TMD & Collinear Observables in the CSS formalism

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November 9, 2018

Workshop on
Parton distributions as a bridge
from low to high energies

November 8 and 9, 2018
[before the Fall CTEQ meeting]
Jefferson Laboratory, Newport News, VA

Jefferson Lab
Thomas Jefferson National Accelerator Facility

C T E Q

PennState
Berks

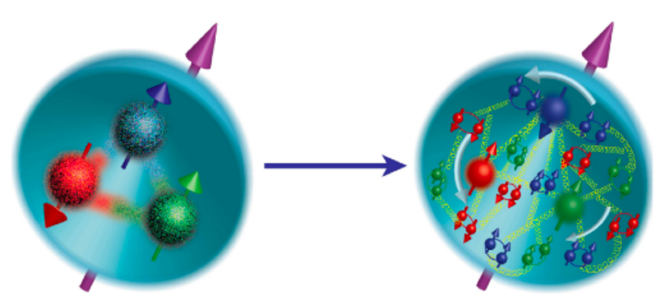
Overview comments

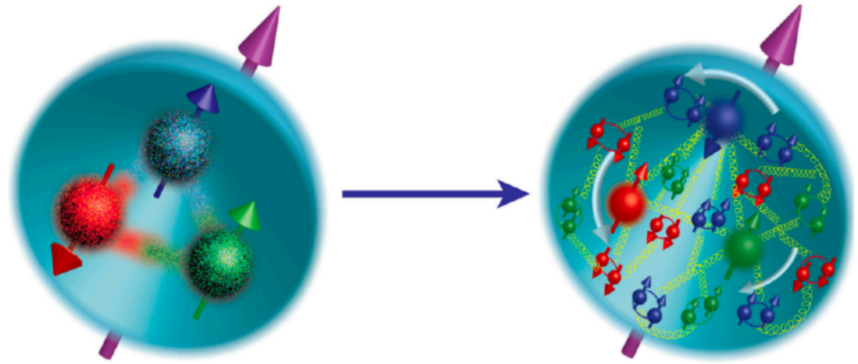
- Report on relating TMD factorization & collinear factorization in studying nucleon structure in CSS formalism, TMDs and collinear pdfs

- Relies on a modification of the so called *W+Y* construction used to “match” the cross section $\sigma\,(\boldsymbol{q}_{\rm T})$ point-by-point, from small $q_T \sim m$ (hadronic), to large $q_T \sim Q$

- Using enhanced version of CSS, able to “re-derive” at @ “LO” the well-known relation between the unpolarized F.T.-TMD & $f_1(x,\mu)$, & the Sivers function, and the (collinear twist-3) Qiu-Sterman function $T_F(x,\mu) \sim f_{1T}^{\perp(1)}(x,\mu)$ @ scale μ
nb … power counting remains open question

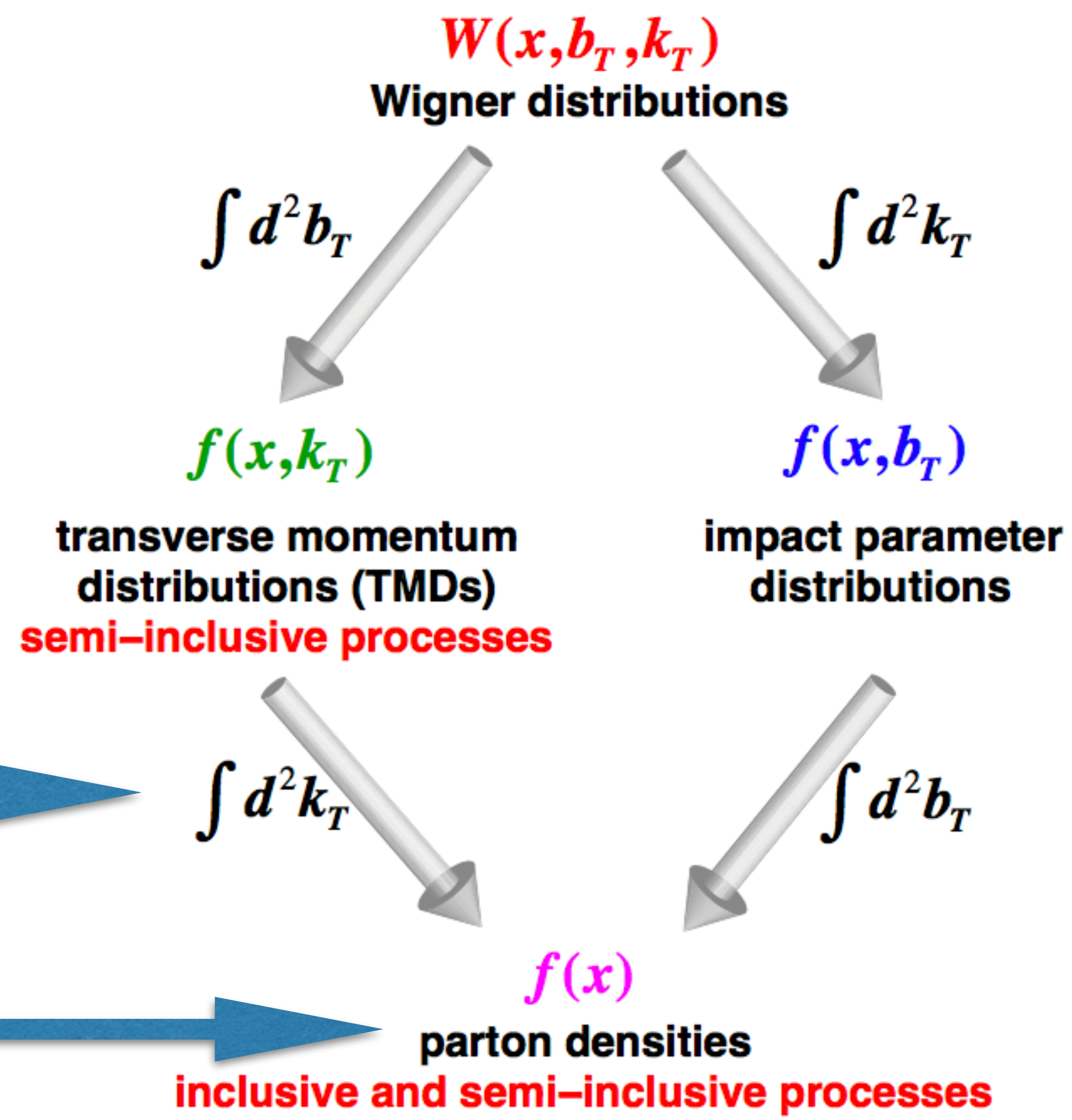
  - **Phys.Rev. D (2016) Collins, Gamberg, Prokudin, Sato, Rogers, Wang**
  - **Phys. Lett B (2018) Gamberg , Metz, Pitonyak, Prokudin**



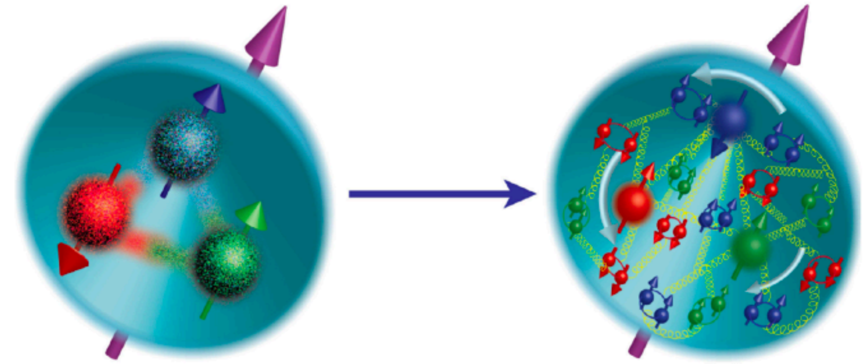


EIC White Paper

arXiv:1212.1701 Accardi et al.



TMD to collinear



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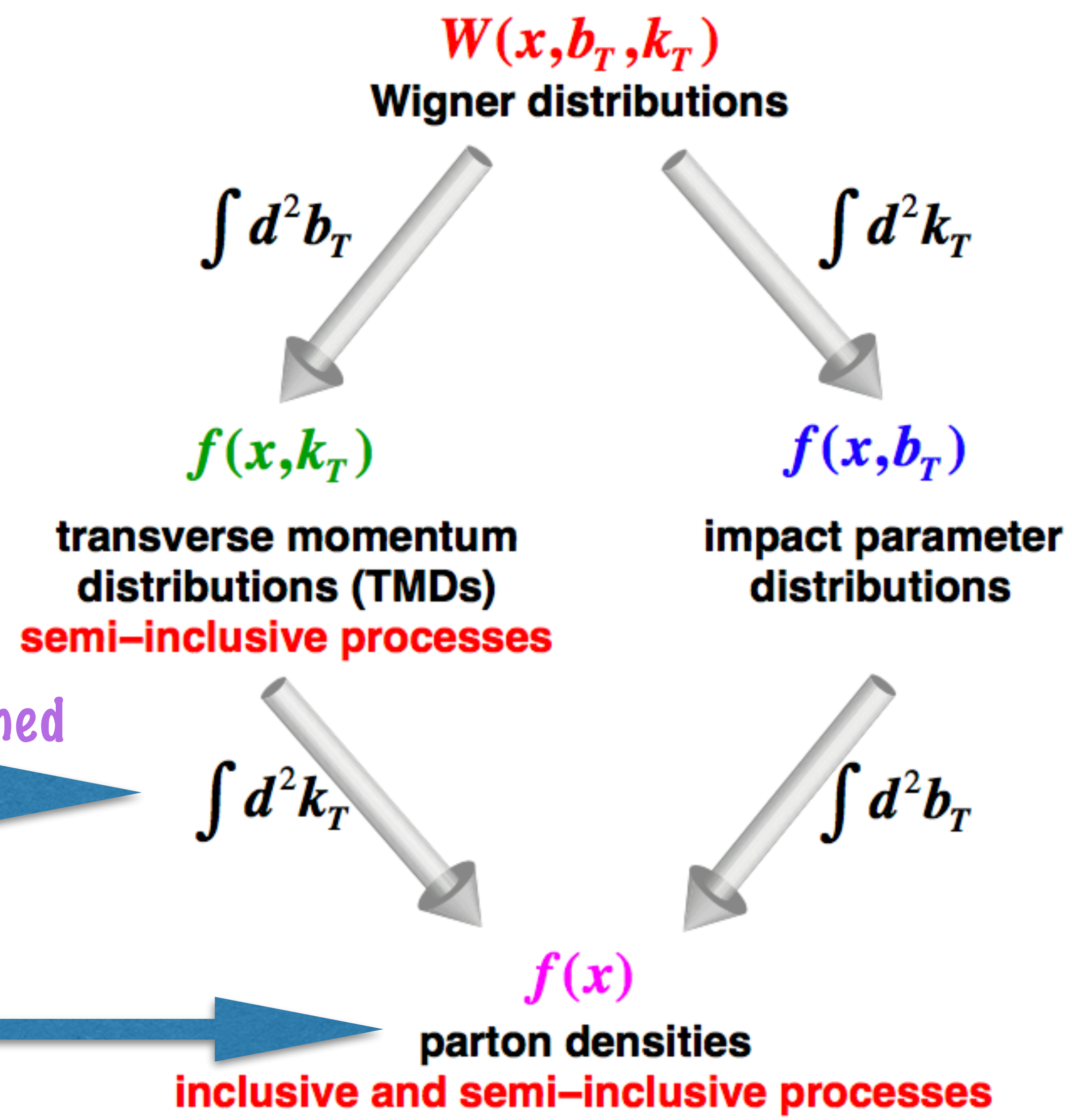


nb CSS TMD factorisation carried out coordinate space: can we can shed some light through CSS ?

TMD to collinear

Must consider UV and IR
Divergences and TMD evolution, CS NPB 1982… CSS 1985
Ji Ma Yuan PRD 2004/5, Collins 2011 Cambridge Press, Aybat Rogers 2011 PRD ….

From closing talk of Davison Soper DIS 2015

Transverse momentum

- The partons in a proton carry momentum components transverse to the beam direction.

- Thus there are transverse momentum dependent (TMD) parton distributions

$$f_{a/A}(x, \boldsymbol{k}_\perp, Q^2)$$

- If you are going into the woods, you have to be careful: there are some subtle issues in the definitions of these.

- On an intuitive level

$$f_{a/A}(x, Q^2) \sim \int d\boldsymbol{k}_\perp \; f_{a/A}(x, \boldsymbol{k}_\perp, Q^2)$$

Some comments on the subject

- Collins QCD Book Ch. 9 &13
- Ji Ma Yuan PRD 2005
- Ji, Qiu, Vogelsang, Yuan PRL, PRD 2006, transverse spin case
- Aybat Rogers PRD 2011
- Aybat Collins Qiu Rogers PRD 2012, transverse spin case
- Vogelsang INT talk 2/27/14
- Collins, Gamberg, Prokudin, Sato, Wang PRD 2016
- Gamberg, Metz, Pitonyak, Prokudin PLB 2018

Moments of TMDs and collinear pdfs

Naive connection of moments of TMDs and collinear pdfs based on matrix elements and a Parton Model picture of "factorization"

.

TMD kinematical CT3 dynamical CT3

$$\int d^2\vec{k}_T \, \frac{\vec{k}_T^2}{2M^2} \quad f_{1T}^{\perp}(x, k_T) \quad = \quad f_{1T}^{\perp(1)}(x) \quad = \quad -\frac{T_F(x,x)}{2M}$$

Qiu & Sterman 1991

Boer, Mulder, Pijlman (2003); Meissner (2009); ...

⋮

TMD kinematical CT3

$$\int d^2\vec{p}_T \, \frac{\vec{p}_T^2}{2z^2 M_h^2} \quad H_1^{\perp}(z, p_T) \quad = \quad H_1^{\perp(1)}(z)$$

Yuan and Zhou (2009)

⋮

Consider the less exotic case

"Parton Model"

TMD — CT2

$$\int d^2\vec{k}_T \quad \boldsymbol{f_1(x, k_T)} \quad = \quad \boldsymbol{f_1(x)}$$

$\vdots$

TMD — CT2

$$\int d^2\vec{p}_T \quad \boldsymbol{D_1(z, p_T)} \quad = \quad \boldsymbol{D_1(z)}$$

$\vdots$

Ignore UV divergences and effects from soft-gluon radiation

Underlies Model building w/ and w/o evolution using TMD and collinear evolution approach Anselmino et al. 2005-2016

$$W_{PM}(q_T, Q) = H_{LO,j',i'}(Q_0) \int d^2k_T f_{j'/A}(x, k_T) d_{B/i'}(z, q_T + k_T)$$

$$\int d^2q_T\, W_{PM}(q_T, Q) = H_{LO,j',i'}(Q_0) f_{j'/A}(x) d_{B/i'}(z)$$

Overview comments Matching

- We modify the "*standard matching prescription*" traditionally used in CSS formalism relating low & high q_T behavior cross section @ moderate Q in particular where studies of TMDs are relevant

Matching studies in CSS related approaches

...

NPB Collins & Soper(1982), & Sterman 1985

NPB (1991) Arnold, Kauffman

PRD (1998) Nadolsky Stump Yuan

PRL (2001) Qiu, Zhang

PRD (2003) Berger, Qiu

NPB (2006) Bozzi, Catani, DeFlorian, Grazzini ...

NPB (2006) Y. Koike, J. Nagashima, W. Vogelsang

arXiv (2014) Sun, Isacson, Yuan-CP,Yuan-F

JHEP (2015) Boglione, Hernandez, Melis Prokudin

PRD (2016) Collins, Gamberg, Prokudin, Rogers, Sato, Wang

PLB (2018) Gamberg , Metz, Pitonyak, Prokudin

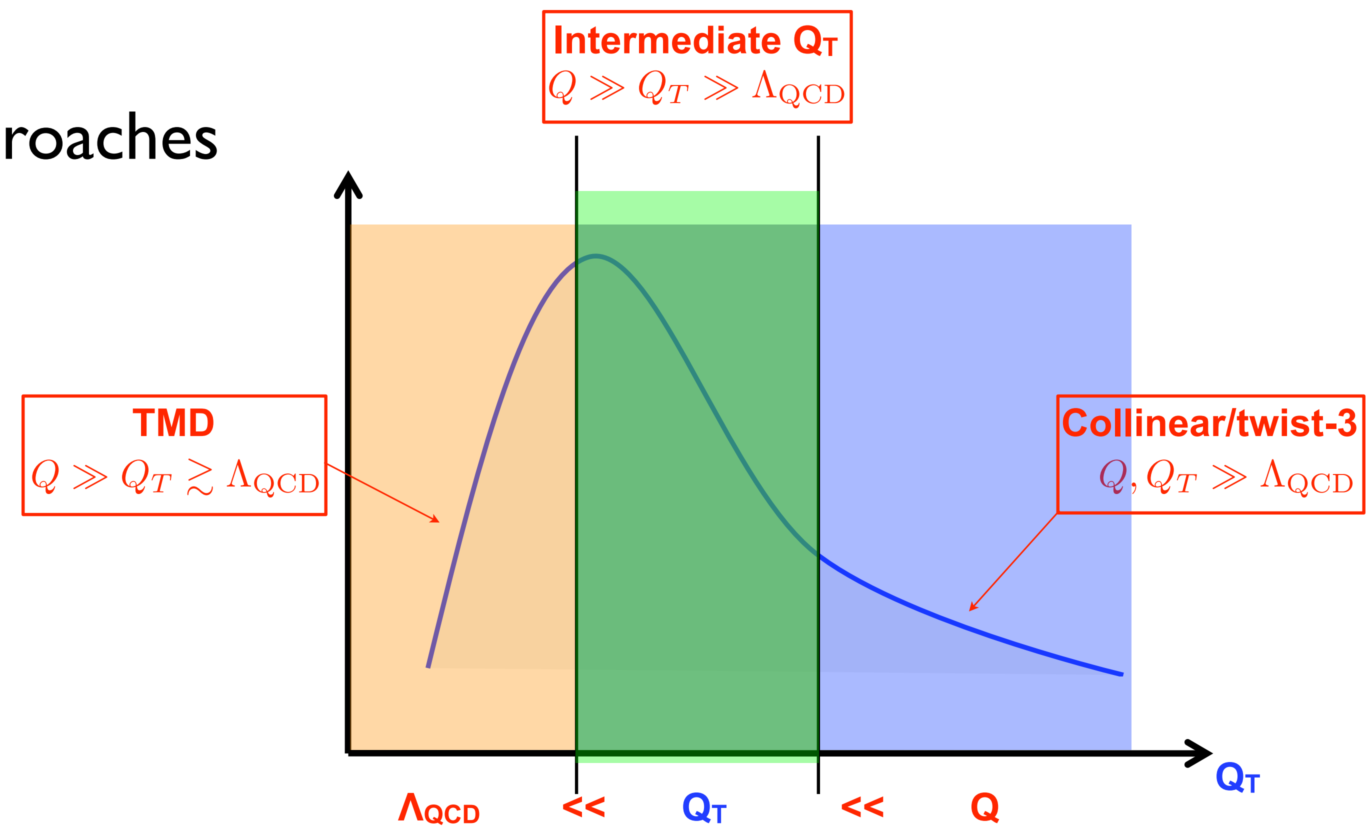
PLB (2018) Echevarria, Kasemets, Lansberg, Pisano, Signori

....

Series of papers on matching TMD and collinear ETQS transv. Spin

Ji, Qiu, Vogelsang, Yuan PRL PRD 2006, ...

Kang, Xiao, Yuan PRL 2011



Alexei's talk pheno & matching...

"CSS Matching-1" *W* + *Y*-schematic

- Collins Soper Sterman NPB 1985
- Collins 2011 Cambridge Press
- Collins, Gamberg, Prokudin, Sato, Rogers, Wang PRD (2016)

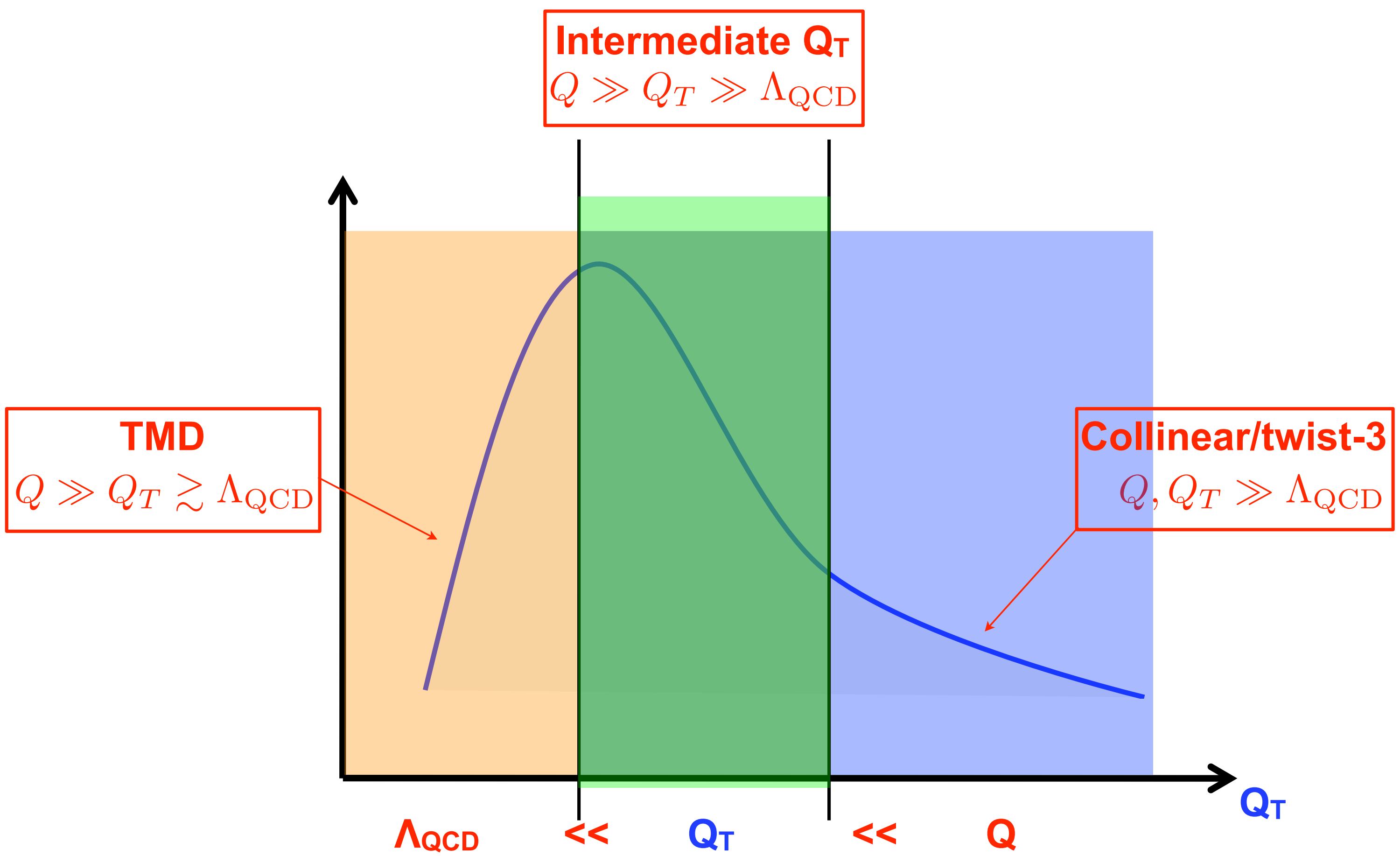
$$d\sigma(m \lesssim q_T \lesssim Q, Q) = W(q_T, Q) + Y(q_T, Q) + O\left(\frac{m}{Q}\right)^c d\sigma(q_T, Q)$$

- The $W + Y$ construction of cross section arise from applying **approximators** $\boldsymbol{T_{\mathrm{TMD}}}$ and $\boldsymbol{T_{\mathrm{coll}}}$ to cross section in "design" regions $m \sim q_T \ll Q$ and $m \ll q_T \sim Q$ respectively, to extract the leading contributions to the TMD & collinear contributions to the cross section

- Uses a subtractive formalism to prevent double counting; resulting in the combination $W + Y$ having a relative error $O(m/Q)^c$ in the range "design" regions $m \sim q_T \ll Q$ and $m \ll q_T \sim Q$

Y-term & Matching
Subtraction formalism of CSS

$$d\sigma(m \lesssim q_T \lesssim Q, Q) = ??\ W(q_T, Q) + FO(q_T, Q)\ ?? \ + \ O\left(\frac{m}{Q}\right)^c d\sigma(q_T, Q)$$



Y-term & Matching
Subtraction formalism of CSS

$$\frac{d\sigma}{dP_T^2} \propto \sum_{jj'} \mathcal{H}_{jj',\,\mathrm{SIDIS}}(\alpha_s(\mu), \mu/Q) \int d^2\boldsymbol{b}_T e^{i\boldsymbol{b}_T\cdot\boldsymbol{P}_T}\ \tilde{F}_{j/H_1}(x, b_T; \mu, \zeta_1)\, \tilde{D}_{H_2/j'}(z, b_T; \mu, \zeta_2) \ +\ Y_{\mathrm{SIDIS}}$$

$$d\sigma(m \lesssim q_T \lesssim Q, Q) = ??\ W(q_T, Q) + FO(q_T, Q)\ ??\ +\ O\left(\frac{m}{Q}\right)^c d\sigma(q_T, Q)$$

<u>If we do, we double count</u>

Subtract out the double counting such that the cross section is matched (SIDIS,DY, $e^+ e^-$) in the "overlap region":Designed s.t. valid to leading order in *m/Q* uniformly in q_T (*<u>see role of "approximations" in TMD factorization</u>*)

$$Y(q_T, Q) = FO(q_T, Q) - ASY(q_T, Q)$$

$$d\sigma(m \lesssim q_T \lesssim Q, Q) = W(q_T, Q) + Y(q_T, Q) + O\left(\frac{m}{Q}\right)^c d\sigma(q_T, Q)$$

JCC Cambridge Press 2011, Collins arXiv: 1212.5974, Catani et al. NPB 06, 15, Collins, Gamberg, Prokudin, Rogers, Sato, Wang PRD 2016 ……

One finds the definition of the *Y* term via "approximators"

$$Y(q_T, Q) \equiv T_{coll}\, d\sigma(q_T, Q) - T_{coll} T_{TMD}\, d\sigma(q_T, Q)$$

$$Y(q_T, Q) = FO(q_T, Q) - ASY(q_T, Q)$$

- It is the difference of the cross section calculated with collinear pdfs and ffs at fixed order FO and the asymptotic contribution of the cross section

- ***nb at small q_T the FO and ASY are dominated by the same diverging terms***

$$\frac{1}{q_T^2} \quad \text{and} \quad \frac{1}{q_T^2} \log \frac{Q^2}{q_T^2}$$

- ***Thus its expected that the Y term is small or zero leaving***

$$d\sigma(q_T \ll Q, Q) \approx W(q_T, Q)$$

Matching $W + Y$-schematic

✦Collins Soper Sterman NPB 1985

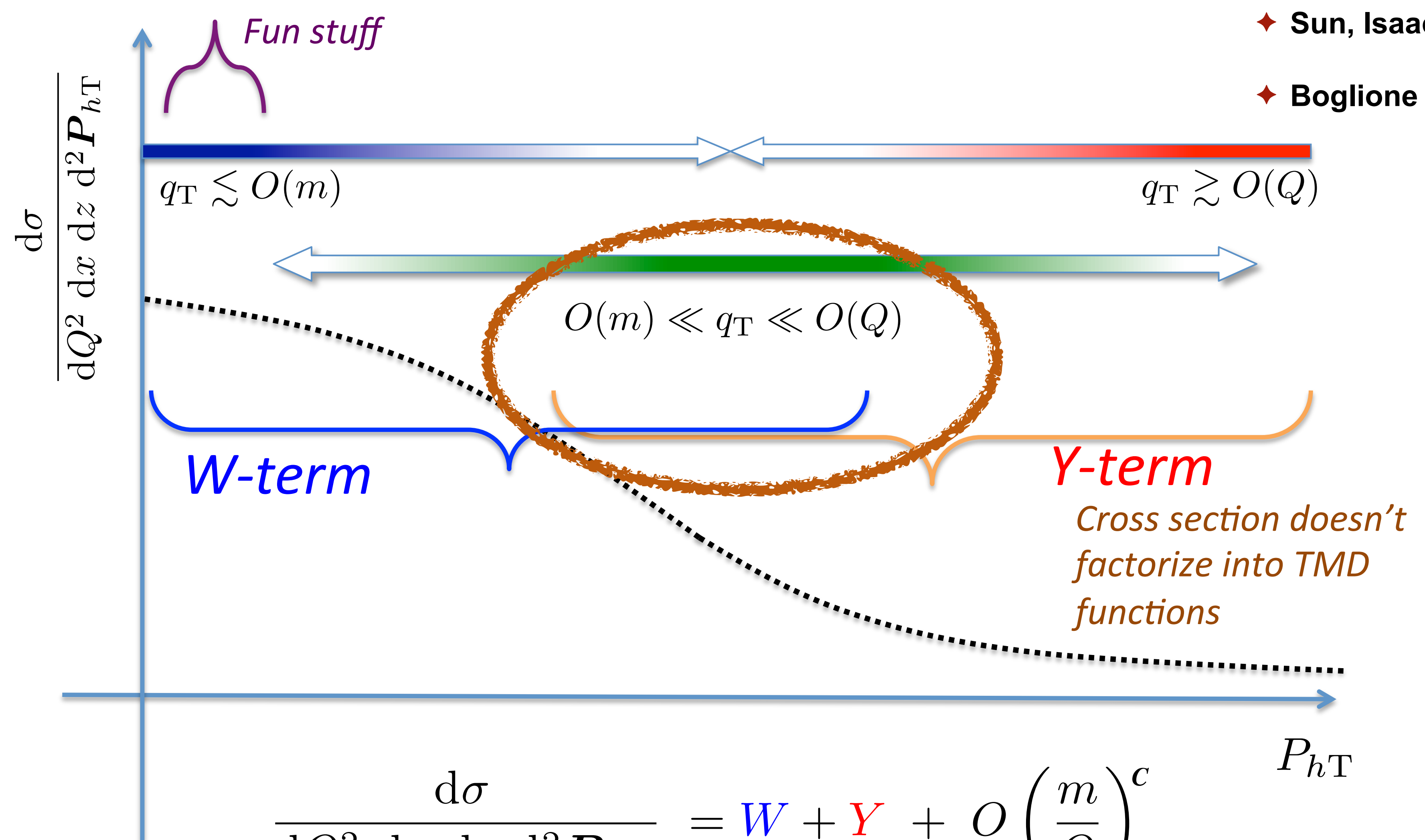
✦Collins 2011 Cambridge Press

- *Designed* with the aim to have a formalism valid to leading power in *m/Q* uniformly in q_T, where *m* is a typical hadronic mass scale
- **&** where broad intermediate range transverse momentum s.t. $m \ll q_T \ll Q$

Implementations/studies

- ✦ **Nadolsky Stump C.P. Yuan PRD 1999 HERA data**
- ✦ **Y. Koike, J. Nagashima, W. Vogelsang NPB (2006) eRHIC**
- ✦ **Sun, Isaacson, C. -P. Yuan , F Yuan arXiv 2014**
- ✦ **Boglione Gonzalez Melis Prokudin JHEP 2015 ….**

From Ted Rogers



$$\frac{d\sigma}{dQ^2\, dx\, dz\, d^2 \boldsymbol{P}_{hT}} = W + Y + O\left(\frac{m}{Q}\right)^c$$

note $P_{hT} = zq_T$

TMD Factorization & Evolution

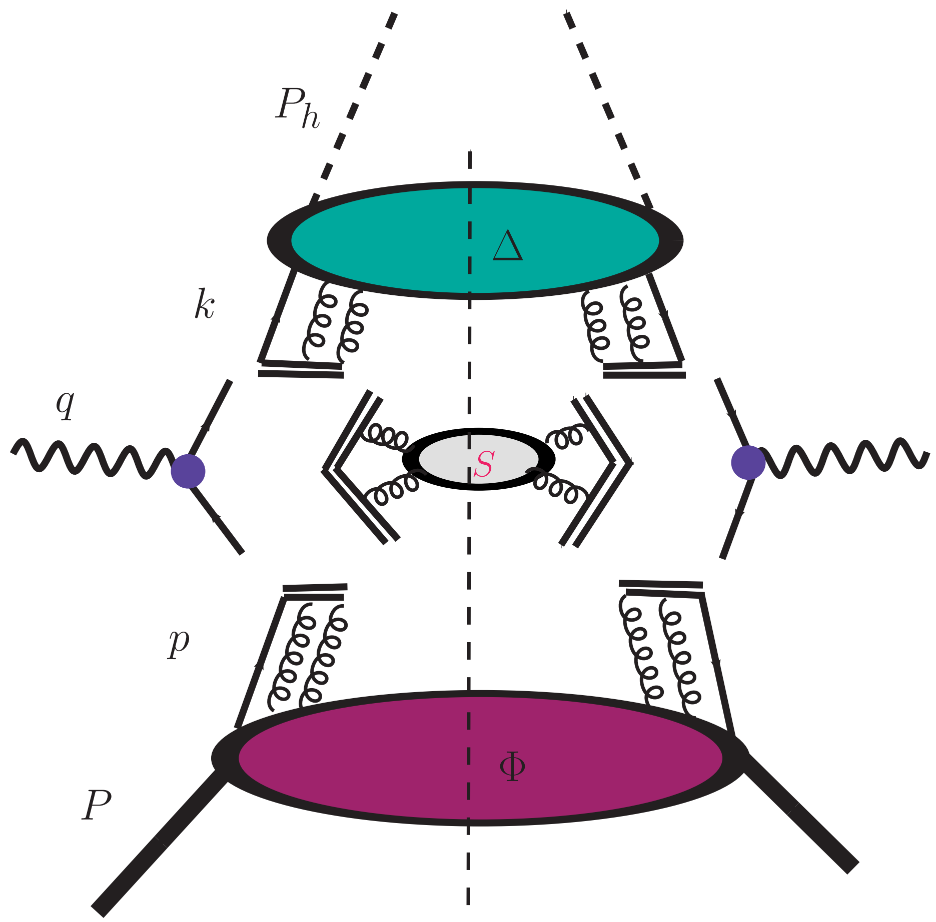
$$\frac{d\sigma}{d\boldsymbol{q}_{\mathrm{T}}^2 dQ^2 \ldots} = W(q_{\mathrm{T}}, Q) + Y(q_{\mathrm{T}}, Q) + O\left(\frac{m}{Q}\right)^{c} \frac{d\sigma}{d\boldsymbol{q}_{\mathrm{T}}^2 dQ^2 \ldots}$$

$$W_{UU}(q_{\mathrm{T}}, Q) = \sum_{jj'} H_{jj'}(\alpha_s(\mu), \mu/Q) \int d^2\boldsymbol{b}_T e^{i\boldsymbol{b}_T\cdot\boldsymbol{q}_T}\ \tilde{f}_{j/H_1}(x, b_T; \mu, \zeta_1)\, \tilde{D}_{H_2/j'}(z, b_T; \mu, \zeta_2)$$

In full QCD, the auxiliary parameters μ and ζ are exactly arbitrary and this is reflected in the the Collins-Soper (CS) equations for the TMD PDF, and the renormalization group (RG) equations

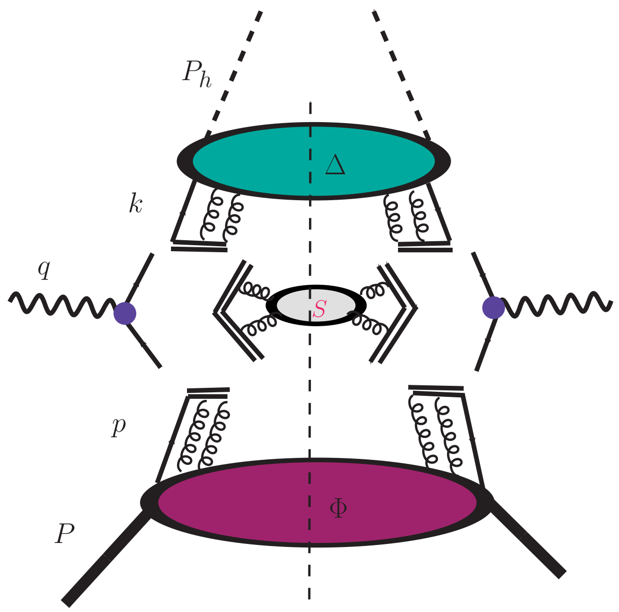
Review elements TMD factorization

★ Collins Soper (81), Collins, Soper, Sterman (85), Boer (01) (09) (13), Ji,Ma,Yuan (04), Collins-Cambridge University Press (11), Aybat Rogers PRD (11), Abyat, Collins, Qiu, Rogers (11), Aybat, Prokudin, Rogers (11), Bacchetta, Prokudin (13), Sun, Yuan (13),Echevarria, Idilbi, Scimemi JHEP 2012, Collins Rogers 2015 ....



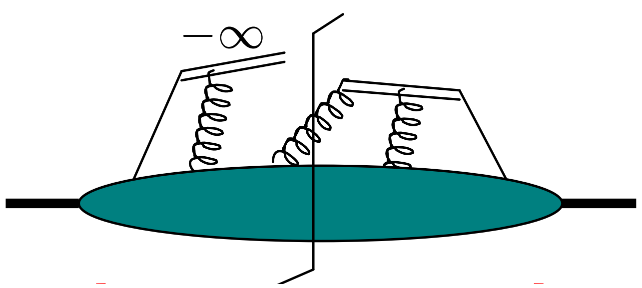
Transverse Momentum Dependent Evolution

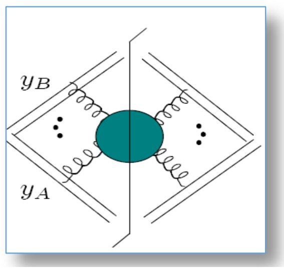
TMD factorization/evolution CSS in b space region analysis & Ward Identities



- ✦ *Collins Soper, NPB 1982*
- ✦ *Collins Soper Sterman NPB 1985*
- ✦ *Ji Ma Yuan PRD PLB ...2004, 2005*
- ✦ *Aybat Rogers PRD 2011*
- ✦ *Aybat Collins Qiu Rogers PRD 2012*
- ✦ *Collins 2011 Cambridge Press*

- TMDs w/Gauge links: color invariant
- TMD PDFs & Soft factor have rapidity/LC divergences
- Rapidity regulator introduced to regulate these divergences



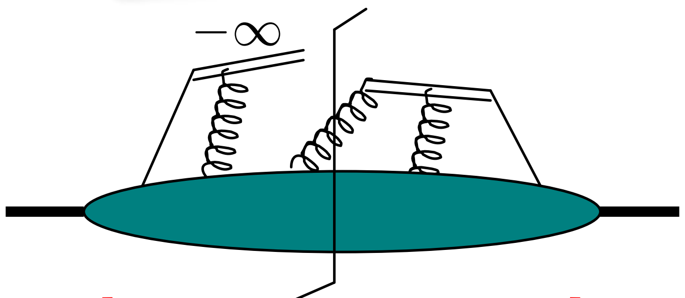


$$\tilde{f}^{\rm sub}_{j/H}(x,b_T;\mu,y_n) = \lim_{\substack{y_A\to+\infty \\ y_B\to-\infty}} \underbrace{\tilde{f}^{\rm unsub}_{j/H}(x,b_T;\mu,y_P-y_B)} \sqrt{\frac{\tilde{S}(b_T;y_A,y_n)}{\tilde{S}(b_T;y_A,y_B)\tilde{S}(b_T;y_n,y_B)}} \times UV_{renorm}$$

$$\Updownarrow$$

$$\tilde{f}^{\rm unsub}_{j/H}(x,b_T;\mu,y_P-y_B) = \int \frac{db^-}{2\pi} e^{-ixP^+b^-} \langle P|\bar{\psi}(0)\gamma^+\mathcal{U}_{[0,b]}\psi(b)|P\rangle\big|_{b^+=0}$$

TMD Evolution follows from independence of rapidity scale



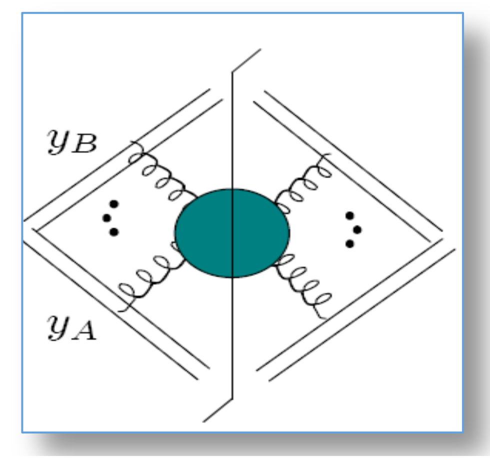
Collins Cambridge press 2011, Aybat & Rogers 2011 PRD

$$\tilde{F}_H^{\text{sub}}(x, b_T; \mu, y_n) = \lim_{\substack{y_A \to \infty \\ y_B \to -\infty}} \tilde{F}_H^{\text{unsub}}(x, b_T; \mu, y_P - y_B) \sqrt{\frac{\tilde{S}(b_T; y_A, y_n)}{\tilde{S}(b_T; y_A, y_B)\tilde{S}(b_T; y_n, y_B)}}$$

From operator definition get

Collins-Soper Equation:

$$- \quad \frac{\partial \ln \tilde{F}(x, b_T, \mu, \zeta)}{\partial \ln \sqrt{\zeta}} = \tilde{K}(b_T; \mu)$$



$$\tilde{K}(b_T; \mu) = \frac{1}{2} \frac{\partial}{\partial y_n} \ln \frac{\tilde{S}(b_T; y_n, -\infty)}{\tilde{S}(b_T; +\infty, y_n)}$$

JCC Soft factor further "repartitioned"
This is done to

1) cancel LC divergences in "unsubtracted" TMDs
2) separate "right & left" movers i.e. full factorization
3) remove double counting of momentum regions

Along with .... Renormalization group Equations

$$\frac{d\tilde{K}}{d\ln\mu} = -\gamma_K(g(\mu))$$

$$\frac{d\ln\tilde{F}(x, b_T; \mu, \zeta)}{d\ln\mu} = -\gamma_F(g(\mu); \zeta/\mu^2)$$

RGE:
get anomalous
for *F* & *K*

Solve Collins Soper & RGE eqs. to obtain "evolved TMDs"

TMD Factorization & Evolution

$$\frac{d\sigma}{d\boldsymbol{q}_{\mathrm{T}}^2 dQ^2 \ldots} = W(q_{\mathrm{T}}, Q) + Y(q_{\mathrm{T}}, Q) + O\left(\frac{m}{Q}\right)^{\mathrm{c}} \frac{d\sigma}{d\boldsymbol{q}_{\mathrm{T}}^2 dQ^2 \ldots}$$

$$W_{UU}(q_{\mathrm{T}}, Q) = \sum_{jj'} H_{jj'}(\alpha_s(\mu), \mu/Q) \int d^2\boldsymbol{b}_T e^{i\boldsymbol{b}_T\cdot\boldsymbol{q}_T}\ \tilde{f}_{j/H_1}(x, b_T; \mu, \zeta_1)\, \tilde{D}_{H_2/j'}(z, b_T; \mu, \zeta_2)$$

In full QCD, the auxiliary parameters μ and ζ are exactly arbitrary and this is reflected in the the Collins-Soper (CS) equations for the TMD PDF, and the renormalization group (RG) equations

CSS solution *F.T.*-TMD

$$\tilde{\boldsymbol{f}}_{\boldsymbol{1}}(\boldsymbol{x}, \boldsymbol{b_T}; \boldsymbol{Q^2}, \boldsymbol{\mu_Q}) \sim \left(\tilde{C}^{f_1}(x/\hat{x}, b_*(b_T); \mu_{b_*}^2, \mu_{b_*}, \alpha_s(\mu_{b_*})) \otimes \boldsymbol{f_1(\hat{x}; \mu_{b_*})}\right)$$

$$\times \exp\left[-S_{pert}(b_*(b_T); \mu_{b_*}, Q, \mu_Q) - S_{NP}^{f_1}(b_T, Q)\right]$$

<u>perturbative Sudakov factor</u>

$$\underbrace{-\ln(Q/\mu_{b_*})\tilde{K}(b_*, \mu_{b_*}) - \int_{\mu_{b_*}}^{\mu_Q} \frac{d\mu'}{\mu'}\left[\gamma(\alpha_s(\mu'); 1) - \gamma_K(\alpha_s(\mu'))\ln(Q/\mu')\right]}_{\text{same for unpol. and pol.}}$$

<u>non-perturbative Sudakov factor</u>

$$g_{f_1}(x, b_T) + g_K(b_T)\ln(Q/Q_0)$$

different for each TMD

universal

$$b_*(b_T) \equiv \sqrt{\frac{b_T^2}{1 + b_T^2/b_{\max}^2}} \qquad \mu_{b_*} = C_1/b_*(b_T)$$

Unpolarized and Sivers evolve in same way

Recall the correlator in *b*-space Bessel Transform

$$\tilde{\Phi}^{[\gamma^+]}(x, \boldsymbol{b}_T) \;=\; \tilde{f}_1(x, \boldsymbol{b}_T^2) - i\, \epsilon_T^{\rho\sigma} b_{T\rho} S_{T\sigma}\, M \tilde{f}_{1T}^{\perp(1)}(x, \boldsymbol{b}_T^2)$$

Boer Gamberg Musch Prokudin JHEP 2011

See lattice studies of Engelhardt et al , Musch 2009-2018

Obeys CS Equation, thus unpolarised and Sivers evolve "similarly"

$$\frac{\partial \tilde{\phi}^i_{f/P}(x, \mathbf{b}_\mathrm{T}; \mu, \zeta_F) \epsilon_{ij} S^j_T}{\partial \ln \sqrt{\zeta_F}} = \tilde{K}(b_T; \mu) \tilde{\phi}^i_{f/P}(x, \mathbf{b}_\mathrm{T}; \mu, \zeta_F) \epsilon_{ij} S^j_T.$$

Idilbi,Ji,Ma,Yuan PRD 2004
Aybat Rogers Collins Qiu PRD 2012
also see Kang Yuan Xiao PRL 2011

TMD Evolution-Solution for unpolarised & Sivers

- TMD/CSS Evolution/Factorization carried out in *b*-space "Bessel transforms"

Boer Gamberg Musch Prokudin 2011 JHEP
Collins Aybat Rogers Qiu 2012 PRD

$$\tilde{\Phi}^{[\gamma^+]}(x, \boldsymbol{b}_T) \;=\; \tilde{f}_1(x, \boldsymbol{b}_T^2) - i\,\epsilon_T^{\rho\sigma} b_{T\rho} S_{T\sigma}\, M \tilde{f}_{1T}^{\perp(1)}(x, \boldsymbol{b}_T^2)$$

Correlator obeys CSS equation so,

$$\boldsymbol{\tilde{f}_1(x, b_T; Q^2, \mu_Q)} \quad \sim \quad \left(\tilde{C}^{f_1}(x/\hat{x}, b_*(b_T); \mu_{b_*}^2, \mu_{b_*}, \alpha_s(\mu_{b_*})) \otimes \boldsymbol{f_1(\hat{x}; \mu_{b_*})}\right)$$

Collins (2011); ...

$$\times\; \exp\left[-S_{pert}(b_*(b_T); \mu_{b_*}, Q, \mu_Q) - S_{NP}^{f_1}(b_T, Q)\right]$$

Qiu & Sterman PRL 1991

$$\boldsymbol{\tilde{f}_{1T}^{\perp(1)}(x, b_T; Q^2, \mu_Q)} \quad \sim \quad \left(\tilde{C}^{f_{1T}^{\perp}}(\hat{x}_1, \hat{x}_2, b_*(b_T); \mu_{b_*}^2, \mu_{b_*}, \alpha_s(\mu_{b_*})) \otimes \boldsymbol{T_F(\hat{x}_1, \hat{x}_2; \mu_{b_*})}\right)$$

$$\times\; \exp\left[-S_{pert}(b_*(b_T); \mu_{b_*}, Q, \mu_Q) - S_{NP}^{f_{1T}^{\perp}}(b_T, Q)\right]$$

Aybat, Collins, Qiu, Rogers (2012); Echevarria, Idilbi, Kang, Vitev (2014); ...

Putting Solution of CSS Eqn. Together
W term

$$\tilde{W}_{\rm UU}(b_T,Q) = \tilde{W}^{\rm OPE}_{\rm UU}(b_*(b_T),Q)\tilde{W}^{\rm NP}_{\rm UU}(b_T,Q)$$

$$= \sum_j H_j(\mu_Q,Q)\, \tilde{f}_1^j(x,b_*(b_T);\mu^2_{b_*},\mu_{b_*})\, \tilde{D}_1^{h/j}(z,b_*(b_T);\mu^2_{b_*},\mu_{b_*})$$

$$\times \exp\left\{\tilde{K}(b_*(b_T);\mu_{b_*})\ln\left(\frac{Q^2}{\mu^2_{b_*}}\right) + \int_{\mu_{b*}}^{\mu_Q}\frac{d\mu'}{\mu'}\left[2\gamma(\alpha_s(\mu');1) - \ln\left(\frac{Q^2}{\mu'^2}\right)\gamma_K(\alpha_s(\mu'))\right]\right\}$$

$$\times \exp\left\{-g_{\rm pdf}(x,b_T;Q_0,b_{max}) - g_{\rm ff}(z,b_T;Q_0,b_{max}) - g_K(b_T;b_{max})\ln\left(\frac{Q^2}{Q_0^2}\right)\right\},$$

$$\tilde{W}^{\rm siv}_{\rm UT}(b_T,Q) = \tilde{W}^{\rm siv,OPE}_{\rm UT}(b_*(b_T),Q)\tilde{W}^{\rm siv,NP}_{\rm UT}(b_T,Q)$$

$$= \sum_j H_j(\mu_Q,Q)\, \tilde{f}_{1T}^{\perp(1)j}(x,b_*(b_T);\mu^2_{b_*},\mu_{b_*})\, \tilde{D}_1^{h/j}(z,b_*(b_T);\mu^2_{b_*},\mu_{b_*})$$

$$\times \exp\left\{\tilde{K}(b_*(b_T);\bar{\mu})\ln\left(\frac{Q^2}{\mu^2_{b_*}}\right) + \int_{\mu_{b*}}^{\mu_Q}\frac{d\mu'}{\mu'}\left[2\gamma(\alpha_s(\mu');1) - \ln\left(\frac{Q^2}{\mu'^2}\right)\gamma_K(\alpha_s(\mu'))\right]\right\}$$

$$\times \exp\left\{-g_{\rm siv}(x,b_T;Q_0,b_{max}) - g_{\rm ff}(z,b_T;Q_0,b_{max}) - g_K(b_T;b_{max})\ln\left(\frac{Q^2}{Q_0^2}\right)\right\},$$

Matching of the small and large $b_{\rm T}$ behaviour of solution to CSS $b_{\rm max}$

$$b_*(b_T) \equiv \sqrt{\frac{b_T^2}{1+b_T^2/b_{max}^2}}, \qquad \mu_{b_*} \equiv \frac{C_1}{b_*(b_T)},$$

Re-factorization collinear pdfs OPE

$$\tilde{f}_1^{j}(x,b_*(b_T);\mu_{b_*}^2,\mu_{b_*}) = \sum_{j'}\int_x^1 \frac{d\hat{x}}{\hat{x}}\,\tilde{C}^{\rm pdf}_{j/j'}(x/\hat{x},b_*(b_T);\mu_{b_*}^2,\mu_{b_*},\alpha_s(\mu_{b_*}))\, f_1^{j'}(\hat{x};\mu_{b_*}) + O((m\,b_*(b_T))^p)\,,$$

$$\tilde{D}_1^{h/j}(z,b_*(b_T);\mu_{b_*}^2,\mu_{b_*}) = \sum_{i'}\int_z^1 \frac{d\hat{z}}{\hat{z}^3}\,\tilde{C}^{\rm ff}_{i'/j}(z/\hat{z},b_*(b_T);\mu_{b_*}^2,\mu_{b_*},\alpha_s(\mu_{b_*}))\, D_1^{h/i'}(\hat{z};\mu_{b_*}) + O((m\,b_*(b_T))^p)\,,$$

★ Collins-Cambridge Univ Press 2011, Aybat Rogers PRD 2011, Collins Rogers PRD 2015 ....

$$\tilde{f}_{1T}^{\perp(1)j}(x,b_*(b_T);\mu_{b_*}^2,\mu_{b_*}) = -\frac{1}{2M_P}\sum_{j'}\int_x^1 \frac{d\hat{x}_1}{\hat{x}_1}\frac{d\hat{x}_2}{\hat{x}_2}\tilde{C}^{\rm siv}_{j/j'}(\hat{x}_1,\hat{x}_2,b_*(b_T);\mu_{b_*}^2,\mu_{b_*},\alpha_s(\mu_{b_*}))T_F^{j'}(\hat{x}_1,\hat{x}_2;\mu_{b_*}) + O((m\,b_*(b_T))^{p'})\,,$$

★ Kang, Xiao, Yuan PRL 2011

★ Abyat, Collins, Qiu, Rogers PRD 2012

TMD Evolution-Solution for unpolarised

With $\mu_b = C_1/b_*$ as hard scale, the b dependence of TMDs is calculated in perturbation theory & related to collinear parton distribution (PDFs) thru OPE

$$\tilde{f}_1(x, b_T; Q^2, \mu_Q) \sim \left(\tilde{C}^{f_1}(x/\hat{x}, b_*(b_T); \mu_{b_*}^2, \mu_{b_*}, \alpha_s(\mu_{b_*})) \otimes f_1(\hat{x}; \mu_{b_*})\right)$$

Collins (2011); ...

$$\times \exp\left[-S_{pert}(b_*(b_T); \mu_{b_*}, Q, \mu_Q) - S_{NP}^{f_1}(b_T, Q)\right]$$

Collins 2011 QCD Aybat Rogers PRD 2011

Regulating small b Modification CSS *FT*-TMD

$$\tilde{f}_1(x, b_T; Q^2, \mu_Q) \quad \sim \quad \left(\tilde{C}^{f_1}(x/\hat{x}, b_*(b_T); \mu_{b_*}^2, \mu_{b_*}, \alpha_s(\mu_{b_*})) \otimes f_1(\hat{x}; \mu_{b_*})\right)$$

$$\times \; \exp\left[-S_{pert}(b_*(b_T); \mu_{b_*}, Q, \mu_Q) - S_{NP}^{f_1}(b_T, Q)\right]$$

<u>perturbative Sudakov factor</u>

$$\underbrace{-\ln(Q/\mu_{b_*})\tilde{K}(b_*, \mu_{b_*}) - \int_{\mu_{b_*}}^{\mu_Q} \frac{d\mu'}{\mu'}\left[\gamma(\alpha_s(\mu'); 1) - \gamma_K(\alpha_s(\mu'))\ln(Q/\mu')\right]}_{\text{same for unpol. and pol.}}$$

<u>non-perturbative Sudakov factor</u>

$$g_{f_1}(x, b_T) + g_K(b_T)\ln(Q/Q_0)$$

different for each TMD

universal

$$b_*(b_T) \equiv \sqrt{\frac{b_T^2}{1 + b_T^2/b_{\max}^2}} \qquad \mu_{b_*} = C_1/b_*(b_T)$$

<u>Note</u>: $b_*(0) = 0$ **and** $(\mu_{b_*})_{b_* \to 0} = \infty$ ➡ problematic large logarithms in S_{pert}

(Bozzi, Catani, de Florian, Grazzini (2006); Collins, Gamberg, Prokudin, Rogers, Sato, Wang (2016))

Parsi Petronzio NPB 1979, Altarelli et al. NPB 1984 CSS NPB250, Bozzi Catani, de Florian Grazzini NPB 2006

$bQ << 1$ contributions to the W term

- Addressed in "q_T resummation" Parsi Petronzio NPB 1979, Altarelli et al. NPB 1984 CSS NPB250, Bozzi Catani, de Florian Grazzini NPB 2006
- Regulate the large logs(Q^2b^2) at small b in the *FT* they Bozzi et al. , replace L=logs(Q^2b^2) with L=logs(Q^2b^2+1) cutting off the $b << 1/Q$ contribution
- Also Kulesza,Sterman,Vogelsang PRD 2002 in threshold resummation studies

We place "another" boundary condition on now small b_T in "TMD CSS analysis" Collins, Gamberg, Prokudin, Rogers, Sato, Wang PRD 2016

"Improved CSS" (Unpolarized) (Collins, Gamberg, Prokudin, Rogers, Sato, Wang (2016))*

Place a lower cut-off on b_T: $b_T \longrightarrow b_c(b_T)$ where $b_c(b_T) = \sqrt{b_T^2 + \left(\frac{b_0}{C_5 Q}\right)^2} = \sqrt{b_T^2 + b'^2_{min}}$,

$\Rightarrow$ $\mu_{b_*} \rightarrow \bar{\mu} \equiv \dfrac{C_1}{b_*(b_c(b_T))}$ so μ_{b_*} is cut off at $\mu_c \approx C_1 Q$

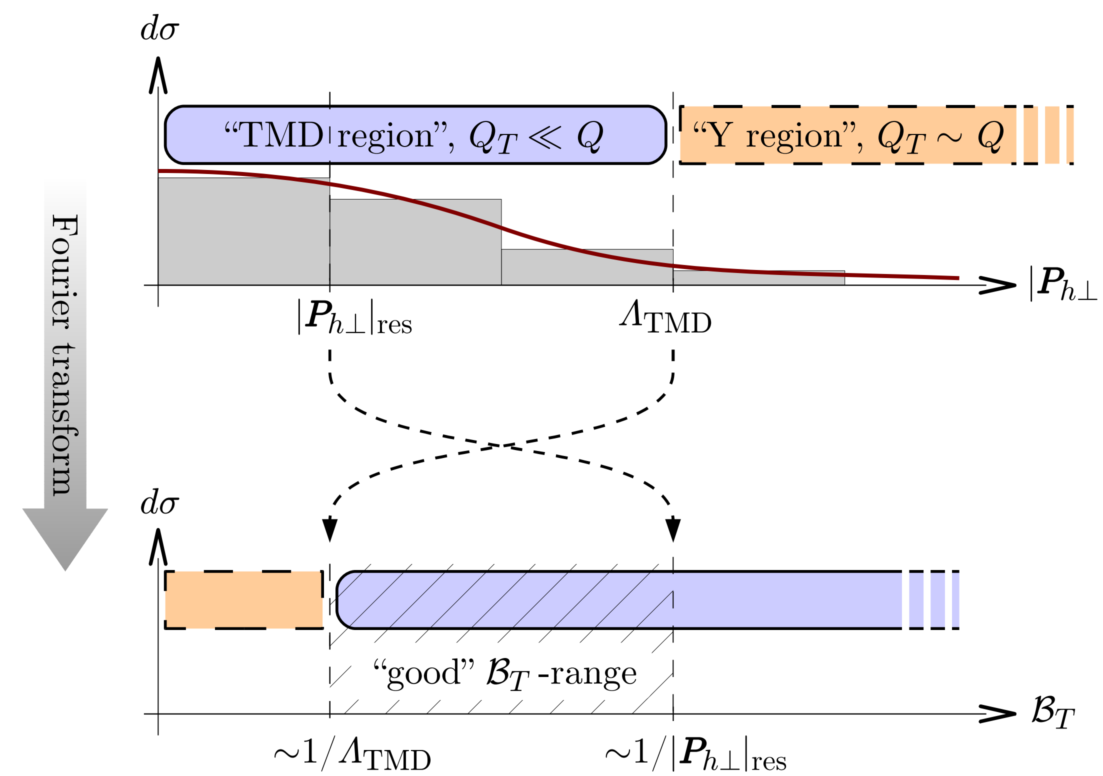
Modification to CSS W Term

B.C. Introduce small b-cuttoff

Similar to Catani et al. NPB 2006 & "Bessel Weighting" ppr. Boer LG Musch Prokudin JHEP 2011

$$b_c(b_T) = \sqrt{b_T^2 + b_0^2/\left(C_5 Q\right)^2} \implies b_c(0) \sim 1/Q$$

Regulate unphysical divergences from in W term



$$\tilde{W}_{New}(q_T, Q; \eta, C_5) = \Xi\left(\frac{q_T}{Q}, \eta\right) \int \frac{d^2 b_T}{(2\pi)^2} e^{i q_T \cdot b_T} \tilde{W}^{OPE}\left(b_*(b_c(b_T)), Q\right) \tilde{W}_{NP}(b_c(b_T)), Q; b_{max})$$

Generalized B.C.

$$b_*(b_c(b_{\rm T})) \longrightarrow \begin{cases} b_{\rm min} & b_{\rm T} \ll b_{\rm min} \\ b_{\rm T} & b_{\rm min} \ll b_{\rm T} \ll b_{\rm max} \\ b_{\rm max} & b_{\rm T} \gg b_{\rm max}\,. \end{cases}$$

Modified *F.T.*-TMDs enhanced CSS

"Improved CSS" (Unpolarized) (Collins, Gamberg, Prokudin, Rogers, Sato, Wang (2016))

Place a lower cut-off on b_T: $b_T \to b_c(b_T)$ where $b_c(b_T) = \sqrt{b_T^2 + b_0^2/(C_5 Q)^2}$

$$\Rightarrow \quad \mu_{b_*} \to \bar{\mu} \equiv \frac{C_1}{b_*(b_c(b_T))} \text{ so } \mu_{b_*} \text{ is cut off at } \mu_c \approx \frac{C_1 C_5 Q}{b_0}$$

$$\tilde{\boldsymbol{f}}_{\boldsymbol{1}}(\boldsymbol{x}, \boldsymbol{b_c(b_T)}; \boldsymbol{Q^2}, \boldsymbol{\mu_Q}) \sim \left(\tilde{C}^{f_1}(x/\hat{x}, b_*(b_c(b_T)); \bar{\mu}^2, \bar{\mu}, \alpha_s(\bar{\mu})) \otimes \boldsymbol{f_1(\hat{x}; \bar{\mu})}\right) \times \exp\left[-S_{pert}(b_*(b_c(b_T)); \bar{\mu}, Q, \mu_Q) - S_{NP}^{f_1}(b_c(b_T), Q)\right]$$

"Improved CSS" (Polarized) (Gamberg, Metz, DP, Prokudin, Phys. Lett B (2018))

$$\tilde{\Phi}^{[\gamma^+]}(x, \vec{b}_T; Q^2, \mu_Q) = \tilde{\boldsymbol{f}}_{\boldsymbol{1}}(\boldsymbol{x}, \boldsymbol{b_T}; \boldsymbol{Q^2}, \boldsymbol{\mu_Q}) - iM\epsilon^{ij} b_T^i S_T^j \tilde{\boldsymbol{f}}_{\boldsymbol{1T}}^{\perp(\boldsymbol{1})}(\boldsymbol{x}, \boldsymbol{b_T}; \boldsymbol{Q^2}, \boldsymbol{\mu_Q})$$

b_T -> $b_c(b_T)$

NO *b_T -> $b_c(b_T)$* replacement – kinematic factor NOT associated with the scale evolution

b_T -> $b_c(b_T)$

Unpolarized and Sivers W term & TMDs

$$\tilde{W}_{\mathrm{UU}}(b_c(b_T),Q) = \sum_j H_j(\mu_Q,Q)\, \tilde{f}_1^j(x,b_c(b_T);Q^2,\mu_Q)\, \tilde{D}_1^{h/j}(z,b_c(b_T);Q^2,\mu_Q)\,,$$

$$\tilde{W}_{\mathrm{UT}}^{\mathrm{siv}}(b_c(b_T),Q) = \sum_j H_j(\mu_Q,Q)\, \tilde{f}_{1T}^{\perp(1)j}(x,b_c(b_T);Q^2,\mu_Q)\, \tilde{D}_1^{h/j}(z,b_c(b_T);Q^2,\mu_Q)\,.$$

Unpolarized *FT* TMD

✦ **Phys. Lett B (2018) Gamberg , Metz, Pitonyak, Prokudin**

$$\tilde{f}_1^j(x,b_c(b_T);Q^2,\mu_Q) = \sum_{j'} \int_x^1 \frac{d\hat{x}}{\hat{x}}\, \tilde{C}^{\mathrm{pdf}}_{j/j'}(x/\hat{x}, b_*(b_c(b_T));\bar{\mu}^2,\bar{\mu},\alpha_s(\bar{\mu}))\, f_1^{j'}(\hat{x};\bar{\mu})$$
$$\times \exp\left\{\tilde{K}(b_*(b_c(b_T));\bar{\mu})\ln\left(\frac{Q}{\bar{\mu}}\right) + \int_{\bar{\mu}}^{\mu_Q}\frac{d\mu'}{\mu'}\left[\gamma(\alpha_s(\mu');1) - \ln\left(\frac{Q}{\mu'}\right)\gamma_K(\alpha_s(\mu'))\right]\right\}$$
$$\times \exp\left\{-g_{\mathrm{pdf}}(x,b_c(b_T);Q_0,b_{max}) - g_K(b_c(b_T);b_{max})\ln\left(\frac{Q}{Q_0}\right)\right\}\,,$$

Sivers *FT* TMD or 1st Bessel moment

$$\tilde{f}_{1T}^{\perp(1)j}(x,b_c(b_T);Q^2,\mu_Q) = -\frac{1}{2M_P}\sum_{j'}\int_x^1 \frac{d\hat{x}_1}{\hat{x}_1}\frac{d\hat{x}_2}{\hat{x}_2}\, \tilde{C}^{\mathrm{siv}}_{j/j'}(\hat{x}_1,\hat{x}_2,b_*(b_c(b_T));\bar{\mu}^2,\bar{\mu},\alpha_s(\bar{\mu}))\, T_F^{j'}(\hat{x}_1,\hat{x}_2;\bar{\mu})$$
$$\times \exp\left\{\tilde{K}(b_*(b_c(b_T));\bar{\mu})\ln\left(\frac{Q}{\bar{\mu}}\right) + \int_{\bar{\mu}}^{\mu_Q}\frac{d\mu'}{\mu'}\left[\gamma(\alpha_s(\mu');1) - \ln\left(\frac{Q}{\mu'}\right)\gamma_K(\alpha_s(\mu'))\right]\right\}$$

Taking small b limit relate TMD and Collinear factorization

- **Relies on modification of *W+Y* construction**
- **Collins, Gamberg, Prokudin, Sato, Rogers, Wang PRD 2016**

$$\frac{d\sigma}{dx\,dy\,d\phi_S\,dz} \equiv 2z^2 \int d^2\boldsymbol{q}_{\rm T}\,\Gamma(\boldsymbol{q}_{\rm T}, Q, S) = 2z^2\,\tilde{W}^{\rm OPE}_{\rm UU}(b'_{min}, Q)_{\rm LO} + O(\alpha_s(Q)) + O((m/Q)^p)$$

$$= \frac{2\alpha_{em}^2}{yQ^2}(1-y+y^2/2)\sum_j e_j^2\, f_1^j(x;\mu_c)\, D_1^{h/j}(z;\mu_c) + O(\alpha_s(Q)) + O((m/Q)^p)$$

- **Gamberg , Metz, Pitonyak, Prokudin PLB 2018**

$$\frac{d\langle P_{h\perp}\,\Delta\sigma(S_T)\rangle}{dx\,dy\,dz} = -4\pi z^3 M_P\,\tilde{W}^{\rm siv,OPE}_{\rm UT}(b'_{min}, Q)_{\rm LO} + O(\alpha_s(Q)) + O((m/Q)^{p'})$$

$$= \frac{2\pi\, z\,\alpha_{em}^2}{yQ^2}(1-y+y^2/2)\sum_j e_j^2\, T_F^j(x,x;\mu_c)\, D_1^{h/j}(z;\mu_c) + O(\alpha_s(Q)) + O((m/Q)^{p'})$$

Agrees with collinear twist-3 result at leading order

Z.-B.Kang,Vitev, Xing,PRD(2013)

Relationship between moments of regularised TMDs and collinear pdfs
LO result-done in b-space w/ the OPE - small b region

Relies on the small b limit with $b_{\rm min}$ cutoff $\quad b_{min} \propto \frac{1}{Q}$

$$\int d^2\boldsymbol{k}_{\rm T}\, f_1^j(x, k_T; Q^2, \mu_Q; C_5) = \tilde{f}_1^j(x, b'_{min}; Q^2, \mu_Q) = f_1^j(x; \mu_c) + O(\alpha_s(Q))$$

$$z^2 \int d^2\boldsymbol{p}_{\rm T}\, D_1^j(z, p_T; Q^2, \mu_Q; C_5) = z^2 \tilde{D}_1^{h/j}(z, b'_{min}; Q^2, \mu_Q) = D_1^{h/j}(z; \mu_c) + O(\alpha_s(Q))\,,$$

Because the evolution kernel is same …

$$\int d^2\boldsymbol{k}_{\rm T} \frac{k_T^2}{2M_P^2}\, f_{1T}^{\perp j}(x, k_T; Q^2, \mu_Q; C_5) = \tilde{f}_{1T}^{\perp(1)j}(x, b'_{min}; Q^2, \mu_Q) = \frac{-1}{2M_P} T_F^j(x, x; \mu_c) + O(\alpha_s(Q))$$

$$\int d^2\boldsymbol{p}_{\rm T}\, \frac{p_T^2}{z^2 2M_h^2}\, H_1^{\perp j}(z, p_T; Q^2, \mu_Q; C_5) = \tilde{H}_1^{\perp(1)j}(z, b'_{min}; Q^2, \mu_Q) = \tilde{H}_1^{\perp(1)j}(z, \mu_c) + O(\alpha_s(Q))$$

✦ **Phys. Lett B (2018) Gamberg , Metz, Pitonyak, Prokudin**

Moments of TMDs and collinear pdfs

Naive connection of moments of TMDs and collinear pdfs based on matrix elements and a Parton Model picture "factorization" preserved

.

TMD | kinematical CT3 | dynamical CT3

$$\int d^2\vec{k}_T\,\frac{\vec{k}_T^2}{2M^2}\quad \boldsymbol{f_{1T}^{\perp}(x,k_T)} \quad = \quad \boldsymbol{f_{1T}^{\perp(1)}(x)} \quad = \quad -\frac{T_F(x,x)}{2M}$$

Qiu & Sterman 1991

Boer, Mulder, Pijlman (2003); Meissner (2009); ...

$\vdots$

TMD | kinematical CT3

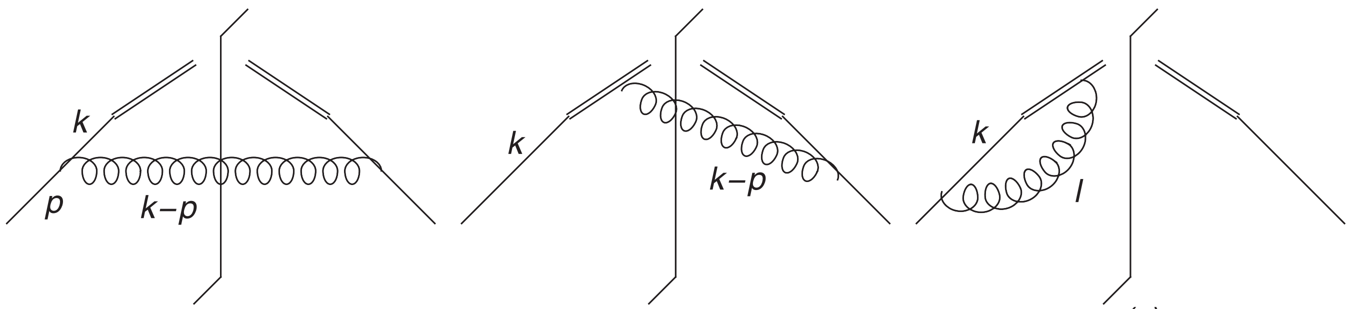
$$\int d^2\vec{p}_T\,\frac{\vec{p}_T^{\,2}}{2z^2M_h^2}\quad \boldsymbol{H_1^{\perp}(z,p_T)} \quad = \quad \boldsymbol{H_1^{\perp(1)}(z)}$$

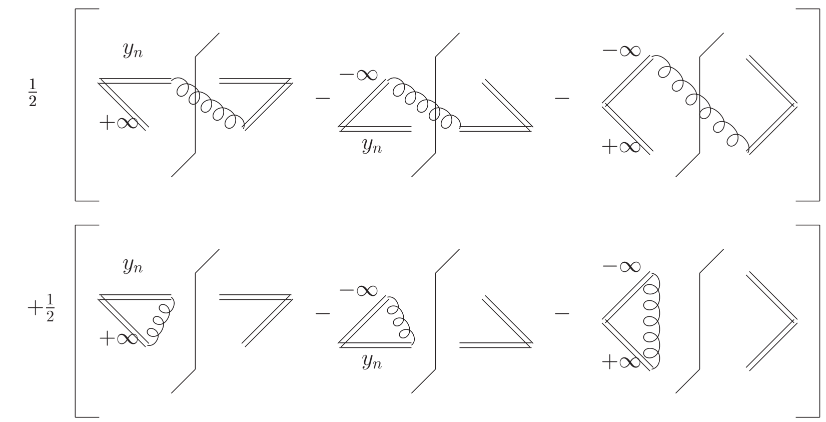
Yuan and Zhou (2009)

$\vdots$

Investigate at NLO

At small b can calculate coefficient function
CS NPB 1982, JCC 2011 Cambridge press, Aybat Roger PRD 2011, Bacchetta & Prokudin NPB 2013





$$\tilde{C}^{[1]}_{j/f}(x, \mathbf{b}_T) = \tilde{F}^{[1]}_{j/f}(x, \mathbf{b}_T) - f^{[1]}_{j/f}(x)$$

$$\begin{aligned}\tilde{C}^{\mathrm{PDF}}_{f/j}(x, b_{\mathrm{T}}; \zeta_{\mathrm{PDF}}, \mu, \alpha_s(\mu)) =&\delta_{fj}\delta(1-x) + \delta_{fj} 2C_{\mathrm{F}}\left\{2\ln\left(\frac{2e^{-\gamma_{\mathrm{E}}}}{\mu b_{\mathrm{T}}}\right)\left[\left(\frac{2}{1-x}\right)_+ - 1 - x\right] + 1 - x \right.\\ &\left. - \delta(1-x)\left[\frac{1}{2}\left[\ln\left(\frac{b_{\mathrm{T}}\mu}{2e^{-\gamma_{\mathrm{E}}}}\right)^2\right]^2 + \ln\left(\frac{b_{\mathrm{T}}\mu}{2e^{-\gamma_{\mathrm{E}}}}\right)^2 \ln\left(\frac{\zeta_{\mathrm{PDF}}}{\mu^2}\right)\right]\right\}\left(\frac{\alpha_s(\mu)}{4\pi}\right)\\ &+ \mathcal{O}\left(\left(\frac{\alpha_s(\mu)}{4\pi}\right)^2\right)\end{aligned}$$

Note the coefficient function is IR safe the collinear divergence is subtracted; we get NLO correction from replacing

Matching the TMD and ETQS description of TSSAs

PHYSICAL REVIEW D **73,** 094017 (2006)

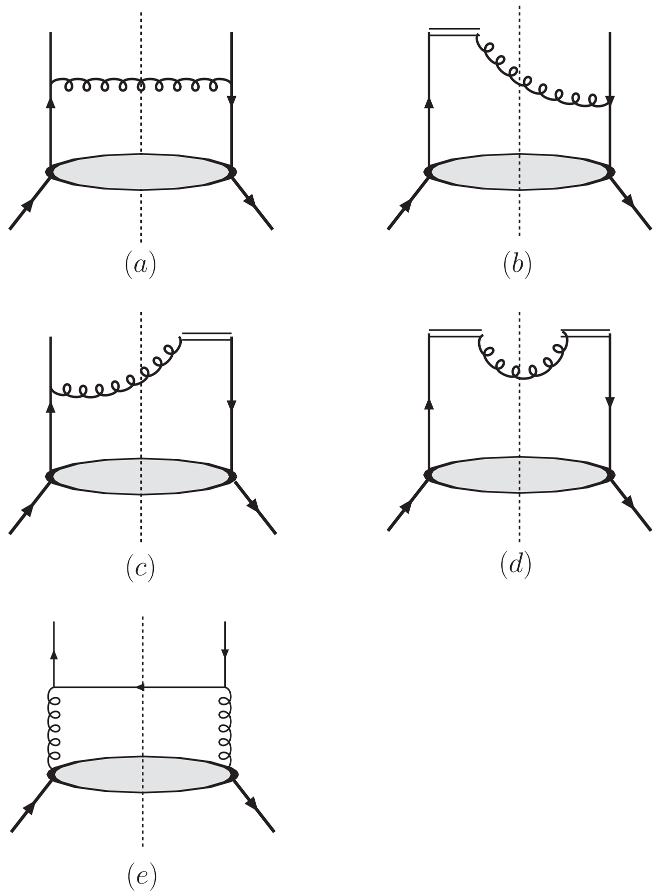


FIG. 8. Feynman diagrams contributing to the spin-independent quark distribution at large transverse momentum.

$$q(z, k_\perp) = \frac{\alpha_s}{2\pi^2}\frac{1}{\vec{k}_\perp^2}C_F\int\frac{dx}{x}q(x) \times\left[\frac{1+\xi^2}{(1-\xi)_+} + \delta(\xi-1)\left(\ln\frac{z^2\zeta^2}{\vec{k}_\perp^2}-1\right)\right] + \frac{\alpha_s}{2\pi^2}\frac{1}{\vec{k}_\perp^2}T_R\int\frac{dx}{x}g(x)[\xi^2+(1-\xi)^2], \quad (36)$$

Inserting the perturbative TMD distribution (36) and the soft function (41) into the factorization formula (40), we find that indeed the unpolarized cross section given by Eqs. (26) and (27) is reproduced, including the quark-gluon scattering piece. Here we use the above normalization condition for the soft function, and the normalization that the integration over the TMD distribution yields the normal Feynman parton distributions,

$$\int d^2\vec{k}_\perp q(z_1, k_\perp) = q(z_1), \qquad \int d^2\vec{k}_\perp \bar{q}(z_2, k_\perp) = \bar{q}(z_2). \quad (42)$$

Ji, Qiu, Vogelsang, Yuan PRL, PRD 2006

Calculation of soft and hard poles for "Sivers like function"

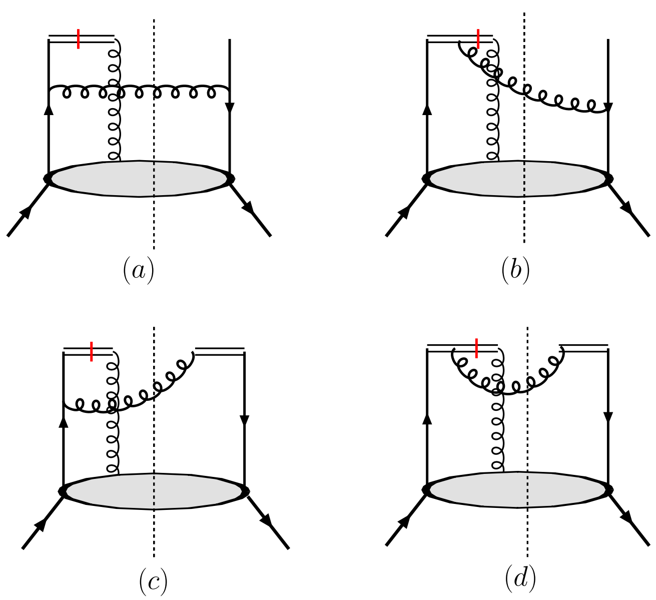


FIG. 9 (color online). Feynman diagrams contributing to the Sivers functions at a large transverse momentum.

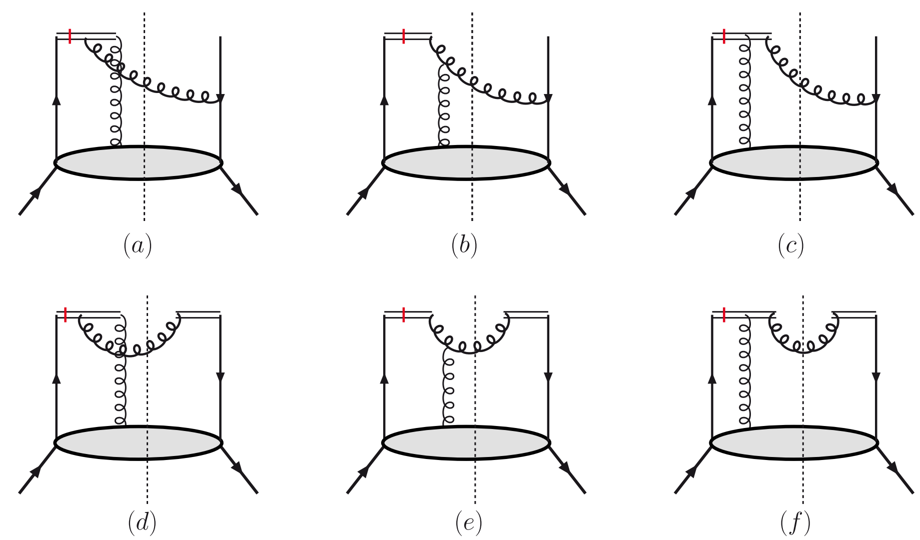


FIG. 10 (color online). Same as Fig. 9, but for the hard-pole contributions.

We have given the first-order perturbative result for the Sivers function in Eq. (39). As was shown in [19] (see also [22,23]), its $k_\perp^2$-moment is related to the twist-three quark-gluon correlation function defined in Eq. (3) of Sec. II:

$$\frac{1}{M_P}\int d^2\vec{k}_\perp \vec{k}_\perp^2 q_T(x, k_\perp) = T_F(x, x). \qquad (44)$$

Ji, Qiu, Vogelsang, Yuan

Goes like

$$f_{j/P}^{\overline{ms}}\left(x,Q^2;\mu_F^2\right) = \int \frac{dz}{z} f^0_{j/P}\left(\frac{x}{z}\right)\left(\delta(1-z) + \frac{\alpha_s}{2\pi} P_{q/a}(z)\left(\ln\frac{Q^2}{\mu_F^2} + \text{finite terms}\right)\right)$$

$$\Longrightarrow$$

$$f_{j/P}^{\overline{ms}}\left(x,Q^2;\mu_F^2\right) = \int \frac{dz}{z} f^0_{j/P}\left(\frac{x}{z}\right)\delta(1-z) + \frac{\alpha_s}{2\pi} C_F \left\{ \ln\frac{Q^2}{\mu_F^2}\left(\frac{1}{C_F} P_{q/a}(z)\right) - \delta(1-x)\left(\frac{3}{2} + \frac{1}{2}\ln^2\frac{Q^2}{\mu_F^2}\right)\right.$$

$$+ \quad \text{finite} \quad \text{terms}\ldots$$

NLO result (LO) splitting function in limit w/ soft gluons
work in progress w/ Pitonyak, Prokudin & et al. TMDc

$$b_c(b_T) = \sqrt{b_T^2 + b_0^2/\left(C_5 Q\right)^2} \implies b_c(0) \sim 1/Q$$

Vogelsang INT talk 2/27/14 gets result in joint resummation;
again collinear divergence is subtracted in similar manner

Relating full q_T cross section to collinear under investigation …

- Vogelsang et al. joint resumption
- Catani's unitarity condition on $W+Y$
- …

Comments

- With our method, the redefined *W* term allowed us to construct a relationship between TMD-factorization formulas and standard collinear factorization formulas, with errors relating the two being suppressed by powers of 1/Q
- Importantly, the exact definitions of the TMD pdfs and ffs are unmodified from the usual ones of factorization derivations. We preserve transverse-coordinate space version of the W term, but only modify the way in which it is used
- **We have applied to transverse polarized phenomena**
- We are able to recover the well-known relations between TMD and collinear quantities expected from the leading order parton model picture operator definition
- We recover the LO collinear twist 3 result from a weighted q_T integral of the differential cross section and derive the well known relation between the TMD Sivers function and the collinear twist 3 Qiu Sterman function from iCSS approach

Backup

Matching and *W* + *Y to collinear Factorization*

$$\int d^2\boldsymbol{q}_T \frac{d\sigma}{d^2\boldsymbol{q}_T \ldots} = \int d^2\boldsymbol{q}_T \, W + \int d^2\boldsymbol{q}_T \, Y$$

A second/third issue is the problem of matching the TMD factorized cross section integrated over q_T to the collinear factorization formalism.

<u>LHS</u>, In QCD the cross section integrated over all q_T; it is of the form of factors of collinear parton densities and/or fragmentation functions at scale Q convoluted with hard scattering that is expanded in powers of $\alpha_S(Q)$

<u>RHS</u>

1) Integral

$$\int d^2\boldsymbol{q}_T \, W(\boldsymbol{q}_T, Q, S) = \tilde{W}_{UU}(b_T \to 0, Q)$$
$$\sim b_T^a \times (\log\text{ corrections}) = 0\,,$$
$$a = 8C_F/\beta_0, \quad \beta_0 = 11 - 2n_f/3$$

2) Using collinear factorization the *Y* term "starts" at NLO $\alpha_s^{[1]}$

b-Dependence driven by perturbative part of ev. Kernel

$$\exp\left[\int_{\mu_{b*}}^{\mu_Q} \frac{d\mu'}{\mu'}\left[2\gamma(\alpha_s(\mu');1) - 2\ln\left(\frac{Q}{\mu'}\right)\gamma_K(\alpha_s(\mu'))\right]\right]$$

$$\begin{aligned}
\tilde{W}(b_T \to 0, Q) &\sim \exp\left[\frac{C_F}{\pi\beta_0}\int_{\ln\mu_b^2}^{\ln\mu_Q^2} \ln\mu'^2\right] = \exp\left[-\frac{C_F}{\pi\beta_0}\ln\left(\frac{\mu_b^2}{\mu_Q^2}\right)\right] \\
&= \exp\left[-\frac{C_F}{\pi\beta_0}\ln\left(\frac{C_1^2}{b_T^2\mu_Q^2}\right)\right] \\
&= b_T^a \quad \text{where, } {\color{red} a = 2C_F/(\pi\beta_0) > 0} \\
&\to 0
\end{aligned}$$

Must regulate the large logs in $b_T Q$

TMD Evolution-Solution for unpolarised

With $\mu_b = C_1/b_*$ as hard scale, the b dependence of TMDs is calculated in perturbation theory and related to their collinear parton distribution (PDFs), fragmentation functions (FFs), or multiparton correlation functions, thru an OPE

$$\tilde{f}_1(x, b_T; Q^2, \mu_Q) \sim \left(\tilde{C}^{f_1}(x/\hat{x}, b_*(b_T); \mu_{b_*}^2, \mu_{b_*}, \alpha_s(\mu_{b_*})) \otimes \boldsymbol{f_1(\hat{x}; \mu_{b_*})}\right)$$

Collins (2011); ...

$$\times \exp\left[-S_{pert}(b_*(b_T); \mu_{b_*}, Q, \mu_Q) - S_{NP}^{f_1}(b_T, Q)\right]$$

Also relation to Parton Model?

Turn off α_s don't get back parton model

$$\tilde{f}(x, b_{\rm T}; \zeta, \mu) \to f_{j/P}(x) \exp\left\{\left(g_{j/P}(x, b_{\rm T}) + g_k(b_{\rm T}) \ln\frac{Q}{Q_0}\right)\right\}$$

$$= f_{j/P}(x) \exp\left\{\left(g_1 + g_2 \ln\frac{Q}{Q_0}\right)\frac{b_{\rm T}^2}{2}\right\}$$

Collins 2011 QCD Aybat Rogers PRD 2011

TMD factorization & evolution from *b*-space rep of SIDIS cross section interpret as a multipole expansion in terms of $b_T\,[\mathrm{GeV}^{-1}]$ conjugate $\boldsymbol{P}_{h\perp}$

$\tilde{W}_{UU}(x, z, b, Q^2)$

Boer, Gamberg, Musch, Prokudin, **JHEP (2011)**

$$
\begin{aligned}
&\frac{d\sigma}{dx_B\, dy\, d\phi_S\, dz_h\, d\phi_h\, |\boldsymbol{P}_{h\perp}| d|\boldsymbol{P}_{h\perp}|} = \\
&\frac{\alpha^2}{x_B y Q^2}\frac{y^2}{(1-\varepsilon)}\left(1+\frac{\gamma^2}{2x_B}\right)\int\frac{d|\boldsymbol{b}_T|}{(2\pi)}|\boldsymbol{b}_T|\Big\{J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UU,T} + \varepsilon\, J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UU,L} \\
&+ \sqrt{2\,\varepsilon(1+\varepsilon)}\,\cos\phi_h\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UU}^{\cos\phi_h} + \varepsilon\cos(2\phi_h)\, J_2(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UU}^{\cos(2\phi_h)} \\
&+ \lambda_e\sqrt{2\,\varepsilon(1-\varepsilon)}\,\sin\phi_h\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{LU}^{\sin\phi_h} \\
&+ S_\parallel\Big[\sqrt{2\,\varepsilon(1+\varepsilon)}\,\sin\phi_h\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UL}^{\sin\phi_h} + \varepsilon\sin(2\phi_h)\, J_2(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UL}^{\sin 2\phi_h}\Big] \\
&+ S_\parallel\lambda_e\Big[\sqrt{1-\varepsilon^2}\, J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{LL} + \sqrt{2\,\varepsilon(1-\varepsilon)}\,\cos\phi_h\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{LL}^{\cos\phi_h}\Big] \\
&+ |\boldsymbol{S}_\perp|\Big[\sin(\phi_h-\phi_S)\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\left(\mathcal{F}_{UT,T}^{\sin(\phi_h-\phi_S)} + \varepsilon\,\mathcal{F}_{UT,L}^{\sin(\phi_h-\phi_S)}\right) \\
&\quad + \varepsilon\sin(\phi_h+\phi_S)\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UT}^{\sin(\phi_h+\phi_S)} \\
&\quad + \varepsilon\sin(3\phi_h-\phi_S)\, J_3(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UT}^{\sin(3\phi_h-\phi_S)}
\end{aligned}
$$

$$\mathcal{F}_{UT,T}^{\sin(\phi_h-\phi_S)} = -\mathcal{P}[\tilde{f}_{1T}^{\perp(1)}\tilde{D}_1]$$

$$\mathcal{F}_{UT}^{\sin(\phi_h+\phi_S)} = -\mathcal{P}[\tilde{h}_1\tilde{H}_1^{\perp(1)}]$$

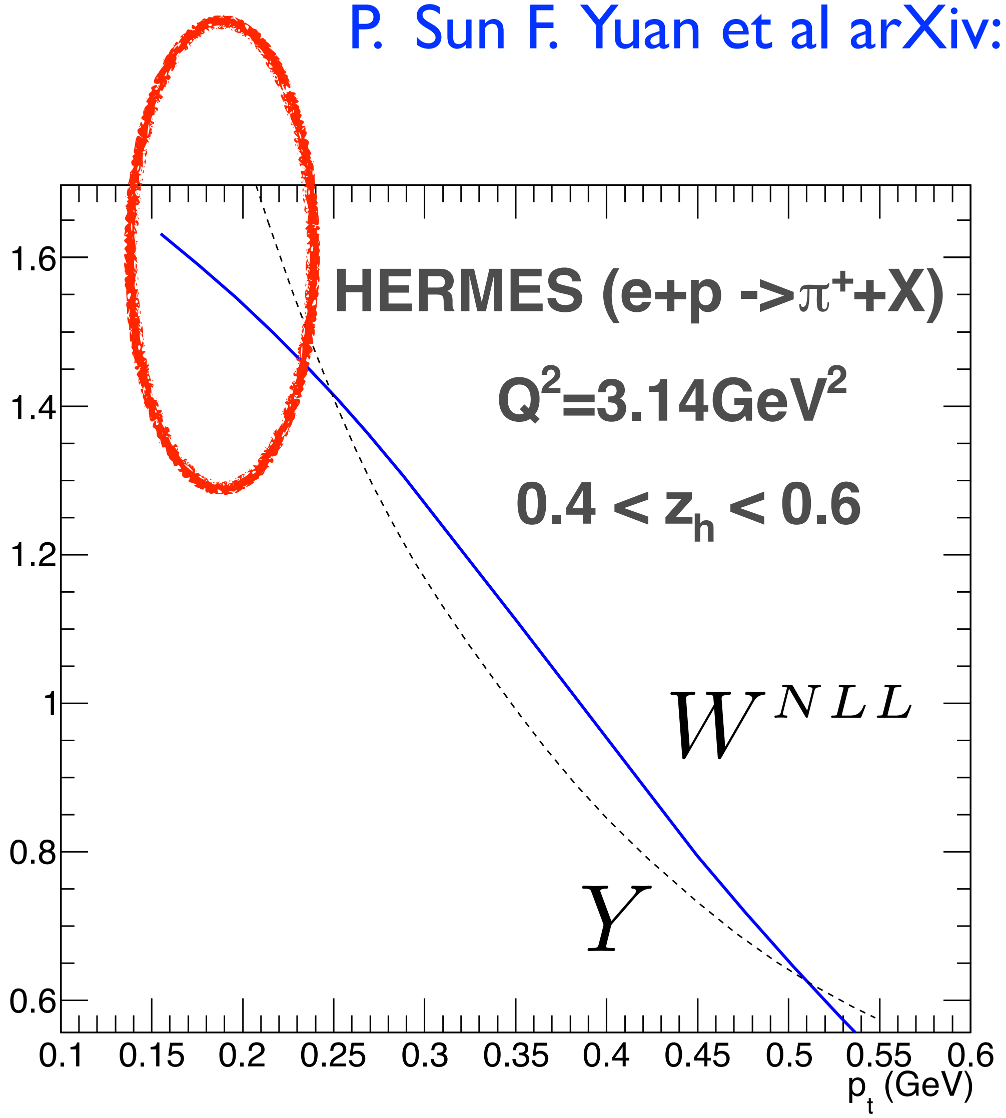
$$\ldots \quad + \quad Y$$

Matching and $W + Y$ -studies low q_T

- At small q_T the Y term is in principle suppressed: it is the difference of the FO perturbative calculation of the cross section and the asymptotic contribution of W for small q_T

- But there can be a difference of of large terms and truncation errors are augmented: **Here the *Y* term is larger than *W* ?!**

P. Sun F. Yuan et al arXiv: 1406.3073



$$Y(q_T, Q) = FO(q_T, Q) - ASY(q_T, Q)$$

Matching and $W + Y$ -studies large q_{T}

Compass Example

Boglione Prokudin et al. JHEP 2015

- When q_T is above some small fraction of Q, W deviates alot from $d\sigma(q_T, Q)$
- Then it becomes negative and "asymptotes" to $\frac{1}{q_T^2} \log \frac{Q^2}{q_T^2}$

Nadolsky et al. PRD 1999, Y. Koike, J. Nagashima, and W. Vogelsang, NPB744, 59 (2006)

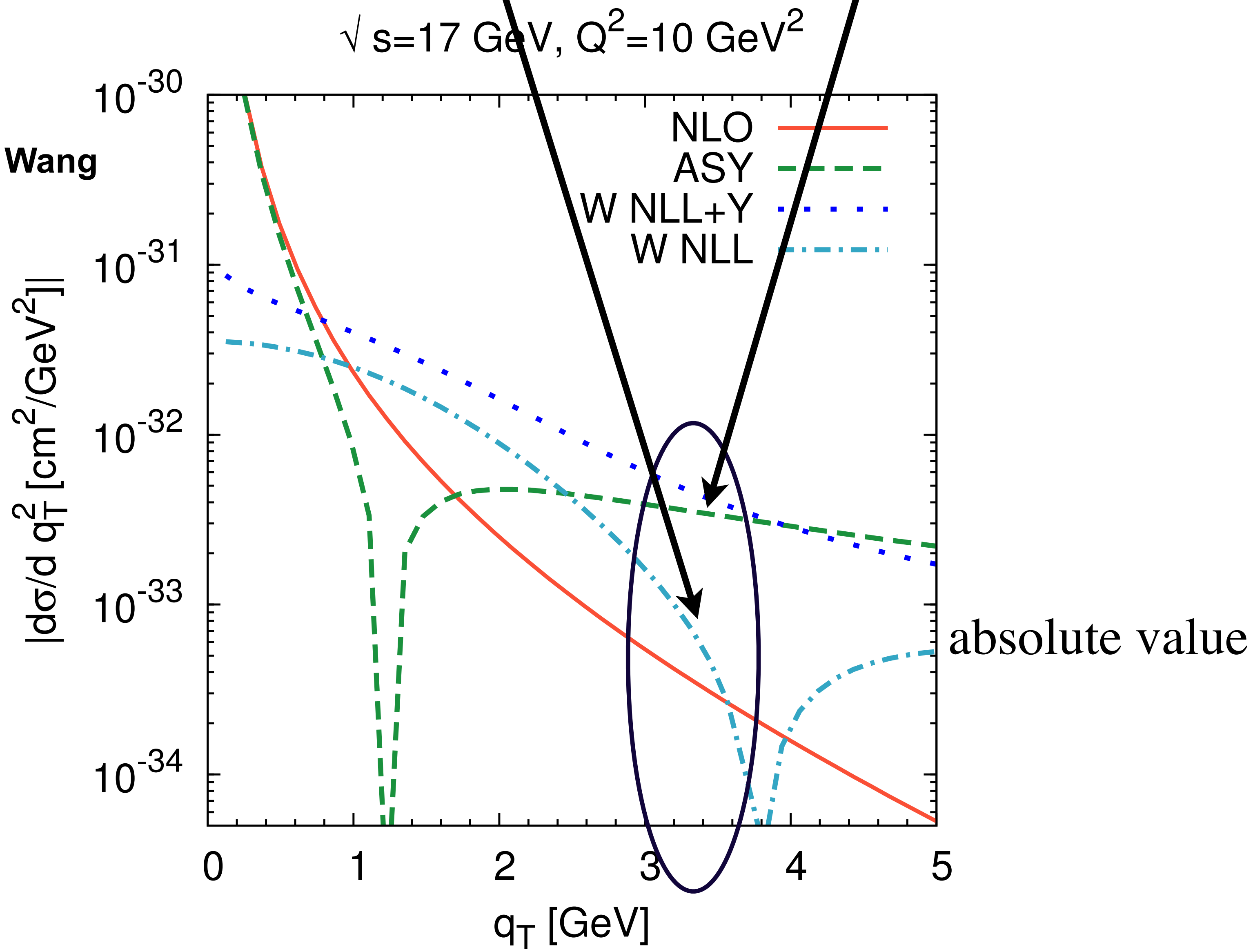
- At large q_T $W+Y$ is then difference of large terms and *truncation errors* can be augmented (ASY!)

PRD 94 2016 Collins, Gamberg, Prokudin, Sato, Rogers, Wang

Matching becomes a challenge COMPASS/Jlab like energies

Gonzalez, Rogers, Sato, Wang arXiv:1808.04396

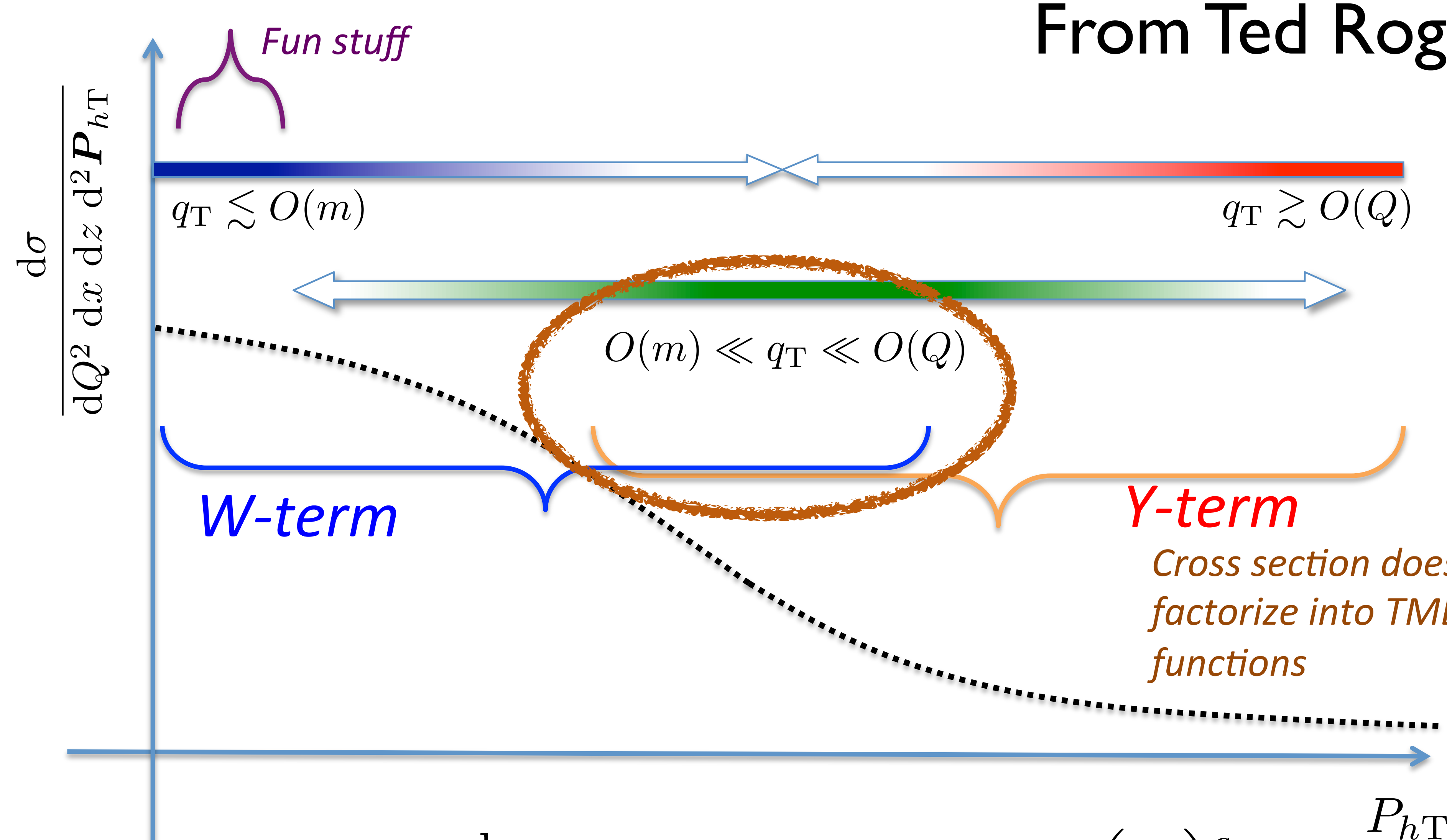
See Alexei's talk



“Matching-1” and $W + Y$-schematic

- However at lower phenomenologically interesting values of Q, neither of the ratios q_T/Q or m/q_T are necessarily very small and matching can be problematic—small “matching region” & resulting in differences of large quantities

From Ted Rogers



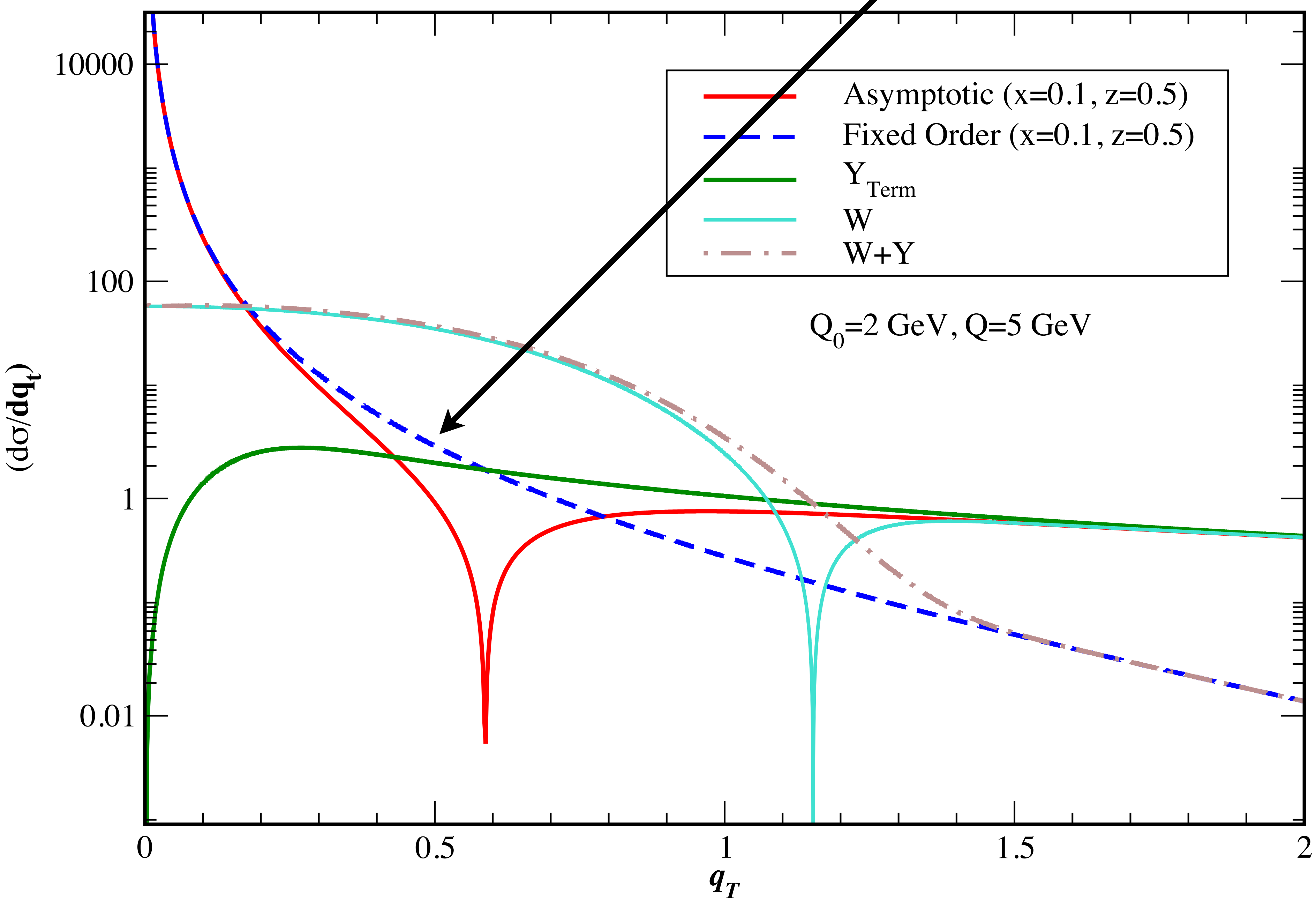
note $P_{hT} = z q_T$

$$\frac{d\sigma}{dQ^2\; dx\; dz\; d^2\boldsymbol{P}_{hT}} = W + Y + O\left(\frac{m}{Q}\right)^c$$

Implementation of Collins, Gamberg, Prokudin, Sato, Rogers, Wang PRD 2016

- Can extend the power suppression error estimate down to $q_T = 0$ to get

$$d\sigma(q_T \lesssim Q, Q) = W(q_T, Q) + Y(q_T, Q) + O\left(\frac{m}{Q}\right)^c d\sigma(q_T, Q)$$



Use analytic expressions for the collinear correlation functions, from GRV ZPC 1992 for up-quark pdf and from KKP NPB 2001 for the up-quark-to-pion ffs.