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FOR GEOMETRY AND PHYSICS

QFT the SM and Electroweak Symmetry Breaking

Luis Alvarez-Gaume Arequipa CLASHEP March 6-13 2019

Based on lecture notes written with M.A.Vázquez-Mozo

Introduction

QFT provides us with language to formulate the more basic laws of Nature

It is a deep and difficult subject. Listening to it from different speakers gives complementary perspectives

Some of its basic assumptions are currently being tested

UV completion of the SM and its behavior at large scales are generating a number of interesting puzzles. Perhaps we are about to shift the paradigm…

We will prime concepts and a general vision, few computations will be done, and the advanced topics are included either for discussion sessions or for further information on a rather big subject. There are excellent textbooks to study QFT, so we will cover only some highlights, whose choice of course depends on my prejudices…

Advanced Topics

Outline not in order…

- Why Quantum Field Theory?
- Quantization
- Kinematical symmetries
- Global symmetries
- Local symmetries
- Discrete symmetries
- Broken symmetries
- Scale symmetries, renormalisation
- Standard Model symmetries
- Amusing examples throughout time permitting

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Range of applications

Non-relativistic theories: Many body Physics, e.g. Quantum Liquids

Interaction of radiation with matter (QED)

Precise description of the strong interactions (QCD)

Successful EW unification leading to the GWS formulation of the SM

Routinely tested at all known accelerators, present and past

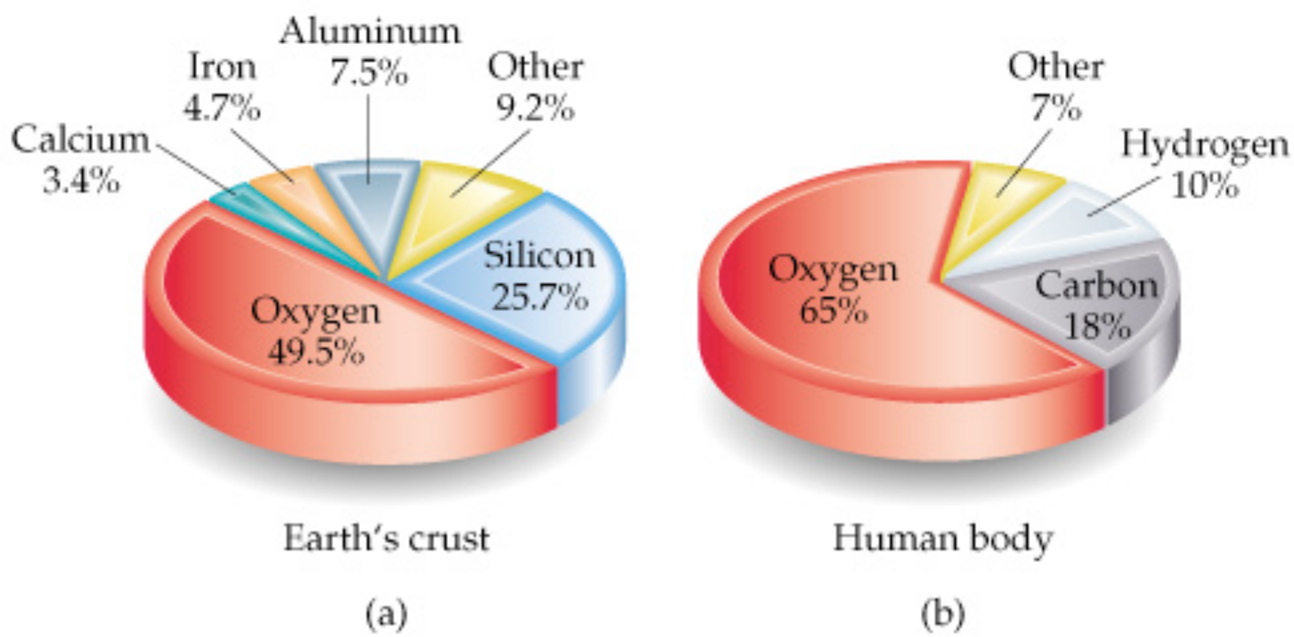
Spectacular application to the Large Scale Structure of the Universe

Basic description of symmetries, unbroken and broken...

CERN

Range of applications: nucleosynthesis

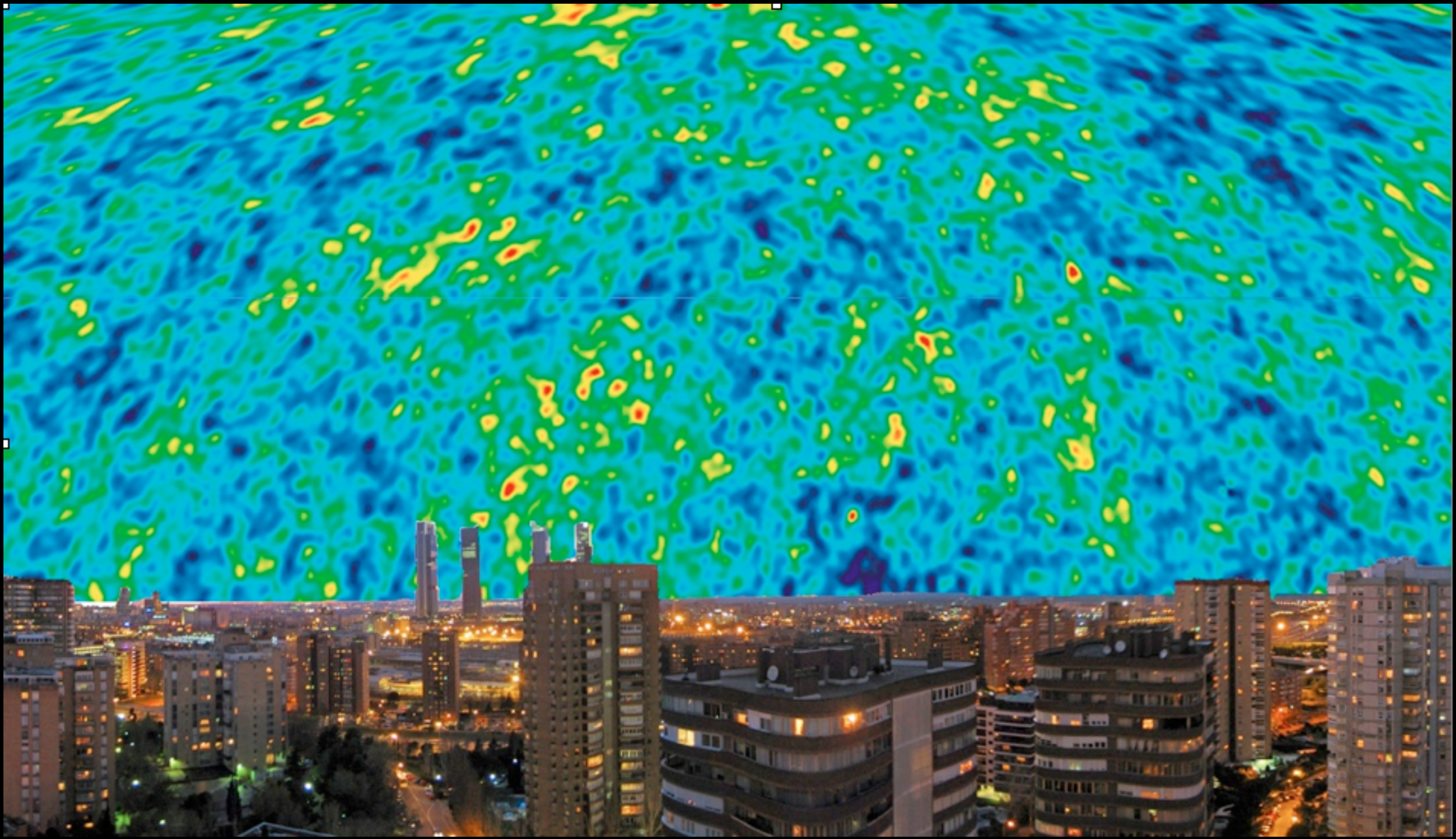
We are nice cosmic product, primordial nucleosynthesis relies on QFT at finite temperature



Only H, D, T, He, Li are produced in the BBN. The rest are cooked in the stars, up to Fe and Ni, the rest, we have learned recently come from kilonovas, the fusion of neutron stars!!

Most of our body is star dust and the rest radioactive waste!

See cosmology lectures



See cosmology lectures

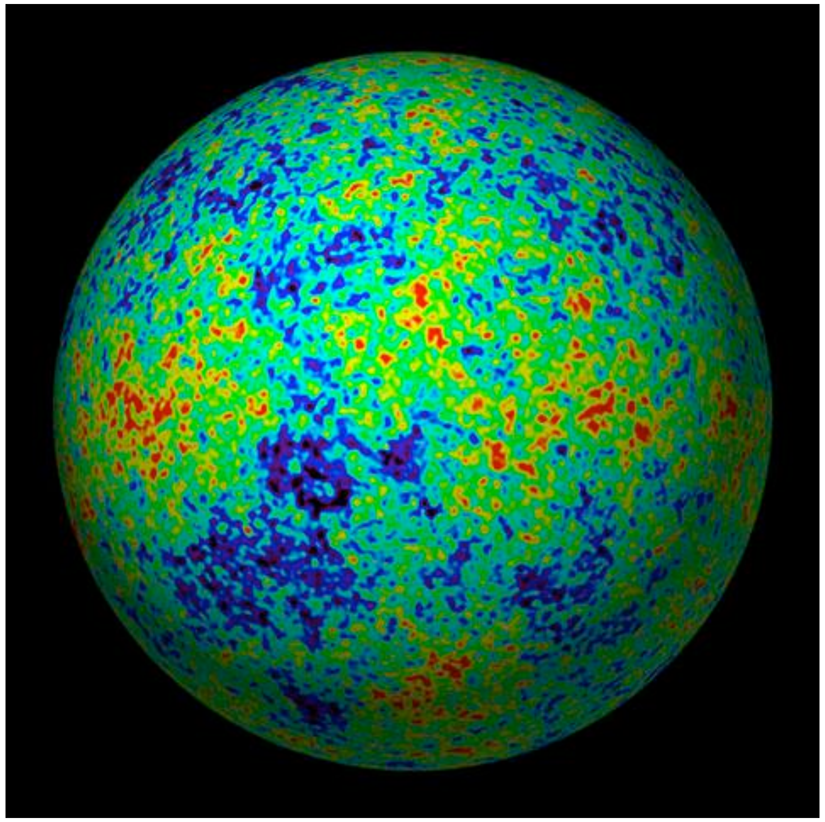
Inflation or the Legend of the Chessboard

2^{0}	2^{1}	2^{2}	2^{3}	2^{4}	2^{5}	2^{6}	2^{7}
2^{8}	2^{9}	2^{10}	2^{11}	2^{12}	2^{13}	2^{14}	2^{15}
2^{16}	2^{17}	2^{18}	2^{19}	2^{20}	2^{21}	2^{22}	2^{23}
2^{24}	2^{25}	2^{26}	2^{27}	2^{28}	2^{29}	2^{30}	2^{31}
2^{32}	2^{33}	2^{34}	2^{35}	2^{36}	2^{37}	2^{38}	2^{39}
2^{40}	2^{41}	2^{42}	2^{43}	2^{44}	2^{45}	2^{46}	2^{47}
2^{48}	2^{49}	2^{50}	2^{51}	2^{52}	2^{53}	2^{53}	2^{55}
2^{56}	2^{57}	2^{58}	2^{59}	2^{60}	2^{61}	2^{62}	2^{63}

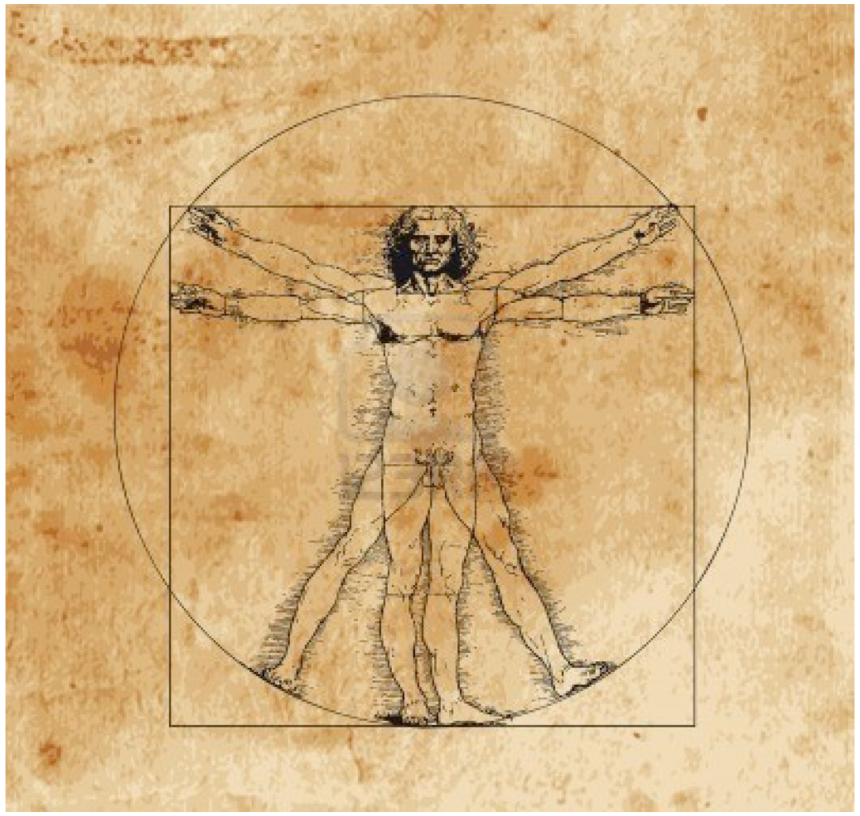
$$2^{64} - 1 \;=\; 18446744073709551615$$

$$1\,\mathrm{m}^3 \;\approx\; 15\,000\,000$$

Inflation: a true free lunch

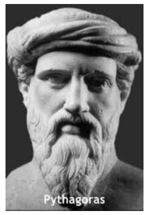


$$= 2^{86} \times$$



The quantum fluctuations are stretched so much that they leave the horizon and become classical. As the Universe expands as a power in time, the fluctuations re-enter the horizon, and this explains (roughly) the structure we see in the CMBR sky. Truly mind blowing. Accelerated expansion soon after the BB

Special relativity



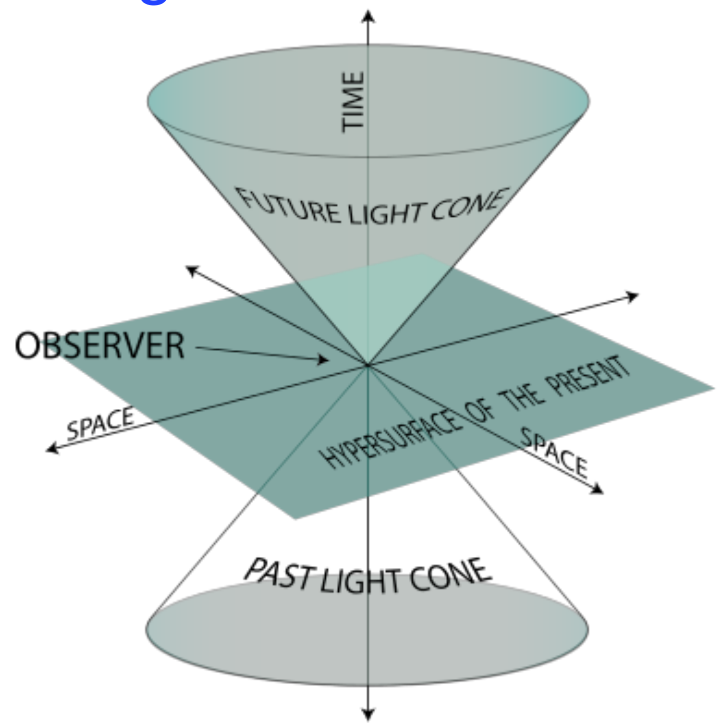
Pythagoras with a minus sign

$$(\Delta L)^2 = (\Delta x)^2 + (\Delta y)^2$$
$$(\Delta \tau)^2 = (\Delta t)^2 - (\Delta x)^2$$

$$t' = \gamma\,(t - \frac{v\,x}{c^2})$$
$$x' = \gamma\,(x - v\,t)$$
$$y' = y$$
$$z' = z$$

$$\sin(\theta) \to \sinh(\zeta)$$
$$\cos(\theta) \to \cosh(\zeta)$$
$$\tan(\theta) \to \tanh(\zeta)$$

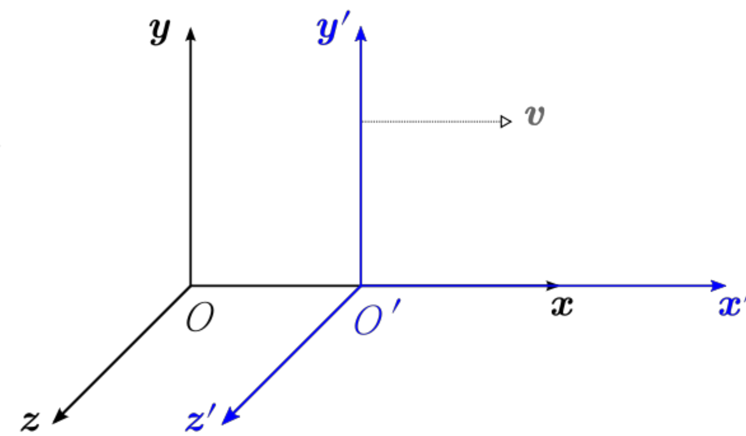
$\tanh(\zeta) = \frac{v}{c}$ (Rapidity)

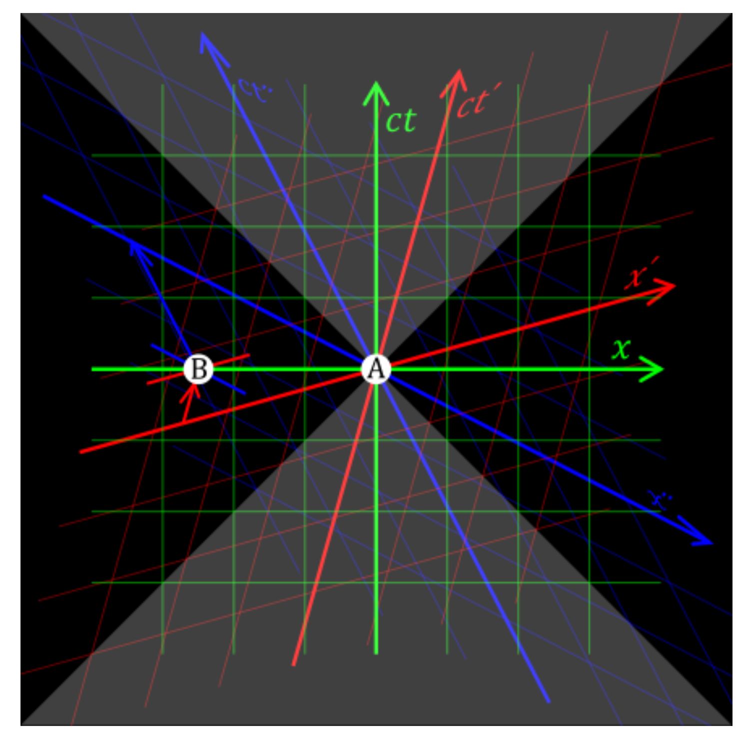


$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$(E, p_x, p_y, p_y)$$

$$E^2 = \mathbf{p}^2 + m^2$$





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Einstein's 1st law

$$E = m\,c^2$$

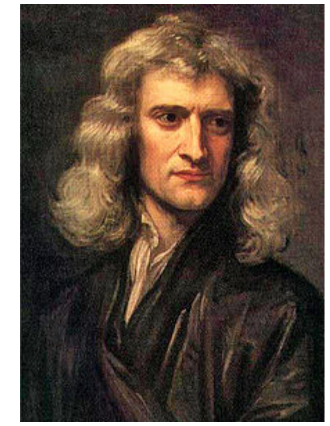
Einstein's 2nd law

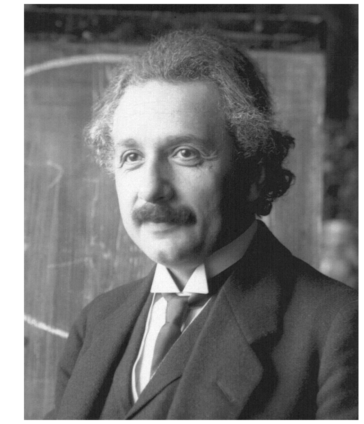
$$m = \frac{E}{c^2}$$

Particle numbers are not conserved. Energy can be converted into particles and vice versa. This is the great difficulty with QM and Relativity. It is also the origin of the existence of antimatter

A few useful remarks

Mechanics reminder





$$
\begin{aligned}
p_N &= m\,v \\
p_E &= m\,v\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}
\end{aligned}
$$

$$
\begin{aligned}
E_N &= \frac{m}{2}\,v^2 \\
E_E &= m\,c^2\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}
\end{aligned}
$$

$$m \to 0, \qquad v \to c \qquad p = \frac{E}{c}$$

Massless particles can only be dealt with in relativity

Mass (inertia) represents resistance to acceleration

Nothing to do with friction

Viscosity is resistance to velocity

Einstein and Heisenberg enrich our lives

Useful basic formulae. A reminder. Just this once, we reintroduce h and c

$$
\begin{aligned}
p^2 &= \left(\frac{E}{c}\right)^2 - \mathbf{p}^2 = m^2c^2 \\
E &= \pm\sqrt{\mathbf{p}^2c^2 + m^2c^4} \approx \pm(mc^2 + \frac{\mathbf{p}^2}{2m} + \ldots) \\
\Delta x \, \Delta p &\geq \frac{\hbar}{2} \\
\lambda &= \frac{h}{mc} \qquad \text{Compton wavelength} \\
E &= \frac{mc^2}{\sqrt{1-\mathbf{v}^2/c^2}} \qquad \mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1-\mathbf{v}^2/c^2}} \\
\Delta p &\geq mc \qquad \Delta E \geq mc^2 \\
(\Delta x)_{\min} &\geq \frac{1}{2}\left(\frac{\hbar}{mc}\right)
\end{aligned}
$$

When the uncertainly in momentum is bigger than mc, the uncertainty in energy is larger than mc^2, hence there is enough energy to produce another particle of the same type. In Relativity mass and energy are interchangeable. Hence we cannot localize a particle below its Compton wavelength. If we do, we will not find a single particle, but rather a fairly complicated quantum state with no well-defined number of particles. Particle number is not conserved. We have an "infinite number of particles" theory.

Particle production by physical processes should be a central part of the theory.

Advanced topic I:

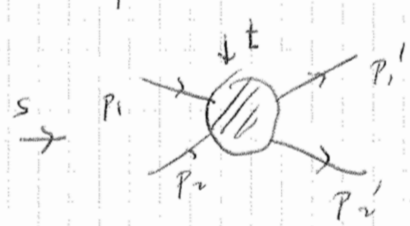
A crash course in relativistic kinematics

Quantum properties of scattering amplitudes and cross sections. Unitarity, optical theorem…

Good for the discussion sessions

Some kinematics

$2 \to 2$ process, not necessarily elastic



$$p_1 + p_2 = p_1' + p_2'$$

$$\left.\begin{aligned} s &= (p_1+p_2)^2 = (p_1'+p_2')^2 \\ t &= (p_1'-p_1)^2 = (p_2'-p_2)^2 \\ u &= (p_1-p_2')^2 = (p_2-p_1')^2 \end{aligned}\right\} \quad u+t+s = \sum_i m_i^2$$

In CM $E_1 + E_2 = \sqrt{s}$ $\quad \vec{p}_1 + \vec{p}_2 = 0$. The on-shell conditions:

$p_i^2 = m_i^2$. In CM we have:

$$p_1 = (E_1, \vec{p}), \quad p_2 = (E_2, -\vec{p}), \quad p_1' = (E_1', \vec{p}'), \quad p_2' = (E_2', -\vec{p}')$$

Later we choose z- as the collision axis, and we represent the momenta as (p^0, p^3, p^1, p^2). The transverse plane to the collision will be the (x,y) plane. Simple kinematics yield:

$$E_1 + E_2 = \sqrt{s}, \qquad E_1^2 - m_1^2 = \vec{p}^{\,2} \qquad E_2^2 - m_2^2 = \vec{p}^{\,2} \qquad \text{and:}$$

$$\boxed{\begin{aligned} E_1 &= \frac{s+m_1^2-m_2^2}{2\sqrt{s}} \qquad & E_1' &= \frac{s+m_1'^2-m_2'^2}{2\sqrt{s}} \\ E_2 &= \frac{s+m_2^2-m_1^2}{2\sqrt{s}} \qquad & E_2' &= \frac{s+m_2'^2-m_1'^2}{2\sqrt{s}} \end{aligned}}$$

The Fermi function is:

$$\lambda(x,y,z) \equiv x^2+y^2+z^2-2xy-2yz-2xz$$

then:

$$\boxed{\begin{aligned} |\vec{p}| &= \frac{1}{2\sqrt{s}}\, \lambda^{1/2}(s, m_1^2, m_2^2) \\ |\vec{p}'| &= \frac{1}{2\sqrt{s}}\, \lambda^{1/2}(s, m_1'^2, m_2'^2) \end{aligned}}$$

We have simple factorization formulae

$$\left.\begin{aligned} |\vec{p}|^2 &= \frac{1}{4s}\left(s^2+m_1^4+m_2^4-2sm_1^2-2sm_2^2-2m_1^2m_2^2\right) \\ &= \frac{1}{4s}\left[\left(s-(m_1+m_2)^2\right)\left(s-(m_1-m_2)^2\right)\right] \\ |\vec{p}'|^2 &= \frac{1}{4s}\left[\left(s-(m_1'+m_2')^2\right)\left(s-(m_1'-m_2')^2\right)\right] \end{aligned}\right\}$$

$$\cancel{s = \frac{2(m_1^2+m_2^2) \pm \left[4(m_1^2+m_2^2)^2 - 8m_1^2m_2^2\right]^{1/2}}{2}}$$

$$\cancel{= m_1^2+m_2^2 \pm \left[m_1\right.}$$

(2)

$$s^2 + m_1^4 + m_2^4 - 2s(m_1^2+m_2^2) - 2m_1^2 m_2^2 = 0$$

$$s^2 - 2s(m_1^2+m_2^2) + (m_1^2-m_2^2)^2 = 0$$

$$s = \frac{1}{2}\left(2(m_1^2+m_2^2) \pm \left(4(m_1^2+m_2^2)^2 - 4(m_1^2-m_2^2)^2\right)^{1/2}\right)$$

$$= m_1^2+m_2^2 \pm \left((m_1^2+m_2^2)^2 - (m_1^2-m_2^2)^2\right)^{1/2} = \begin{cases}(m_1+m_2)^2\\ (m_1-m_2)^2\end{cases}$$

Also:

$$\left.\begin{array}{l} t = (p_1'-p_1)^2 = m_1^2+m_1'^2 - 2p_1p_1' = m_1^2+m_1'^2 - 2(E_1E_1' - |p||p'|\cos\theta_s) \\ u = (p_1-p_2')^2 = m_1^2+m_2'^2 - 2p_1p_2' = m_1^2+m_1'^2 - 2(E_1E_2' + |p||p'|\cos\theta_s) \end{array}\right\}$$

θ_s = angle between $\vec{p}, \vec{p}\,'$ in the CM s-channel.

The simplest case corresponds to $m_i = m$, $i = 1,4$, $|p'| = |p|$, then:

$$E_i = \frac{\sqrt{s}}{2} \qquad |\vec{p}|^2 = |\vec{p}\,'|^2 = \frac{1}{4s}\, s(s-4m^2) = \frac{1}{4}(s-4m^2)$$

implying:

$$\left.\begin{array}{l} s = 4(\vec{p}^{\,2}+m^2) \\ t = -2|\vec{p}|^2(1-\cos\theta_s) \\ u = -2|\vec{p}|^2(1+\cos\theta_s) \end{array}\right\} \quad s+t+u = 4m^2$$

The s-channel physical region is $s \geq 4m^2$, $t, u \leq 0$. Similarly for the t, u - channels.

Cross section

For on-states $\langle f|, |i\rangle$ $\langle f|f'\rangle = \delta_{ff'}$, the S-matrix element is defined such that:

$$P_{fi} = |\langle f|S|i\rangle|^2 = \overline{\langle f|S|i\rangle}\,\langle f|S|i\rangle = \langle i|S^\dagger|f\rangle\langle f|S|i\rangle$$

For a complete set of states $|f\rangle$:

$$\sum_f |f\rangle\langle f| = 1$$

Conservation of probability yields

$$1 = \sum_f |\langle f|S|i\rangle|^2 = \sum_f \langle i|S^\dagger|f\rangle\langle f|S|i\rangle = \langle i|S^\dagger S|i\rangle$$

Unitarity then follows:

$$\boxed{1 = SS^\dagger = S^\dagger S.}$$

(3)

The most common case of study is:

$$2 \to n$$

S is related to the transition matrix:

$$\left\{ \begin{aligned} \langle f | S | i \rangle &= \langle p_1' \dots p_n' | S | p_1 p_2 \rangle = \\ &= \delta_{fi} + i (2\pi)^4 \delta(P^f - P^i) \langle f | T | i \rangle \end{aligned} \right\}$$

The transition rate per-unit time, per-unit volume is:

$$\boxed{R_{fi} = (2\pi)^4 \delta(P^f - P^i) \, |\langle f | T | i \rangle|^2}$$

where we have used some "short-cuts" by working in finite space-time volume, in taking the square of the δ-function. For finite volume V and time T:

$$\left((2\pi)^4 \delta(P_f - P_i) \right)^2 = (2\pi)^4 \delta(P_f - P_i) . (2\pi)^4 \int \frac{d^4x}{(2\pi)^4} e^{i x . (P_f - P_i)}$$

$$= (2\pi)^4 \delta(P_f - P_i) \, V . T$$

hence the formula for the rate. The total cross section for:

$$1 + 2 \to n\text{- particles}$$

is:

$$\sigma_{1+2\to n} = \frac{1}{4 |\bar{p}_1| \sqrt{s}} \sum_{f_n} (2\pi)^4 \delta(P_f - P_i) \, |\langle f_n | T | i \rangle|^2$$

More explicitly

$$\boxed{\sigma_{1+2\to n} = \frac{1}{4 |\bar{p}_1| \sqrt{s}} \sum \int \prod_{i=1}^{n} \frac{d^3 p_i'}{(2\pi)^3 2E_i} (2\pi)^4 \delta(P_f - P_i) \cdot |\langle p_1' \dots p_n' | T | p_1 p_2 \rangle|^2}$$

And some symmetry factors should be included if there are identical particles in the final state. It is important that $|\bar{p}_1|$ is the incoming momentum in the CM system. Then the above formula gives the total cross section in any frame. It is Lorentz invariant. We already know that:

$$|\bar{p}_1|^2 s = \frac{1}{4} \left(s - (m_1 + m_2)^2 \right) \left(s - (m_1 - m_2)^2 \right)$$

(4)

another useful form is:

$$\boxed{|\vec{p}_1|^2 s = (P_1\cdot P_2)^2 - m_1^2 m_2^2}$$

$$P_1\cdot P_2 = \frac{1}{2}\left((P_1+P_2)^2 - m_1^2 - m_2^2\right) = \frac{1}{2}(s - m_1^2 - m_2^2)$$

$$(P_1\cdot P_2)^2 = \frac{1}{4}\left(s^2 + m_1^4 + m_2^4 - 2sm_1^2 - 2sm_2^2 + 2m_1^2m_2^2\right) =$$

$$= s|\vec{p}_1|^2 + m_1^2 m_2^2 \qquad \text{q.e.d.}$$

Since $\sigma_{12\to n}$ is relativistically invariant, we can construct differential cross sections $d\sigma_{12\to n}/d\omega$. ω typically represents a momentum transfer between initial and final particles, or a scattering angle, or energy transfer, or the energy of a final particle. To calculate $d\sigma/d\omega$, we first write ω as a function $\omega = \omega(P_i, P_f')$ of the various initial and final momenta, and then include $\delta(\omega - \omega(P_i, P_f'))$ in the integration to obtain a differential cross section.

One example is obtained when we have two final particles, and we fix $t = (P_1 - P_3)^2$. This leads to:

$$\frac{d\sigma_{12\to 34}}{dt} = \frac{1}{4|p_1|\sqrt{s}} \int \frac{d^3P_3}{(2\pi)^3 2E_3}\frac{d^3P_4}{(2\pi)^3 2E_4}$$

$$(2\pi)^4\,\delta(P_f - P_i)\,|\langle P_3P_4|T|P_1P_2\rangle|^2\,\delta\left(t - (P_1-P_3)^2\right)$$

In the simpler case of equal masses (of the incoming particles at least):

$$|\vec{p}_1|^2 = \frac{1}{4}(s - 4m^2)$$

The overall numerical factor is:

$$\frac{1}{4\cdot 2\cdot 2\cdot (2\pi)^6}(2\pi)^4 = \frac{1}{64\pi^2}$$

In the CM:

$$\delta^4(P_f - P_i) = \delta(E_3 + E_4 - \sqrt{s})\,\delta^3(P_3 + P_4)$$

hence we can eliminate the d^3P_4 integral. Let's not pay attention for the moment to $|T|^2$. The kinematic integrals we are left with are:

(5)

$$\int \frac{d^3 p_3}{E_3 E_4}\, \delta(E_3+E_4-\sqrt{s})\, \delta\left(t-(P_1-P_3)^2\right) = \{ P_3 = p' \}$$

$$= \int \frac{p'^2 dp'\, d\varphi'\, d\cos\theta'}{E_3 E_4}\, \delta\left(\sqrt{p'^2+m_3^2}+\sqrt{p'^2+m_4^2}-\sqrt{s}\right) . \delta\left(t-(P_1-P_3)^2\right)$$

The 1st δ-function is solved using:

$$\delta(f(x)) = \sum_{x_0} \frac{1}{|f'(x_0)|}\, \delta(x-x_0)$$

for the second we need to relate θ' to t.

$$t = (P_1-P_3)^2 = m_1^2+m_3^2 - 2P_1\cdot P_3 = m_1^2+m_3^2 - 2\,(E_1E_3 - pp'\cos\theta')$$

Hence we have:

$$\delta\left(\sqrt{p'^2+m_3^2}+\sqrt{p'^2+m_4^2}-\sqrt{s}\right)\delta\left(2pp'\cos\theta' - 2E_1E_3 + m_1^2+m_3^2 - t\right)$$

For the 1st we compute the derivative:

$$\frac{\partial}{\partial p'}\left(\sqrt{p'^2+m_3^2}+\sqrt{p'^2+m_4^2}-\sqrt{s}\right) = \frac{p'}{E_3}+\frac{p'}{E_4} = \frac{p'(E_3+E_4)}{E_3E_4} = \frac{p'\sqrt{s}}{E_4E_4}$$

Assuming independence on φ' $\int d\varphi' \to 2\pi$. In the integral over $d\cos\theta'$, the δ-function yields $\frac{1}{2pp'}$. Putting all factors together, we have

$$\underbrace{\frac{2\pi\, p'^2}{E_3E_4}}_{\text{measure}} \cdot \underbrace{\frac{1}{\frac{p'\sqrt{s}}{E_4E_4}}}_{\text{1st } \delta\text{-f}^n} \frac{1}{2pp'} = \frac{\pi}{p\sqrt{s}}$$

Then:

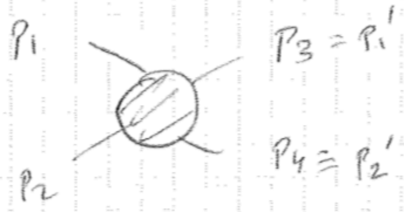
$$\boxed{\frac{d\sigma_{12\to 34}}{dt} = \frac{1}{64\pi p^2 s}\, |T|^2}$$ with t, s fixed and defined for on-shell particles.

With equal incoming masses $4\vec{p}^{\,2} = (s-4m^2)$, in this case:

$$\boxed{\frac{d\sigma_{12\to 34}}{dt} = \frac{1}{16\pi s(s-4m^2)}\, |T|^2} \qquad m_1 = m_2 = m.$$

We assume either no spin, or that $|T|^2$ includes the relevant sums and averages.

Advanced Topic I-6



(5a)

$$|\vec{p}_1|^2 = \frac{1}{4s}\left[\left(s-(m_1+m_2)^2\right)\left(s-(m_1-m_2)^2\right)\right]$$

$$|\vec{p}_3|^2 = \frac{1}{4s}\left[\left(s-(m_3+m_4)^2\right)\left(s-(m_3-m_4)^2\right)\right]$$

$$t = m_1^2+m_3^2 - 2(E_1E_3 - \vec{p}_1\cdot\vec{p}_3) = m_1^2+m_3^2 - 2(E_1E_3 - |\vec{p}_1||\vec{p}_3|\cos\theta_s)$$

$$u = m_1^2+m_4^2 - 2(E_1E_4 + \vec{p}_1\cdot\vec{p}_3) = m_1^2+m_4^2 - 2(E_1E_4 + |\vec{p}_1||\vec{p}_3|\cos\theta_s)$$

The physical region in the s-channel is:

$$s \geq (m_1+m_2)^2, \qquad -1 \leq \cos\theta_s \leq +1$$

The bdry of the physical region as a f'n of s,t is very complicated. For simplicity consider equal masses

$$m_i = m, \quad i=1,4; \qquad |\vec{p}_1| = |\vec{p}_3| = |\vec{p}|:$$

$$E_i = \frac{\sqrt{s}}{2}, \qquad |\vec{p}|^2 = \frac{1}{4}(s-4m^2), \qquad s = 4(\vec{p}^{\,2}+m^2)$$

$$t = -2\vec{p}^{\,2}(1-\cos\theta_s)$$

$$\boxed{\begin{aligned} s &= 4(\vec{p}^{\,2}+m^2)\\ t &= -2|\vec{p}|^2(1-\cos\theta_s)\\ u &= -2|\vec{p}|^2(1+\cos\theta_s)\end{aligned}}$$

s-channel:

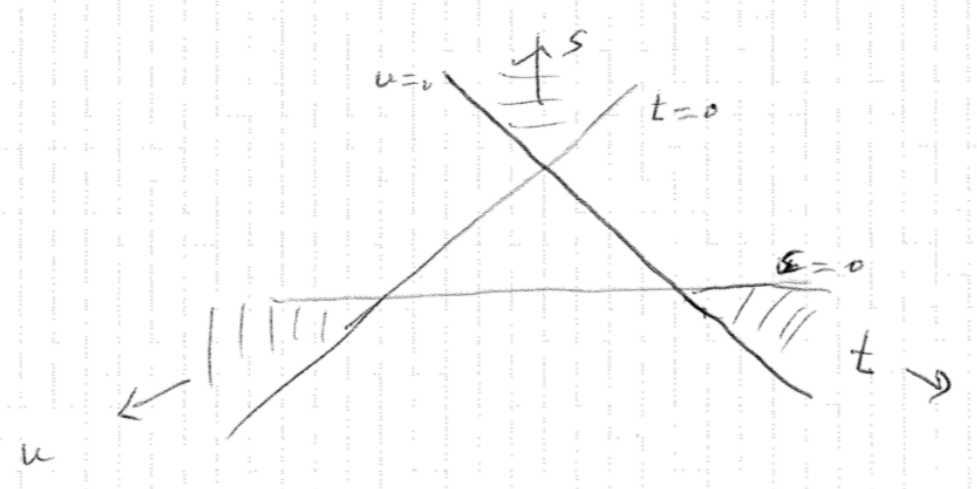
$s \geq 4m^2 \qquad u,t \leq 0$

t-channel

$t \geq 4m^2, \qquad s,u \leq 0$

u-channel

$u \geq 4m^2, \qquad s,t \leq 0$



(5b)

Crossing. Partial waves

Crossing has only been proved in perturbation theory.

At fixed s, t is linear in $\cos\theta_s \equiv z_s$. We can use (s, z_s) as independent variables. Hence:

$$\boxed{A(s,t)_{1+2\to 3+4} = 16\pi \sum_{\ell=0}^{\infty} (2\ell+1) A_\ell(s) P_\ell(z_s)}$$

(no spin is included). We can use phase shifts instead. They are by definition:

$$A_\ell = \frac{\eta_\ell(s)\, e^{2i\delta_\ell(s)} - 1}{2i\,\rho(s)} \quad ; \quad \rho(s) = 2|\vec{p}_1|\sqrt{s}$$

with our choice of normalization for the total cross section. Below the 1st inelastic threshold, $\eta_\ell(s) = 1$, and:

$$A_\ell = e^{i\delta_\ell(s)} \frac{\sin\delta_\ell}{\rho(s)}$$

Unitarity requires $0 \le \eta_\ell \le 1 \Rightarrow \operatorname{Im}\delta_\ell \ge 0$. From the property of the Legendre polynomials:

$$\int_{-1}^{+1} dz\, P_\ell(z) P_m(z) = \frac{2}{2\ell+1}\delta_{\ell,m}\,; \quad A_\ell(s) = \frac{1}{32\pi}\int_{-1}^{+1} dz_s\, P_\ell(z_s)\, A(s, t(s,z_s))$$

Pomeranchuk's Theorem.

Under strong assumptions $\dfrac{\sigma^{tot}(pp)}{\sigma^{tot}(p\bar{p})} \to 1$ as $s \to \infty$.

Unitarity + Optical Theorem. (6)

The forward elastic scattering amplitude is related to the total cross section. This follows from unitarity. Recall:

$$S_{fi} = \delta_{fi} + (2\pi)^4 \delta(P_f - P_i)\, i T_{fi}$$

More formally: $S = 1 + iT$, $SS^\dagger = S^\dagger S = 1$.

$$(1+iT)^\dagger (1+iT) = 1 - iT^\dagger + iT + T^\dagger T = 1 \Rightarrow$$

$$\boxed{T - T^\dagger = i T^\dagger T}$$

with our cavalier way to handle δ-functions we obtain:

$$\langle j|T|i\rangle - \langle j|T^\dagger|i\rangle = (2\pi)^4 i \sum_f \delta^4(P_f - P_i) \langle j|T^\dagger|f\rangle\langle f|T|i\rangle$$

$$\boxed{\langle j|T|i\rangle - \langle j|T^\dagger|i\rangle = (2\pi)^4 i \sum_f \delta^4(P_f - P_i) \langle j|T^\dagger|f\rangle\langle f|T|i\rangle}$$

When $i = j$ (forward elastic scattering):

$$2\,\mathrm{Im}\langle i|T|i\rangle = \sum_f (2\pi)^4 \delta(P_f - P_i)\, |\langle f|T|i\rangle|^2$$

Using the form of the total cross section, the RHS is nothing but $4|\vec{p}_1|\sqrt{s}\,\sigma_{12}^{tot}$

$$\boxed{2\,\mathrm{Im}\langle i|T|i\rangle = 4|\vec{p}_1|\sqrt{s}\,\sigma_{12}^{TOT}}$$

i.e.

$$\boxed{\sigma_{12}^{TOT} = \frac{1}{2|\vec{p}_1|\sqrt{s}}\,\mathrm{Im}\langle i|T|i\rangle}$$

$|i\rangle = |f\rangle$ implies $E_1 = E_3$, $E_2 = E_4$ $|p_1| = |p_3|$, $m_1 = m_3$, $m_2 = m_4$

$t = 0$:

$$\boxed{\sigma_{12}^{TOT} = \frac{1}{2p_1\sqrt{s}}\,\mathrm{Im}\, A(s, t=0)}$$

A elastic scattering amplitude.

Using sophisticated complex analysis Froissart-Martin obtained a bound for the growth of $\sigma^{TOT}(s)$ for hadrons.

$$\left|\ \sigma^{TOT}(s) \le \frac{\pi}{m_\pi^2} \log^2(s/s_0)\ \right|$$

Beating a dead horse...

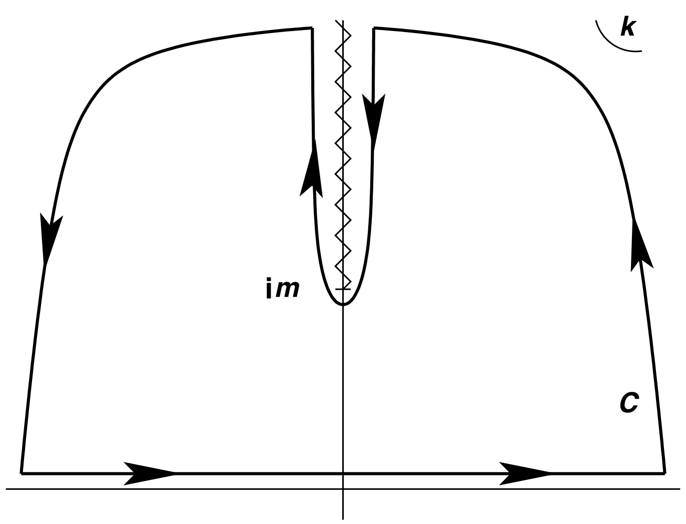
If we still insist against all odds, and decide to violate locality, but to eliminate once and for all the negative energy states by choosing our free Hamiltonian as follows:

$$H = \sqrt{-\nabla^2 + m^2}$$

$$\psi(0,\mathbf{x}) = \delta(\mathbf{x})$$

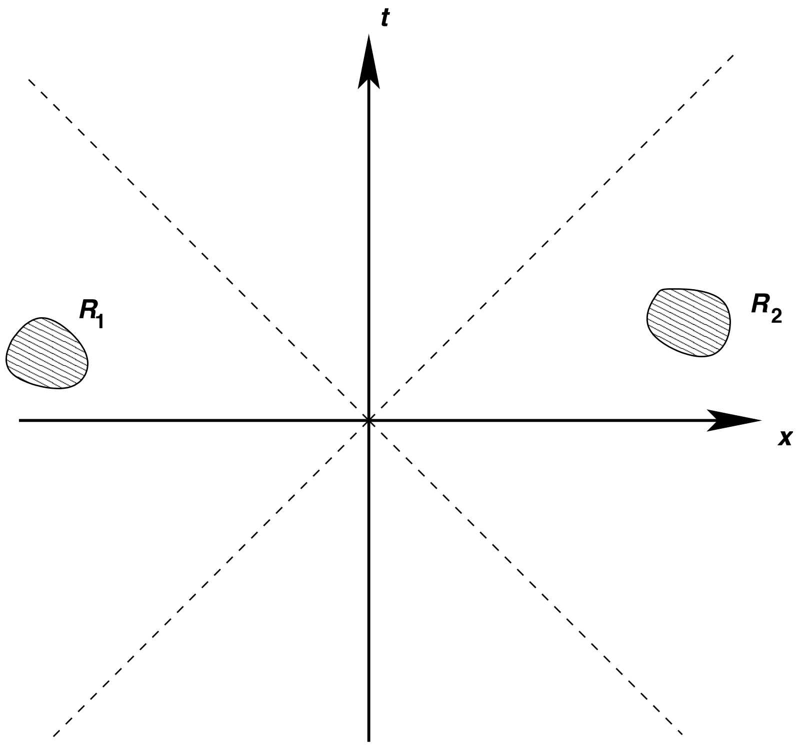
$$\psi(t,\mathbf{x}) = e^{-it\sqrt{-\nabla^2+m^2}}\delta(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}-it\sqrt{k^2+m^2}}.$$

$$\psi(t,\mathbf{x}) = \frac{1}{2\pi^2|\mathbf{x}|}\int_{-\infty}^{\infty} k\,dk\, e^{ik|\mathbf{x}|}\, e^{-it\sqrt{k^2+m^2}}.$$



Oops!! we have violated causality! For any t>0 and any |x|, this wave function does not vanish!...

Relativistic causality



Microscopic causality, Locality in Special Relativity imposes important constraints into what are observables. The light-cone decrees the causal structure of space-time. Physical measurements should be compatible with it

$$[\mathscr{O}(x), \mathscr{O}(y)] = 0, \qquad \text{if } (x-y)^2 < 0.$$

- The world is Quantum
- Particle Wave Duality
- Special Relativity
- Microscopic Causality

LQFT

A simple but illuminating exercise

Recall the harmonic oscillator, just one degree of freedom but now in the Heisenberg representation, the one better adapted for relativistic invariance

$$
\begin{aligned}
L &= \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 \\
p &= \frac{\partial L}{\partial \dot{x}} = \dot{x}(t) \\
\text{e.o.m.} \quad \ddot{x}(t) &+ \omega^2 x(t) = 0 \\
x(t) &= f(t)\, a + f^*(t)\, a^\dagger \qquad f(t) = \sqrt{\frac{\hbar}{2\,\omega}} e^{-i\omega t} \\
[x(t), \dot{x}(t)] &= i\hbar \\
H &= \hbar\omega\,(a^\dagger a + \frac{1}{2}) \\
a &= \sqrt{\frac{\omega}{2\hbar}} \left(x + \frac{i}{\omega} p \right) \qquad [a, a^\dagger] = 1 \\
a\,\psi_0(x) &= 0, \qquad \psi_0(x) = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{\omega x^2}{2\,\hbar}}
\end{aligned}
$$

Consider the free E&M field

$$
\begin{aligned}
L &= \frac{1}{2}\int dV\,(\mathbf{E}^2 - \mathbf{B}^2) \\
H &= \frac{1}{2}\int dV\,(\mathbf{E}^2 + \mathbf{B}^2) \\
[A_i(\mathbf{x},0), E_j(\mathbf{y},0)] &= \delta_{ij}\,\delta(x-y)
\end{aligned}
$$

Expanding in plane waves, we will end up with an infinite collection of harmonic oscillators. Now the analogue of x(0) is the whole function A(x,0), hence the ground state looks something like:

$$
\Psi\,[\mathbf{A}(\mathbf{x},0)] \sim \exp\left(-\frac{1}{2\hbar}\int d\mathbf{x}\,d\mathbf{y}\,\mathbf{A}(\mathbf{x},0)\,D(\mathbf{x}-\mathbf{y})\,\mathbf{A}(\mathbf{y},0)\right)
$$

We have a d.o.f. in each space point, we get a probability amplitude for any possible configuration of the vector potential. This is what makes QFT so different. This is why in QM the vacuum, the ground state, has nothing to do with the "philosopher's vacuum", a purely classical concept where nothing happens

From classical to quantum fields

In scattering experiments we observe asymptotic free particles characterized by their energy-momentum charge and other quantum numbers. Consider just E,p. In the NR-case we describe the one-particle states by kets carrying a unitary rep. of the rotation group.

$$|\mathbf{p}\rangle \in \mathscr{H}_1, \qquad \langle \mathbf{p}|\mathbf{p}'\rangle = \delta(\mathbf{p}-\mathbf{p}') \qquad \int d^3p\,|\mathbf{p}\rangle\langle\mathbf{p}| = \mathbf{1}. \qquad \mathscr{U}(R)|\mathbf{p}\rangle = |R\mathbf{p}\rangle \qquad \widehat{P}^i = \int d^3p\,|\mathbf{p}\rangle\, p^i\,\langle\mathbf{p}|$$

To deal with multi-particle states it is convenient to introduce creation and annihilation operators, this leads to the Fock space of states, built out of the vacuum by acting with creation operators:

$$|\mathbf{p}\rangle = a^\dagger(\mathbf{p})|0\rangle, \qquad a(\mathbf{p})|0\rangle = 0 \qquad \langle 0|0\rangle = 1$$

$$[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = \delta(\mathbf{p}-\mathbf{p}'), \qquad [a(\mathbf{p}), a(\mathbf{p}')] = [a^\dagger(\mathbf{p}), a^\dagger(\mathbf{p}')] = 0.$$

We need relativistic invariance, hence we need to find ways to count states in an invariant way. This is necessary also when we deal with decay rates and cross sections. We need to count final states in a way consistent with Lorentz invariance. We can easily construct such an invariant phase space volume:

$$\int \frac{d^4p}{(2\pi)^4}(2\pi)\delta(p^2-m^2)\,\theta(p^0)\,f(p)$$ to integrate over p0, we use a nice identity:

$$\delta[g(x)] = \sum_{x_i=\text{zeros of } g} \frac{1}{|g'(x_i)|}\delta(x-x_i) \qquad \delta(p^2-m^2) = \frac{1}{2p^0}\delta\left(p^0-\sqrt{\mathbf{p}^2+m^2}\right) + \frac{1}{2p^0}\delta\left(p^0+\sqrt{\mathbf{p}^2+m^2}\right)$$

$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}$$ with $E_{\mathbf{p}} \equiv \sqrt{\mathbf{p}^2+m^2}$ and $(2E_{\mathbf{p}})\delta(\mathbf{p}-\mathbf{p}')$ are invariant

Exercise: Is the phase space a relativistic invariant? $\dfrac{\Delta\mathbf{x}\Delta\mathbf{p}}{(2\pi\hbar)^3}$

...continued

Now proceed by imitation of the NR case, with the non-trivial result that we have a unitary representation of the Lorentz group

$$|p\rangle = (2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{p}}}\,|\mathbf{p}\rangle \qquad \langle p|p'\rangle = (2\pi)^3(2E_{\mathbf{p}})\delta(\mathbf{p}-\mathbf{p}') \qquad \widehat{P}^\mu = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}|p\rangle p^\mu \langle p| \qquad \mathscr{U}(\Lambda)|p\rangle = |\Lambda^\mu{}_\nu p^\nu\rangle \equiv |\Lambda p\rangle$$

$$\langle 0|0\rangle = 1$$

$$\alpha(\mathbf{p}) \equiv (2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{p}}}a(\mathbf{p}) \qquad [\alpha(\mathbf{p}),\alpha^\dagger(\mathbf{p}')] = (2\pi)^3(2E_{\mathbf{p}})\delta(\mathbf{p}-\mathbf{p}'),$$

$$\alpha^\dagger(\mathbf{p}) \equiv (2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{p}}}a^\dagger(\mathbf{p}) \qquad [\alpha(\mathbf{p}),\alpha(\mathbf{p}')] = [\alpha^\dagger(\mathbf{p}),\alpha^\dagger(\mathbf{p}')] = 0.$$

$$|f\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}} f(\mathbf{p})\alpha^\dagger(\mathbf{p})|0\rangle$$

Let us construct some observable in this theory. It will be an operator depending on space time, and satisfying some simple conditions:

- Hermiticity $$\phi(x)^\dagger = \phi(x).$$
- Microcausality $$[\phi(x),\phi(y)] = 0, \qquad (x-y)^2 < 0.$$
- Translational invariance $$e^{i\widehat{P}\cdot a}\phi(x)e^{-i\widehat{P}\cdot a} = \phi(x-a)$$
- Lorentz invariance $$\mathscr{U}(\Lambda)^\dagger\phi(x)\mathscr{U}(\Lambda) = \phi(\Lambda^{-1}x).$$
- Linearity $$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left[f(\mathbf{p},x)\alpha(\mathbf{p}) + g(\mathbf{p},x)\alpha^\dagger(\mathbf{p})\right].$$

We have obtained from first principles the quantization of the Klein-Gordon field. There are more straightforward ways, but the procedure shows how to implement the basis principles of the theory, Lorentz invariance, locality and positivity of the spectrum

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left[e^{-iE_{\mathbf{p}}t+i\mathbf{p}\cdot\mathbf{x}}\alpha(\mathbf{p}) + e^{iE_{\mathbf{p}}t-i\mathbf{p}\cdot\mathbf{x}}\alpha^\dagger(\mathbf{p})\right]$$

+ve energy

-ve energy

Some important properties

$$[\phi(t,\mathbf{x}),\partial_t\phi(t,\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y}).$$

$$[\phi(x),\phi(x')] = i\Delta(x-x')$$

$$(\partial_\mu\partial^\mu + m^2)\phi(x) = 0$$

$$i\Delta(x-y) = -\mathrm{Im}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-iE_{\mathbf{p}}(t-t')+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}$$
$$= \int \frac{d^4p}{(2\pi)^4}(2\pi)\delta(p^2-m^2)\varepsilon(p^0)e^{-ip\cdot(x-x')}$$

$$\Delta(x-y) = 0 \qquad \text{for } (x-y)^2 < 0$$

The construction is free of paradoxes. It satisfies the KG equation because the +ve and -ve energy plane waves satisfy it. Of course with a free field we do not go very far…

We should design more powerful techniques leading to similar properties for more general theories where interactions can take place.

There are two general approaches: the canonical-formalism, and the Feynman path integral. We will briefly introduce the first, just as a reminder.

Canonical quantization

Remember: PHYSICS is where the ACTION is!

Proceed by analogy with ordinary QM

$$S[x,\dot{x}] = \int dt\, L(x,\dot{x})$$

$$L = \sum_i \frac{1}{2} m_i \dot{\mathbf{x}}_i^2 - V(\mathbf{x})$$

$$S[\phi(x)] \equiv \int d^4x \mathscr{L}(\phi, \partial_\mu \phi) = \int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \right)$$

$$\mathbf{x}_a,\, \dot{\mathbf{x}}_a \longleftrightarrow \phi(\mathbf{x},0),\, \dot{\phi}(\mathbf{x},0)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$$

$$\partial_\mu \left[\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \right] - \frac{\partial \mathscr{L}}{\partial \phi} = 0 \qquad \Longrightarrow \qquad (\partial_\mu \partial^\mu + m^2)\phi = 0$$

canonical momenta

$$p = \frac{\partial L}{\partial \dot{x}}$$

$$\pi(x) \equiv \frac{\partial \mathscr{L}}{\partial(\partial_0 \phi)} = \frac{\partial \phi}{\partial t}$$

$$H = \sum_i p_i\, \dot{x}^i - L$$

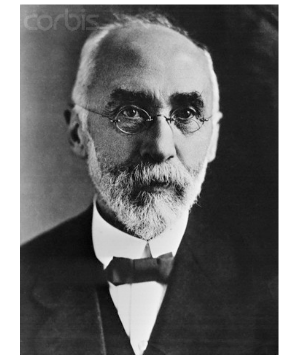
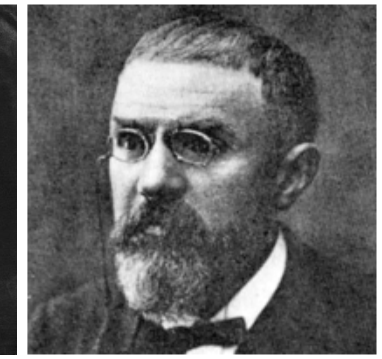
$$H \equiv \int d^3x \left(\pi \frac{\partial \phi}{\partial t} - \mathscr{L} \right) = \frac{1}{2}\int d^3x \left[\pi^2 + (\nabla\phi)^2 + m^2 \right].$$

$$[q^i, p_j] = i\hbar$$

$$[\phi(t,\mathbf{x}), \partial_t \phi(t,\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y}).$$

Expanding in solutions to the KG equations and performing the canonical quantisation, we recover the algebra of creation and annihilation operator we had before, but we get a surprise, the zero point energy (Casimir effect)

Lorentz and Poincaré Groups


In trying to systematically construct viable QFTs it is useful to understand the representations of the Lorentz (and Poincaré) groups.

The Hilbert space of states has to carry a unitary representation of the Lorentz group, so that quantum amplitudes are consistent with Unitarity and Relativistic Invariance. The fields themselves however, transform under finite dimensional representations. They are much easier to study. Just recall the usual rotation group SU(2). The Lorentz group, also known as SO(3,1) preserves the Minkowski metric

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu \qquad \mu, \nu = 0, 1, 2, 3$$

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad\qquad \eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta}$$

$$\det\Lambda = \pm 1 \qquad\qquad (\Lambda^0_0)^2 - \sum_{i=1}^{3} (\Lambda^i_0)^2 = 1$$

- $\mathfrak{L}^\uparrow_+$: proper, orthochronous transformations with $\det\Lambda = 1, \Lambda^0{}_0 \geqslant 1$.
- $\mathfrak{L}^\uparrow_-$: improper, orthochronous transformations with $\det\Lambda = -1, \Lambda^0{}_0 \geqslant 1$.
- $\mathfrak{L}^\downarrow_-$: improper, non-orthochronous transformations with $\det\Lambda = -1, \Lambda^0{}_0 \leqslant -1$.
- $\mathfrak{L}^\downarrow_+$: proper, non-orthochronous transformations with $\det\Lambda = 1, \Lambda^0{}_0 \leqslant -1$.

$$\mathfrak{L}^\uparrow_+ \xrightarrow{\mathscr{P}} \mathfrak{L}^\uparrow_-, \qquad \mathfrak{L}^\uparrow_+ \xrightarrow{\mathscr{T}} \mathfrak{L}^\downarrow_-, \qquad \mathfrak{L}^\uparrow_+ \xrightarrow{\mathscr{P}\mathscr{T}} \mathfrak{L}^\downarrow_+$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Lorentz and Poincaré Groups

$$R(\mathbf{e},\varphi) = e^{-i\varphi\,\mathbf{e}\cdot\mathbf{J}}$$

$$B(\mathbf{u},\lambda) = e^{-i\lambda\,\mathbf{u}\cdot\mathbf{M}}$$

Rotations and boosts generate Lorentz transformation, hence six parameter and six generators of infinitesimal transformations. The algebra is easy to obtain and "diagonalise"

$$\begin{aligned}[J_i,J_j] &= i\varepsilon_{ijk}J_k,\\ [J_i,M_k] &= i\varepsilon_{ijk}M_k,\\ [M_i,M_j] &= -i\varepsilon_{ijk}J_k\end{aligned}$$

$$J_k^{\pm} = \frac{1}{2}(J_k \pm iM_k)$$

$$\begin{aligned}[J_i^{\pm},J_j^{\pm}] &= i\varepsilon_{ijk}J_k^{\pm},\\ [J_i^{+},J_j^{-}] &= 0.\end{aligned}$$

The representations of each SU(2) are labelled by a single integer or half integer "angular" momentum s=0, 1/2, 1, 3/2, … Under parity

$$(\mathbf{s}_+,\mathbf{s}_-)$$

Representation	Type of field
$(\mathbf{0},\mathbf{0})$	Scalar
$(\frac{1}{2},\mathbf{0})$	Right-handed spinor
$(\mathbf{0},\frac{1}{2})$	Left-handed spinor
$(\frac{1}{2},\frac{1}{2})$	Vector
$(\mathbf{1},\mathbf{0})$	Selfdual antisymmetric 2-tensor
$(\mathbf{0},\mathbf{1})$	Anti-selfdual antisymmetric 2-tensor

$$\begin{aligned}\mathbf{J} &\xrightarrow{P} \mathbf{J}\\ \mathbf{M} &\rightarrow -\mathbf{M}\\ \mathbf{J}^{\pm} &\rightarrow \mathbf{J}^{\mp}\\ (\mathbf{s_1},\mathbf{s_2}) &\rightarrow (\mathbf{s_2},\mathbf{s_1})\end{aligned}$$

$$\mathbf{J} = \mathbf{J}^{+} + \mathbf{J}^{-}$$

$$(\mathbf{s}_+,\mathbf{s}_-) = \sum_{\mathbf{j}=|\mathbf{s}_+-\mathbf{s}_-|}^{\mathbf{s}_++\mathbf{s}_-} \mathbf{j}$$

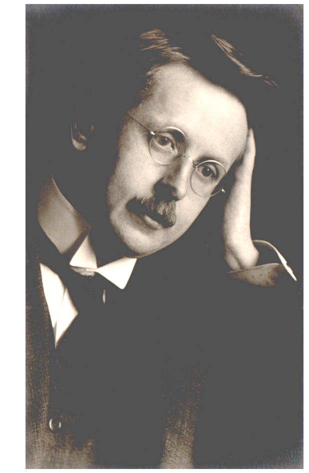
Luis Alvarez-Gaume Arequipa CLASHEP March 6-13 2019

CERN

Weyl spinors

The simplest representations have fundamental physical importance, they are called Weyl spinors. Clearly they are representations of the connected component of SO(3,1), but not of parity, since parity interchanges the representations

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



$$J_i^+ = \frac{1}{2}\sigma_i, \qquad J_i^- = 0 \qquad \text{for} \qquad (\tfrac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}),$$

$$J_i^+ = 0, \qquad J_i^- = \frac{1}{2}\sigma_i \qquad \text{for} \qquad (\mathbf{0}, \tfrac{\mathbf{1}}{\mathbf{2}}).$$

$$u_\pm \longrightarrow e^{-\frac{i}{2}(\theta \mathbf{n} \mp i\beta)\cdot\sigma} u_\pm \qquad\qquad u_\pm \longrightarrow e^{i\theta} u_\pm$$

Consider for simplicity this global symmetry: fermion number

$$\sigma_\pm^\mu = (\mathbf{1}, \pm\sigma_i) \qquad \begin{array}{c} u_+^\dagger \sigma_+^\mu u_+ \\ u_-^\dagger \sigma_-^\mu u_- \end{array}$$

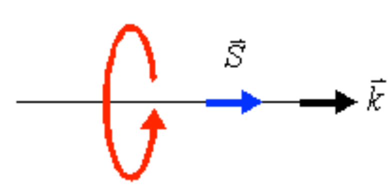
$$\mathscr{L}^\pm_{\text{Weyl}} = i u_\pm^\dagger \left(\partial_t \pm \sigma\cdot\nabla\right) u_\pm = i u_\pm^\dagger \sigma_\pm^\mu \partial_\mu u_\pm$$

$$(\partial_0 \pm \sigma\cdot\nabla)\, u_\pm = 0$$

$$u_\pm(x) = u_\pm(k) e^{-ik\cdot x}$$
$$k^2 = k_0^2 - \mathbf{k}^2 = 0$$

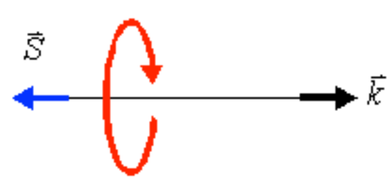
$$(|\mathbf{k}| \mp \mathbf{k}\cdot\sigma)\, u_\pm = 0$$

$$u_+ : \qquad \frac{\sigma\cdot\mathbf{k}}{|\mathbf{k}|} = 1,$$



positive helicity, right handed antineutrinos

$$u_- : \qquad \frac{\sigma\cdot\mathbf{k}}{|\mathbf{k}|} = -1$$



negative helicity, left handed, neutrinos

Charge conjugation and Majorana masses

We know that under parity, the L,R Weyl spinors are exchanged. Another way to exchange them is via complex conjugation, later to be related to charge conjugation

$$
\begin{aligned}
M_L &= e^{-\frac{i}{2}\theta\cdot\sigma-\frac{1}{2}\beta\cdot\sigma} & \det M_L &= 1 & \det M &= \epsilon_{ab} M_{a1}\, M_{b2} \\
M_R &= e^{-\frac{i}{2}\theta\cdot\sigma+\frac{1}{2}\beta\cdot\sigma} & \det M_R &= 1 & \det M\,\epsilon_{ab} &= \epsilon_{cd} M_{ca}\, M_{db}
\end{aligned}
\qquad \epsilon = i\,\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
$$

Using $\sigma^* = -\sigma_2\,\sigma\,\sigma_2$

$$
\begin{aligned}
\psi_L^c &= \sigma_2\,\psi_L^* \quad \text{transforms like} \quad \psi_R \\
\psi_R^c &= \sigma_2\,\psi_R^* \quad \text{transforms like} \quad \psi_L
\end{aligned}
$$

- We can express any theory fully in terms of L or R fermions.
- Charge conjugation and parity exchange L and R
- A parity invariant theory requires L,R spinors at the same time
- We can construct a mass for pure L spinors if we ignore fermion number
- Fermions anticommuting

$$
\mathscr{L}_{\text{Weyl}}^{\pm} = iu_{\pm}^{\dagger}\sigma_{\pm}^{\mu}\partial_{\mu}u_{\pm} + \frac{m}{2}\left(\varepsilon_{ab}u_{\pm}^{a}u_{\pm}^{b} + \text{h.c.}\right)
$$

$$
\epsilon_{ab}u^a\,u^b = u^1\,u^2 - u^2\,u^1
$$

Most general Majorana mass, Takagi factorisation

$$
\frac{1}{2}\left(M_{IJ}\,\epsilon_{ab}\,u^{a,I}\,u^{b,J} + \text{h.c.}\right),
$$

$$
I,J = 1,\ldots N_F, \quad M_{IJ} = M_{JI} \text{ complex}
$$

$$
M = U \begin{pmatrix} m_1 & \ldots & 0 \\ 0 & \ddots & 0 \\ 0 & \ldots & m_{N_F} \end{pmatrix} U^T
$$

m_i are positive square roots of $MM^\dagger$

This is the most general fermion mass matrix!!! It includes CKM, in fact it is more general

Weyl + parity: Dirac

$$(\tfrac{1}{2},\mathbf{0})\oplus(\mathbf{0},\tfrac{1}{2})$$

$$P: u_\pm \longrightarrow u_\mp \qquad \psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix} \qquad \left.\begin{aligned} i\sigma_+^\mu \partial_\mu u_+ &= m u_- \\ i\sigma_-^\mu \partial_\mu u_- &= m u_+ \end{aligned}\right\} \Longrightarrow \quad i\begin{pmatrix} \sigma_+^\mu & 0 \\ 0 & \sigma_-^\mu \end{pmatrix}\partial_\mu \psi = m \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}\psi$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma_-^\mu \\ \sigma_+^\mu & 0 \end{pmatrix} \qquad \overline{\psi} \equiv \psi^\dagger \gamma^0 = \psi^\dagger \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \qquad \mathscr{L}_{\text{Dirac}} = \overline{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi$$

DIRACOLOGY

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \qquad \gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \qquad P_\pm = \tfrac{1}{2}(1 \pm \gamma_5)$$

$$P_+\psi = \begin{pmatrix} u_+ \\ 0 \end{pmatrix} \qquad P_-\psi = \begin{pmatrix} 0 \\ u_- \end{pmatrix}$$

$$\begin{aligned} \mathrm{Tr}\,\gamma^\mu\gamma^\nu &= 4\,\eta^{\mu\nu} \\ \mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta &= 4\,\eta^{\mu\nu}\eta^{\alpha\beta} - 4\,\eta^{\mu\alpha}\eta^{\beta\nu} + 4\,\eta^{\mu\beta}\eta^{\alpha\nu} \\ \mathrm{Tr}\,\gamma_5\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu &= 4\,i\,\epsilon^{\alpha\beta\mu\nu} \end{aligned}$$

We look for +ve and -ve energy solutions as usual

$$u(k,s)e^{-ik\cdot x}$$

$$(\not{k} - m)u(k,s) = 0$$

$$v(k,s)e^{ik\cdot x}$$

$$(\not{k} + m)v(k,s) = 0$$

$$k^2 = m^2$$

$$\begin{aligned} \overline{u}(\mathbf{k},s)u(\mathbf{k},s) &= 2m, \\ \overline{u}(\mathbf{k},s)\gamma^\mu u(\mathbf{k},s) &= 2k^\mu, \\ \sum_{s=\pm\frac{1}{2}} u_\alpha(\mathbf{k},s)\overline{u}_\beta(\mathbf{k},s) &= (\not{k}+m)_{\alpha\beta} \end{aligned}$$

$$\begin{aligned} \overline{v}(\mathbf{k},s)v(\mathbf{k},s) &= -2m, \\ \overline{v}(\mathbf{k},s)\gamma^\mu v(\mathbf{k},s) &= 2k^\mu, \\ \sum_{s=\pm\frac{1}{2}} v_\alpha(\mathbf{k},s)\overline{v}_\beta(\mathbf{k},s) &= (\not{k}-m)_{\alpha\beta} \end{aligned}$$

We repeat the bosonic arguments, except for the fact that we have now anti-commutation relations between electron and positron creation-annihilation operators

$$\widehat{\psi}_\alpha(t,\vec{x}) = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}\left[u_\alpha(\vec{k},s)\,\widehat{b}(\vec{k},s)e^{-i\omega_{\vec{k}}t+i\vec{k}\cdot\vec{x}} + v_\alpha(\vec{k},s)\,\widehat{d}^{\,\dagger}(\vec{k},s)e^{i\omega_{\vec{k}}t-i\vec{k}\cdot\vec{x}}\right].$$

$$\{\widehat{\psi}_\alpha(t,\mathbf{x}),\widehat{\psi}^\dagger_\beta(t,\mathbf{y})\} = \delta(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta}$$

$$\begin{aligned}
\{b(\mathbf{k},s),b^\dagger(\mathbf{k}',s')\} &= (2\pi)^3(2\omega_\mathbf{k})\delta(\mathbf{k}-\mathbf{k}')\delta_{ss'},\\
\{b(\mathbf{k},s),b(\mathbf{k}',s')\} &= \{b^\dagger(\mathbf{k},s),b^\dagger(\mathbf{k}',s')\} = 0.\\
\{d(\mathbf{k},s),d^\dagger(\mathbf{k}',s')\} &= (2\pi)^3(2\omega_\mathbf{k})\delta(\mathbf{k}-\mathbf{k}')\delta_{ss'},\\
\{d(\mathbf{k},s),d(\mathbf{k}',s')\} &= \{d^\dagger(\mathbf{k},s),d^\dagger(\mathbf{k}',s')\} = 0.
\end{aligned}$$

$$\hat{H} = \frac{1}{2}\sum_{s=\pm\frac{1}{2}}\int\frac{d^3k}{(2\pi)^3}\left[b^\dagger(\mathbf{k},s)b(\mathbf{k},s) - d(\mathbf{k},s)d^\dagger(\mathbf{k},s)\right].$$

$$\widehat{H} = \sum_{s=\pm\frac{1}{2}}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}\left[\omega_{\vec{k}}b^\dagger(\vec{k},s)b(\vec{k},s) + \omega_{\vec{k}}d^\dagger(\vec{k},s)d(\vec{k},s)\right] - \;2\int d^3k\,\omega_{\vec{k}}\delta(\vec{0}).$$

We have a conserved charge and current

$$j^\mu = \overline{\psi}\gamma^\mu\psi, \qquad \partial_\mu j^\mu = 0 \qquad Q = e\int d^3x\, j^0$$

The two-point function or Feynman propagator is:

$$S_{\alpha\beta}(x_1,x_2) = \langle 0|T\Big[\psi_\alpha(x_1)\overline{\psi}_\beta(x_2)\Big]|0\rangle$$

$$T\Big[\psi_\alpha(x)\overline{\psi}_\beta(y)\Big] = \theta(x^0-y^0)\psi_\alpha(x)\overline{\psi}_\beta(y) - \theta(y^0-x^0)\overline{\psi}_\beta(y)\psi_\alpha(x).$$

Introducing gauge fields

The canonical gauge field is the electromagnetic field. The first one that was understood as a gauge field. For some time this symmetry sounded like a luxury. In fact the classical theory can be formulated exclusively in terms of the E,B field that are manifestly gauge invariant. This is not so in the quantum theory, where we need to use the vector and scalar potentials. There are new, non-local observables. They are responsible for the Bohm-Aharonov effect and the quantisation of electric charge (if there is a single monopole in the Universe, (Dirac)).

What we have learned is that all fundamental interactions known to us are mediated by suitable generalisations of the EM field. They are gauge theories. In fact it seems as though Nature abhors global symmetries. It appears that all the known global symmetries are just low-energy accidents. All symmetries in the UV should be local. To this we have to add now the scalar interaction induced by the Higgs particle.

We do not know why this should be so. String Theory is the only theory where this fact finds an explanation. Unfortunately there is no evidence for it at this moment...

E&M in Quantum Mechanics

Classical EM

$$\nabla \cdot \mathbf{E} = 0$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$$
$$\nabla \times \mathbf{B} = \frac{\partial}{\partial t}\mathbf{E}$$

$$\mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t}$$
$$\mathbf{B} = \nabla \times \mathbf{A}.$$

$$\partial_\mu F^{\mu\nu} = j^\mu \qquad j^\mu = (\rho, \mathbf{j})$$
$$\varepsilon^{\mu\nu\sigma\eta}\partial_\nu F_{\sigma\eta} = 0, \qquad A^\mu = (\varphi, \mathbf{A})$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Classical EM in relativistic form

Coupling to QM requires the gauge potentials and a non-trivial transformation of the wave function, this gives subtle consequences to gauge symmetry

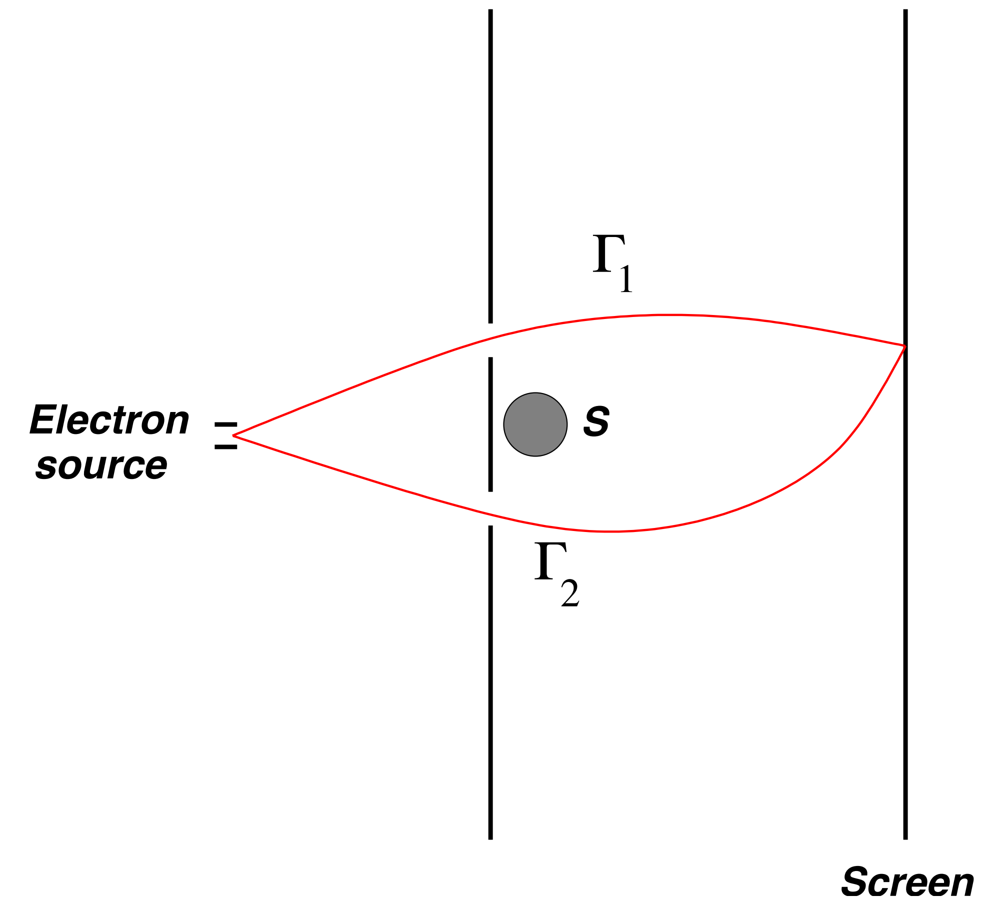
$$i\frac{\partial}{\partial t}\Psi = \left[-\frac{1}{2m}(\nabla - ie\mathbf{A})^2 + e\varphi\right]\Psi$$
$$\Psi(t,\mathbf{x}) \longrightarrow e^{-ie\,\varepsilon(t,\mathbf{x})}\Psi(t,\mathbf{x})$$
$$\varphi(t,\mathbf{x}) \to \varphi(t,\mathbf{x}) + \frac{\partial}{\partial t}\varepsilon(t,\mathbf{x}), \qquad \mathbf{A}(t,\mathbf{x}) \to \mathbf{A}(t,\mathbf{x}) + \nabla\varepsilon(t,\mathbf{x}).$$
$$A_\mu \longrightarrow A_\mu + \partial_\mu \varepsilon$$

Advanced Topic II:

The quantization of charge, and magnetic monopoles

Non-local observables

$$\begin{aligned}\Psi &= e^{ie\int_{\Gamma_1}\mathbf{A}\cdot d\mathbf{x}}\Psi_1^{(0)} + e^{ie\int_{\Gamma_2}\mathbf{A}\cdot d\mathbf{x}}\Psi_2^{(0)} \\ &= e^{ie\int_{\Gamma_1}\mathbf{A}\cdot d\mathbf{x}}\left[\Psi_1^{(0)} + e^{ie\oint_{\Gamma}\mathbf{A}\cdot d\mathbf{x}}\Psi_2^{(0)}\right]\end{aligned}$$



$$U = \exp\left[ie\oint_{\Gamma}\mathbf{A}\cdot d\mathbf{x}\right]$$

This is the Aharonov-Bohm effect. The phase factor, and its non-abelian generalisation are known as “Wilson loops” or holonomies of the gauge field. Note that classically there would be no effect. The Lorentz force equation only involves E,B hence the electrons would not see the solenoid at all!!

$$m\frac{du^{\mu}}{d\tau} = eF^{\mu\nu}u_{\nu}$$

Magnetic monopoles:Dirac and charge quantisation

Advanced Topic II-3

$$\nabla\cdot\mathbf{E} = 0$$
$$\nabla\cdot\mathbf{B} = 0$$
$$\nabla\times\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$$
$$\nabla\times\mathbf{B} = \frac{\partial}{\partial t}\mathbf{E}$$

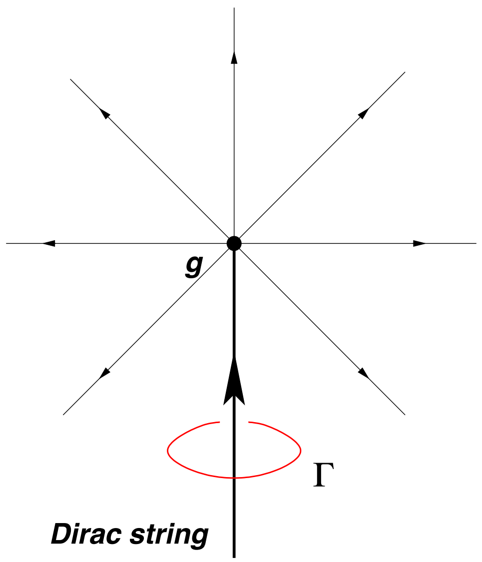
$$\mathbf{E} - i\mathbf{B} \longrightarrow e^{i\theta}(\mathbf{E} - i\mathbf{B})$$

For angle = 90 E and B get exchanged

The symmetry extend to matter if we have magnetic sources:

$$\rho - i\rho_m \longrightarrow e^{i\theta}(\rho - i\rho_m), \qquad \mathbf{j} - i\mathbf{j}_m \longrightarrow e^{i\theta}(\mathbf{j} - i\mathbf{j}_m).$$

Consider a magnetic pole:



$$\nabla\cdot\mathbf{B} = g\,\delta(\mathbf{x}). \qquad B_r = \frac{1}{4\pi}\frac{g}{|\mathbf{x}|^2}, \qquad B_\varphi = B_\theta = 0$$

$$A_\varphi = \frac{1}{4\pi}\frac{g}{|\mathbf{x}|}\tan\frac{\theta}{2}, \qquad A_r = A_\theta = 0.$$

The Dirac string can be changed by gauge transformations, in doing QM it has to be unobservable. Then we can do a "A-B" like argument (Dirac did it 20 years earlier). We should not forget the fact that there is a factor of $\hbar c$

$$e^{ieg} = 1 \qquad eg = 2\pi n$$

$$q_1 g_2 - q_2 g_1 = 2\pi n.$$

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Electromagnetic Fields and Photons

Ignoring sources, the E&M field is a "free field"

$$\mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

The electric field is the momentum p and the vector potential the "coordinate" q

$$\mathscr{L}_{\mathrm{Maxwell}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad A_\mu \longrightarrow A_\mu + \partial_\mu \varepsilon$$

$$\partial_\mu F^{\mu\nu} = 0 \qquad 0 = \partial_\mu \partial^\mu A^\nu - \partial_\nu\left(\partial_\mu A^\mu\right) = \partial_\mu \partial^\mu A^\nu$$

To be able to invert, we need to fix the gauge: $\partial_\mu A^\mu = 0.$

As usual, we look for plane wave solutions
Residual gauge transformation used to fully fix the gauge

$$\varepsilon_\mu(\mathbf{k},\lambda)e^{-i|\mathbf{k}|t + i\mathbf{k}\cdot\mathbf{x}}$$

$$k^\mu \varepsilon_\mu(\mathbf{k},\lambda) = 0$$

$$\epsilon_\mu(\mathbf{k},\lambda) \to \epsilon_\mu(\mathbf{k},\lambda) + k_\mu\,\chi(\mathbf{k}), \;\; k^2 = 0$$

$$k^2 = k_\mu k^\mu = (k^0)^2 - \mathbf{k}^2 = 0$$

Now, as usual we expand the field in oscillator and apply CCR. After fully fixing the gauge there are only two physical polarisations. Gauge invariance seems more a redundancy rather than a symmetry in the description of the theory

$$\widehat{A}_\mu(t,\mathbf{x}) = \sum_{\lambda=\pm1}\int \frac{d^3k}{(2\pi)^3}\frac{1}{2|\mathbf{k}|}\Big[\varepsilon_\mu(\mathbf{k},\lambda)\widehat{a}(\mathbf{k},\lambda)e^{-i|\mathbf{k}|t+i\mathbf{k}\cdot\mathbf{x}} + \varepsilon_\mu(\mathbf{k},\lambda)^*\,\widehat{a}^\dagger(\mathbf{k},\lambda)e^{i|\mathbf{k}|t-i\mathbf{k}\cdot\mathbf{x}}\Big].$$

$$[\widehat{a}(\mathbf{k},\lambda),\widehat{a}^\dagger(\mathbf{k}',\lambda')] = (2\pi)^3(2|\mathbf{k}|)\delta(\mathbf{k}-\mathbf{k}')\delta_{\lambda\lambda'}$$

If we keep all four polarisation by partial gauge fixing, then we get negative probabilities (Gupta-Bleuler, BRST)

$$\delta_{\lambda,\lambda'} \to -\eta_{\lambda,\lambda'}$$

Coupling matter

We imitate the coupling in the Schrödinger equation, this is what used to be called minimal coupling. We make derivatives covariant with respect to space-time dependent changes of phases in the wave-function

$$i\frac{\partial}{\partial t}\Psi = \left[-\frac{1}{2m}(\nabla - ie\mathbf{A})^2 + e\varphi\right]\Psi$$

$$\Psi(t,\mathbf{x}) \longrightarrow e^{-ie\,\varepsilon(t,\mathbf{x})}\Psi(t,\mathbf{x})$$

$$A_\mu \longrightarrow A_\mu + \partial_\mu \varepsilon$$

$$D_\mu\left[e^{ie\varepsilon(x)}\psi\right] = e^{ie\varepsilon(x)}D_\mu\psi.$$

$$D_\mu = \partial_\mu - ieA_\mu.$$

The rigid phase rotation invariance of the Dirac Lagrangian for electrons is transformed into local phase rotations, a physically more satisfactory concept. This defines the coupling of the electron to the E&M field:

$$\mathscr{L}_{\mathrm{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\not{D} - m)\psi, \qquad \mathscr{L}^{(\mathrm{int})}_{\mathrm{QED}} = -eA_\mu\overline{\psi}\gamma^\mu\psi.$$

$$\psi \longrightarrow e^{ie\varepsilon(x)}\psi, \qquad A_\mu \longrightarrow A_\mu + \partial_\mu\varepsilon(x).$$

This is QED, the best tested theory in the history of science, an example is the gyromagnetic ratio of the electron,

$$g\frac{e}{8m}[\gamma^\mu,\gamma^\nu]\,F_{\mu\nu}$$

$$g/2 = 1.00115965218085(76)$$
$$\alpha^{-1} = 137.035999070(98)$$

$$\vec{\mu} = g_\mu \frac{e\hbar}{2m_\mu c}\vec{s}, \qquad \underbrace{g_\mu = 2}_{\text{Dirac}}(1 + a_\mu)$$

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Group Theory reminder

For the SM all group we will need are:

$$G: \quad U(1),\ SU(2),\ SU(3) \qquad [T^a, T^b] = if^{abc}T^c$$

$$G_{SM} = SU(3) \times SU(2) \times U(1)$$

$$g \in G \qquad g = e^{i\epsilon^a T^a} \qquad \mathrm{tr}(T^a T^b) = T_2(R)\,\delta^{ab}$$

$$\det g = 1 \Rightarrow \mathrm{tr}\, T^a = 0$$ (for SU(2), SU(3) not for U(1) of course)

U(1) is of course the simplest, just phase multiplication, i.e. as in QED

SU(2): angular momentum, isospin, and also weak isospin

$$[T^a, T^b] = i\varepsilon^{abc}T^c \qquad T^{\pm} = \frac{1}{\sqrt{2}}(T^1 \pm iT^2), \qquad T^3$$

$$[T^3, T^{\pm}] = \pm T^{\pm} \qquad [T^+, T^-] = T^3$$

$$T^a = \frac{1}{2}\sigma^a$$ For spin ½ $$\mathrm{tr}\frac{\sigma^a}{2}\frac{\sigma^a}{2} = \frac{1}{2}\delta^{ab} \qquad a, b = 1, 2, 3$$

$$J^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad J^2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad J^3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ For spin 1

For SU(3) the generators are the eight Gell-Mann 3x3 traceless hermitean matrices chosen to satisfy:

$$\mathrm{tr}\frac{\lambda^a}{2}\frac{\lambda^a}{2} = \frac{1}{2}\delta^{ab}; \;\; a, b = 1, \ldots, 8$$

SU(3) of color, an exact gauge symmetry, also flavor SU(3), which is global (see later)

More about SU(3)

There are very few representations we will need for color SU(3):

$$\mathbf{3},\ \bar{\mathbf{3}},\ \mathbf{8}$$

quarks

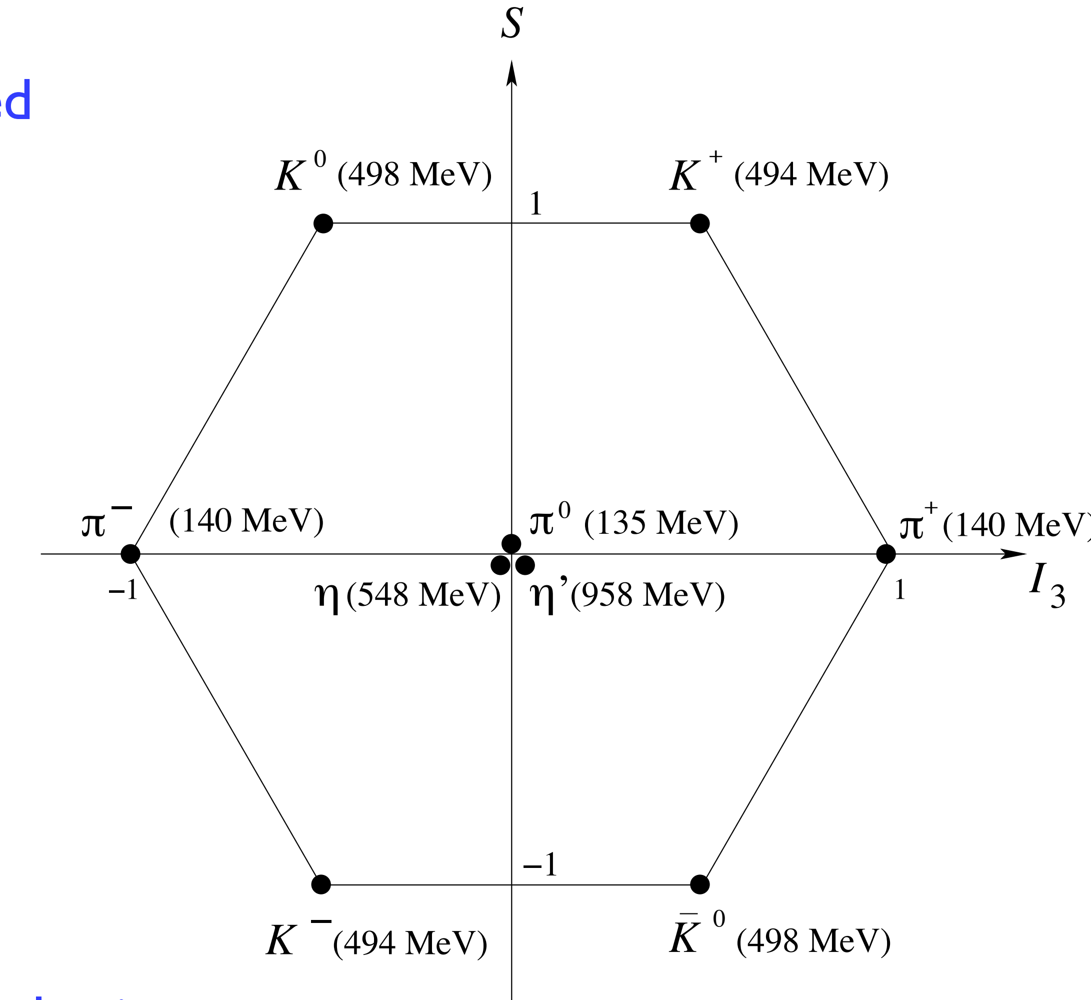
antiquarks

gluons

For flavor SU(3) more needed: mesons, baryons

$$\mathbf{3},\ \bar{\mathbf{3}},\ \mathbf{8},\ \mathbf{10},\ \overline{\mathbf{10}},\ \mathbf{27}\ldots$$

A remarkable fact about the SM and QCD in particular is the fact that once we write the most general Lagrangian compatible with color gauge symmetry, flavor appears as an approximate global symmetry of the problem, although it was theorised earlier.



pseudo-scalar meson octet

$$Q = I_3 + \frac{B+S}{2},$$

$$|\Delta^{++}; s_z = \tfrac{3}{2}\rangle = |uuu\rangle \otimes |\uparrow\uparrow\uparrow\rangle \equiv |u\uparrow, u\uparrow, u\uparrow\rangle.$$

$$|uud\rangle_S = \frac{1}{\sqrt{6}}\Big(|uud\rangle + |udu\rangle - 2|duu\rangle\Big),$$

$$|uud\rangle_A = \frac{1}{\sqrt{2}}\Big(|uud\rangle - |udu\rangle\Big).$$

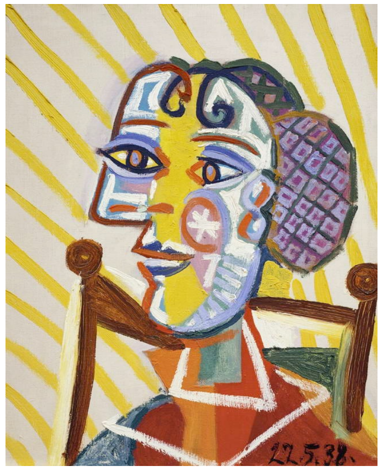
$$|\Uparrow\rangle_S = \frac{1}{\sqrt{6}}\Big(|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle - 2|\uparrow\uparrow\downarrow\rangle\Big);$$

$$|\Uparrow\rangle_A = \frac{1}{\sqrt{2}}\Big(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle\Big).$$

$$|p\uparrow\rangle = \frac{1}{\sqrt{2}}\Big(|uud\rangle_S \otimes |\Uparrow\rangle_A + |uud\rangle_A \otimes |\Uparrow\rangle_S\Big);$$

$$|p\downarrow\rangle = \frac{1}{\sqrt{2}}\Big(|uud\rangle_S \otimes |\Downarrow\rangle_A + |uud\rangle_A \otimes |\Downarrow\rangle_S\Big).$$

Gauge theories and their quantisation



Imagine we have a theory with a global symmetry

$$\psi \to g\,\psi \qquad \bar{\psi} \to \bar{\psi} g^{\dagger} \qquad \mathcal{L} = \bar{\psi} i \partial\!\!\!/ \,\psi$$

Imitating electromagnetism:

$$\partial_\mu \to D_\mu \psi = \left(\partial_\mu + ieA_\mu^a T^a\right)\psi \equiv (\partial_\mu + ieA_\mu)\psi \qquad D_\mu\psi \to g D_\mu \psi$$

We can read off the gauge field transformations

$$A_\mu \to \frac{1}{ie} g\,\partial_\mu g^{-1} + g A_\mu\, g^{-1}$$

$$g \approx 1 + \epsilon \qquad A_\mu \to A_\mu + \frac{1}{ie} D_\mu \epsilon \qquad D_\mu \epsilon + ie[A_\mu, \epsilon]$$

$$[D_\mu, D_\nu] = ieT^a F^a_{\mu\nu}, \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - ef^{abc} A^b_\mu A^c_\nu$$

$$F_{\mu\nu} \equiv T^a F^a_{\mu\nu} \to g F_{\mu\nu}\, g^{-1}$$

Nonabelian gauge fields have self-couplings unlike photons. This is responsible for confinement, among other things

General Gauge Theory

General gauge theory Lagrangian:

$$\mathscr{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu\, a} + i\overline{\psi}\not{D}\psi + (D_\mu\phi)^\dagger D^\mu\phi$$
$$-\overline{\psi}\left[M_1(\phi) + i\gamma_5 M_2(\phi)\right]\psi - V(\phi).$$

We need to provide the gauge group and the matter representations for bosons and fermions and off we go

Quantising a gauge theory is no joke. There are plenty of subtleties. We give you just a taste

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We can define chromoelectric and magnetic fields as in QED

$$\begin{aligned} F^a_{0i} &= \partial_0 A^a_i - \partial_i A^a_0 - ief^{abc}A^b_0 A^c_i \equiv E^a_i \\ F^a_{ij} &= \epsilon_{ijk}B^a_k, \qquad F^a_{0i} = \partial_0 A^a_0 - D_i A^a_0 \end{aligned}$$

The canonical variables are

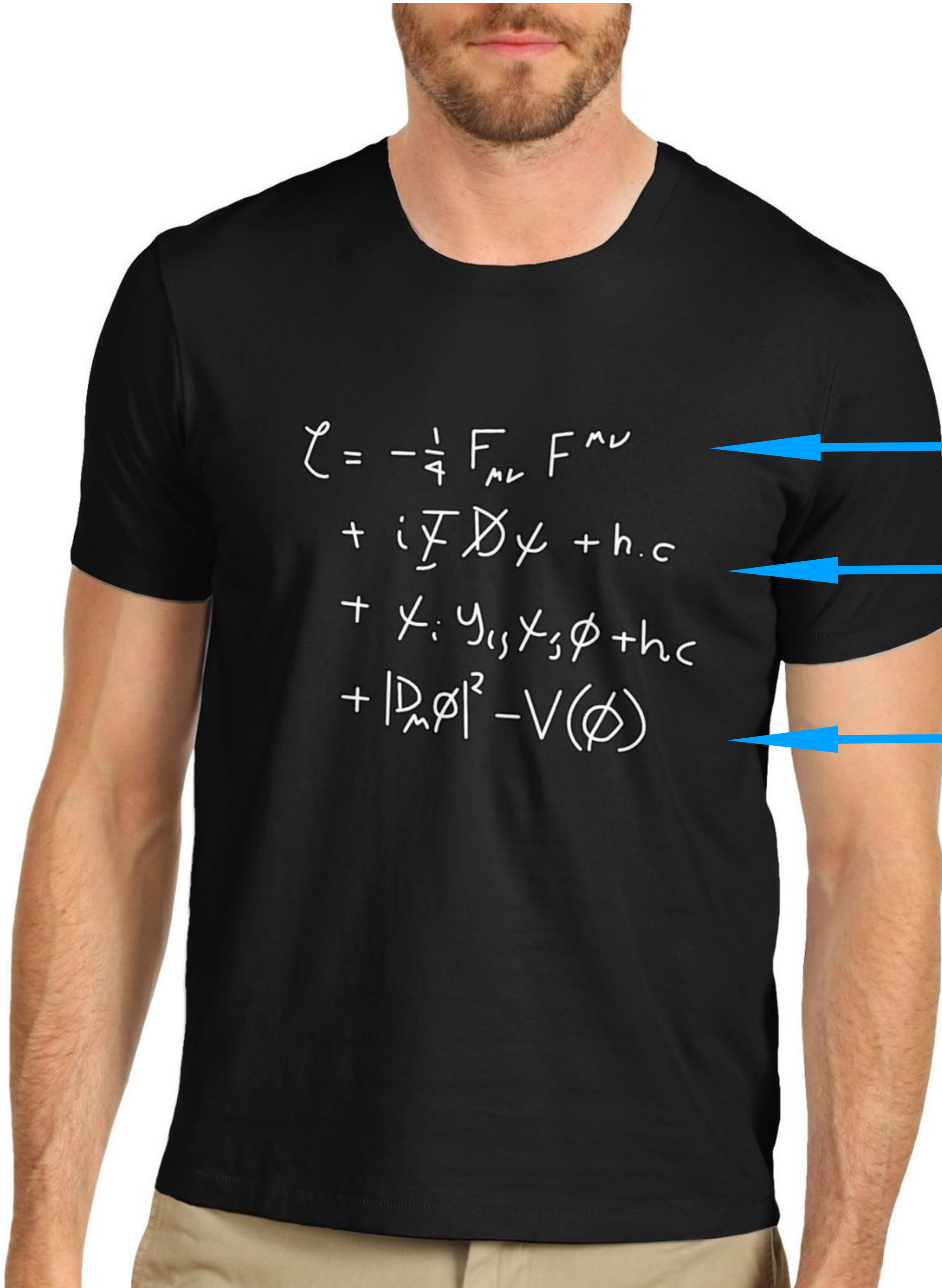
$$\mathbf{A}^a, \mathbf{E}^a$$

$$\mathcal{L} = \mathbf{E}^a\,\partial_0\mathbf{A}^a - \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) - A^a_0\,(\mathbf{D}\cdot\mathbf{E})^a$$

A^a_0 implements a constraint

We can read off the Hamiltonian density

The SM on a t-shirt



Force

Matter

Vacuum

+ gravity!

$$H = \int d^3x \left(\frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) + A_0^a \, (\mathbf{D} \cdot \mathbf{E})^a \right)$$

$$[A_i^a(\mathbf{x}, 0), E_j^b(\mathbf{y}, 0)] \; = \; i \, \delta_{ij} \, \delta^{ab} \delta(\mathbf{x} - \mathbf{y})$$

We can fix the gauge A_0=0 so that we only have time-independent gauge transformations in the Hamiltonian theory, but we are missing one of the equations of motion, Gauss' law that has to be implemented as a constraint.

$$(\mathbf{D} \cdot \mathbf{E})^a = 0$$

Cannot be implemented at the operator level. It generates gauge transformations

$$[Q(\epsilon), A_i^a] \; = \; i(D\epsilon)^a \qquad U(\epsilon) = \exp(i \int d^3x \, \epsilon^a(\mathbf{x}) \, (\mathbf{D} \cdot \mathbf{E})^a), \qquad U \, H \, U^{-1} = H$$

Gauss' law becomes a condition on the physical states:

$$U(\epsilon)|\mathrm{phys}\rangle = |\mathrm{phys}\rangle$$

$$\mathbf{D} \cdot \mathbf{E} \, |\mathrm{phys}\rangle = 0$$

Each gauge configuration sits in an orbit and we need choose only one element, this is done by "fixing" the gauge for the t-independent gauge transf.

$$\boxed{\text{WE HAVE 2-DIM G PHYSICAL DEGREES OF FREEDOM}}$$

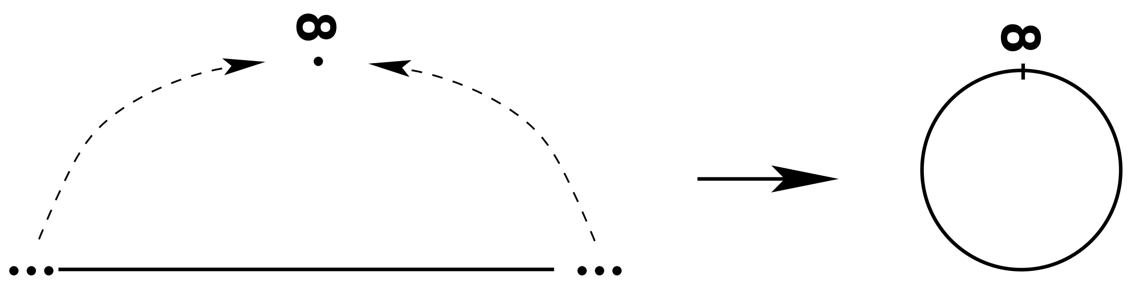
Some remarks: Vacuum structure

- Gauge symmetry is more a redundant description of the d.o.f.

- Gauss' law implements gauge invariance under gauge t. connected to the identity. Consider finite-E configurations

$$g(\mathbf{x}) = e^{i\alpha(\mathbf{x})} \to 1 \qquad |\mathbf{x}| \to \infty$$

$$\alpha(\mathbf{x}) \to 0 \qquad |\mathbf{x}| \to \infty$$



There are others, and Gauss' law cannot impose invariance

$$g(\mathbf{x}) : S^3 \to G, \quad g(\infty) = 1 \qquad \pi_3(G) = Z \text{ the integers}$$

$$g : S^1 \longrightarrow U(1), \qquad g(x) = e^{i\alpha(x)}$$

$$\alpha(2\pi) = \alpha(0) + 2\pi n$$

$$\oint_{S^1} g(x)^{-1} dg(x) = 2\pi n$$

$$n = \frac{1}{24\pi^2} \int_{S^3} d^3x\, \varepsilon_{ijk} \mathrm{Tr}\left[(g^{-1}\partial_i g)\, (g^{-1}\partial_i g)\, (g^{-1}\partial_i g) \right]$$

You cannot comb a sphere

A surprise: CP violation

❖ Gauge invariance only requires that under non-trivial transformations, a phase is generated. This is a vacuum angle! In fact it violates CP.

❖ It can be measured by looking for an edm of the neutron. So far no result:

❖ The strong CP problem, axions, invisible axions, axion cosmology, dark matter...

$$g_1 \in \mathcal{G}/\mathcal{G}_0 \quad \text{the generator}$$

$$\mathscr{U}(g_1)|\text{phys}\rangle = e^{i\theta}|\text{phys}\rangle.$$

$$S = -\frac{1}{4}\int d^4x\, F^a_{\mu\nu}F^{\mu\nu\, a} - \frac{\theta g^2_{\text{YM}}}{32\pi^2}\int d^4x\, F^a_{\mu\nu}\widetilde{F}^{\mu\nu\, a}$$

$$\widetilde{F}^a_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\sigma\lambda}F^{\sigma\lambda\, a} \qquad F^a_{\mu\nu}\widetilde{F}^{\mu\nu\, a} = 4\,\mathbf{E}^a\cdot\mathbf{B}^a$$

$$\frac{g^2_{\text{YM}}}{32\pi^2}\int d^4x\, F^a_{\mu\nu}\widetilde{F}^{\mu\nu\, a} = \frac{1}{24\pi^2}\int d^3x\, \varepsilon_{ijk}\text{Tr}\left[(g\partial_i g^{-1})(g_+\partial_j g^{-1})(g_+\partial_k g^{-1})\right].$$

❖ There are two general procedures to obtain computational rules in QFT: The canonical formalism and the Path Integral formulation.

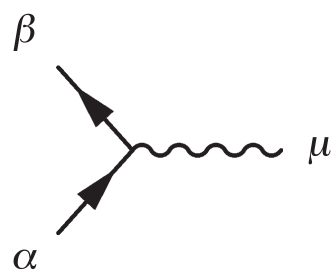
❖ You may recall that one used the Interaction Representation, Wick's theorem, T-products, Gaussian integrations…

❖ In the end we get a collection of well-defined rules that allow us to compute the probability amplitude associates to a given scattering process, out of which we can evaluate the decay width, differential and total cross section and many other quantities that can be observed for instance in collider experiments. The next few pages provide simply a reminder

QED Feynman rules

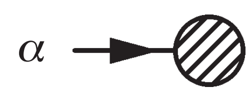
$$\Longrightarrow \quad \left(\frac{i}{\not p - m + i\varepsilon}\right)_{\beta\alpha}$$

$$\Longrightarrow \quad \frac{-i\eta_{\mu\nu}}{p^2 + i\varepsilon}$$



$$\Longrightarrow \quad -ie\gamma^{\mu}_{\beta\alpha}(2\pi)^4\delta^{(4)}(p_1+p_2+p_3).$$

Incoming fermion:

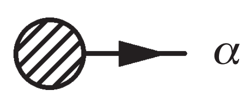


$\Longrightarrow \quad u_\alpha(\mathbf{p}, s)$

Incoming antifermion:

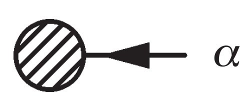
$\Longrightarrow \quad \overline{v}_\alpha(\mathbf{p}, s)$

Outgoing fermion:



$\Longrightarrow \quad \overline{u}_\alpha(\mathbf{p}, s)$

Outgoing antifermion:



$\Longrightarrow \quad v_\alpha(\mathbf{p}, s)$

Incoming photon:

$\Longrightarrow \quad \varepsilon_\mu(\mathbf{p})$

Outgoing photon:

$\Longrightarrow \quad \varepsilon_\mu(\mathbf{p})^*$

Integrate over loop momenta

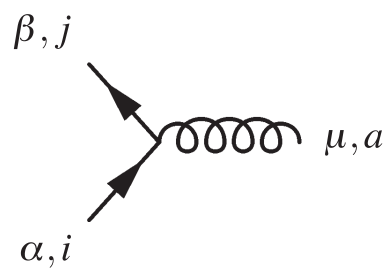
$$\int \frac{d^d p}{(2\pi)^4};$$

A minus sign has to be included for every fermion loop and for every positron line that goes from the initial to the final state. With some extra effort we can derive the Feynman rules for QCD-like theories. They appear in the next page. The quark and anti-quark factors are similar to the electron positron ones, except that we need to include color quantum numbers. The real difference comes with the gluon or non-abelian vector bosons interactions, the are quite involved and contain a large amount of interesting physics perturbatively and specially non-perturbatively.

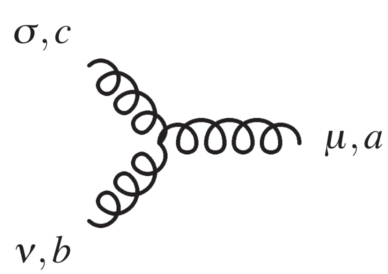
Standard Model Feynman rules

$$\Longrightarrow \left(\frac{i}{\not{p} - m + i\varepsilon}\right)_{\beta\alpha} \delta_{ij}$$

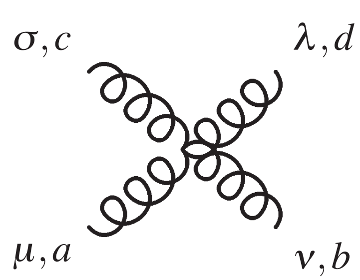
$$\Longrightarrow \frac{-i\eta_{\mu\nu}}{p^2 + i\varepsilon} \delta^{ab}$$



$$\Longrightarrow -ig\gamma^{\mu}_{\beta\alpha} t^a_{ij}$$



$$\Longrightarrow g f^{abc}\left[\eta^{\mu\nu}(p_1^{\sigma} - p_2^{\sigma})\text{permutations}\right]$$



$$\Longrightarrow -ig^2\Big[f^{abe} f^{cde}\left(\eta^{\mu\sigma}\eta^{\nu\lambda} - \eta^{\mu\lambda}\eta^{\nu\sigma}\right) + \text{permutations}\Big]$$

Although the rules seem to be those for QCD, notice that we could always include in the group theory factors t^a-{ij} chiral projectors and make the group not simple but semi-simple as in the case of the SM: SU(3)xSU(2)xU(1). If we work in nice renormalizable gauges, the only difference is that we have to include the Feynman rules for the couplings of the scalar sector. Something we will do later.

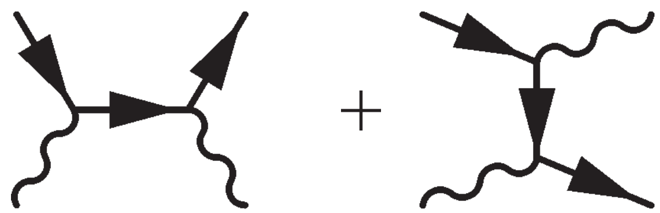
$$t^a_{ij} \to t^a_{ij} \, \frac{1}{2}(1 \pm \gamma_5)$$

With this simple trick the hard part, which is the coupling of the W,Z, and photons can be read simply from the rules in the LHS

One example: Thomson Scattering

$$\gamma(k,\varepsilon)+e^-(p,s)\longrightarrow\gamma(k',\varepsilon')+e^-(p',s')$$

We work in the NR approximation for simplicity but keeping explicitly the dependence on the photon polarisations. We can guess that the answer has to be a pure number times the classical electron radius



$$= (ie)^2\overline{u}(\mathbf{p}',s')\not{\varepsilon}'(\mathbf{k}')^*\frac{\not{p}+\not{k}+m_e}{(p+k)^2-m_e^2}\not{\varepsilon}(\mathbf{k})u(\mathbf{p},s)$$
$$+\ (ie)^2\overline{u}(\mathbf{p}',s')\not{\varepsilon}(\mathbf{k})\frac{\not{p}-\not{k}'+m_e}{(p-k')^2-m_e^2}\not{\varepsilon}'(\mathbf{k}')^*u(\mathbf{p},s)$$

$$p^2=m_e^2=p'^2 \qquad |\mathbf{p}|,|\mathbf{k}|,|\mathbf{p}'|,|\mathbf{k}'|\ll m_e \qquad \not{a}\not{b}=-\not{b}\not{a}+2(a\cdot b)\mathbf{1}$$

$$k^2=0=k'^2 \qquad (p+k)^2-m_e^2\approx 2m_e|\mathbf{k}|, \qquad (p-k')^2-m_e^2\approx -2m_e|\mathbf{k}'|$$

$$(\not{k}-m)u(k,s)=0$$

$$\overline{u}(\mathbf{k},s)\gamma^\mu u(\mathbf{k},s)=2k^\mu$$

$$\langle f|\widehat{S}|i\rangle = \langle f|i\rangle + (2\pi)^4\delta^{(4)}\left(\sum_{\text{final}} p_i' - \sum_{\text{initial}} p_j\right) i\mathscr{M}_{i\to f}$$

$$d\sigma = \frac{|\mathscr{M}_{i\to f}|^2}{4E_1E_2|\mathbf{v}_1-\mathbf{v}_2|}(2\pi)^4\,\delta^{(4)}\left(p_1+p_2-\sum_{j=1}^{n} p_j'\right) d\Phi_k.$$

$$\begin{aligned} F_{\text{coll}} &= 4E_1E_2|\mathbf{v}_1-\mathbf{v}_2| = 4E_1E_2\left|\frac{\mathbf{p}_1}{E_1}-\frac{\mathbf{p}_2}{E_2}\right| \\ &= 4|E_2\mathbf{p}_1 - E_1\mathbf{p}_2| = 4\Big(E_2|\mathbf{p}_1| + E_1|\mathbf{p}_2|\Big) \\ &= 4\sqrt{(p_1\cdot p_2)^2 - m_1^2m_2^2}. \end{aligned}$$

Square the amplitude, sum over final electron polarisations, and sum over the initial ones. We will consider unpolarised incoming photons and study how the outgoing photons can gain some degree of polarisation

$$\sum_{s=\pm\frac{1}{2}} u_\alpha(\mathbf{k},s)\overline{u}_\beta(\mathbf{k},s) = (\not{k}+m)_{\alpha\beta}\,;$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2m_e^2}\overline{|i\mathscr{M}_{i\to f}|^2} = \left(\frac{e^2}{4\pi m_e}\right)^2\left|\varepsilon(\mathbf{k})\cdot\varepsilon'(\mathbf{k}')^*\right|^2$$

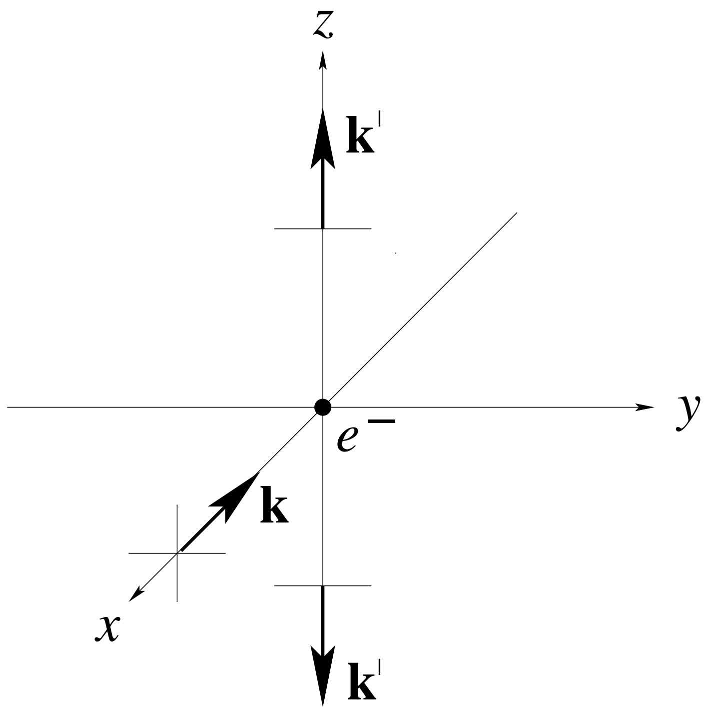
$$\frac{d\sigma}{d\Omega} = \frac{3}{8\pi}\sigma_T\left|\varepsilon(\mathbf{k})\cdot\varepsilon'(\mathbf{k}')^*\right|^2;$$

$$\sigma_T = \frac{e^4}{6\pi m_e^2} = \frac{8\pi}{3}r_{\text{cl}}^2$$

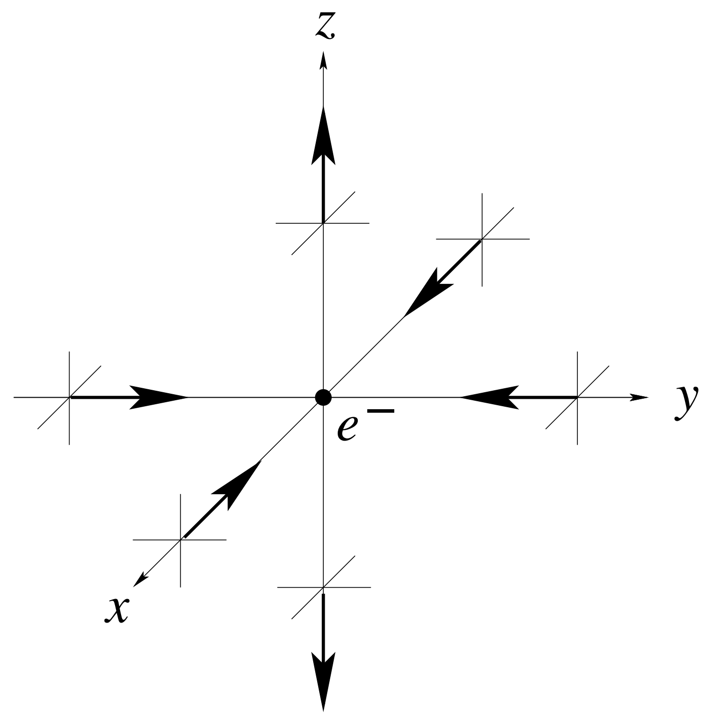
$$\begin{aligned}\frac{1}{2}\sum_{a=1,2}\left|\varepsilon(\mathbf{k},a)\cdot\varepsilon'(\mathbf{k}')^*\right|^2 &= \frac{1}{2}\left(\delta_{ij}-\frac{k_ik_j}{|\mathbf{k}|^2}\right)\varepsilon_j'(\mathbf{k}')\varepsilon_i'(\mathbf{k}')^* \\ &= \frac{1}{2}\Big[1-|\hat{\mathbf{k}}\cdot\varepsilon'(\mathbf{k}')|^2\Big],\end{aligned}$$

We want to monitor the polarisation of the outgoing photons even when the incoming ones are not polarised

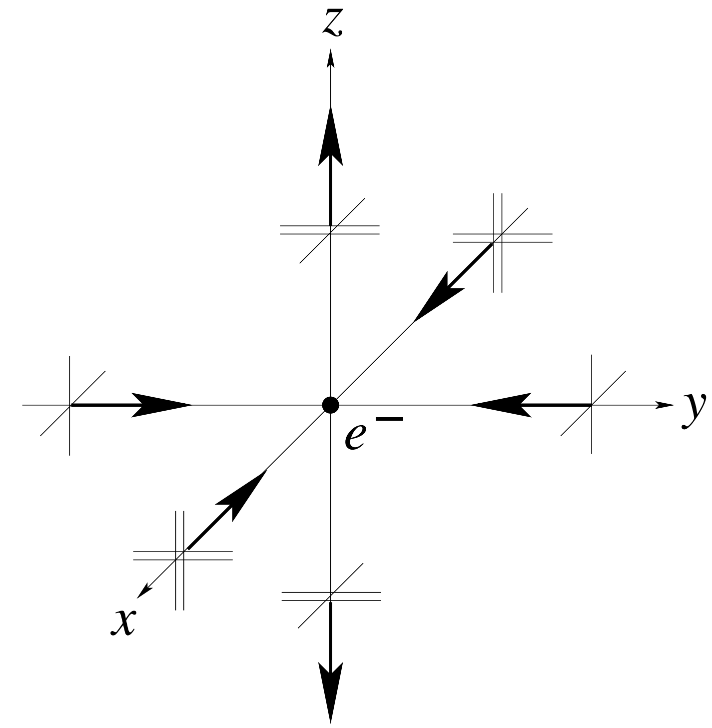
Advanced Topic IV-2



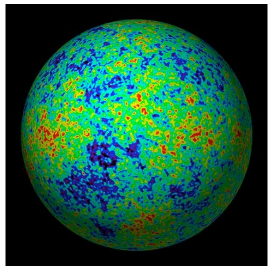
How we can get polarised light



An isotropic incoming distribution of light does not generate polarisation



A incoming light with a quadrupole perturbation generates net polarisation

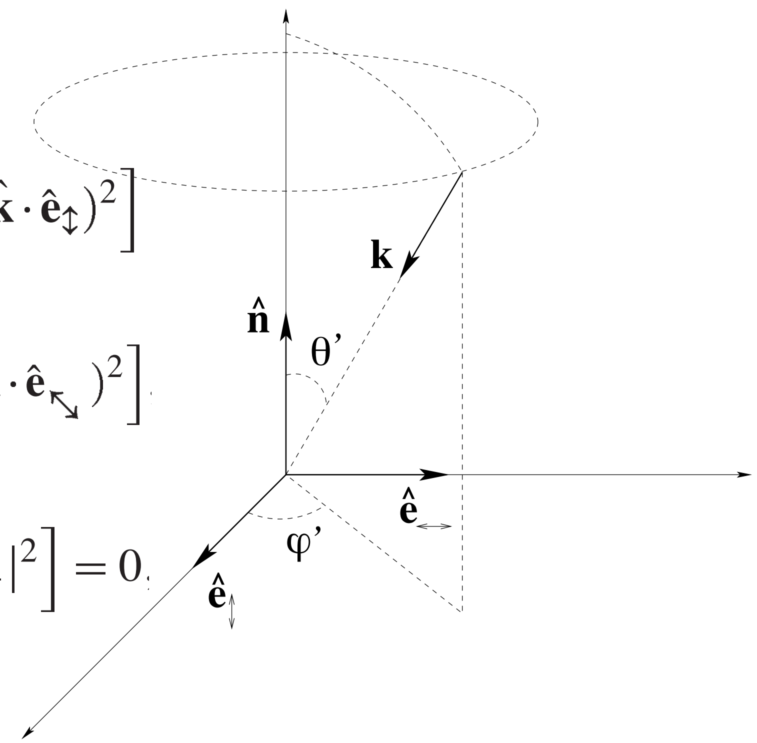


Stokes parameters:

$$Q(\hat{\mathbf{n}}) \sim \sum_{a=1,2} \int d\Omega(\hat{\mathbf{k}})\, f(\hat{\mathbf{k}},\hat{\mathbf{n}}) \Big[|\varepsilon(\mathbf{k},a)\cdot\hat{\mathbf{e}}_{\leftrightarrow}|^2 - |\varepsilon(\mathbf{k},a)\cdot\hat{\mathbf{e}}_{\updownarrow}|^2 \Big] \qquad -\frac{1}{2}\int d\Omega(\hat{\mathbf{k}})\, f(\hat{\mathbf{k}},\hat{\mathbf{n}}) \Big[(\hat{\mathbf{k}}\cdot\hat{\mathbf{e}}_{\leftrightarrow})^2 - (\hat{\mathbf{k}}\cdot\hat{\mathbf{e}}_{\updownarrow})^2 \Big]$$

$$U(\mathbf{u}) \sim \sum_{a=1,2} \int d\Omega(\hat{\mathbf{k}})\, f(\hat{\mathbf{k}},\mathbf{u}) \Big[|\varepsilon(\mathbf{k},a)\cdot\hat{\mathbf{e}}_{\nearrow}|^2 - |\varepsilon(\mathbf{k},a)\cdot\hat{\mathbf{e}}_{\nwarrow}|^2 \Big] = -\frac{1}{2}\int d\Omega(\hat{\mathbf{k}})\, f(\hat{\mathbf{k}},\hat{\mathbf{n}}) \Big[(\hat{\mathbf{k}}\cdot\hat{\mathbf{e}}_{\nearrow})^2 - (\hat{\mathbf{k}}\cdot\hat{\mathbf{e}}_{\nwarrow})^2 \Big].$$

$$V(\hat{\mathbf{u}}) \sim \sum_{a=1,2} \int d\Omega(\hat{\mathbf{k}})\, f(\hat{\mathbf{k}},\mathbf{u}) \Big[|\varepsilon(\mathbf{k},a)\cdot\hat{\mathbf{e}}_{+}|^2 - |\varepsilon(\mathbf{k},a)\cdot\hat{\mathbf{e}}_{-}|^2 \Big] = \int d\Omega(\hat{\mathbf{k}})\, f(\hat{\mathbf{k}},\mathbf{u}) \Big[|\hat{\mathbf{k}}\cdot\hat{\mathbf{e}}_{+}|^2 - |\hat{\mathbf{k}}\cdot\hat{\mathbf{e}}_{-}|^2 \Big] = 0,$$



$$\hat{\mathbf{e}}_{\pm} = -\frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_{\varphi} \pm i\hat{\mathbf{e}}_{\theta})$$

Quadrupole distribution

Finally we reach the punch line. No circular polarisation is generated by Thomson scattering, and we can write the combination:

$$Q(\hat{\mathbf{n}}) \pm iU(\hat{\mathbf{n}}) \sim -\int d\Omega(\theta', \varphi') f(\theta', \varphi'; \hat{\mathbf{n}}) \sin^2\theta' e^{\pm 2i\varphi'}$$

$$Y_2^{\pm 2}(\theta', \varphi') = 3\sqrt{\frac{5}{96\pi}} \sin^2\theta' e^{\pm 2i\varphi'}$$

One of the obvious generators of quadrupole anisotropies are gravitational waves. Inflation predicts primordial gravitation waves, the measurement of polarisation in the CMB offers an amazing window to obtain this information. The simple computation of Thomson scattering has unexpected consequences

$$Q(\hat{\mathbf{n}}) \pm iU(\hat{\mathbf{n}}) = -\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \Big(E_{\ell m} \pm iB_{\ell m}\Big)\, {}_{\pm 2}Y_\ell^m(\hat{\mathbf{n}})$$

$$\langle E^*_{\ell m} E_{\ell' m'} \rangle = C_\ell^{EE} \delta_{\ell\ell'} \delta_{mm'}, \qquad \langle B^*_{\ell m} B_{\ell' m'} \rangle = C_\ell^{BB} \delta_{\ell\ell'} \delta_{mm'}$$

Quantum mechanical realisation of Symmetries (Wigner's theorem). In a QM theory physical symmetries are maps among states that preserve probability amplitudes (their modulus). They can be unitary or anti-unitary

$$|\alpha\rangle \longrightarrow |\alpha'\rangle, \qquad |\beta\rangle \longrightarrow |\beta'\rangle$$

$$|\langle\alpha|\beta\rangle| = |\langle\alpha'|\beta'\rangle|.$$

$$\langle \mathscr{U}\alpha|\mathscr{U}\beta\rangle = \langle\alpha|\beta\rangle$$ unitary

$$\langle \mathscr{U}\alpha|\mathscr{U}\beta\rangle = \langle\alpha|\beta\rangle^*$$ anti-unitary T-reversal, CPT

For continuous symmetries we have Noether's celebrated theorem: If under infinitesimal transformations, AND WITHOUT USING THE EQUATIONS OF MOTION you can show that:

$$S[\phi] = \int d^4x\, \mathcal{L}(\phi, \partial_\mu\phi)$$

$$\delta_\varepsilon \mathscr{L} = \partial_\mu K^\mu$$

then there is a conserved current in the theory

Noether's Theorem

In formulas:

$$\begin{aligned}\delta_\varepsilon \mathscr{L} &= \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\partial_\mu \delta_\varepsilon \phi + \frac{\partial \mathscr{L}}{\partial \phi}\delta_\varepsilon \phi \\ &= \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\delta_\varepsilon \phi\right) + \left[\frac{\partial \mathscr{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\right)\right]\delta_\varepsilon \phi \\ &= \partial_\mu K^\mu.\end{aligned}$$

$$\partial_\mu J^\mu = 0 \qquad \boxed{J^\mu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\delta_\varepsilon \phi - K^\mu}$$

With a conserved charge that generates the symmetry:

$$Q \equiv \int d^3x J^0(t,\mathbf{x}) \qquad \frac{dQ}{dt} = \int d^3x\, \partial_0 J^0(t,\mathbf{x}) = -\int d^3x\, \partial_i J^i(t,\mathbf{x}) = 0,$$

$$\delta\phi = i[\phi, Q].$$

Space-time translations -- Energy-Momentum
Lorentz transformation-- Angular momentum and CM motion
Phase rotation -- abelian and non-abelian charges

Useful examples

Massive Dirac fermions:

$$\mathscr{L} = i\overline{\psi}_j \partial\!\!\!/ \psi_j - m\overline{\psi}_j \psi_j \qquad \psi_i \longrightarrow U_{ij}\psi_j \qquad U \in U(N) \;\; N \text{the number of fermions}$$

$$U = \exp(i\alpha^a T^a), \qquad (T^a)^\dagger = T^a$$

$$j^{\mu a} = \overline{\psi}_i T^a_{ij} \gamma^\mu \psi_j \qquad \partial_\mu j^\mu = 0 \qquad Q^a = \int d^3x\, \psi_i^\dagger T^a_{ij} \psi_j$$

$$[Q^a, H] = 0. \qquad \mathscr{U}(\alpha) = e^{i\alpha^a Q^a}.$$

When U is the identity, we have fermion number, or charge

In the m=0 we have more symmetry: CHIRAL SYMMETRY, rotate L,R fermions independently

$$\mathcal{L} = i\bar{\psi}_{jL} \partial\!\!\!/ \psi_{Lj} + i\bar{\psi}_{jR} \partial\!\!\!/ \psi_{Rj}$$

$$\psi_{L,R} \to U_{L,R}\, \psi_{L,R} \qquad U(N)_L \times U(N)_R$$

Wigner-Weyl mode

Imagine we have a symmetry that is a symmetry of the ground state

$$[Q^a, H] = 0. \qquad \mathscr{U}(\alpha)|0\rangle = |0\rangle \qquad Q^a|0\rangle = 0$$

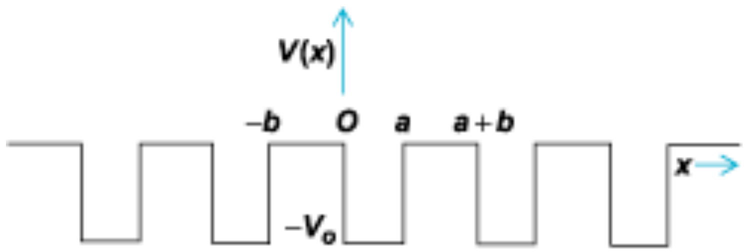
Then the states of the theory fall into multiplets of the symmetry group

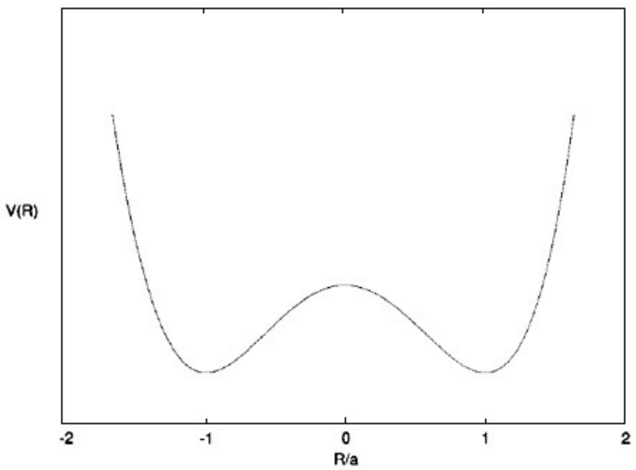
$$\mathscr{U}(\alpha)\phi_i\mathscr{U}(\alpha)^{-1} = U_{ij}(\alpha)\phi_j$$

$$|i\rangle = \phi_i|0\rangle$$

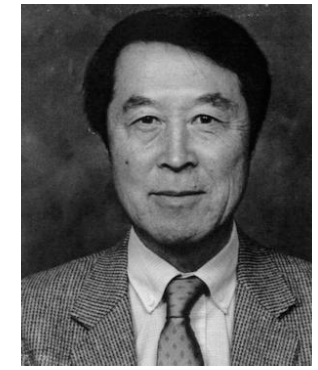
$$\mathscr{U}(\alpha)|i\rangle = \mathscr{U}(\alpha)\phi_i\mathscr{U}(\alpha)^{-1}\mathscr{U}(\alpha)|0\rangle = U_{ij}(\alpha)\phi_j|0\rangle = U_{ij}(\alpha)|j\rangle$$

The spectrum of the theory is classified in terms of multiplets of the symmetry group. This is the case of the Hydrogen atom. The Hamiltonian is rotational invariant, the ground state is an s-wave state, hence all excited states fall into degenerate representations of the rotation group: 1s, 2s, 2p, 2s, 3p, 3d,… In QM (finite number of d.o.f.) this is always the case (tunnelling, band theory in solids)





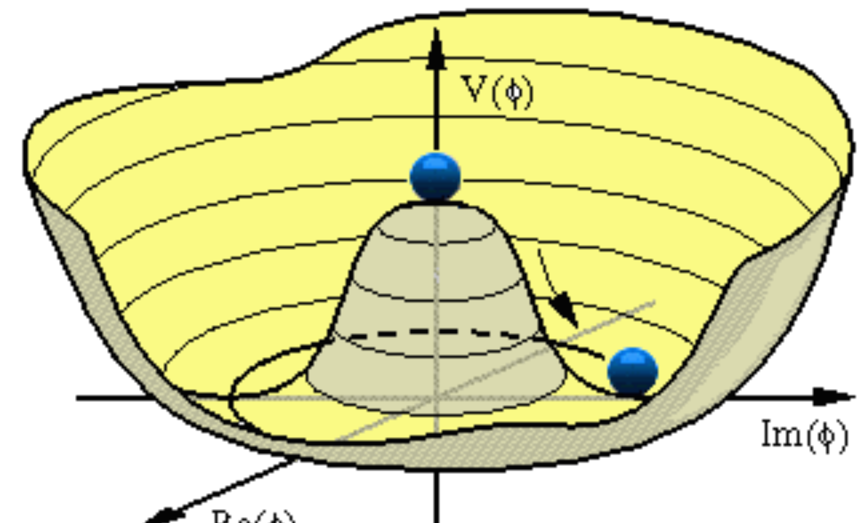
Nambu-Goldstone mode



Sometimes also called hidden symmetry. The symmetry is spontaneously broken by the vacuum

$$[Q^a, H] = 0. \qquad Q^a|0\rangle \neq 0.$$

Consider a collection of N scalar fields with a global symmetry group G



$$\mathscr{L} = \frac{1}{2}\partial_\mu \varphi^i \partial^\mu \varphi^i - V(\varphi^i) \qquad \delta\varphi^i = \varepsilon^a (T^a)^i_j \varphi^j$$

$$H[\pi^i, \varphi^i] = \int d^3x \left[\frac{1}{2}\pi^i\pi^i + \frac{1}{2}\nabla\varphi^i \cdot \nabla\varphi^i + V(\varphi^i)\right]$$

$$\mathscr{V}[\varphi^i] = \int d^3x \left[\frac{1}{2}\nabla\varphi^i \cdot \nabla\varphi^i + V(\varphi^i)\right]$$

The minima satisfy

$$\langle\varphi^i\rangle \qquad V(\langle\varphi^i\rangle) = 0, \qquad \nabla\varphi = \mathbf{0} \qquad \left.\frac{\partial V}{\partial\varphi^i}\right|_{\varphi^i = \langle\varphi^i\rangle} = 0$$

$$T^a = \{H^\alpha, K^A\} \qquad (H^\alpha)^i_j \langle\varphi^j\rangle = 0 \qquad (K^A)^i_j \langle\varphi^j\rangle \neq 0.$$

unbroken broken

Nambu-Goldstone mode

The masses are given by the second derivatives of the potential (assuming canonical normalisation)

$$M^2_{ij} \equiv \left. \frac{\partial^2 V}{\partial \varphi^i \partial \varphi^j} \right|_{\varphi = \langle \varphi \rangle}$$

Invariance

$$\delta V(\varphi) = \varepsilon^a \frac{\partial V}{\partial \varphi^i} (T^a)^i_j \varphi^j = 0 \qquad \frac{\partial^2 V}{\partial \varphi^i \partial \varphi^k} (T^a)^i_j \varphi^j + \frac{\partial V}{\partial \varphi^i} (T^a)^i_k = 0$$

$$M^2_{ik} (T^a)^i_j \langle \varphi^j \rangle = 0. \qquad M^2_{ik} (K^A)^i_j \langle \varphi^j \rangle = 0$$

For every broken generator there is a massless scalar field

The argument works at the full quantum level

The fields acquiring a VEV need not be elementary

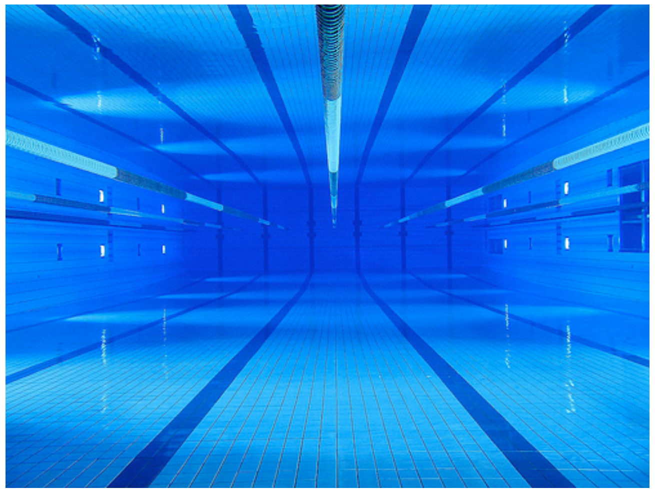
Simplest example:

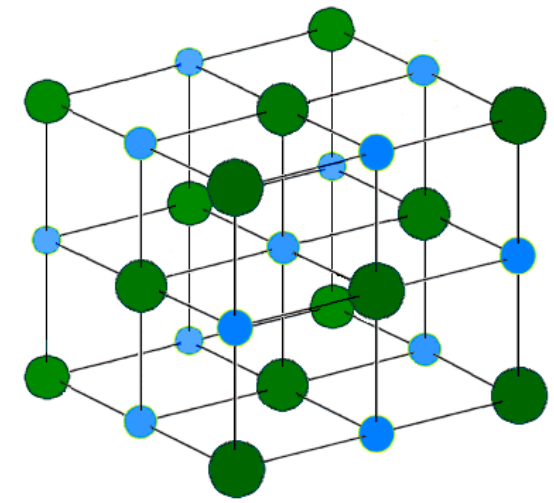
$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi$$

$$\phi \to \phi + c$$

Its own NG-boson

Phonons are NG bosons





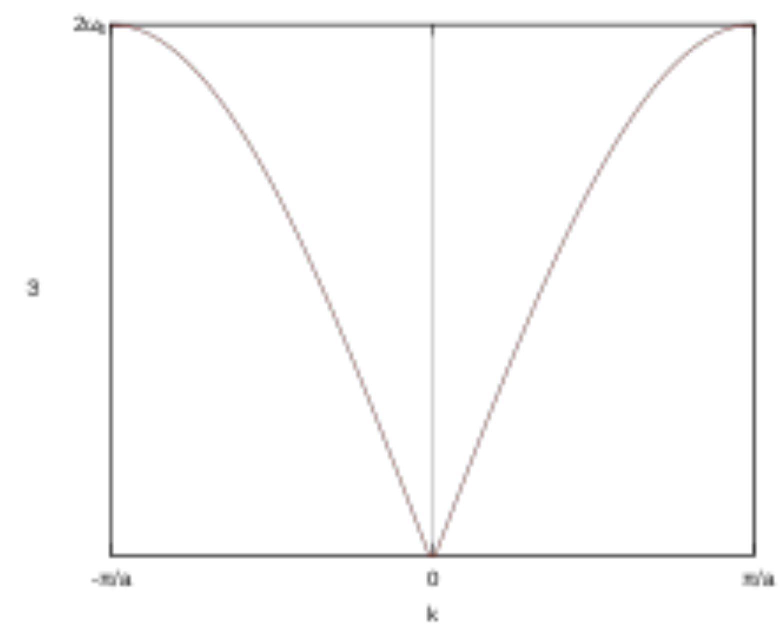
A liquid is translationally invariant

The crystal after solidification has discrete translational symmetry

The low energy excitation of the lattice contain acoustic phonons

Their dispersion relation is as for NG bosons

They propagate at the speed of sound



$$\omega(k) = 2\omega|\sin(ka/2)|$$

Order parameters

❖ The notion of symmetry breaking is intimately connected with the theory of phase transitions in CMP

❖ It is quite frequent that in going from one phase to another the symmetry of the ground state (vacuum) changes

❖ In real physical systems this is what we see with magnetic domains in magnetic material below the Curie point

❖ In going from one phase to the other, some parameters change in a noticeable way. These are the order parameters.

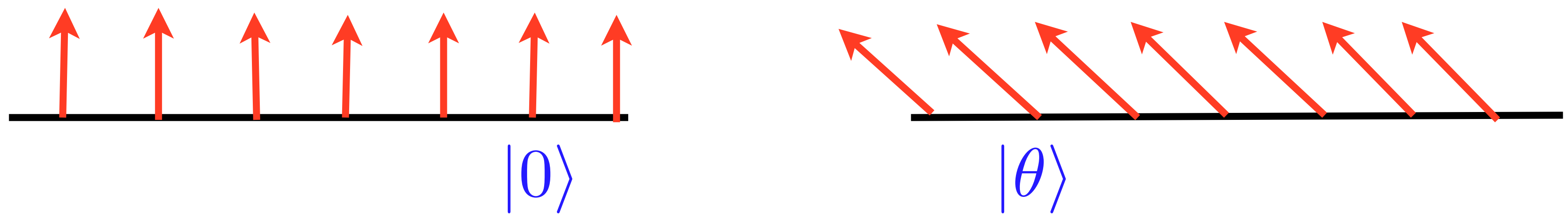
❖ In liquid-solid transition it is the density

❖ In magnetic materials it is the magnetization

❖ In the Ginsburg-Landau theory of superconductivity, the Cooper pairs acquire a VEV. This breaks U(1) inside the superconductor and thus explains among other things the Meissner effect. The Cooper pairs are pairs of electrons bound by the lattice vibrations. In ordinary superconductors their size is several hundred Angstroms.

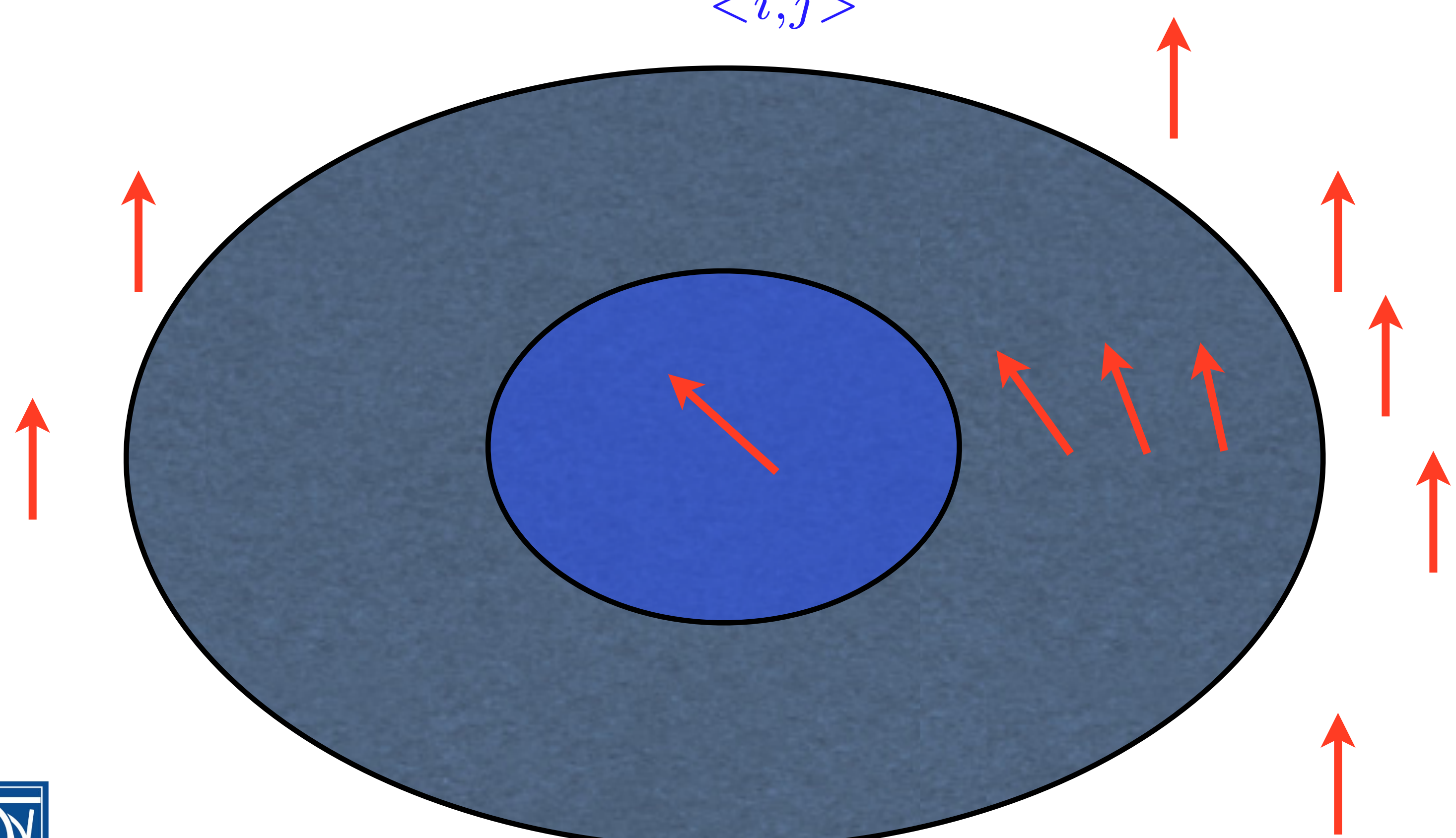
❖ The order parameters need not be elementary fields...

By abuse of language we often hear, or say that in theories with SSB there is vacuum degeneracy. This is fact is not the case, at least in LQFT. In understanding this we will also understand why there are massless states in theories with SSB. N is the volume in the example. The Heisenberg model of magnetism. H is rotational invariant above the critical temperature, and magnetised below it



$$H = -J \sum_{<i,j>} \mathbf{s}_i \cdot \mathbf{s}_j$$

$$\begin{aligned} \langle 0|\theta\rangle &= (\cos(\theta/2))^N \\ &\to 0 \qquad N \to \infty \end{aligned}$$



By making the transitions very slowly we can manage to make this configuration to have as small an energy as we wish. Hence we have a continuum spectrum above zero. This is the sign of a massless particle, the NG-boson

No Goldstone bosons in finite volume

This simple example contains the ingredients of the general case. Consider a theory in a box of side L and PBCs, the plane waves solutions are easy to write down

$$\Phi = (\phi_1, \phi_2) \qquad \zeta(x) = \frac{1}{\sqrt{2}}[\varphi_1(x) + i\varphi_2(x)] \equiv \frac{1}{\sqrt{2}}[a + h(x)]\, e^{i\theta(x)}.$$

$$\mathscr{L} = \frac{1}{2}\partial_\mu \Phi \cdot \partial^\mu \Phi - \frac{\lambda}{4}\left(\Phi^2 - a^2\right)^2$$

$$= \partial_\mu \zeta^* \partial^\mu \zeta - \lambda\left(|\zeta|^2 - \frac{a^2}{2}\right)^2 = \frac{a^2}{2}\partial_\mu\theta\partial^\mu\theta + \cdots, \qquad \partial_\mu \partial^\mu \theta = 0 \qquad \partial_\mu \partial^\mu \phi = 0$$

$$\varphi_{\mathbf{k}}(t,\mathbf{x}) = \frac{1}{\sqrt{V}} e^{-i|\mathbf{k}|t + i\mathbf{k}\cdot\mathbf{x}}, \quad \mathbf{k} = \frac{2\pi}{L}\mathbf{n} \qquad \varphi(t,\mathbf{x}) = \varphi_0 + \pi_0 t + \sum_{\mathbf{k}\neq 0} \frac{1}{\sqrt{2V|\mathbf{k}|}}\left[\alpha(\mathbf{k}) e^{-i|\mathbf{k}|t + i\mathbf{k}\cdot\mathbf{x}} + \alpha^\dagger(\mathbf{k}) e^{i|\mathbf{k}|t - i\mathbf{k}\cdot\mathbf{x}}\right].$$

$$[\varphi(t,\mathbf{x}_1), \dot\varphi(t,\mathbf{x}_2)] = i\delta(\mathbf{x}_1 - \mathbf{x}_2) = \frac{i}{V} + \frac{i}{V}\sum_{\mathbf{k}\neq 0} e^{i\mathbf{k}\cdot(\mathbf{x}_1 - \mathbf{x}_2)}$$

$$[\varphi_0, \pi_0] = \frac{i}{V}. \qquad a = \frac{1}{\sqrt{2}}\left(\varphi_0 + iV^{\frac{1}{3}}\pi_0\right), \quad a^\dagger = \frac{1}{\sqrt{2}}\left(\varphi_0 - iV^{\frac{1}{3}}\pi_0\right), \qquad :H: = \frac{V}{2}\pi_0^2 + \sum_{\mathbf{k}\neq 0}|\mathbf{k}|\alpha^\dagger(\mathbf{k})\alpha(\mathbf{k}).$$

$$[a, a^\dagger] = V^{-\frac{2}{3}}. \qquad Q = \int d^3x\, \partial_0\varphi = V\pi_0 = \frac{V^{\frac{2}{3}}}{i\sqrt{2}}\left(a - a^\dagger\right). \qquad e^{-i\xi Q}\varphi(x) e^{i\xi Q} = \varphi(x) + \xi,$$

$$|\xi\rangle \sim e^{i\xi Q}|0\rangle = e^{-\frac{1}{\sqrt{2}}\xi V^{\frac{2}{3}}(a^\dagger - a)}|0\rangle. \qquad \boxed{\langle 0|\xi\rangle = e^{-\frac{1}{4}\xi^2 V^{\frac{2}{3}}}\langle 0|0\rangle.}$$

http://carlossicoli.free.fr/A/Alvarez-Gaume_L.,_Vazquez-Mozo_M.A.-An_Invitation_to_Quantum_Field_Theory_-Springer(2011).pdf

Pions

In HEP they provide the only observed NG bosons

The order parameter is not an elementary field

To find other NG bosons in the SM we have to go to the Higgs sector, and there they are "eaten" to provide masses for the W and Z vector bosons

In QCD there are no fundamental scalars. Consider just two flavors u,d. We have chiral symmetry

$$\begin{pmatrix} u_{L,R} \\ d_{L,R} \end{pmatrix} \longrightarrow M_{L,R} \begin{pmatrix} u_{L,R} \\ d_{L,R} \end{pmatrix}$$

$$G = \underbrace{SU(2)_L \times SU(2)_R}_{SU(2)_V} \times U(1)_B \times U(1)_A$$

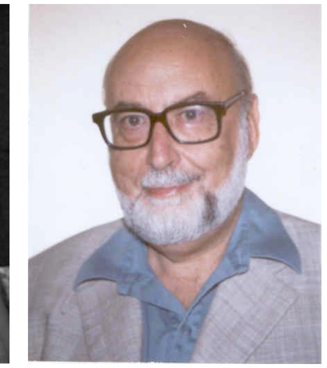
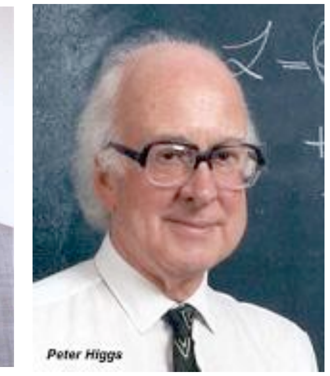
$$q_\alpha^f \qquad f = u, d, \;\; \alpha = 1, 2, 3$$

$$\langle \bar{q}^f \cdot q^{f'} \rangle = \Lambda^3_{\chi SB} \, \delta^{ff'}$$

$$\bar{q}^f \cdot q^{f'} \simeq \Lambda^3_{\chi SB} \, e^{i\pi^a \sigma^a / f_\pi}$$

These are the pions.
This is an IR property of QCD, not accessible to Pert. Th.
Low-E pion theorems, chiral Lagrangians….

The BEH mechanism

Notice we say the mechanism, not necessary the particle! In gauge theories one cannot just add a mass for the gauge bosons. This badly destroys the gauge symmetry and the theory is inconsistent.

BEH showed that in gauge theories with SSB the NG bosons are "eaten" by the gauge bosons to become massive but preserving the basic properties of the gauge symmetry. Ex. Abelian Higgs model

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi)^\dagger(D^\mu\varphi) - \frac{\lambda}{4}\left(\varphi^\dagger\varphi - \mu^2\right)^2 \qquad \varphi \longrightarrow e^{i\alpha(x)}\varphi, \qquad A_\mu \longrightarrow A_\mu + \partial_\mu\alpha(x)$$

$$\langle\varphi\rangle = \mu e^{i\vartheta_0} \longrightarrow \mu e^{i\vartheta_0 + i\alpha(x)} \qquad \varphi(x) = \left[\mu + \frac{1}{\sqrt{2}}\sigma(x)\right]e^{i\vartheta(x)}$$

Take the unitary gauge

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e^2\mu^2 A_\mu A^\mu + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{2}\lambda\mu^2\sigma^2 - \lambda\mu\sigma^3 - \frac{\lambda}{4}\sigma^4 + e^2\mu A_\mu A^\mu\sigma + e^2 A_\mu A^\mu\sigma^2.$$

$$m_\gamma^2 = 2e^2\mu^2$$

The simplest example is the GL and BCS theory of superconductivity, in this case the "Higgs" particle is composite, it is an object of charge made of two bound electrons that get a "VEV" (Cooper pairs) that get a VEV in the superconducting state. The photon is massive in this state. This explains among other things the Meissner effect.

Gauge couplings: color

There are three gauge groups in the theory, the color group SU(3) and the electroweak group SU(2)xU(1) of weak isospin and hypercharge. Y and T3 mix to generate electric charge and the photon

$$SU(3)_c \times SU(2) \times U(1)_Y \to SU(3) \times U(1)_Q$$

QCD by itself is a perfect theory in many ways

$$\mathscr{L}_{\mathrm{QCD}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + \sum_{f=1}^{6} \overline{Q}^f \left(i\not{D} - m_f \right) Q^f . \qquad Q^f_i \longrightarrow U(g)_{ij} Q^f_j , \qquad \text{with} \quad g \in \mathrm{SU}(3)$$

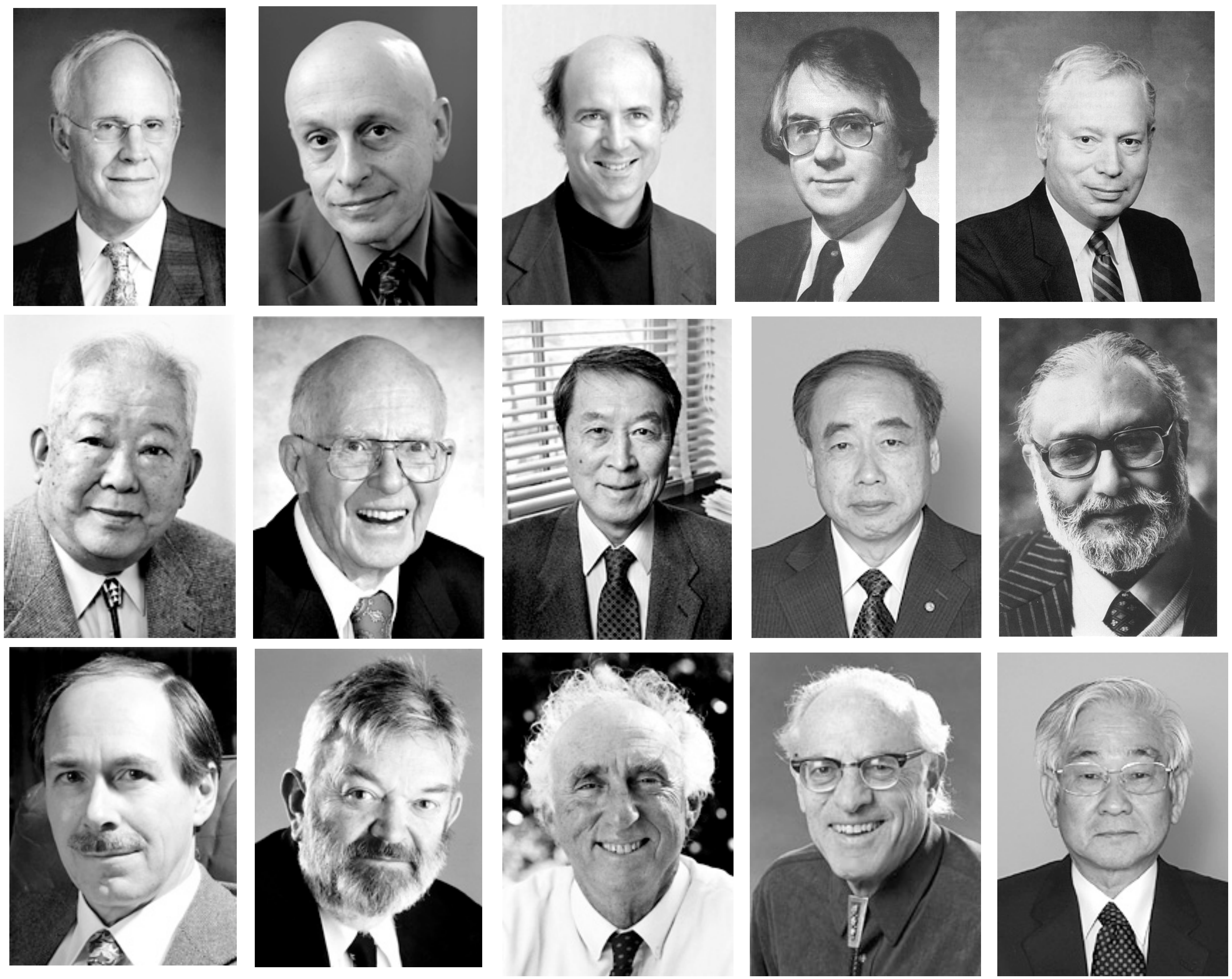
Isospin as an approximate symmetry:

$$\mathscr{L} = (\overline{u}, \overline{d}) \begin{pmatrix} i\not{D} - \frac{m_u + m_d}{2} & 0 \\ 0 & i\not{D} - \frac{m_u + m_d}{2} \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix} - \frac{m_u - m_d}{2} (\overline{u}, \overline{d}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}$$

Once the electroweak sector is included the story of the masses is far more complicated (see later)

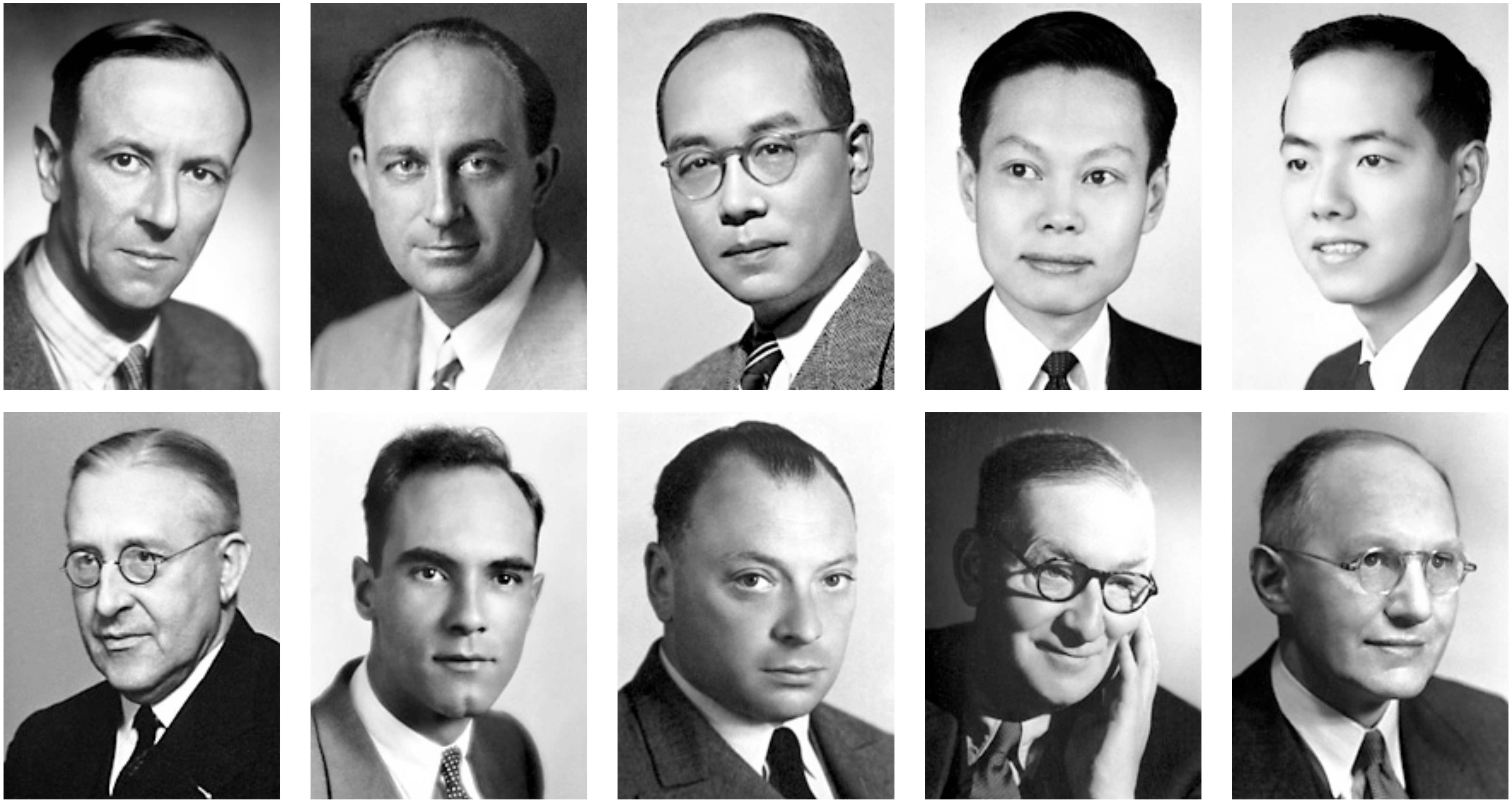
The physics that led to the SM is the combined results of many people over more than a hundred years, some of the them were awarded the Nobel Prize in Physics. We can think of the beginning of the SM Odyssey with the discovery of the electron by Thomson in 1897.

Who is who in the Standard Model

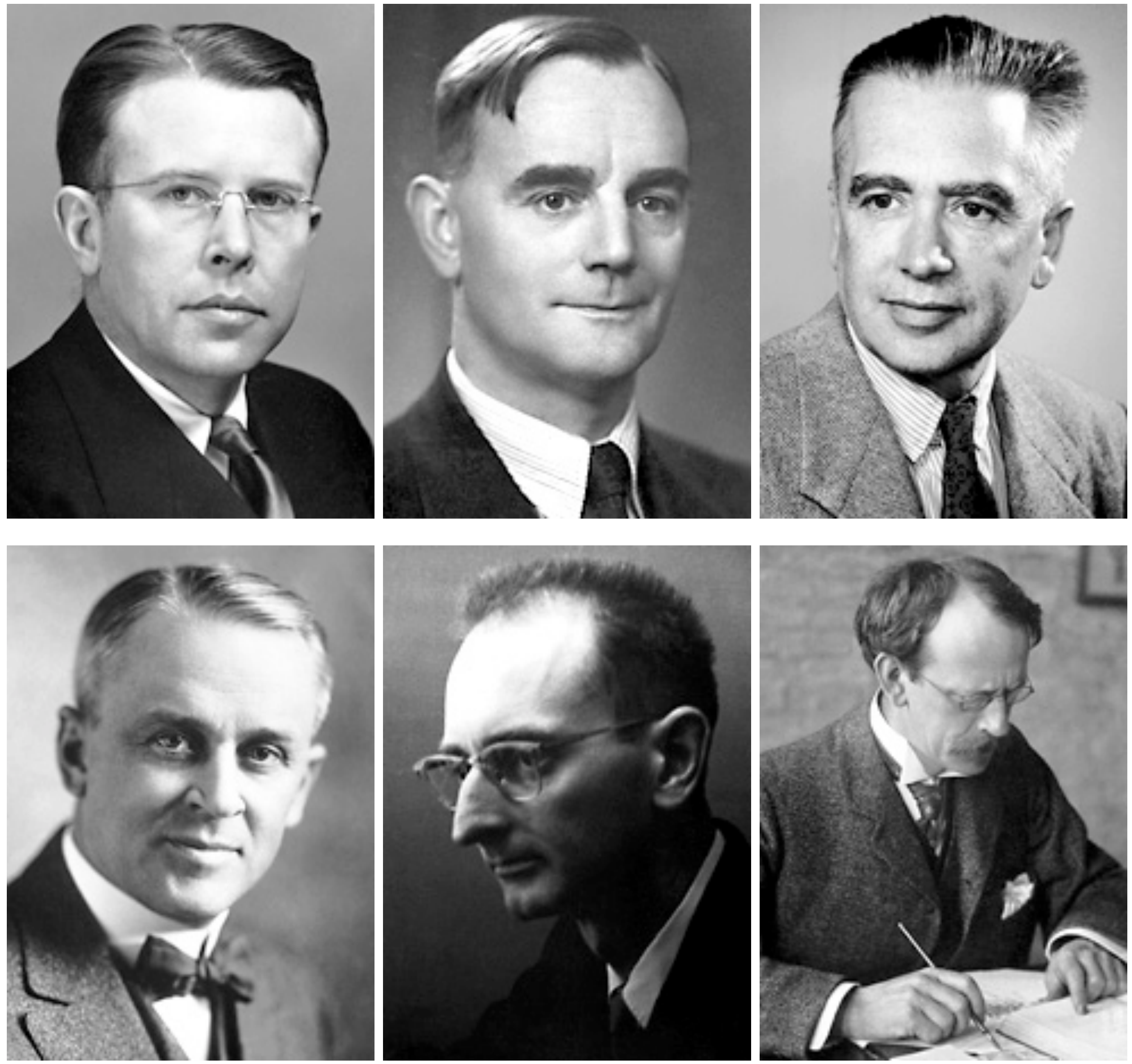


Who is who in the Standard Model

Who is who in the Standard Model



Who is who in the Standard Model



How many did you recognize?

Who is who in the Standard Model

If you do it more carefully (Werner Riegler), what you find is:

87 Nobel Prices related to the Development of the Standard Model

31 for Standard Model Experiments
13 for Standard Model Instrumentation and Experiments
3 for Standard Model Instrumentation
21 for Standard Model Theory
9 for Quantum Mechanics Theory
9 for Quantum Mechanics Experiments
1 for Relativity

Fermion quantum numbers

The fundamental fermions come in three families with the same quantum numbers with respect to the gauge group

Leptons					
i (generation)	1	2	3	T^3	Y
$\mathbf{L}^i$	$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L$	$\begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L$	$\begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L$	$\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$	$-\frac{1}{2}$
ℓ^i_R	e^-_R	μ^-_R	τ^-_R	0	-1

Quarks					
i (generation)	1	2	3	T^3	Y
$\mathbf{Q}^i$	$\begin{pmatrix} u \\ d \end{pmatrix}_L$	$\begin{pmatrix} c \\ s \end{pmatrix}_L$	$\begin{pmatrix} t \\ b \end{pmatrix}_L$	$\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$	$\frac{1}{6}$
U^i_R	u_R	c_R	t_R	0	$\frac{2}{3}$
D^i_R	d_R	s_R	b_R	0	$-\frac{1}{3}$

In principle one could add sterile neutrinos, right handed neutrinos who are singlets under the gauge group. They would generate Dirac masses for the known neutrinos

The EW group has four generators

$$\mathbf{W}_\mu = W_\mu^+ T^- + W_\mu^- T^+ + W_\mu^3 T^3, \qquad \mathbf{B}_\mu = B_\mu Y.$$

$$Q = T^3 + Y.$$

$$[Q, T^\pm] = \pm T^\pm, \qquad [Q, T^3] = [Q, Y] = 0.$$

$$A_\mu = B_\mu \cos\theta_w + W_\mu^3 \sin\theta_w,$$

$$Z_\mu = -B_\mu \sin\theta_w + W_\mu^3 \cos\theta_w$$

$$\begin{aligned} D_\mu &= \partial_\mu - ig\mathbf{W}_\mu - ig'\mathbf{B}_\mu \\ &= \partial_\mu - igW_\mu^+ T_R^- - igW_\mu^- T_R^+ - igW_\mu^3 T_R^3 - ig' B_\mu Y_R, \end{aligned}$$

$$e = g\sin\theta_w = g'\cos\theta_w.$$

$$\begin{aligned} D_\mu = {} & \partial_\mu - igW_\mu^+ T_R^- - igW_\mu^- T_R^+ - iA_\mu (g\sin\theta_w T_R^3 + g'\cos\theta_w Y_R) \\ & - iZ_\mu (gT_R^3 \cos\theta_w - g' Y_R \sin\theta_w). \end{aligned}$$

$$\begin{aligned} \mathscr{L}_{\text{gauge}} = {} & -\frac{1}{2} W_{\mu\nu}^+ W^{-\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{ig}{2} \cos\theta_w W_\mu^+ W_\nu^- Z^{\mu\nu} \\ & + \frac{ie}{2} W_\mu^+ W_\nu^- F^{\mu\nu} - \frac{g^2}{2} \Big[(W_\mu^+ W^{+\mu})(W_\mu^- W^{-\mu}) - (W_\mu^+ W^{-\mu})^2 \Big] \end{aligned}$$

$$\begin{aligned} \mathscr{L}_{\text{matter}} = \sum_{i=1}^{3} \Big(& i\overline{\mathbf{L}}^j \not{D} \mathbf{L}^j + i\overline{\ell}_R^j \not{D} \ell_R^j \\ & + i\overline{\mathbf{Q}}^j \not{D} \mathbf{Q}^j + i\overline{U}_R^j \not{D} U_R^j + i\overline{D}_R^j \not{D} D_R^j \Big) \end{aligned}$$

$$W_{\mu\nu}^\pm = \partial_\mu W_\nu^\pm - \partial_\nu W_\mu^\pm \mp ie\Big(W_\mu^\pm A_\nu - W_\nu^\pm A_\mu\Big) \mp ig\cos\theta_w \Big(W_\mu^\pm Z_\nu - W_\nu^\pm Z_\mu\Big)$$

$$Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu$$

$$\frac{g}{2\cos\theta_w} Z_\mu \overline{\nu}_\ell \gamma^\mu \nu_\ell, \qquad \frac{g}{\cos\theta_w} \left(-\frac{1}{2} + \sin^2\theta_w\right) Z_\mu \overline{\ell}_L \gamma^\mu \ell_L.$$

Higgs couplings

The Higgs couplings responsible for the masses of the leptons and the current algebra masses of the quarks are:

$$\mathbf{H} = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} \qquad \widetilde{\mathbf{H}} \equiv i\sigma^2 \mathbf{H}^* = \begin{pmatrix} H^{0*} \\ H^{+*} \end{pmatrix} \qquad \mathscr{L}_{\text{Higgs}} = (D_\mu \mathbf{H})^\dagger D^\mu \mathbf{H} - V(\mathbf{H}, \mathbf{H}^\dagger), \qquad V(\mathbf{H},\mathbf{H}^\dagger) = \frac{\lambda}{4}\left(\mathbf{H}^\dagger \mathbf{H} - \mu^2\right)^2$$

$$Y(\mathbf{H}) = \tfrac{1}{2}$$

$$\mathscr{L}^{(\ell)}_{\text{Yukawa}} = -\sum_{i,j=1}^{3} \left(C^{(\ell)}_{ij} \overline{\mathbf{L}}^i \mathbf{H} \ell^j_R + C^{(\ell)*}_{ji} \overline{\ell}^i_R \mathbf{H}^\dagger \mathbf{L}^j \right)$$

$$\mathscr{L}^{(q)}_{\text{Yukawa}} = -\sum_{i,j=1}^{3} \left(C^{(q)}_{ij} \overline{\mathbf{Q}}^i \mathbf{H} D^j_R + C^{(q)*}_{ji} \overline{D}^i_R \mathbf{H}^\dagger \mathbf{Q}^j \right) - \sum_{i,j=1}^{3} \left(\widetilde{C}^{(q)}_{ij} \overline{\mathbf{Q}}^i \widetilde{\mathbf{H}} U^j_R + \widetilde{C}^{(q)*}_{ji} \overline{U}^i_R \widetilde{\mathbf{H}}^\dagger \mathbf{Q}^j \right).$$

The most general Lagrangian compatible with the gauge symmetry and up to dimension 4, so that the theory is renormalizable. Once H gets its VEV the masses are generated from the Yukawa couplings. Use unitary gauge. The gauge fields get masses from the kinetic term

$$\mathbf{H}(x) = e^{i\mathbf{a}(x)\cdot\frac{\sigma}{2}} \begin{pmatrix} 0 \\ \mu + \frac{1}{\sqrt{2}} h(x) \end{pmatrix} = \begin{pmatrix} 0 \\ \mu + \frac{1}{\sqrt{2}} h(x) \end{pmatrix}$$

Fundamental property

In a theory with a single scalar doublet, using SU(2) x U(1) transformations, it is ALWAYS possible to represent the doublet in the form:

$$\mathbf{H}(x) = e^{i\mathbf{a}(x)\cdot\frac{\sigma}{2}}\begin{pmatrix} 0 \\ \mu + \frac{1}{\sqrt{2}}h(x) \end{pmatrix}$$

In other words, EM is always preserved. If we have several scalar doublets, or other representations, the scalar potential has to be chosen judiciously.

For authoritative treatment of the scalar sector of the SM, please attend Prof. M. Carena's lectures. The next few pages provide a quick introduction to some of the basic formulas and ideas

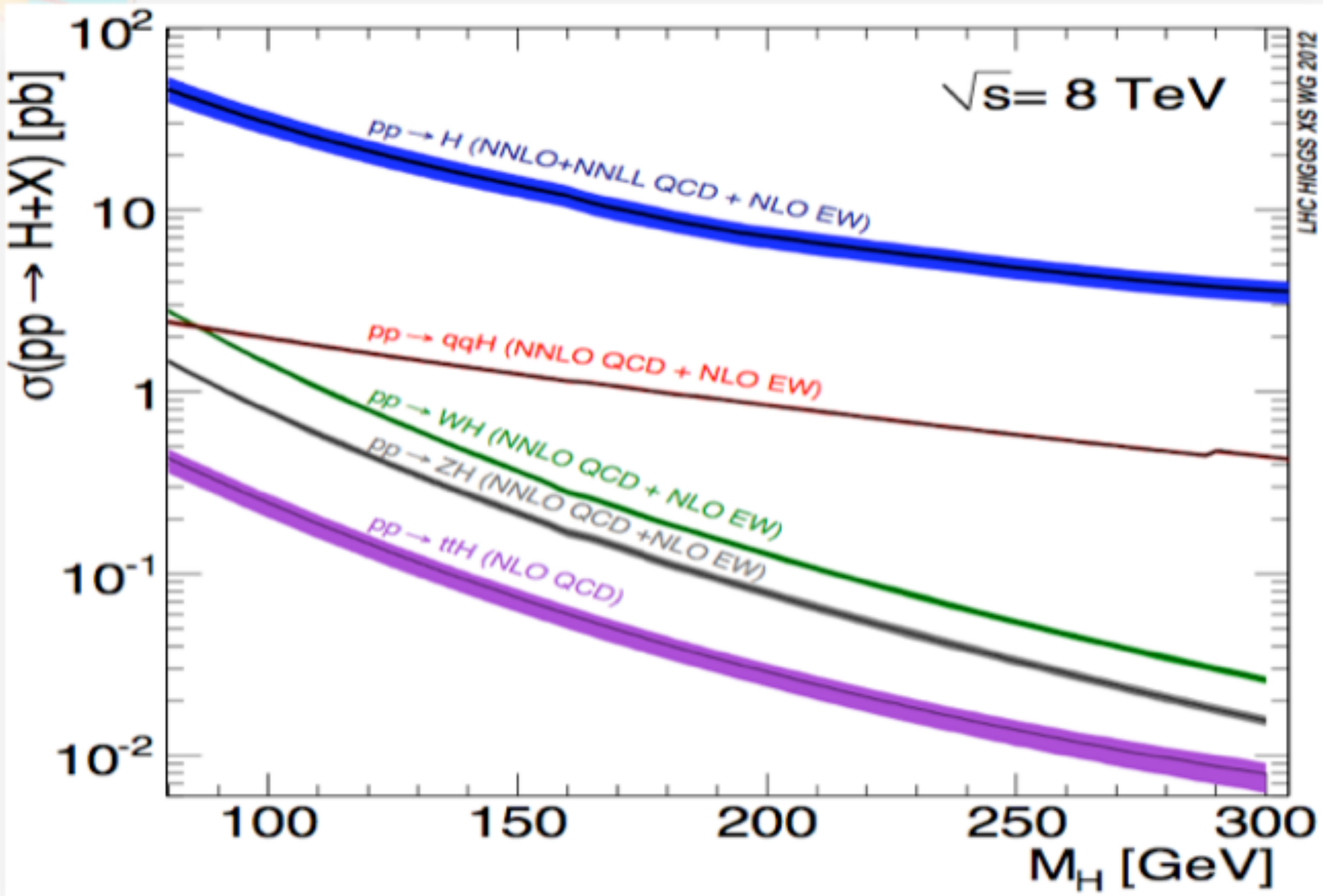
A simplified view of searching for the Higgs...

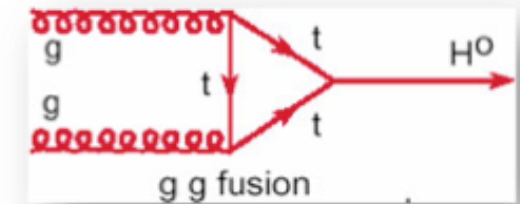
Picture courtesy of J.J. Gómez Cadenas

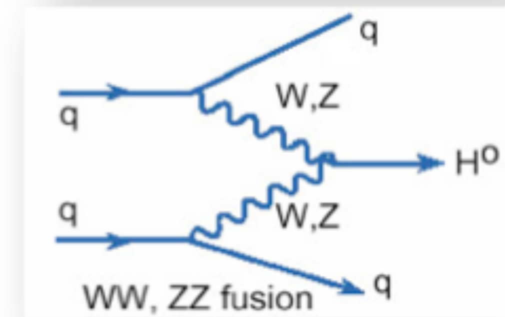
In fact looking for the Higgs has been a substantially harder job that the man on the left searching for his needle...

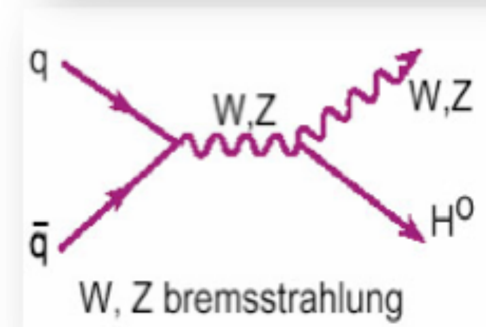
Higgs boson production

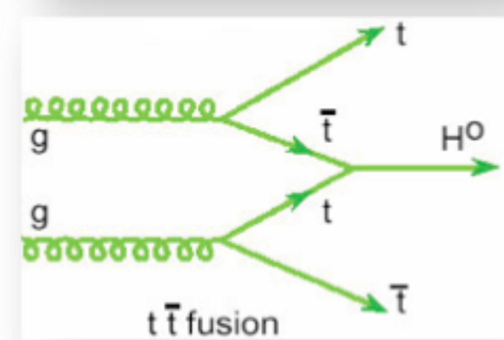
July 4th 2012 The Status of the Higgs Search J. Incandela for the CMS COLLABORATION











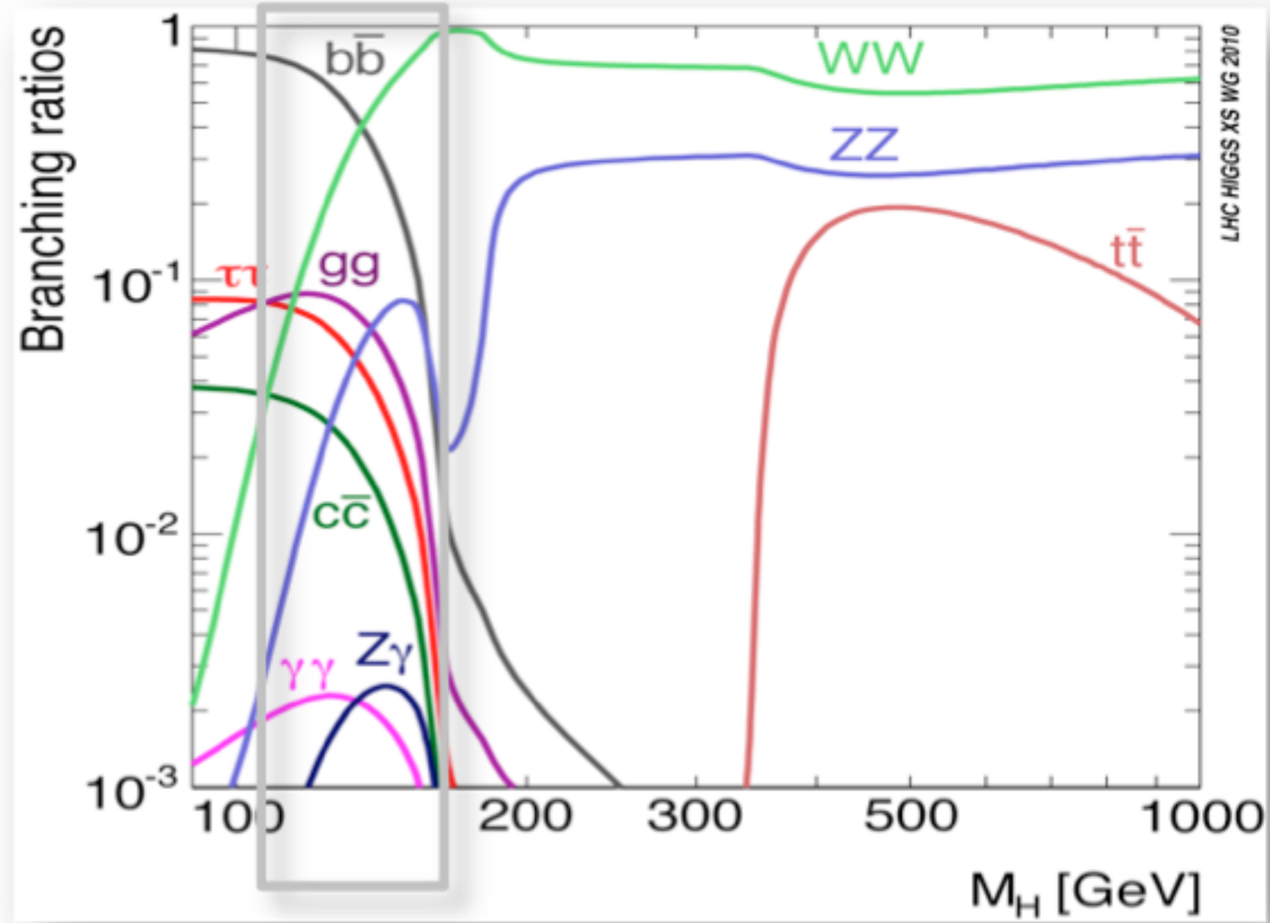
- $\sqrt{s}$=8 TeV: 25-30% higher σ than $\sqrt{s}$=7 TeV at low m_H
- All production modes to be exploited
  - gg VBF VH ttH
  - Latter 3 have smaller cross sections but better S/B in many cases

Courtesy J. Incancella

Higgs boson decays

5 decay modes exploited

- High mass: WW, ZZ
- Low mass: $b\bar{b}$, $\tau\tau$, WW, ZZ, $\gamma\gamma$
- Low mass region is very rich but also very challenging:
  main decay modes ($b\bar{b}$, $\tau\tau$) are hard to identify in the huge background
- Very good mass resolution (1%): $H \rightarrow \gamma\gamma$ and $H \rightarrow ZZ \rightarrow 4l$



Courtesy J. Incancella

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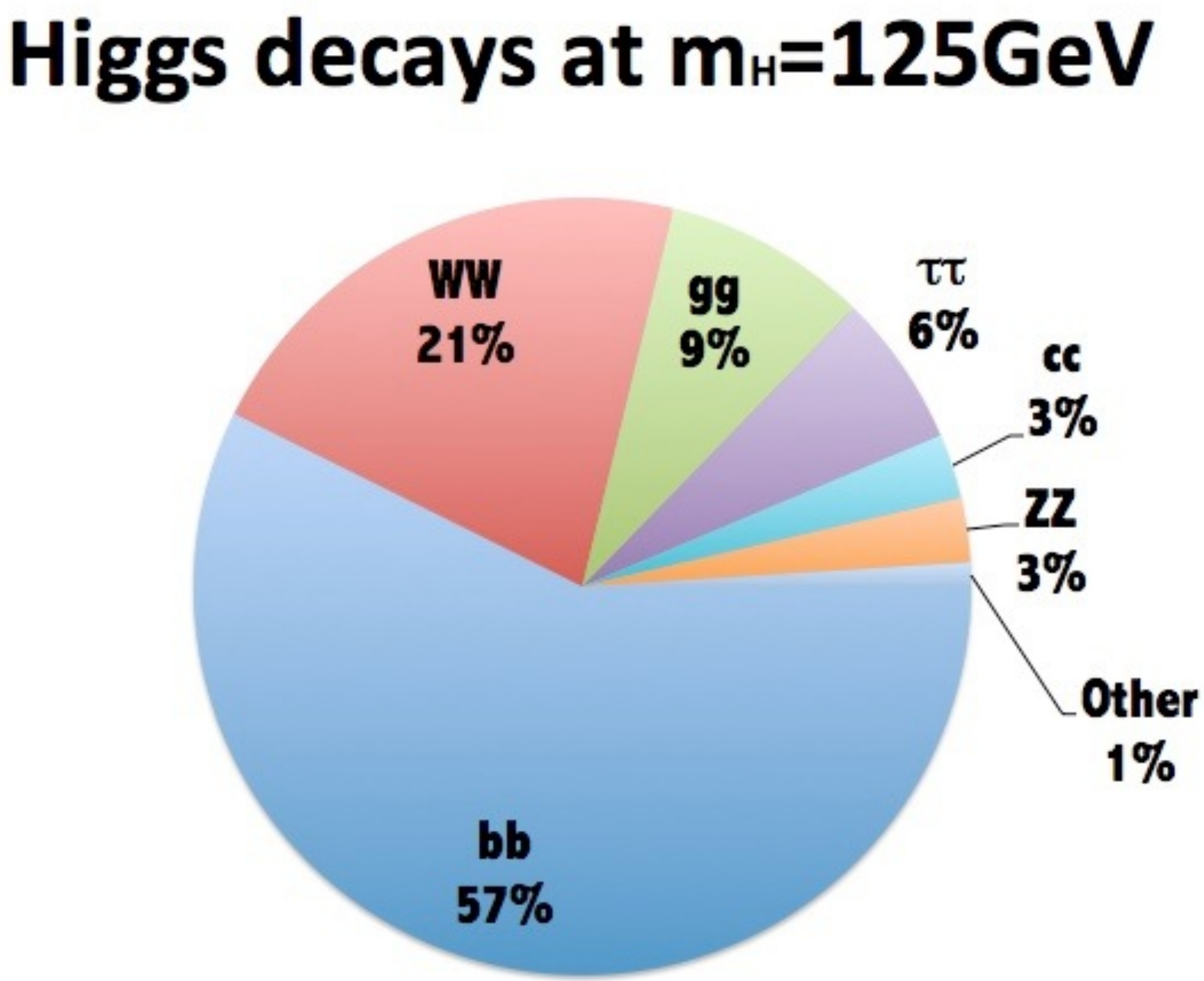
Luis Alvarez-Gaume Arequipa CLASHEP March 6-13 2019

July 4th 2012 The Status of the Higgs Search J. Incandela for the CMS COLLABORATION



No point in upgrading these plots, see M. Carena's lectures

$$\mathscr{L}^{(\ell)}_{\text{mass}} = -(\overline{e}_L, \overline{\mu}_L, \overline{\tau}_L)\, M^{(\ell)} \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix} + \text{h.c.} \qquad \mathscr{L}^{(q)}_{\text{mass}} = -(\overline{d}_L, \overline{s}_L, \overline{b}_L)\, M^{(q)} \begin{pmatrix} d_R \\ s_R \\ b_R \end{pmatrix} - (\overline{u}_L, \overline{c}_L, \overline{t}_L)\, \widetilde{M}^{(q)} \begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix} + \text{h.c.}$$

$$M^{(\ell,q)}_{ij} = \mu C^{(\ell,q)}_{ij}, \qquad \widetilde{M}^{(q)}_{ij} = \mu \widetilde{C}^{(q)}_{ij}$$

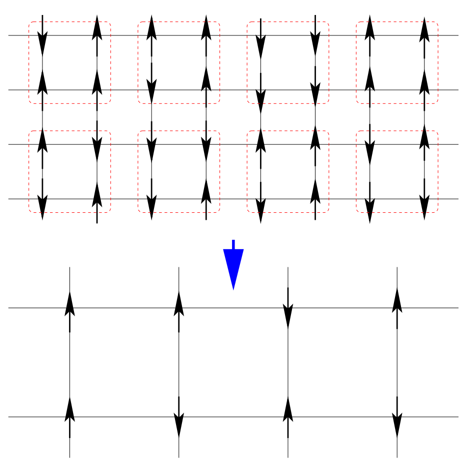
$$V_L^{(\ell)\dagger} M^{(\ell)} V_R^{(\ell)} = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \qquad V_L^{(q)\dagger} M^{(q)} V_R^{(q)} = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} \qquad \widetilde{V}_L^{(q)\dagger} \widetilde{M}^{(q)} \widetilde{V}_R^{(q)} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}$$

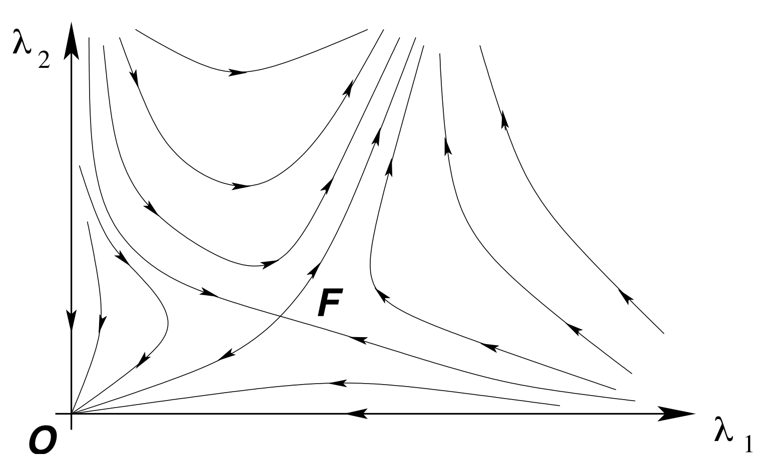
In the quark sector going from gauge to mass eigenstates leaves a matrix of phases in the charged currents, the CKM matrix. Not for neutral currents GIM. Similar arguments work for neutrinos, as you will see in Pilar Hernandez lectures.

$$j^\mu_+ = (\overline{u}_L, \overline{c}_L, \overline{t}_L)\gamma^\mu \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} = (\overline{u}'_L, \overline{c}'_L, \overline{t}'_L)\gamma^\mu \widetilde{V}_L^{(q)\dagger} V_L^{(q)} \begin{pmatrix} d'_L \\ s'_L \\ b'_L \end{pmatrix} \qquad V \equiv \widetilde{V}_L^{(q)\dagger} V_L^{(q)}$$

Scale invariance, renormalisation

$\times \lambda^{-1}$





Renormalisation deals with the scale dependence of the the physics even if the original theory is scale invariant.

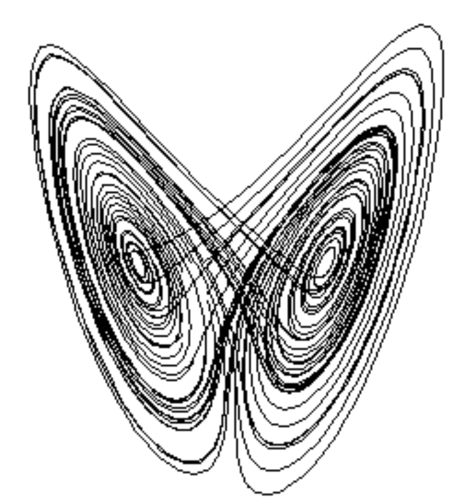
Virtual phenomena can get more complicated or simplify as we move to larger or shorter distances

$$x^\mu \longrightarrow \lambda x^\mu, \qquad \phi(x) \longrightarrow \lambda^{-\Delta}\phi(\lambda^{-1}x),$$

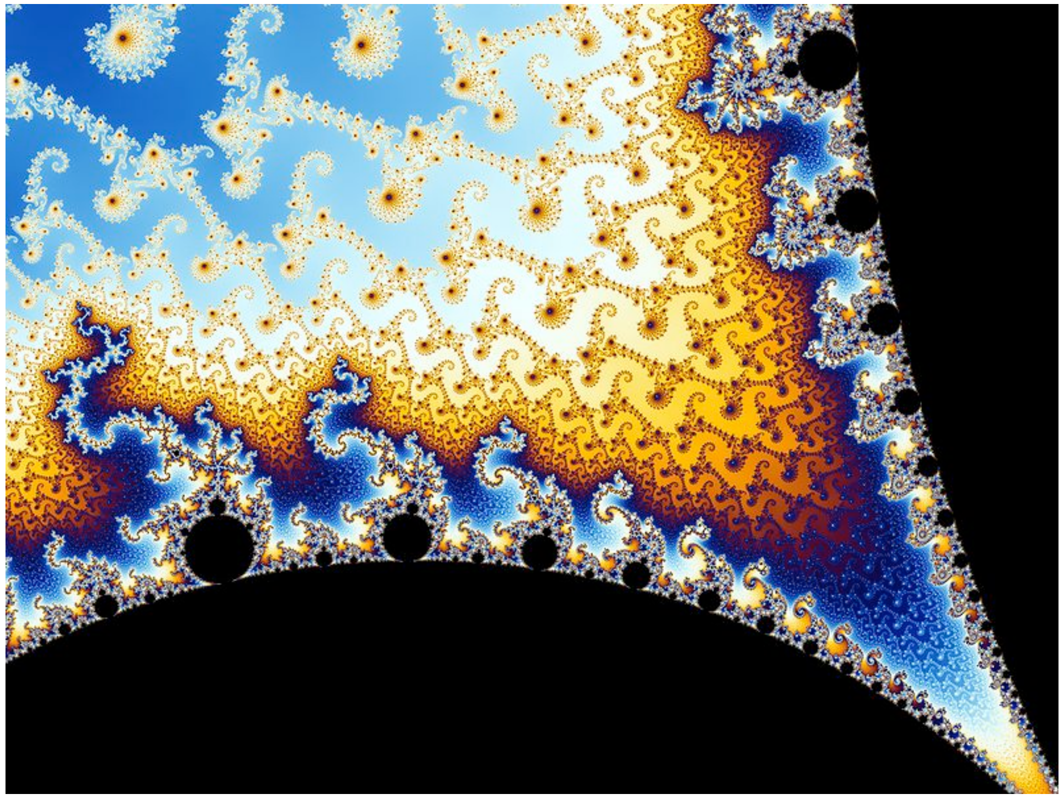
$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{g}{4!}\phi^4, \qquad \mathscr{L} \longrightarrow \lambda^{-4}\mathscr{L}[\phi]$$

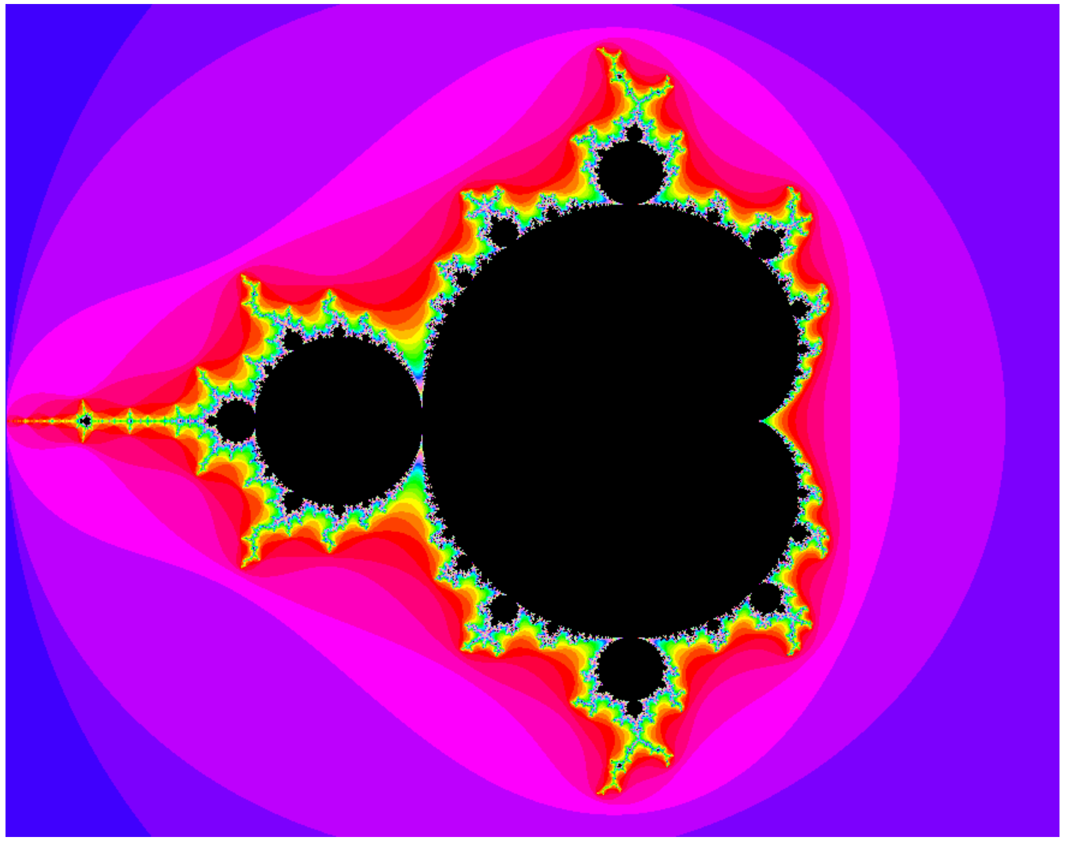
$$H \xrightarrow{\mathscr{R}} H^{(1)} \xrightarrow{\mathscr{R}} H^{(2)} \xrightarrow{\mathscr{R}} \ldots \xrightarrow{\mathscr{R}} H_\star.$$

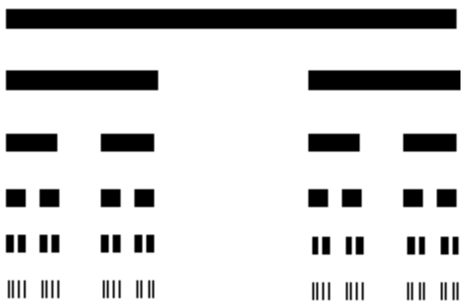
In relativistic QFT we seem to get only fixed points, no limit cycles nor strange attractors

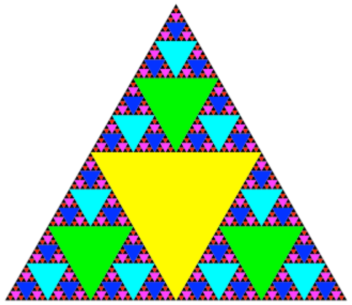


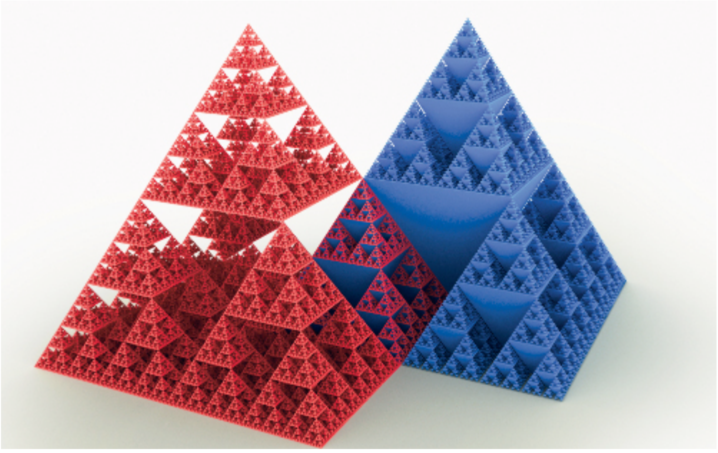
Scale invariance, fractals





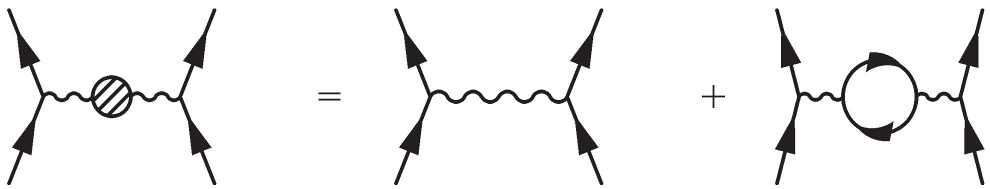






Why not in QFT? It would be rather remarkable if in a field theory we found strange attractors at high or low energies. Lorentz or Poincaré invariance play an important role in determining the possible limit structures. Only fixed points?

Fixed points, beta functions. Coupling constants run



$$\eta_{\alpha\beta}\left(\overline{v}_e\gamma^\alpha u_e\right)\left\{\frac{e^2}{4\pi q^2}\left[1+\frac{e^2}{12\pi^2}\log\left(\frac{q^2}{\Lambda^2}\right)\right]\right\}\left(\overline{v}_\mu\gamma^\beta u_\mu\right)$$

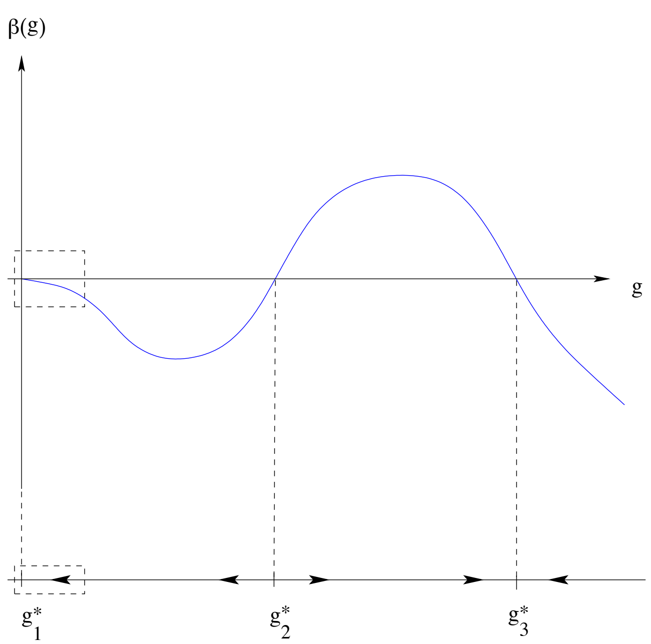
$$e(\mu)^2 = e(\Lambda)^2_{\text{bare}}\left[1+\frac{e(\Lambda)^2_{\text{bare}}}{12\pi^2}\log\left(\frac{\mu^2}{\Lambda^2}\right)\right]$$

$$\beta(g) = \mu\frac{dg}{d\mu}.$$

$$\beta(e)_{\text{QED}} = \frac{e^3}{12\pi^2}.$$

$$\beta(g) = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right).$$

At one loop



$$\beta(g^*) = 0.$$

IR free (QED)

$$\beta'(g)|_{g^*} > 0 \quad , \qquad \mu\frac{dg}{d\mu} = \beta'(g-g^*)+\ldots$$

$$\mu\uparrow \quad , \qquad g\uparrow$$

UV free (QCD) but IR complicated, confinement

$$\beta'(g)|_{g^*} < 0 \quad , \qquad \mu\frac{dg}{d\mu} = \beta'(g-g^*)+\ldots$$

$$\mu\uparrow \quad , \qquad g\downarrow$$

There is a dynamically generated scale. This is what makes QCD really hard

$$\langle \mathbf{p}^2\rangle = \Lambda^2_{\text{QCD}}. \qquad \Lambda_{\text{QCD}} \gg m_u, m_d$$

The Higgs mechanism, how much does it contribute to your weight?

The Higgs mechanism is not responsible for most of the mass of the observable matter in the universe…. You are a macroscopic quantum object!!

The mass parameters obtained for the light quarks are too small to explain the masses of protons and neutrons that make up nuclei. From elementary nuclear physics we know:

$$M(Z,A) = Z\,m_p + (A-Z)\,m_n + \Delta M(Z,A)$$

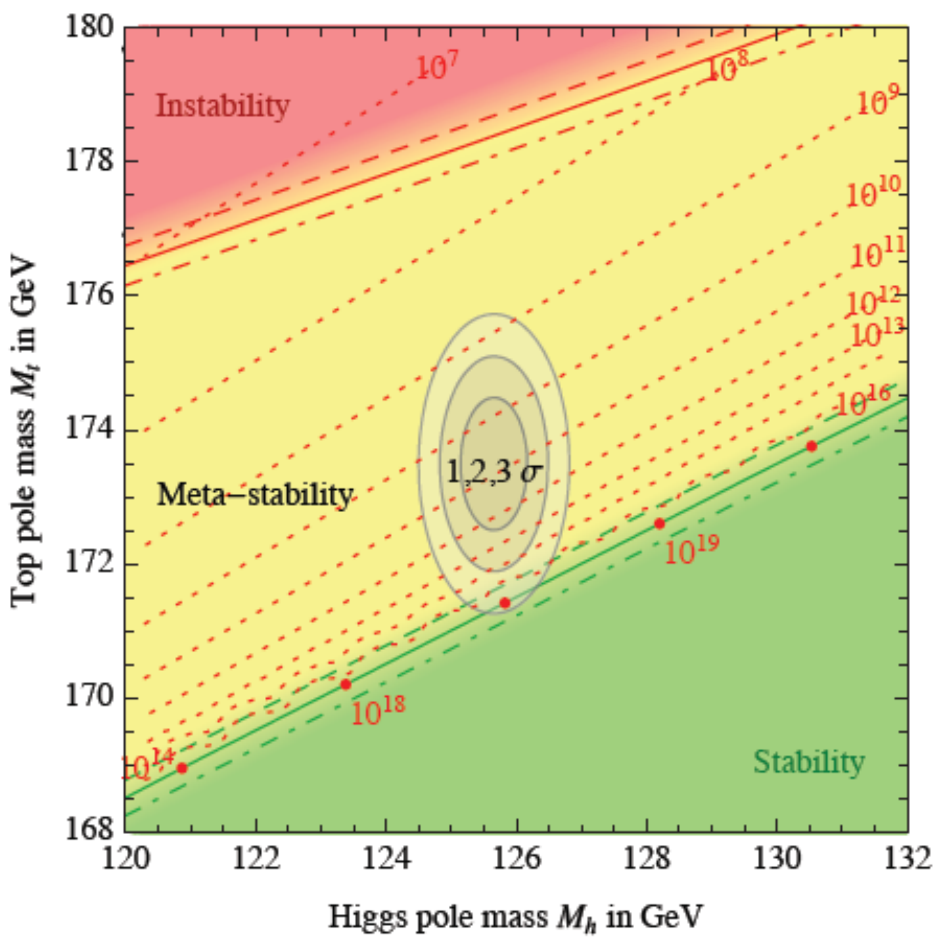
The largest contribution come from the fact that quarks and gluons are highly relativistic objects confined in a space of the order of a fermi. A purely quantum phenomenon due to QCD: the confinement of colour. A new scale is generated dynamically. Generated with the breaking of scale invariance. Most of the mass of nucleons come from this. Even if the mass parameters of the u,d quarks was set to zero, we would still have nucleons. What makes the study of the strong interactions hard is the fact that:

$$\Lambda >> m_u,\, m_d$$

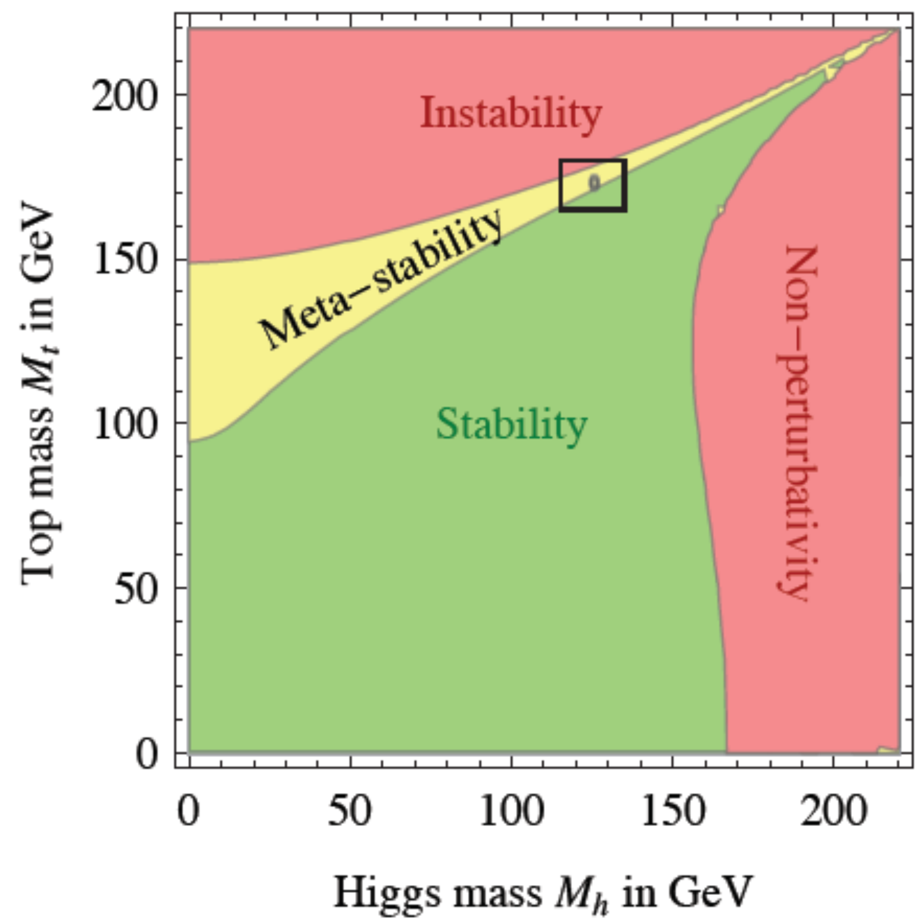
A large fraction of our mass has its origin in this quantum phenomenon of confinement. We are indeed macroscopic quantum objects! There is also a beautiful analogy with the BEH mechanism, but of a more subtle type as a dual superconductor.

An unexpected result of the LHC

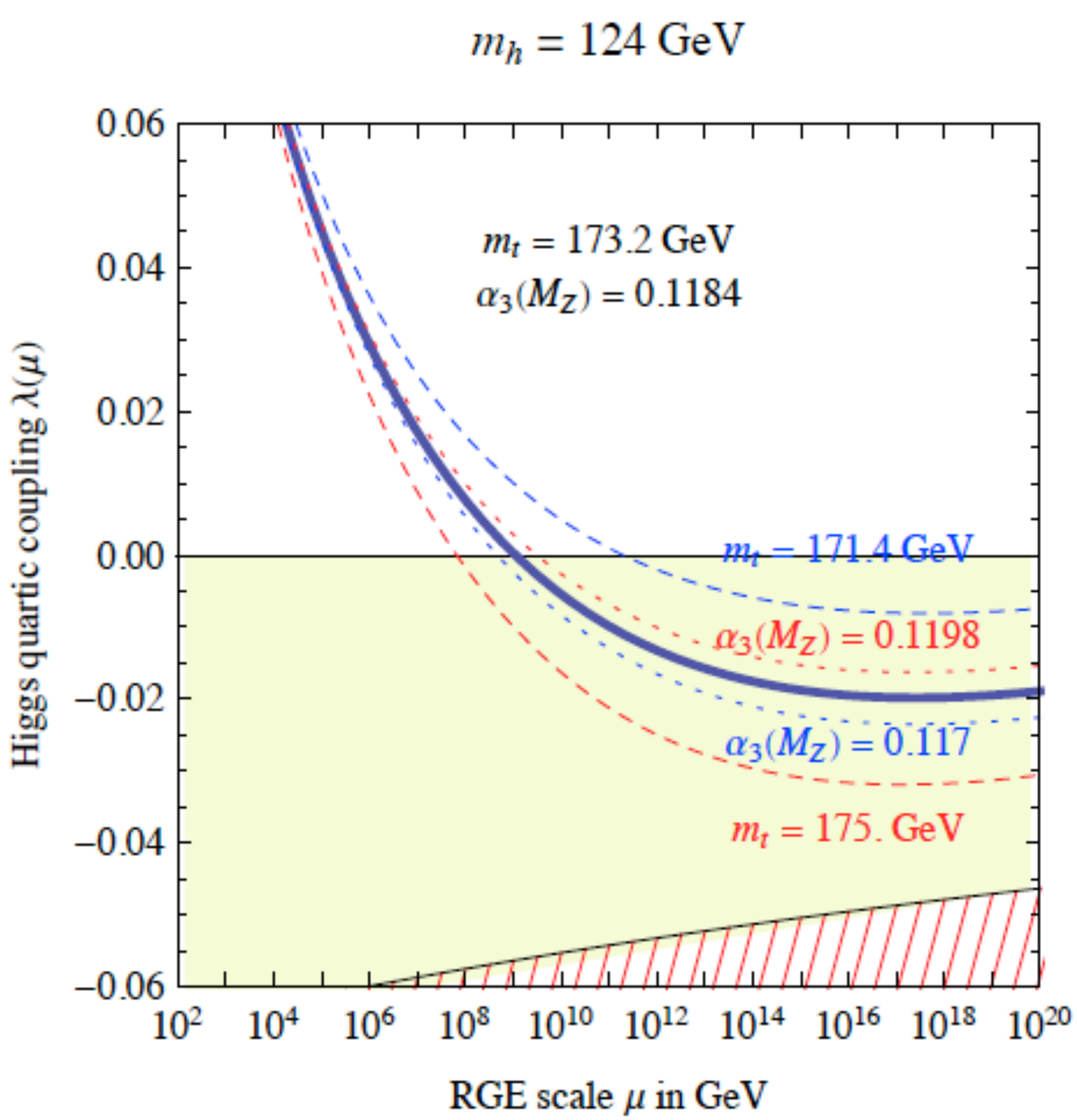
The Planck Chimney

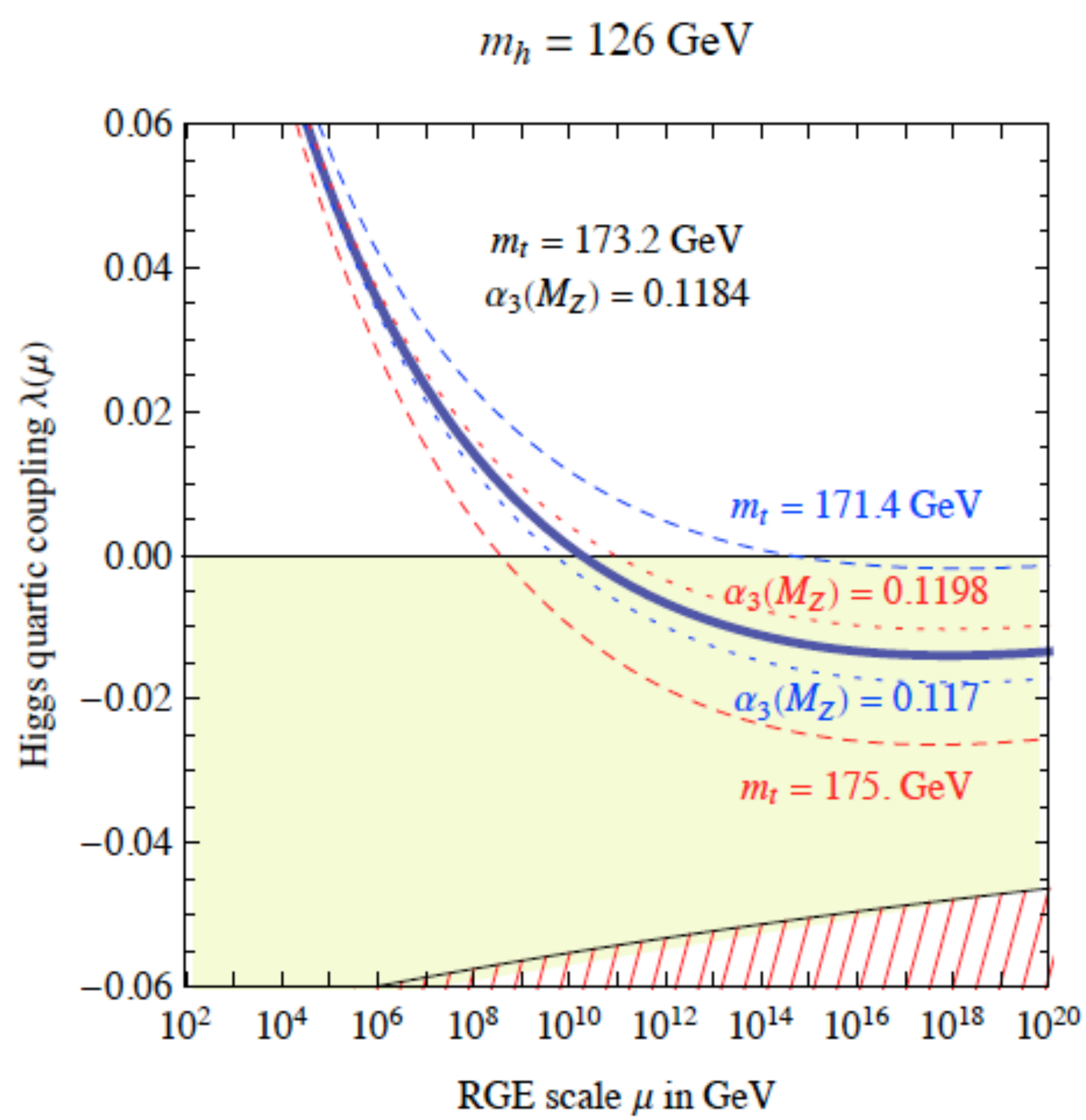


Instability
Meta–stability
Stability
1,2,3 σ
Top pole mass M_t in GeV
Higgs pole mass M_h in GeV

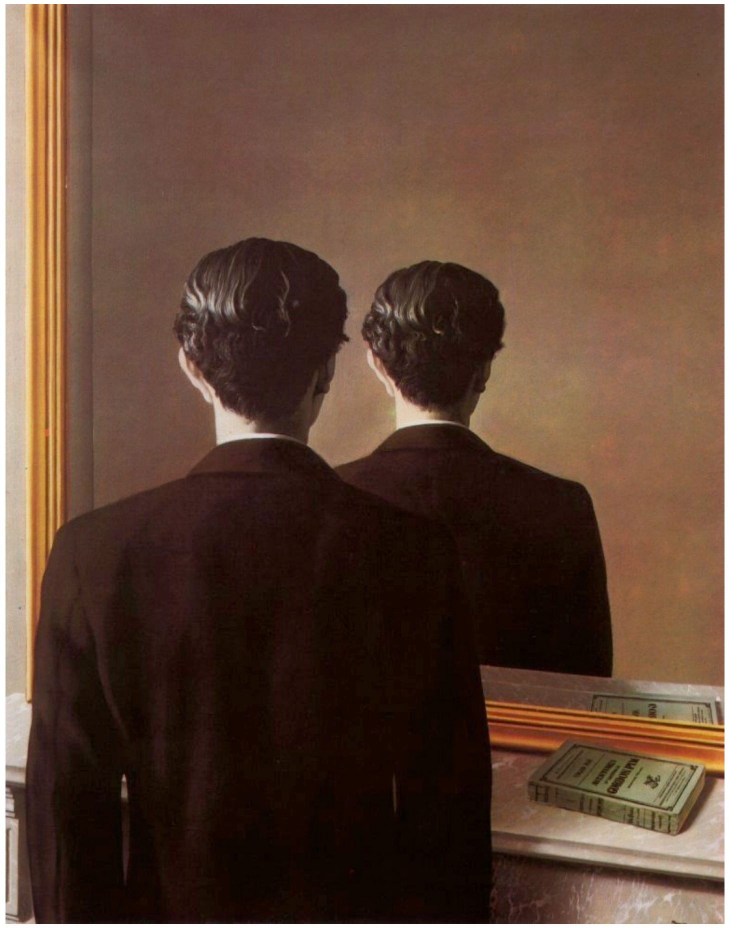


Instability
Meta–stability
Stability
Non–perturbativity
Top mass M_t in GeV
Higgs mass M_h in GeV


$m_h = 124$ GeV
0.06
0.04
0.02
0.00
−0.02
−0.04
−0.06
Higgs quartic coupling $\lambda(\mu)$
$m_t = 173.2$ GeV
$\alpha_3(M_Z) = 0.1184$
$m_t = 171.4$ GeV
$\alpha_3(M_Z) = 0.1198$
$\alpha_3(M_Z) = 0.117$
$m_t = 175.$ GeV
10^2 10^4 10^6 10^8 10^{10} 10^{12} 10^{14} 10^{16} 10^{18} 10^{20}
RGE scale μ in GeV


$m_h = 126$ GeV
0.06
0.04
0.02
0.00
−0.02
−0.04
−0.06
Higgs quartic coupling $\lambda(\mu)$
$m_t = 173.2$ GeV
$\alpha_3(M_Z) = 0.1184$
$m_t = 171.4$ GeV
$\alpha_3(M_Z) = 0.1198$
$\alpha_3(M_Z) = 0.117$
$m_t = 175.$ GeV
10^2 10^4 10^6 10^8 10^{10} 10^{12} 10^{14} 10^{16} 10^{18} 10^{20}
RGE scale μ in GeV

Discrete symmetries



- In the classical world, we have invariance uncer P,C,T. All we had was E&M and gravity.

- In QFT they are not guaranteed in fact P,C,T, CP are broken symmetries. The only one that survives so far is CPT. It has several important consequences. CP violation is fundamental in the generation of matter. In the SM we need at least three families

- The existence of antiparticles with the same mass and decay rate

- The connection between spin and statistics

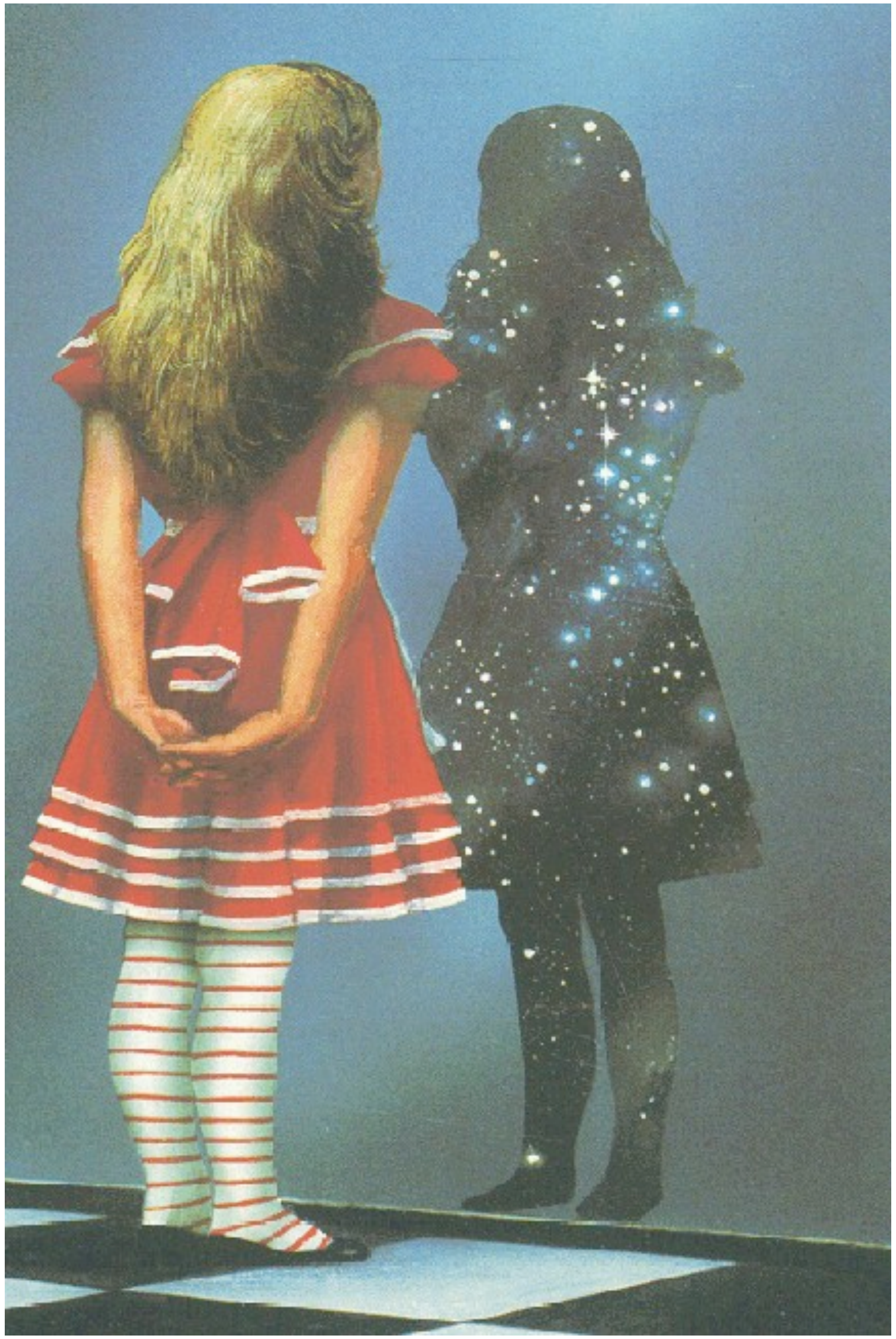
- T-reversal and CPT are the only ones implemented by anti-unitary operators

$$\begin{tikzcd} \mathbf{q}_0, \mathbf{p}_0 \arrow[r, "\mathrm{T}"] \arrow[d, "t"] & \mathbf{q}_0, -\mathbf{p}_0 \\ \mathbf{q}(t), \mathbf{p}(t) \arrow[r, "\mathrm{T}"] & \mathbf{q}(t), -\mathbf{p}(t) \arrow[u, "t"'] \end{tikzcd}$$

Discrete symmetries

Parity
Time reversal
Charge Conjugation
CPT

We need to break CP to make more matter than antimatter. It could be the neutrinos! That is really weird and extraordinary… We do not really know what we will find through the looking glass



Anomalous Symmetries

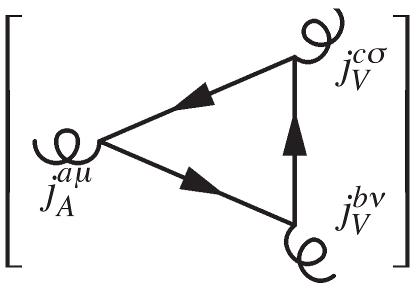
Sometimes symmetries of the classical Lagrangian do not survive quantisation. There are three examples we can cite:

❖Global chiral symmetries, responsible for the electromagnetic decay of the neutral pion

❖Gauged chiral symmetries. This happens when left and right multiplets have different representations of the gauge group. At the one-loop level we find a non-trivial conditions among the quantum numbers necessary to maintain gauge invariance. It suffices to satisfy this condition at the one-loop level

❖Scale invariance. The behaviour of the theory under scale transformation. Rather how physics depends on scales is far more interesting that just dimensional analysis.

$$\langle 0|T\left[j_A^{a\mu}(x)\, j_V^{b\nu}(x')\, j_V^{c\sigma}(0)\right]|0\rangle =$$



$$\Big[\;\cdots\;\Big]_{\text{symmetric}} \propto \pm \, \mathrm{tr}\left[\tau^a_{i,\pm}\{\tau^b_{i,\pm},\tau^c_{i,\pm}\}\right]$$

$$\sum_{i=1}^{N_+} \mathrm{tr}\left[\tau^a_{i,+}\{\tau^b_{i,+},\tau^c_{i,+}\}\right] - \sum_{j=1}^{N_-} \mathrm{tr}\left[\tau^a_{j,-}\{\tau^b_{j,-},\tau^c_{j,-}\}\right] = 0.$$

Anomaly cancellation condition, it has highly non-trivial implications for the family structure

Anomalous Symmetries

quarks: $\begin{pmatrix} u^\alpha \\ d^\alpha \end{pmatrix}_{L,\frac{1}{6}}$ $\quad u^\alpha_{R,\frac{2}{3}}$ $\quad d^\alpha_{R,\frac{2}{3}}$

leptons: $\begin{pmatrix} \nu_e \\ e \end{pmatrix}_{L,-\frac{1}{2}}$ $\quad e_{R,-1}$

$(3,2)^L_{\frac{1}{6}} \qquad (1,2)^L_{-\frac{1}{2}}$

$(3,1)^R_{\frac{2}{3}} \qquad (3,1)^R_{-\frac{1}{3}} \qquad (1,1)^R_{-1}$

Anomalies cancel generation by generation. In fact the hypercharge assignments is completely determined if we also impose the traceless-ness of any U(1)

$$\sum_{\text{left}} Y_+^3 - \sum_{\text{right}} Y_-^3 = 3\times 2\times\left(\frac{1}{6}\right)^3 + 2\times\left(-\frac{1}{2}\right)^3 - 3\times\left(\frac{2}{3}\right)^3$$
$$- 3\times\left(-\frac{1}{3}\right)^3 - (-1)^3 = \left(-\frac{3}{4}\right) + \left(\frac{3}{4}\right) = 0.$$

$\mathrm{SU}(3)^3$	$\mathrm{SU}(2)^3$	$\mathrm{U}(1)^3$
$\mathrm{SU}(3)^2\,\mathrm{SU}(2)$	$\mathrm{SU}(2)\,\mathrm{U}(1)$	
$\mathrm{SU}(3)^2\,\mathrm{U}(1)$	$\mathrm{SU}(2)\,\mathrm{U}(1)^2$	
$\mathrm{SU}(3)\,\mathrm{SU}(2)^2$		
$\mathrm{SU}(3)\,\mathrm{SU}(2)\,\mathrm{U}(1)$		
$\mathrm{SU}(3)\,\mathrm{U}(1)^2$		

Deriving quantum numbers

$$SU(N)_c \times SU(2) \times U(1) \qquad \text{S.M. anomaly}$$

$$\begin{array}{ccc} (N,2)^L_{q_L} & \oplus & (1,2)^L_{l_L} \\ (N,1)^R_{u_R} & \oplus & (N,1)^R_{d_R} \oplus (1,1)^R_{e_R} \end{array}$$

Single family

Anomaly conditions, we will normalize e_R=-1 as in the SM

$$U(1)SU(2)^2 \qquad 2N\, q_L + 2\, l_L = 0$$
$$U(1)SU(N)^2 \qquad 2\, q_L - (u_R + d_R) = 0$$
$$U(1)^3 \qquad 2\, N\, q_L^3 + 2\, l_L^3 - N\, u_R^3 - N\, d_R^3 - e_R^3 \; = \; 0$$
$$U(1) \qquad 2\; q_L + 2\, l_L - N(u_R + d_R) - e_R \; = 0$$

A simple computation now yields a (nearly) unique solution:

$$q_L \; = \; \frac{1}{2\,N} \qquad l_L \; = \; -\frac{1}{2} \qquad e_R \; = \; -1$$

$$\begin{aligned} u_R \; + \; d_R \; &= \; 1/N \\ u_R\, d_R \; &= \; -\frac{1}{4}(1 - \frac{1}{N^2}) \end{aligned} \qquad \begin{aligned} u_R \; &= \; \frac{N+1}{2N} \\ d_R \; &= \; -\frac{N-1}{2N} \end{aligned}$$

For N=3 we obtain the hypercharges of the SM!!

Farewell

- QFT is a vast and complex subject
- SM is a major achievement
- It summarizes our knowledge of the fundamental laws of Nature
- But also our ignorance
- Many puzzles and unanswered questions remain
- We may be at the end of a cycle. Perhaps the symmetry paradigm has been exhausted.
- Naturalness, a red herring. Higgs or not Higgs
- Gravity into the picture finally?
- Hopefully we are entering a golden decade

Thank you