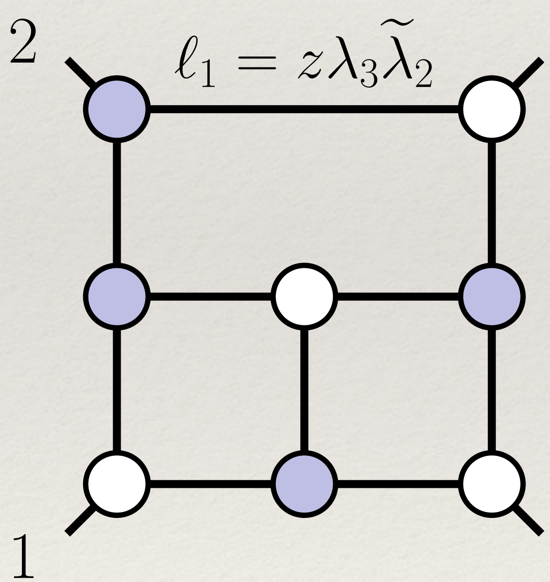
- fix c's with homogeneous cuts: geometric interpretation

[Bern,EH,Litsey,Stankowicz,Trnka]

$$\text{Res}\ \mathscr{A}^{(L)} = 0$$

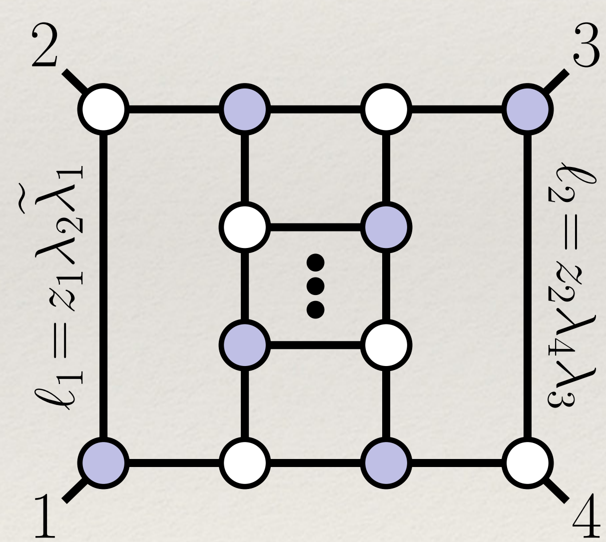
iv) gravity [EH,Trnka]

- Does there exist an analogous story in gravity?
- Gravity is nonplanar —> term-by-term analysis?
  - analytic properties that single out gravity?



$\sim \mathcal{O}(1/z)$

pole at infinity



$\overset{z\gg 1}{\sim} \frac{dz}{z^{4-L}}$

non-logarithmic poles at infinity

drastically different properties than in YM!

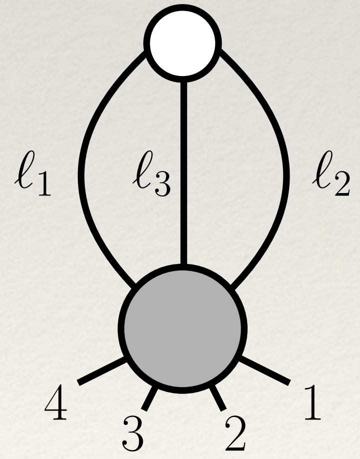
iv-1) gravity in the IR [EH,Trnka]

- Gravity on-shell diagrams:

$$\Omega^{\mathcal{N}=8\text{ sugra}} = \left[\frac{d\alpha_1}{\alpha_1^3}\cdots\frac{d\alpha_r}{\alpha_r^3}\prod_v \Delta_v\right]\delta(C\cdot\mathcal{Z})$$

on-shell diagrams vanish in collinear region

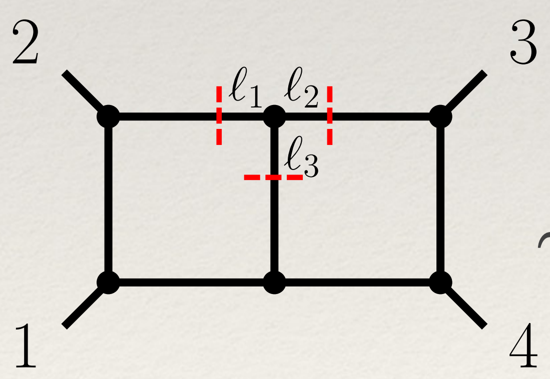
- Gravity on-shell functions, i.e. more general cuts:



near $\langle \ell_1 \ell_2\rangle = 0$:

$$\mathcal{M} \sim \frac{[\ell_1\ell_2]}{\langle \ell_1\ell_2\rangle} \times \text{regular}$$

$\leftrightarrow$

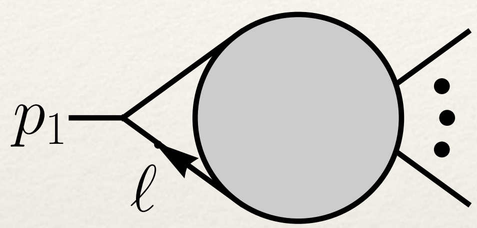


$$\sim \frac{1}{\langle \ell_1\ell_2\rangle[\ell_1\ell_2]}$$

gravity properties are “global” in nature!

iv-2) mild-IR behavior of gravity amplitudes

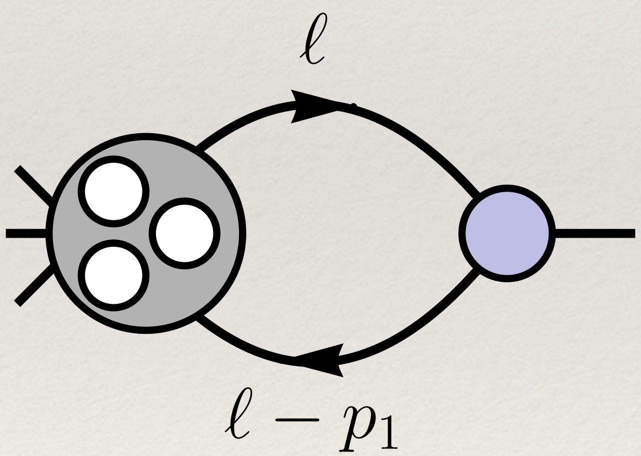
- collinear region of loop momentum:



$$\ell^2 = 0 \Rightarrow (\ell - p_1)^2 = \langle \ell 1 \rangle [\ell 1]$$

$$\ell^2 = 0 = \langle \ell 1 \rangle = [\ell 1] \Rightarrow \boxed{\ell^\mu = \alpha p_1^\mu}$$

- Gravity on-shell functions vanish there!



$$1 \sim \frac{\langle \ell 1 \rangle}{[\ell 1]} \times \text{regular} \quad \xrightarrow{\langle \ell 1 \rangle \to 0} \quad 0$$

nontrivial cancelations even at L=1

homogeneous constraint!

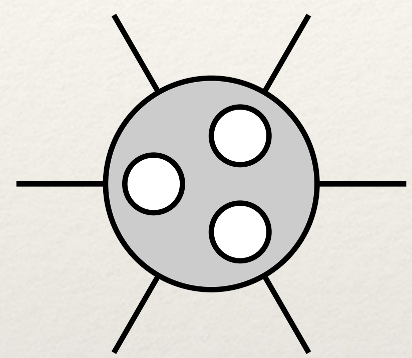
- L=1, 4pt: sum of 6 boxes

$$\mathscr{A}^{(L)} \sim \frac{1}{\epsilon^{2L}} \quad \text{vs.} \quad \mathscr{M}^{(L)} \sim \frac{1}{\epsilon^{L}}$$

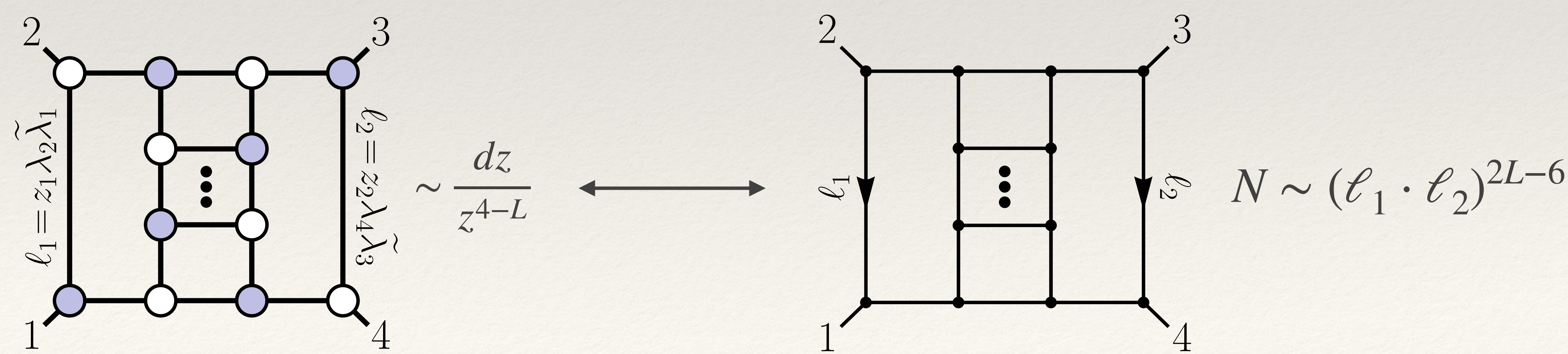
gravity on-shell functions vanish in collinear region <—> soft IR-behavior of Amplitude

iv-3) gravity in the UV

- no off-shell definition of ℓ: no invariant probe of $\ell \to \infty$

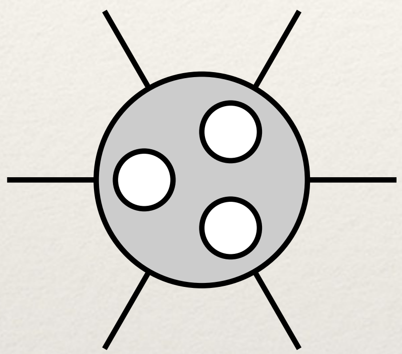


- study cuts that make ℓ well defined, then probe $\ell \to \infty$
- maximal cuts: dictate diagram scaling!



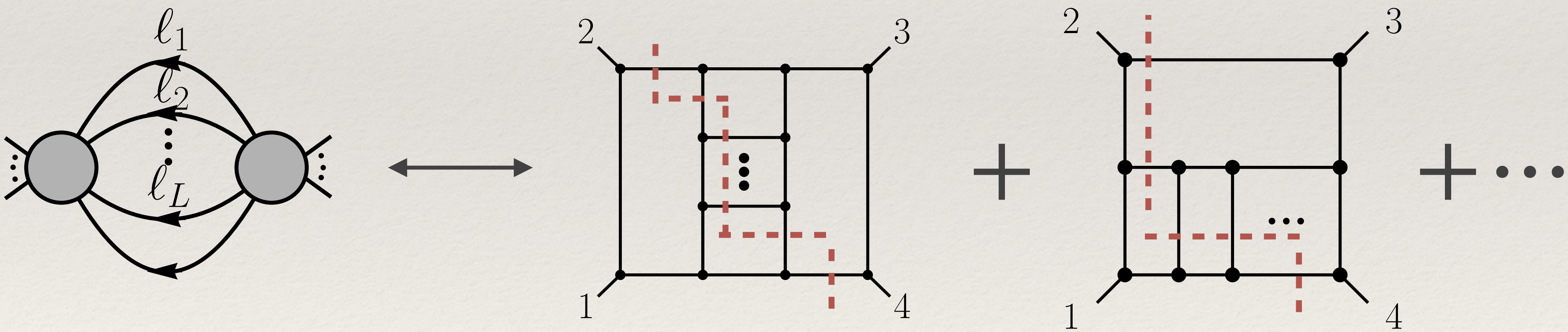
iv-3) gravity in the UV

- Can we do better than maximal cuts?



- get as close as possible to off-shell $\mathscr{I}$

- multi-unitarity cut! L+1 props on-shell

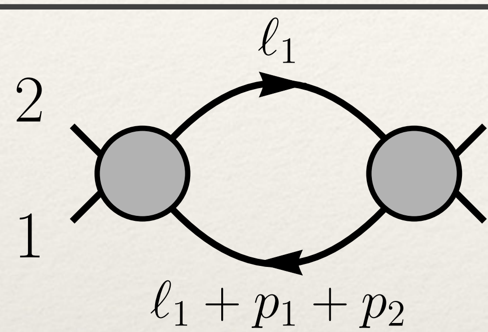


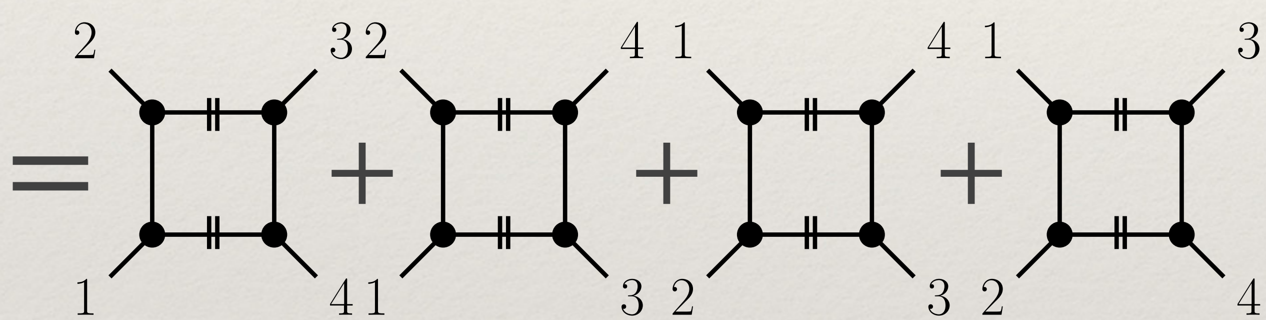
- interesting cancellation when $a < \max(b_i)$ as $\ell_i(z) \overset{z\to\infty}{\to} \infty$

$$\mathscr{M}(z)|_{\text{cut}} \sim z^a = z^{b_1} + z^{b_2} + z^{b_3} + \cdots$$

iv-3) gravity in the UV

- L=1



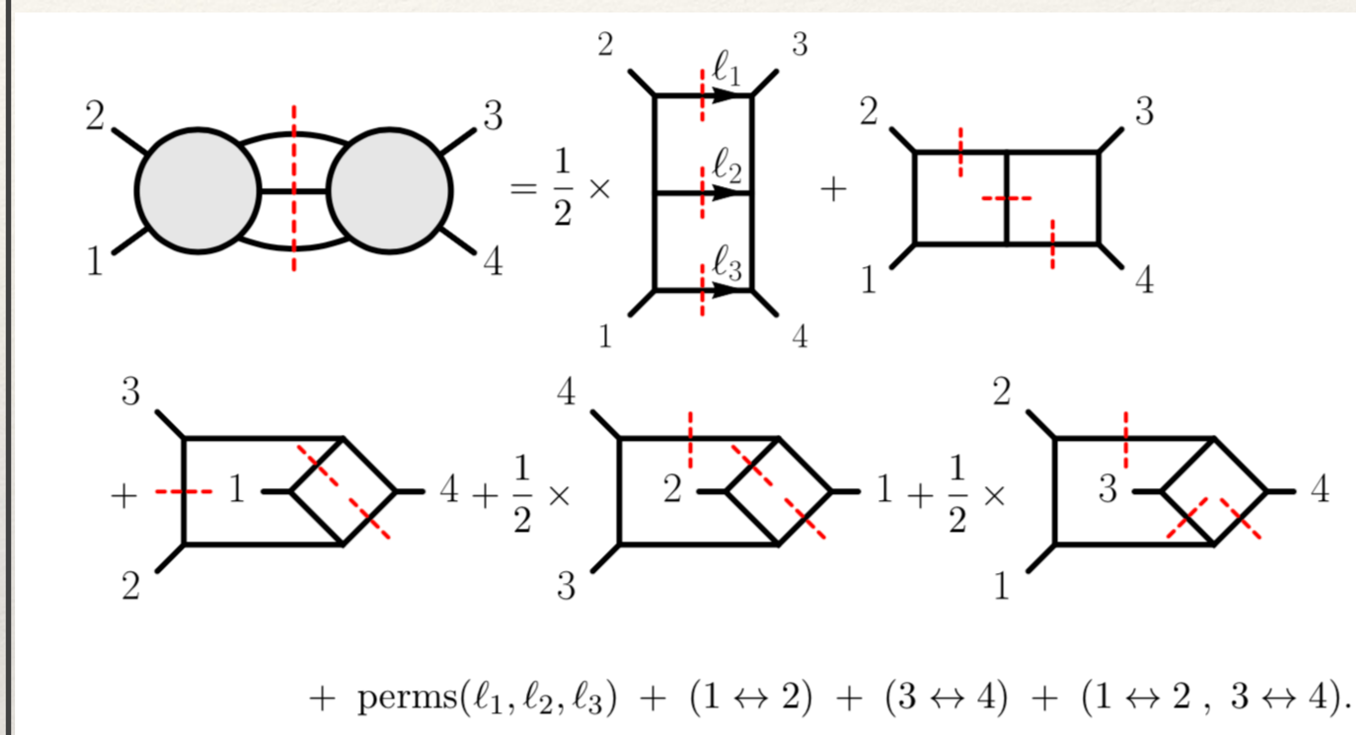


$$\sim \frac{1}{(\ell_1\cdot 2)(\ell_1\cdot 3)}+\frac{1}{(\ell_1\cdot 1)(\ell_1\cdot 3)}+\frac{1}{(\ell_1\cdot 2)(\ell_1\cdot 4)}+\frac{1}{(\ell_1\cdot 1)(\ell_1\cdot 4)}$$

$$= \frac{s_{12}^2}{(\ell_1\cdot 1)(\ell_1\cdot 2)(\ell_1\cdot 3)(\ell_1\cdot 4)}$$

- cancelation in d-dim

- L=2



[Bern,Enciso,Parra-Martinez,Zeng]

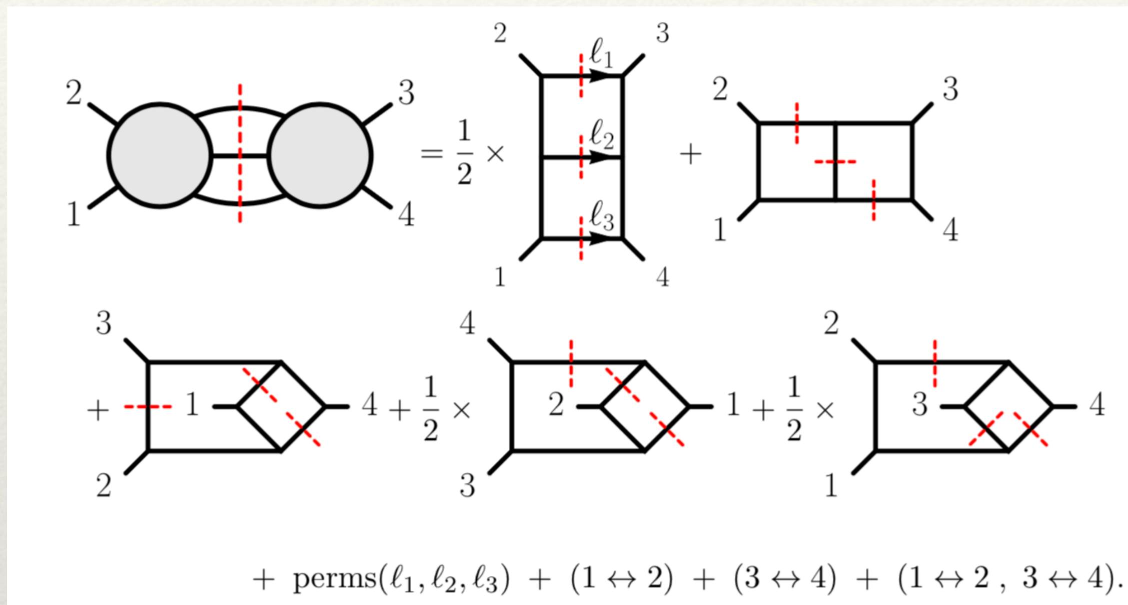
- half-max. sugra in d=5
- no cancelation!

[EH, Trnka]

- d=4 special! spinor-helicity

iv-3) gravity in the UV

- some details about L=2, d=4

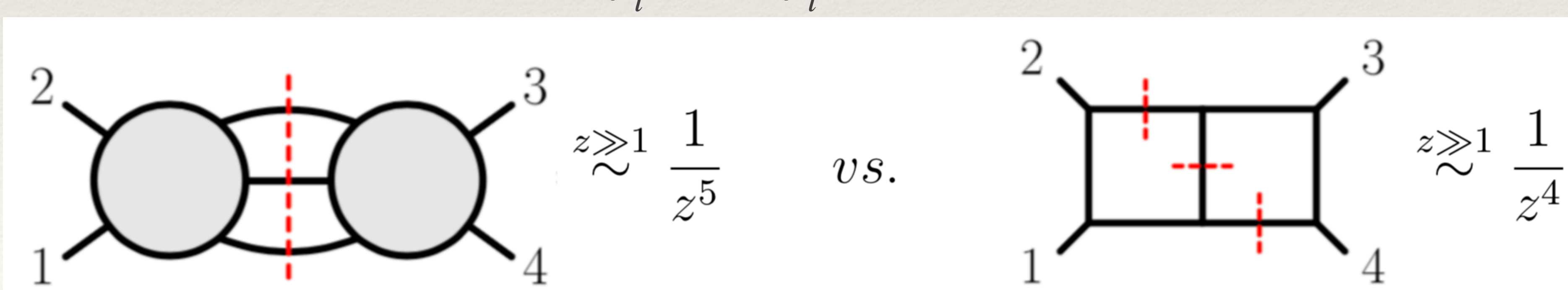


- probing infinity: $\ell_1^2 = 0 \Rightarrow \ell_i = \lambda_{\ell_i} \tilde{\lambda}_{\ell_i}$

holomorphic shift

$$\lambda^\alpha_{\ell_i} \mapsto \lambda^\alpha_{\ell_i} + z\sigma_i\, \eta^\alpha$$

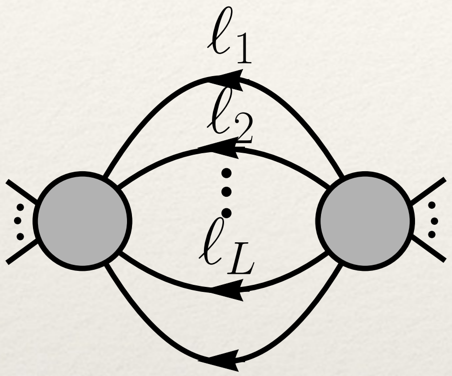
constant reference spinor



cancelation in d=4 for N=8 sugra!

iv-3) gravity in the UV

- ideally, would like L-loop, d=4 test:



susy state sum

$$= \int d\tilde{\eta}\, \mathcal{M}_L^{(0),k_L} \times \mathcal{M}_R^{(0),k_R}, \quad k_L + k_R - (L+1) = k$$

- probing infinity: $\ell_i^2 = 0 \Rightarrow \ell_i = \lambda_{\ell_i} \tilde{\lambda}_{\ell_i}$

$$\lambda_{\ell_i}^{\alpha} \mapsto \lambda_{\ell_i}^{\alpha} + z\sigma_i\, \eta^{\alpha}$$

holomorphic shift

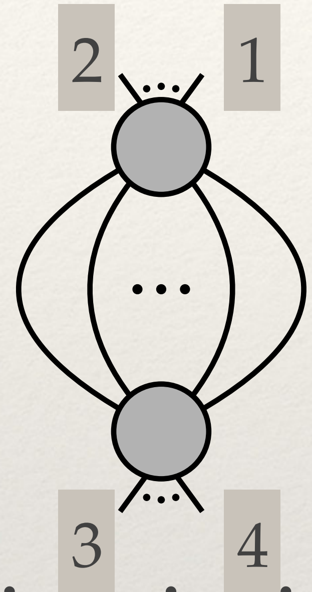
constant reference spinor

- technical challenge:

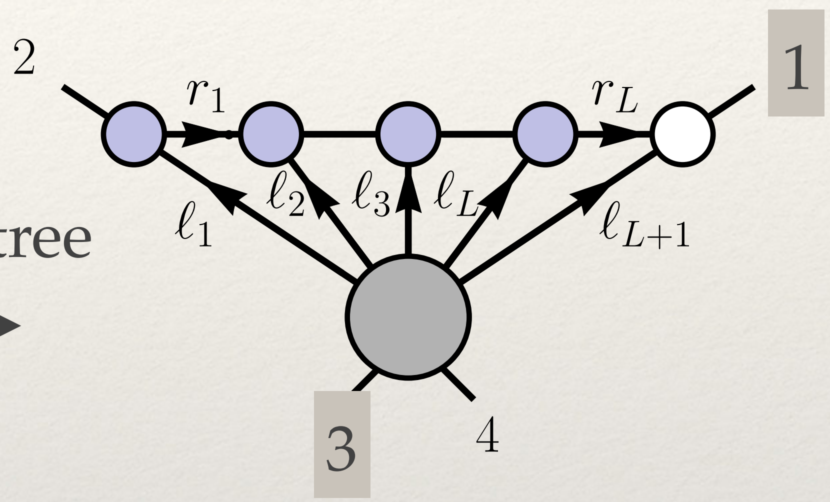
need good control over higher point, higher k gravity trees!

iv-3) gravity in the UV

- intermediate work-around:



deeper cut: forces n-pt MHV-tree

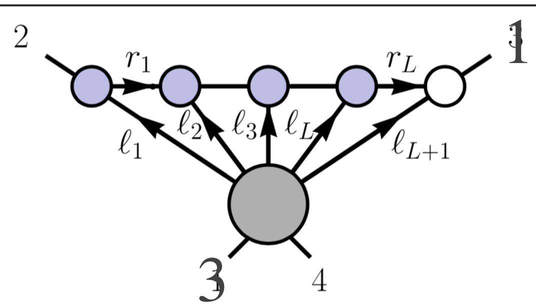


- probing infinity: $\ell_i = \lambda_{x_i}\widetilde{\lambda}_2\,,\quad i = 1,\ldots,L-1$

holomorphic shift

$\lambda_{x_i} \mapsto \lambda_{x_i} + \alpha\ \eta$ ← constant reference spinor

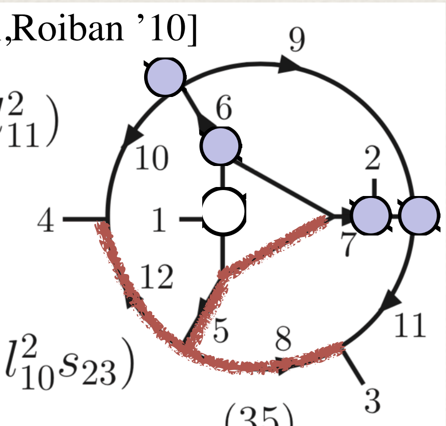
$\alpha \to \infty \Rightarrow \ell_i \to \infty$

$L =$	2	3	4	L
	α^{-2}	α^{-3}	α^{-4}	α^{-L}
worst diagram	α^{-2}	α^{-1}	α^{0}	?

[Bern,Carrasco,Dixon,Johansson,Roiban '10]

BCJ- YM numerator:

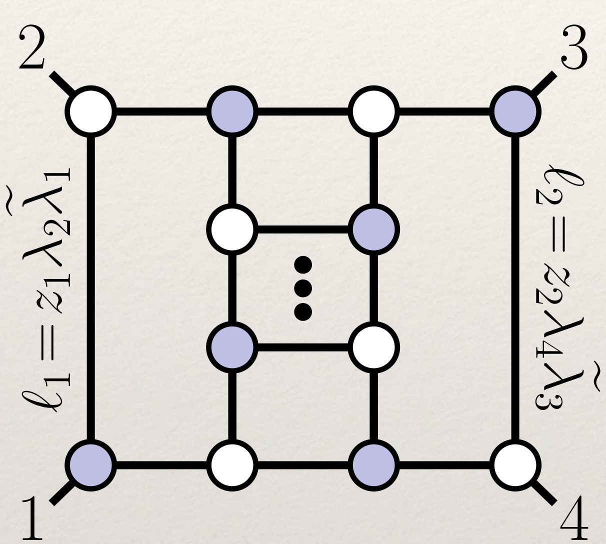
$$s_{23}(s_{10,11}(s_{17}-l_6^2)-l_5^2 l_{11}^2) + s_{12}s_{15}(s_{23}-s_{78}+l_9^2) + s_{16}(s_{12}s_{79}+l_{10}^2 s_{23}) - l_7^2(s_{12}s_{1,10}-l_{12}^2 s_{12}+l_{10}^2 s_{23})$$



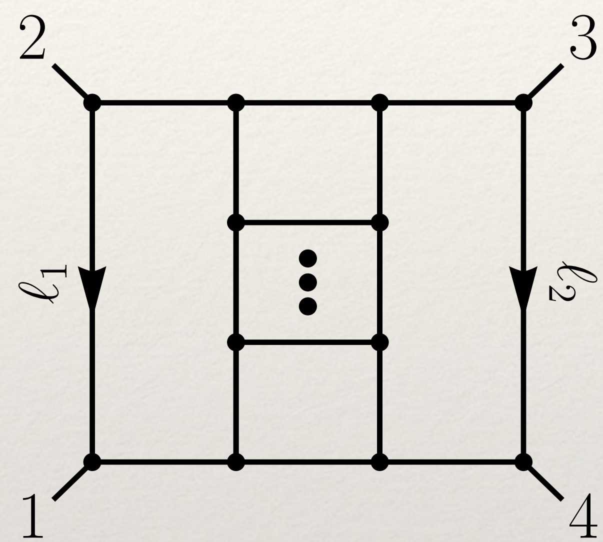
(35)

iv-3) gravity in the UV

- different all-loop cut where diagram scaling is know!

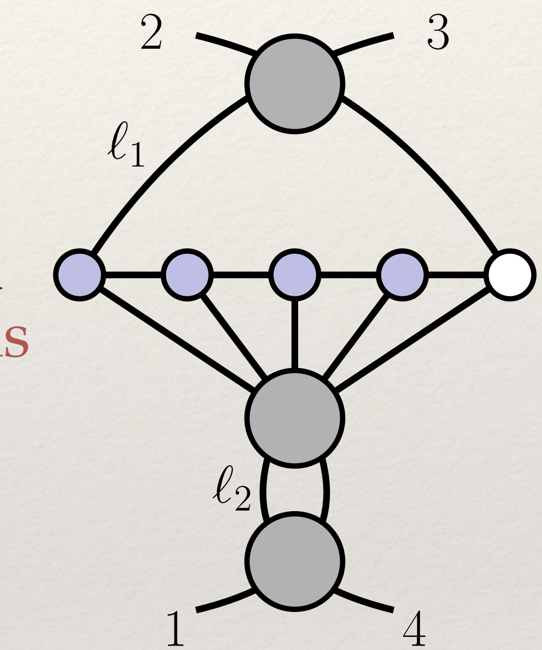


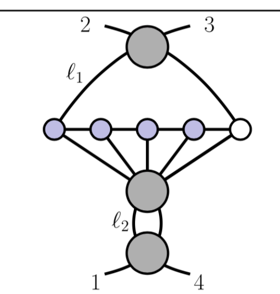
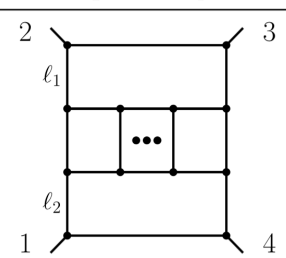
$\sim \dfrac{dz}{z^{4-L}}$



$N \sim (\ell_1 \cdot \ell_2)^{2L-6}$

allow for cancelations



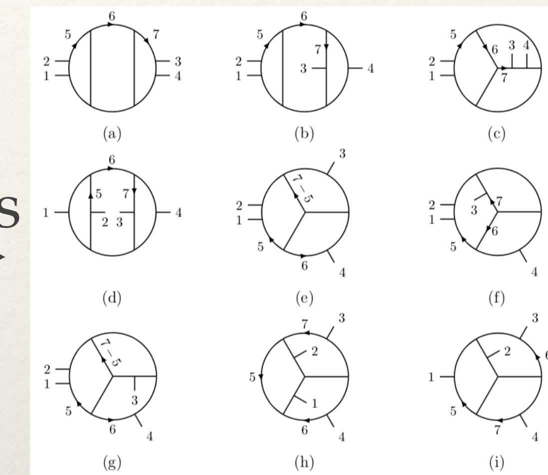
$L =$	3	4	5	L
	α^{-10}	α^{-8}	α^{-8}	α^{-8}
	α^{-5}	α^{-4}	α^{-3}	α^{L-8}

Massive cancellations between diagrams !

iv-4) uniqueness of gravity from analytic properties

- remember YM-strategy:
  - dlog (IR)
  - no poles @ $\ell \to \infty$ (UV)

} construct integrand basis that has these properties term-by-term →

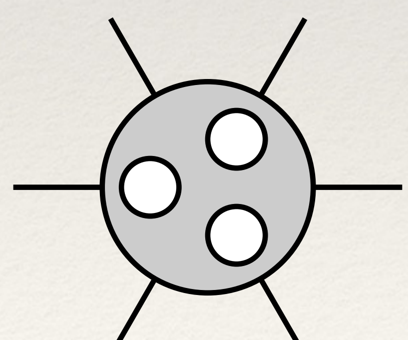


additional homogenous information

hom. analytic properties

$\cong$

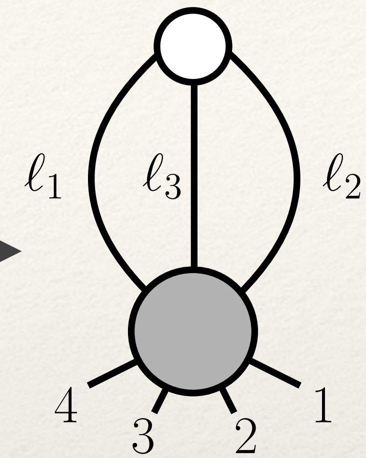
geometry



can uniquely reconstruct the YM integrand

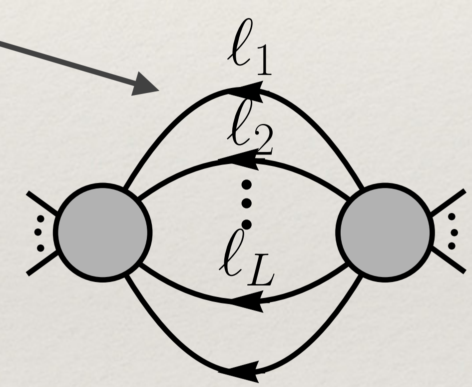
iv-4) uniqueness of gravity from analytic properties

- Gravity is completely different:
  - dlog (IR) →
  - no poles @ $\ell \to \infty$ (UV)



near $\langle \ell_1 \ell_2 \rangle = 0$:

$$\mathcal{M} \sim \frac{[\ell_1 \ell_2]}{\langle \ell_1 \ell_2 \rangle} \times \text{regular}$$



additional homogenous information

Improved large-z scaling

Uniquely reconstruct the gravity?

in progress [Edison,EH,Langer,Parra-Martinez,Trnka]

- 2-loop 4pt, 1-loop 5pt, …

hom. analytic properties

geometry?

stay tuned!

v) Conclusions

- new geometric formulations of QFT
  - Grassmannian, Amplituhedron in planar N=4 sYM
  - geometry ⟷ canonical differential forms with logarithmic singularities
- hints that these geometric structures persist in nonplanar N=4 sYM
  - same analytic properties, dlog + no poles at infinity [manifest term-by-term]
- Gravity has still a lot of surprises in store for us:
  - IR-properties (vanishing collinear) & UV-conditions (improved large z-scaling) are global in nature
  - do we have the full list of homogeneous constraints that “define” gravity?
  - Can we “geometrize” these properties?

THANK YOU FOR YOUR ATTENTION