UV completion on the world line

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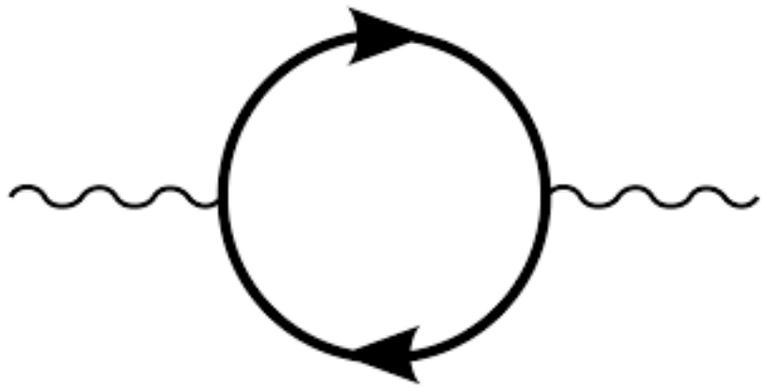
w/ Nicola Dondi (CP3) arXiv:1905.04258
w/ Daniel Lewis

Overview

- Structure of string amplitudes and their finiteness:
  a) Modular invariance b) World-sheet Green's function
- Particle theory: UV completion on the world-line
- Calculating amplitudes
- Conclusions

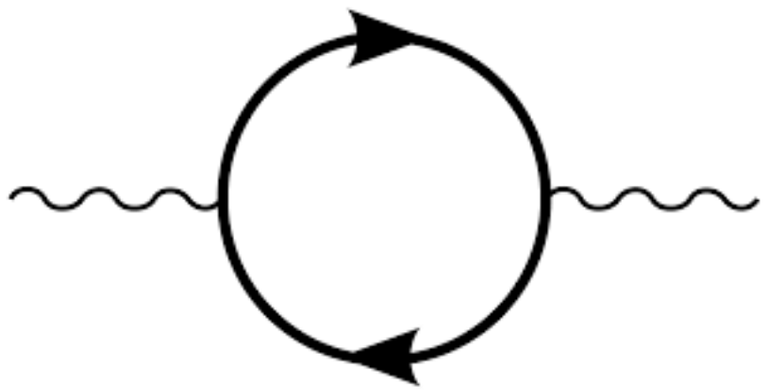
Structure of string amplitudes

Field theory vacuum polarization diagram:



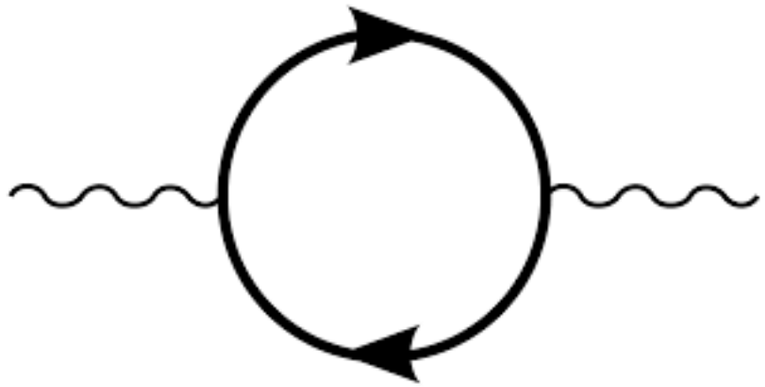
Write it in a “stringy” way: with the propagator written as a Schwinger integral over t:

$$\Delta(p^2) = \frac{1}{p^2 + m^2} = \int_0^\infty dt\ e^{-t(p^2+m^2)}$$



$$\Pi^{\mu\nu} \approx \frac{bg_{YM}^2}{16\pi^2}(p_1^\mu p_2^\nu - p_1 \cdot p_2 \eta^{\mu\nu}) \int_0^1 dx \int \frac{dt}{t} e^{t(\frac{p_1 \cdot p_2}{2} x(1-x) - m^2)}$$

$$t = t_1 + t_2 \; ; \; x = \frac{t_1}{t_1 + t_2}$$



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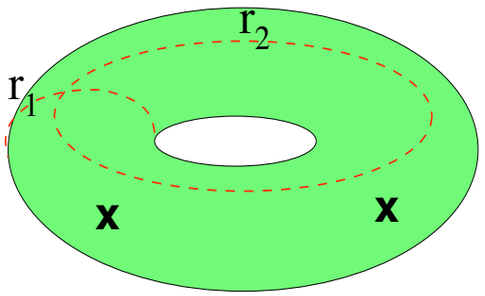

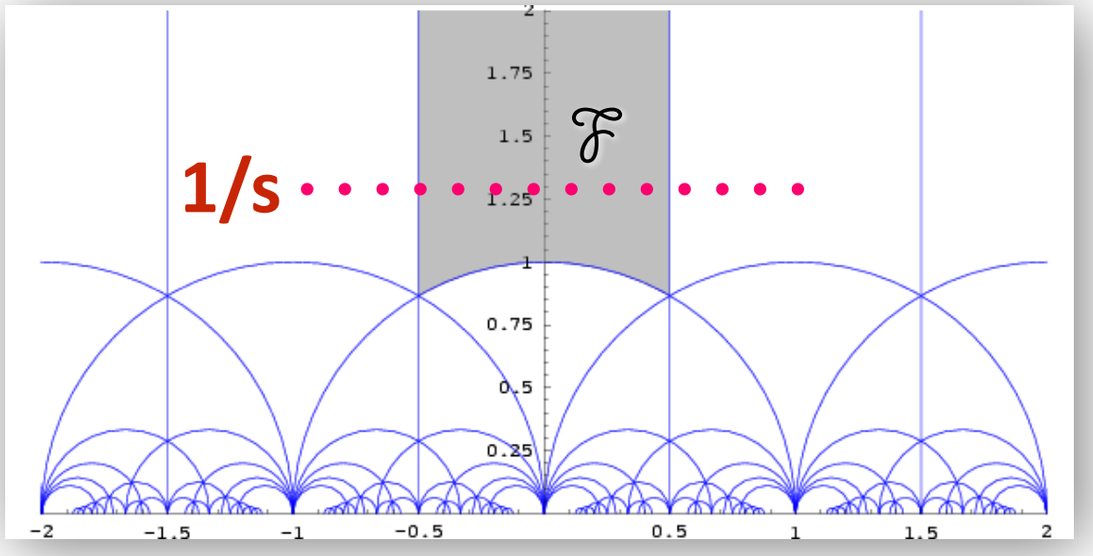

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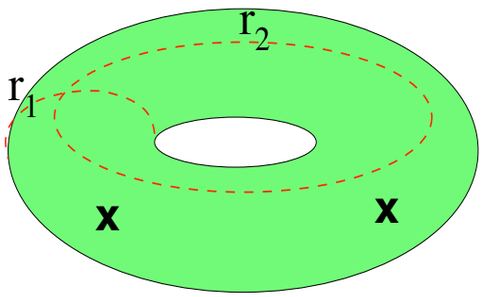

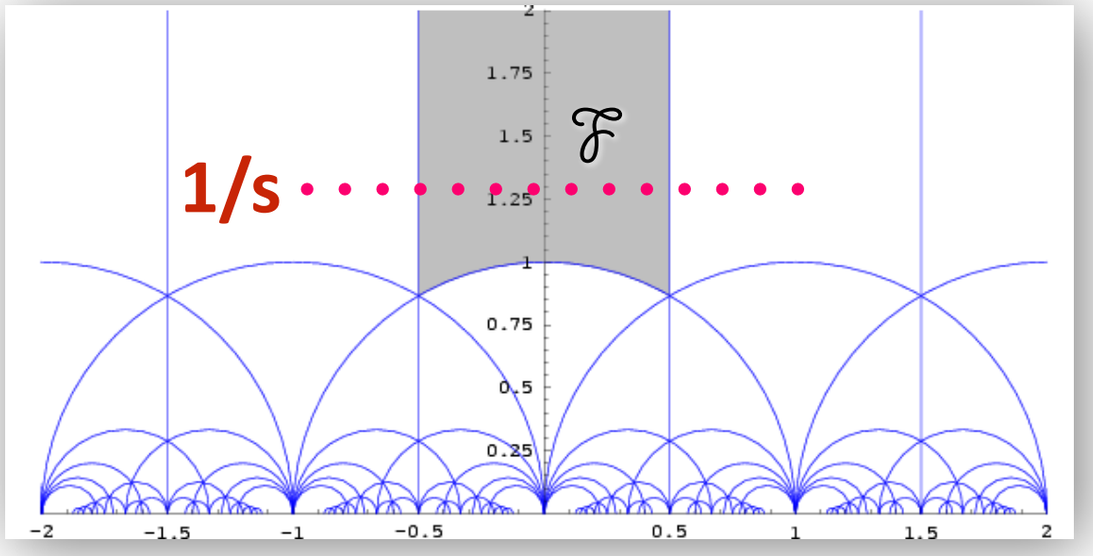

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$$\times\int\frac{d^2z}{\tau_2}\left(4\pi i\partial_\tau\log(\frac{\vartheta_{\alpha\beta}(0|\tau)}{\eta(\tau)}\right)|\vartheta_1(z)|^{2p_1\cdot p_2}\exp\left[-p_1\cdot p_2\frac{2\pi}{\tau_2}\Im(z)^2\right]\quad \mathrm{Tr}\left[\frac{k}{4\pi^2}\partial_{\bar z}^2\log\vartheta_1(\bar z)+Q^2\right]$$

$$\approx \frac{g_{YM}^2}{16\pi^2}\ (p_1^\mu p_2^\nu - p_1\cdot p_2\eta^{\mu\nu})\int\frac{d\tau_2}{\tau_2}\int\frac{d^2z}{\tau_2}\ \frac{1}{4\pi^2}\mathrm{Tr}\left(4\pi i\partial_\tau\log\frac{\vartheta_{\alpha\beta}(0|\tau)}{\eta(\tau)}\left[-\frac{1}{4\pi\tau_2}+Q^2\right]\right)\ e^{-p_1\cdot p_2 G_{12}(\tau,z)}$$

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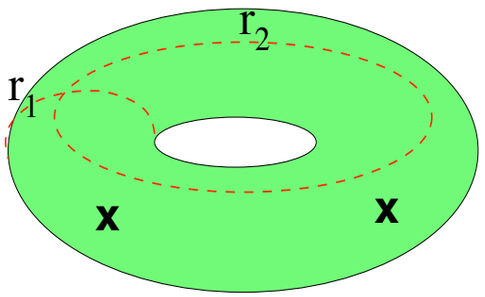

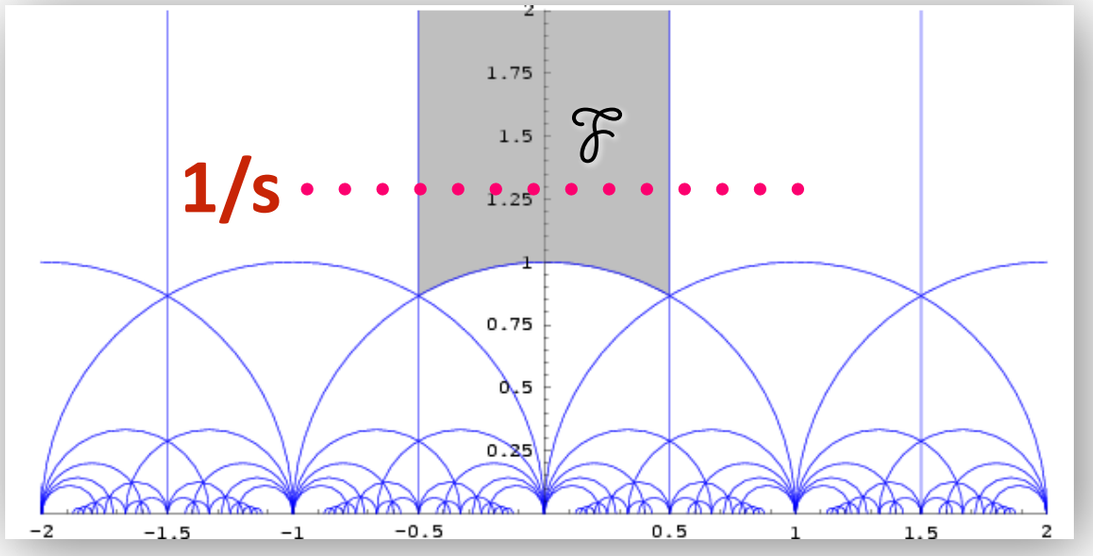

a) PARTITION FUNCTION

$$\Pi^{\mu\nu} \approx \frac{g_{YM}^2}{16\pi^2}(p_1^\mu p_2^\nu - p_1\cdot p_2\eta^{\mu\nu})\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\frac{1}{4\pi^2|\eta(\tau)|^4}\sum_{\alpha,\beta,Z_2}\mathcal{Z}_{B_{int}}^{Z_2}\mathcal{Z}_F^{\alpha,\beta,Z_2}$$

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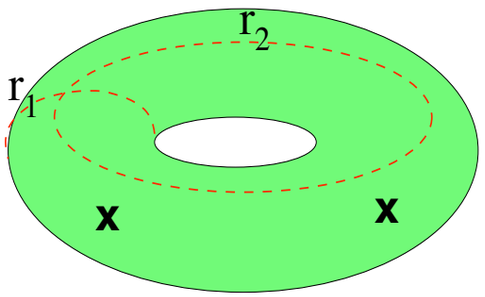

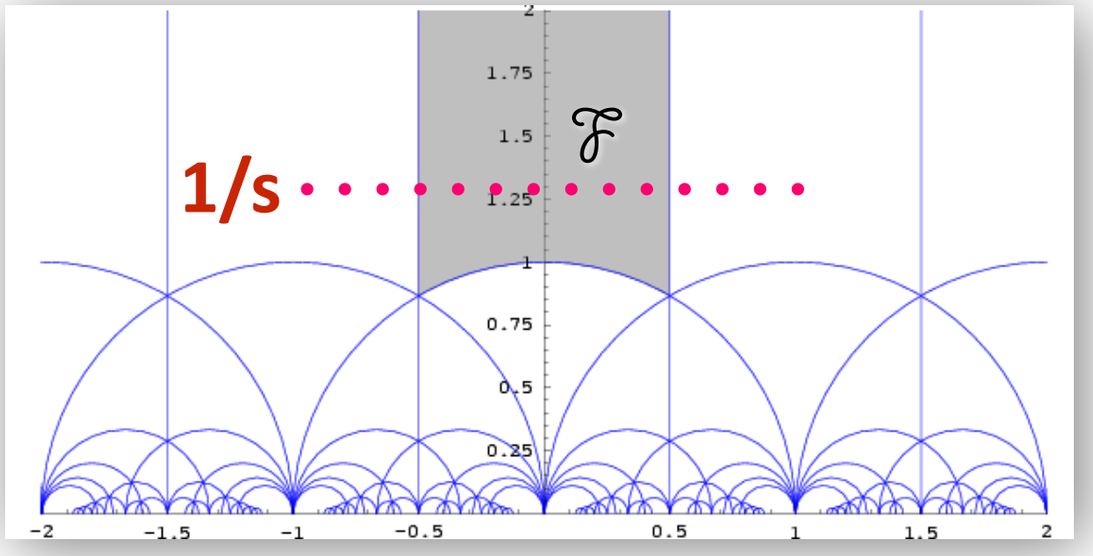

b) WORLD SHEET GREEN'S FUNCTION — softens the UV behaviour

$$\Pi^{\mu\nu} \approx \frac{g_{YM}^2}{16\pi^2}(p_1^\mu p_2^\nu - p_1\cdot p_2\eta^{\mu\nu})\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\frac{1}{4\pi^2|\eta(\tau)|^4}\sum_{\alpha,\beta,Z_2}\mathcal{Z}_{B_{int}}^{Z_2}\mathcal{Z}_F^{\alpha,\beta,Z_2}$$

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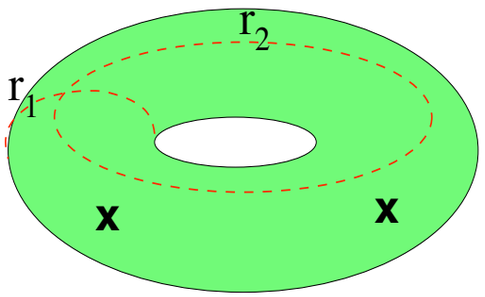

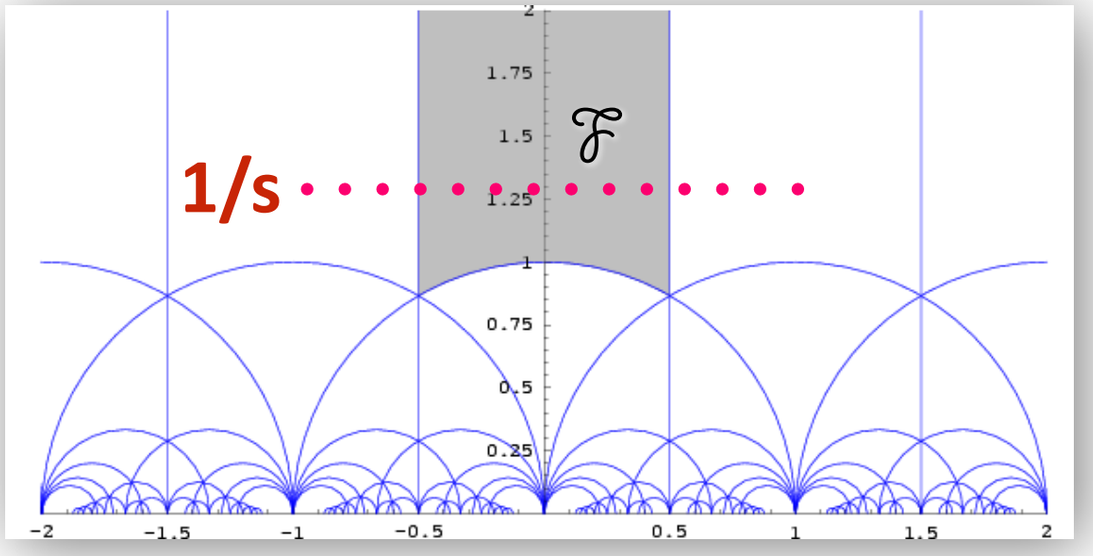

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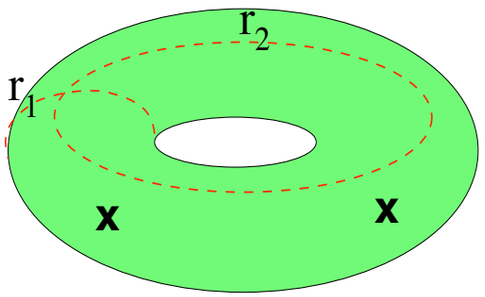

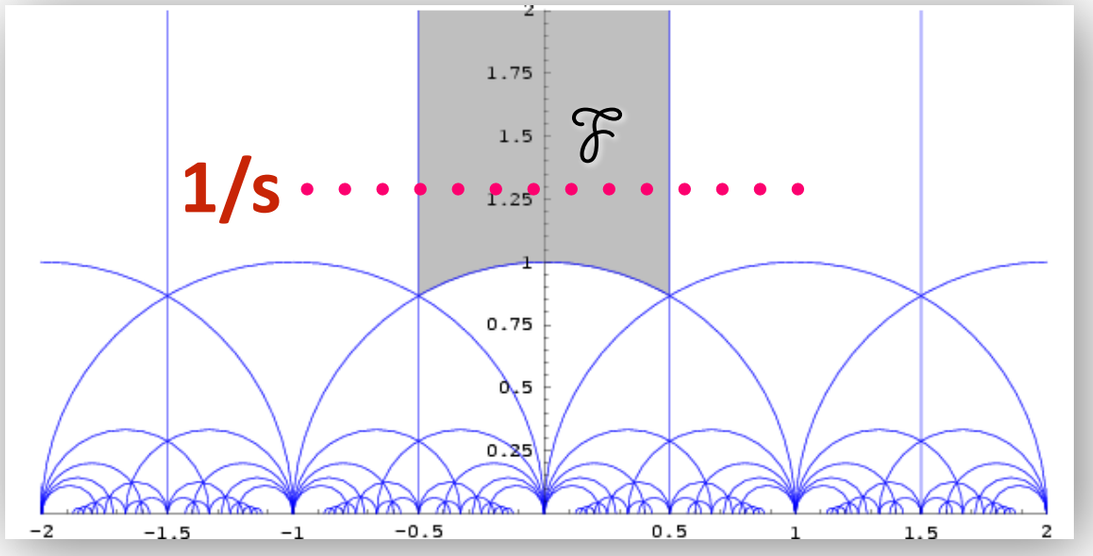

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$$G_{12}(\tau,x,y) \overset{\tau_2\to\infty}{\longrightarrow} \pi\alpha'\tau_2 x(1-x)+\ldots$$

$$G_{12}(\tau,x,y) \overset{\tau_2\to 0}{\longrightarrow} \pi\alpha'\frac{1}{\tau_2}y(1-y)+\ldots$$

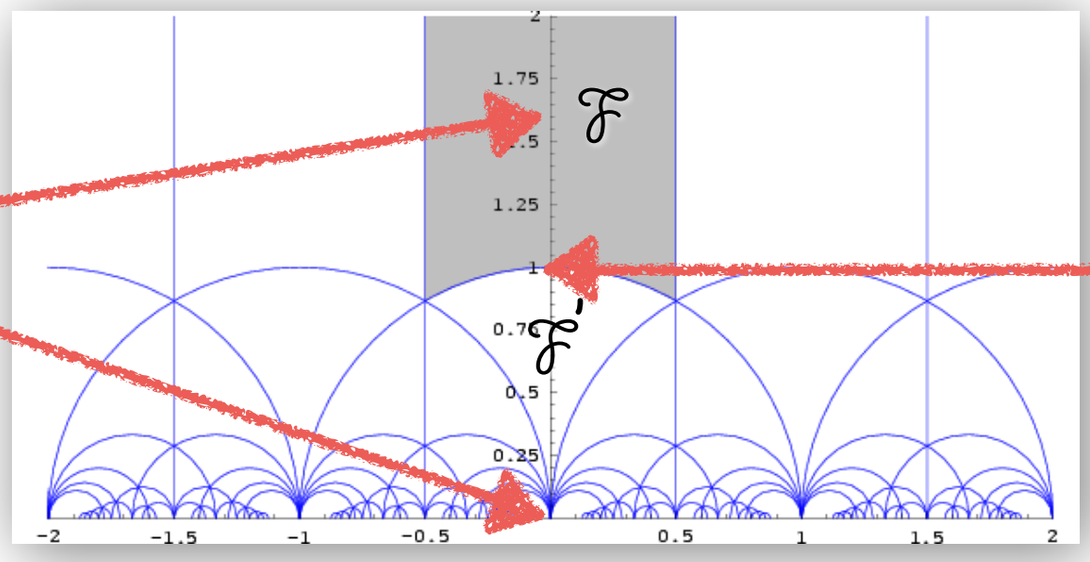
Heuristically we have

$$\Pi^{\mu\nu} \approx \frac{g_{YM}^2}{16\pi^2}(p_1^\mu p_2^\nu - p_1\cdot p_2\eta^{\mu\nu})\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\ \text{Big-Mess}(\tau)\times e^{-p_1\cdot p_2 G_{12}(\tau)}$$

The particle limit of the world-sheet Green's function gives the natural logarithmic running with s, plus threshold. But modular invariance means we could just as well include the integration over the UV cusp too:

$$\Pi^{\mu\nu} \approx \frac{g_{YM}^2}{16\pi^2}(p_1^\mu p_2^\nu - p_1\cdot p_2\eta^{\mu\nu})\int_{\mathcal{F}'}\frac{d^2\tau}{\tau_2^2}\ \text{Big-Mess}(-1/\tau)\times e^{-p_1\cdot p_2 G_{12}(-1/\tau)}$$

Logarithmic in s:
UV = IR



Finite threshold comes from saddle behaviour around here. (Position may depend on Mandelstam variables)

- This behaviour is quite generic: e.g. 4 point one-loop coupling (which was examined by Gross & Mende) also has a saddle that obeys:

$$\exp(-\pi\hat{\tau}_2) = -\frac{t}{s}$$

where t,s are now Mandelstam variables. But small angle scattering t/s<<1 puts the saddle in the IR particle-like regime. So we can actually mimic this behaviour with a modified *world-line* Green's function, so exponential is of the form:

$$\exp(-\pi\tau_2\frac{tu}{2s} + 4ue^{\pi\tau_2 u/s})$$

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This first sub-leading term in G yields all the stringy behaviour

$$\exp(-\pi\tau_2\frac{tu}{2s} + 4ue^{\pi\tau_2 u/s})$$

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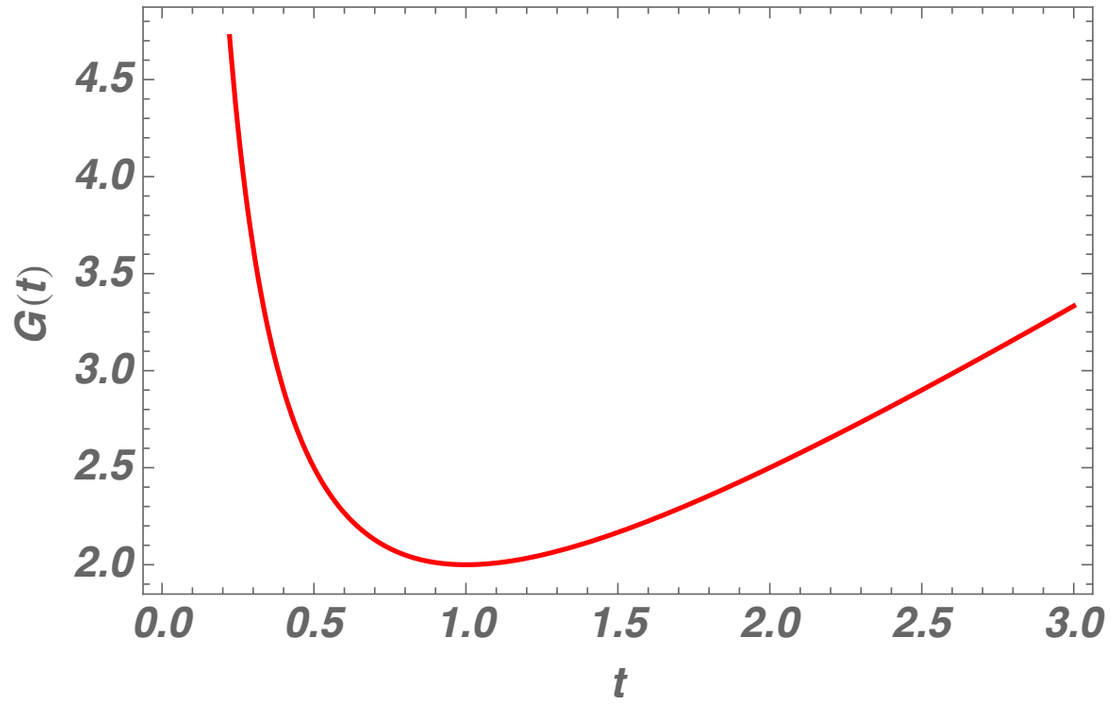
Conclusion: string theory amplitudes are regulated by a saddle point in the world-sheet propagator, because by the time we integrate over everything but $\tau_2 \equiv t$, we have an exponent like



UV completion on the world-line

- ***Define a world-line theory with a G with the same properties as string theory:***

Although the WL formalism emerges in the particle limit of string theory, a first quantised particle theory can be built in this formalism.

Feynman;
Affleck, Alvarez, Manton;
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Normally would have e.g. the tree-level propagator in a scalar theory:

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More generally the Schwinger proper-time always come from G(t) on the worldline. So we can imagine more general WL theories that give

$$\Delta(p^2) \ = \ \int_0^\infty \mathrm{d}t\, e^{-T(t)(p^2+m^2)}$$

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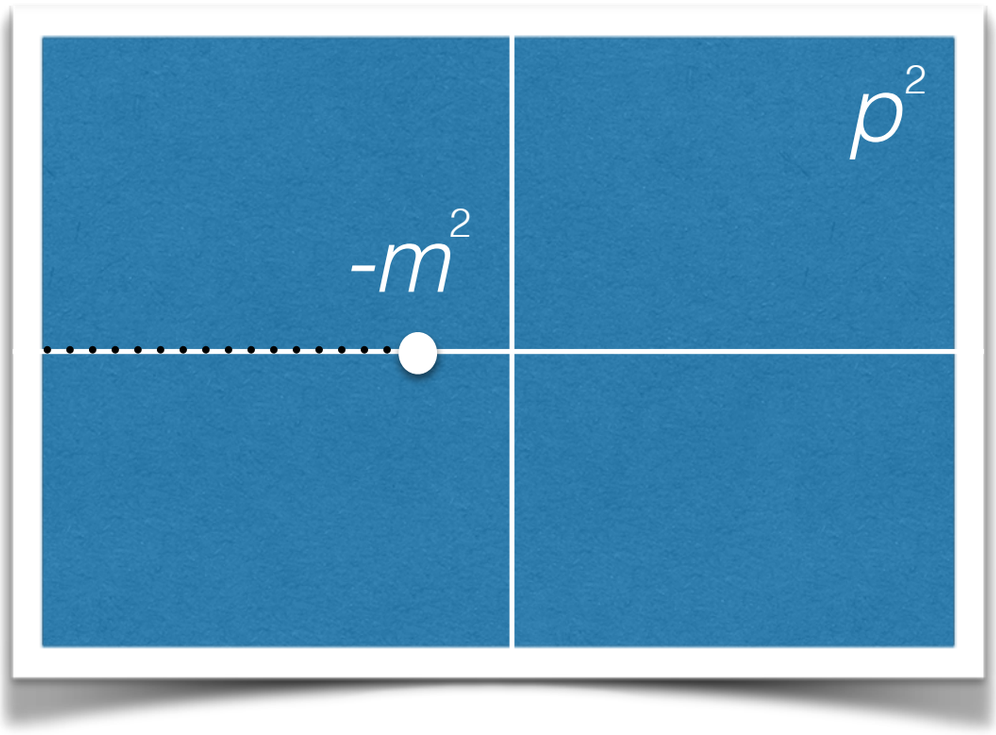
To get the correct IR behaviour we need to choose a T(t) satisfying:

$$T(t) \;\overset{\tau\to\infty}{\longrightarrow}\; t\ .$$

We also need to avoid producing ghosts:

Th'm: *Any theory for which* $tT(t^{-1})$ *is entire and* $Re(T) > 0$ *for all* $t > 0$ *is ghost-free.*

$\Delta(p^2)$ is holomorphic in right half-plane, with *generically* a branch-cut in left-half going from the pole



e.g. the “trivial case”: $T(t) = t + 1$

$$\Delta(p^2) = e^{-(p^2+m^2)}/(p^2 + m^2)$$

This is the only case which gives an entire function multiplying the usual propagator. It turns out to be the infinite derivative model of Siegel et al, derived from string field-theory.

We can see that this theory is indistinguishable from just imposing a cut-off on proper time:

$$\Delta(p^2) \;=\; \int^{\infty} \mathrm{d}T \quad e^{-T(p^2+m^2)}$$

e.g. 2: mimick string amplitudes, by copying the only Modular transformation that matters:

$$T(t) \quad = \quad T(t^{-1})$$

Unique example with correct IR behaviour: $T \; = \; t + t^{-1}$
gives a different infinite derivative field theory:

$$\Delta(p^2) \; = \; 2K_1(2(p^2+m^2))$$

$$\longrightarrow \; \begin{cases} \frac{1}{p^2+m^2} & ; \quad p^2 \ll 1 \; , \\ \frac{\sqrt{\pi}e^{-2(p^2+m^2)}}{\sqrt{p^2+m^2}} & ; \quad p^2 \gg 1 \; . \end{cases}$$

Importantly still only single pole: ghost-free (but has the same exponential behaviour as the Siegel et al. theory at large momentum.)

NB: this is not target-space duality which would be invariance under $T \to 1/T$

Interpretation 1: substitute for *T(t)* in path integral — path integrals are weighted with a function that has a lower limit. (The Siegel model has a Heaviside-function weighting: we are simply generalising this kind of theory):

$$\Delta(p^2) \;=\; \int_2^\infty \mathrm{d}T \frac{T}{\sqrt{T^2-4}}\; e^{-T(p^2+m^2)}$$

Interpretation 2: Fourier Transforming to target space, we see the Bessel function has introduced a minimal length:

$$\Delta(x,y) \;=\; \int_0^\infty \mathrm{d}t\, \frac{1}{(4\pi T)^{d/2}} e^{-\left[\frac{(x-y)^2}{4T}+Tm^2\right]}$$

Solutions to diffusion equation with (in our 2nd example) $D(t) = (1-1/t^2)$

The initial data is sampled with Gaussians that spread with proper-time. But now the *T=0* "delta-function" is never reached — minimal smearing gives minimal length.

Perturbation theory

Much of this follows standard WL techniques, with the difference coming only at the end (with the weighting of paths)

Generic trees: written like the string version (or rather vice-versa)

e.g. scalar QED: write as a *world-line theory*, with Wilson line for photon emission

$$\Delta(x,y) = \int_0^\infty \mathrm{d}t e^{-Tm^2} \int_{x(0)=x}^{x(T)=y} \mathcal{D}x e^{-S[x,A_\mu]} ,$$

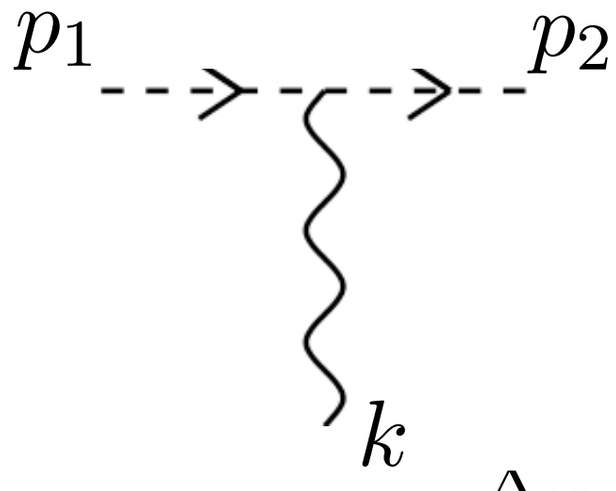
$$S[x,A_\mu] = \int_0^T \mathrm{d}\tau \, \frac{\dot{x}^2}{4} + iq\,\dot{x}\cdot A(x) ,$$

expand photon as plane waves: $A_\mu(x(\tau)) = \sum_{i=1}^{n} \varepsilon_{i,\mu} e^{ik_i\cdot x}$

$$\mathcal{A}^{(n)} = q^n \delta^4(p_1+p_2+\textstyle\sum_i k_i) \displaystyle\int_0^\infty \mathrm{d}t\, e^{-T(p_1^2+m^2)}$$
$$\times \int_0^T \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_n \, e^{(p_1-p_2)\cdot\sum_i(-\tau_i k_i - i\varepsilon_i)} e^{(k_i\cdot k_j G_{ij} - 2i\varepsilon_i\cdot k_j \dot{G}_{ij} + \varepsilon_i\cdot\varepsilon_j \ddot{G}_{ij})}$$

with $G_{ij} = \frac{1}{2}|\tau_i - \tau_j|$, and extract term in n-polarization vectors.

e.g. gauge coupling …

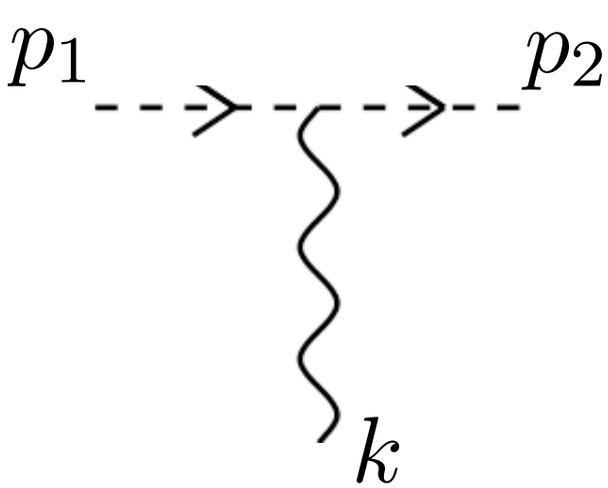


$$\mathcal{A}^{(1)} \;=\; iq\,\delta^4(p_1+p_2+k)\;\varepsilon\cdot(p_1-p_2)\,\frac{\Delta_{12}}{p_1^2-p_2^2}$$

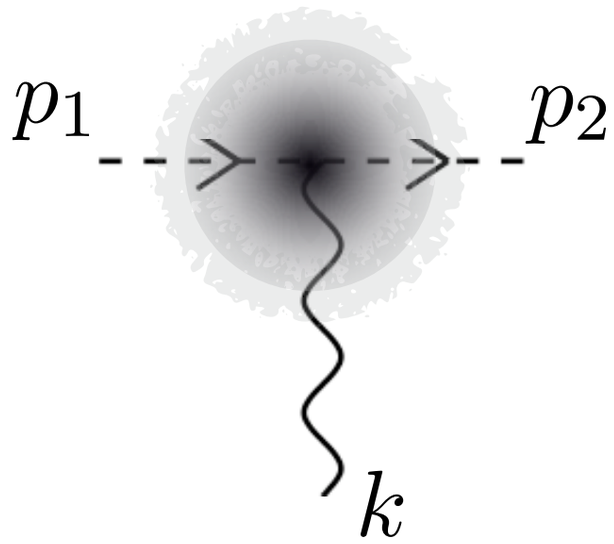
$$\frac{\Delta_{12}}{p_1^2-p_2^2} \to \begin{cases} \frac{-1}{(p_1^2+m^2)(p_2^2+m^2)} & ;\, p^2 \ll 1, \\ \frac{\sqrt{\pi}}{p_1^2-p_2^2}\left(\frac{e^{-2(p_1^2+m^2)}}{\sqrt{p_1^2+m^2}} - \frac{e^{-2(p_2^2+m^2)}}{\sqrt{p_2^2+m^2}}\right) & ;\, p^2 \gg 1, \end{cases}$$

Note at large momentum we can no longer truncate the propagators from the vertex:

Interpretation 3: Vertices are smeared on a scale of order 1



$p^2 \ll 1$



$p^2 \gg 1 \equiv M_s^2$

Generic one-loop diagrams: again like the string version

$$\mathcal{A}_{1\ell}^{(n)}(\{p_i\}) \;=\; \int \mathrm{d}t\, e^{-m^2 T(t)} \int \mathcal{D}x\; V[p_1]...V[p_n] e^{-S[x,0]}$$

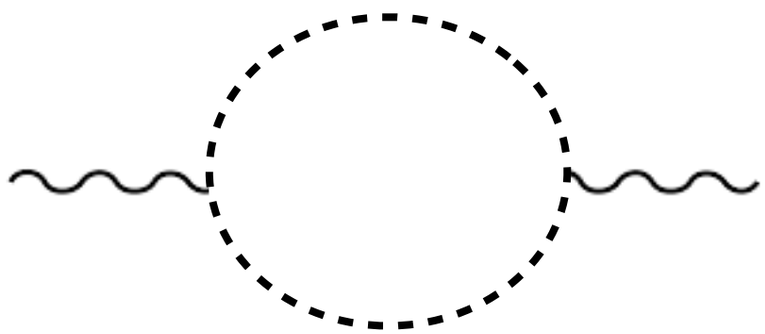
$$V_A[p] \;=\; \int_0^T \mathrm{d}\tau \varepsilon\cdot\dot{x}\, e^{ip\cdot x}$$

Can always rearrange it so propagators are treated democratically: e.g. 2 point

$$\mathcal{A}^{(2)}_{1-loop}(\{p_i\}) = (p^\mu p^\nu - g^{\mu\nu}p^2)\frac{1}{(4\pi)^{d/2}}\int \frac{\mathrm{d}t_1\,\mathrm{d}t_2}{(T_1+T_2)^{d/2}} \times e^{-m^2(T_1+T_2)+s\frac{T_1T_2}{T_1+T_2}}$$

Dominated by the saddle at *t=1*: but this is not surprising, because we built it in. All UV sensitive amplitudes are dominated by saddles.

Can do threshold corrections: note they are proportional to the gauge beta function, so we can have unification with $M_s \ll M_{GUT}$ (admittedly without knowing why $M_{Pl} \gg M_s$)

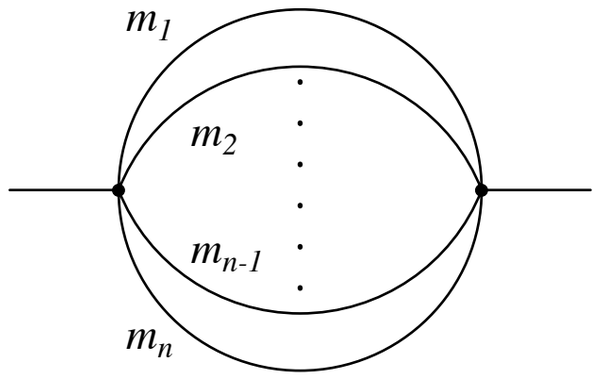


$$\mathcal{A}^{(2)}_{1-loop}(\{p_i\}) = (p^\mu p^\nu - g^{\mu\nu}p^2)\frac{b_s}{(4\pi)^{d/2}}\int \frac{\mathrm{d}t_1\,\mathrm{d}t_2}{(T_1+T_2)^{d/2}} \times e^{-m^2(T_1+T_2)+s\frac{T_1T_2}{T_1+T_2}}$$

$$\begin{aligned}\frac{16\pi^2}{g^2_{EFT}(s)} - \frac{16\pi^2}{g^2_{EFT}(M_s)} &= -\,b_s \log\left(\frac{s}{M_s^2}\right) \\ &= \frac{16\pi^2}{g^2(s)} - \frac{16\pi^2}{g^2_{\rm tree}} + \Theta\end{aligned}$$

$$\Theta = b_s \log\left(4e^{\,\gamma_E+1-\frac{\sqrt{\pi}}{2}}\right)$$

N-loop 2 point Sunset diagram — (feel the power!)



$$\mathcal{A}_\ell^{(2)}(p_1,p_2) \;=\; \frac{\delta^4(p_1+p_2)}{(\ell+1)!}\int\prod_i^{\ell+1}\mathrm{d}t_i\,\frac{1}{\left[(4\pi)^\ell\sum W_\ell^{\ell+1}\right]^{d/2}}e^{-m^2\sum_i^{\ell+1}T_i-p_1^2\left[T_{\ell+1}-T_{\ell+1}^2\frac{W_{\ell-1}^{\ell}}{W_\ell^{\ell+1}}\right]}$$

W_L^j is the sum of all words of length *L* that can be made with the symbols $\{T_1,\ldots,T_j\}$

$$\mathcal{A}_\ell^{(2)}(s)\sim\frac{1}{(16\pi^2)^{\ell+1}(\ell+1)!}e^{-2s/(\ell+1)}$$

Conclusions

- The finiteness of string amplitudes can be understood from the lowest corrections to the World-Line Green's function of the effective particle theory
- Inspired by this we propose a new class of UV-complete world-line theories
- They correspond to infinite derivative field theories, but have nicer properties — e.g. amplitudes dominated by saddle points
- In particular with this formalism no need to worry about Wick rotation — calculate amplitudes in world-line formalism just like you would in strings. All divergences are IR ones.
- Some interesting features: e.g. universal gauge thresholds — mirror unification?
- Many open questions: Gravity? Macrocausality? Unitarity at level of S-matrix? Microscopic understanding.