Tunneling in real time

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The aim:

- Show that **vacuum tunneling** rates can be recovered in **Minkowski space-time** (we focus on bosonic fields, QM or 4D QFT in a planar limit).
- Provide an **optical theorem interpretation of vacuum decay**

The novelty:

- There was no understanding of the behaviour of fluctuation determinants under rotations of the time-contour
- Tunneling calculations had not been connected with the optical theorem
- Insight on relation between effective potential at the minimum and normalization of quantum state connected to false-vacuum

The plan:

- Vacuum-to-vacuum transitions and decay rates
- Picard-Lefschetz theory and complex saddle points
- Bounce and fluctuation determinant for arbitrarily rotated time contours

Why do we care?

Why do we care?

Tunneling calculations relevant in solid state and particle physics

Consistency check of usual assumptions

Development of functional techniques for rotated time contours

New insights?

Vacuum-to-vacuum transitions and decay rates

Vacuum transition amplitudes

$$Z[T] = \langle q|e^{-iHT(1-i\epsilon)}|q\rangle = \sum_n |\langle q|n\rangle|^2 e^{-iE_nT(1-i\epsilon)} \sim |\langle q|0\rangle|^2 e^{-iE_0T}$$

Analytic continuation to Euclidean time [Callan, Coleman]

$$Z_E[T_E] = \langle q|e^{-HT_E}|q\rangle = \sum_n |\langle q|n\rangle|^2 e^{-E_nT} \sim |\langle q|0\rangle|^2 e^{-E_0T_E}$$

Decay rate per unit volume of vacuum state

$$\gamma = \mathrm{Im}\left(-\frac{2}{V}E_0\right) = \mathrm{Im}\left(-\frac{2i}{VT}\log Z[T]\right) = \mathrm{Im}\left(\frac{2}{VT_E}\log Z_E[T_E]\right)$$

Vacuum transition amplitudes

$$Z[T] = \langle q| e^{-iHT(1-i\epsilon)} |q\rangle = \int [d\phi] e^{iS[\phi]}$$

$$Z_E[T_E] = \langle q| e^{-HT_E} |q\rangle = \int [d\phi] e^{-S_E}$$

How come Z_E is complex, when Euclidean path integral is real?

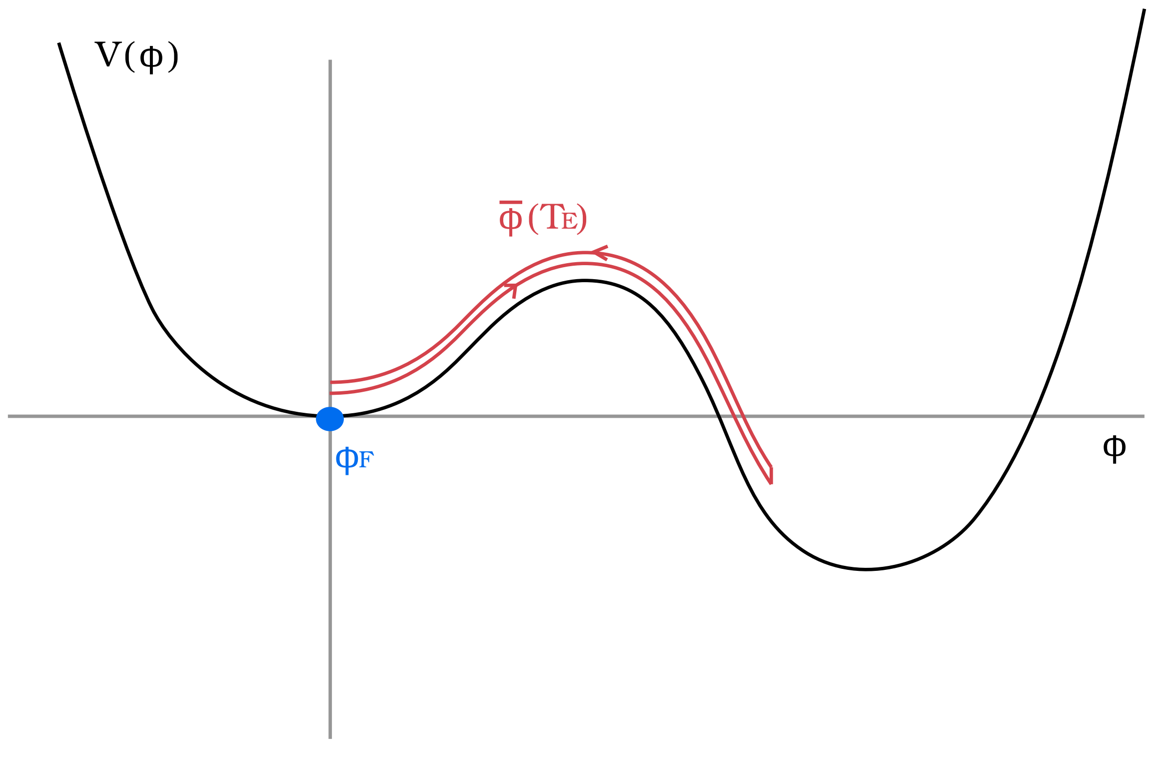
Is it really true that, with path integral definition, $Z[T] = Z_E[iT]$??

Do the tunneling rates agree?

The Euclidean calculation

Expand path integral around saddle point $\bar{\phi}$ (the "bounce") [Callan & Coleman]

$$\left.\frac{\delta S_E}{\delta \phi}\right|_{\phi=\bar{\phi}} = 0, \qquad \lim_{t\to\pm T/2} \bar{\phi} = \phi_F$$



For infinite T one has multi-bounce saddle-points correspoding to the field bouncing n-times back and forth from the false-vacuum, for all n

The Euclidean calculation

Sum over multi-bounce saddles gives exponential of single-bounce result:
[Callan & Coleman]

$$Z_E[T_E] = Z_E^{\mathrm{FV}} + e^{Z_E^{1\ \mathrm{bounce}}[T_E]}$$

$$Z_E^{1\ \mathrm{bounce}}[T_E] = \int_0 [d\phi] e^{-S_E[\bar{\phi}+\phi]} = e^{-S_E[\bar{\phi}]} \int_0 [d\phi] e^{-\frac{1}{2}\phi \mathfrak{M}_E \phi} \qquad (= e^{-S_E[\bar{\phi}]} (\det \mathfrak{M}_E)^{-1/2} ?)$$

$\mathfrak{M}_E$ has: 4 zero modes: Trade zero modes by integration over center of bounce

1 negative eigenvalue, divergent integral? Define integration on negative mode direction by analytic continuation. This gives desired imaginary part

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$$Z_E^{1\ \mathrm{bounce}}[T_E] = \frac{1}{2} Z_E^{\mathrm{Gaussian,\ 1\ bounce}}[T_E] = e^{-S_E[\bar{\phi}]} \left(\frac{S_E[\bar{\phi}]}{2\pi}\right)^2 \mathcal{V}T_E \quad \left(\frac{i}{2} |\det{}' \mathfrak{M}_E|^{-1/2}\right)$$

$$\gamma = \mathrm{Im}\left(\frac{2}{\mathcal{V}T_E} \log Z_E[T_E]\right) = \mathrm{Im}\left(\frac{2}{\mathcal{V}T_E} Z_E^{1\ \mathrm{bounce}}[T_E]\right) = e^{-S_E[\bar{\phi}]} \left(\frac{S_E[\bar{\phi}]}{2\pi}\right)^2 \quad |\det{}' \mathfrak{M}_E|^{-1/2}$$

Picard-Lefschetz theory and complex saddle points

Real integrals from complex steepest-descent paths

$$\int d^n x e^{\mathcal{I}(x)}$$

Cauchy theorem:

Contours along real x can be deformed to the complex plane without changing the value of the integral.

Interested in complex contours ending in convergence regions

$$h \equiv \mathrm{Re}\mathcal{I}(x) < 0$$

Picard-Lefschetz theory:

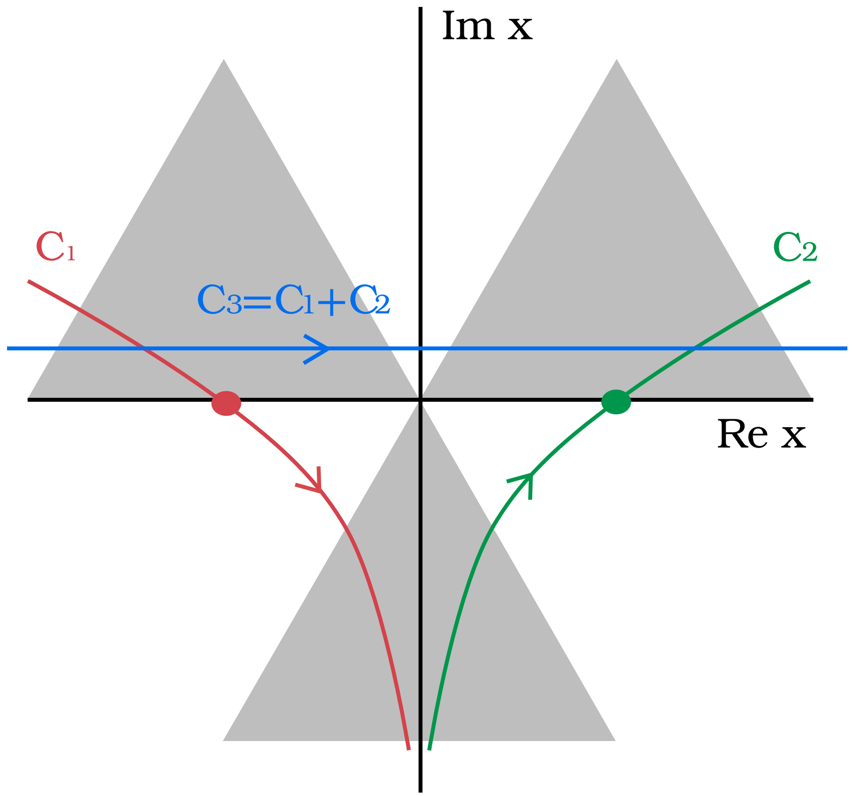
Can define basis of convergent contours

Elements of basis are steepest-descent flows ending in saddle points (which can be complex)

Deformed path = Sum over complex steepest-descent flows attached to saddle points

Real integrals as sum of complex steepest-descent paths

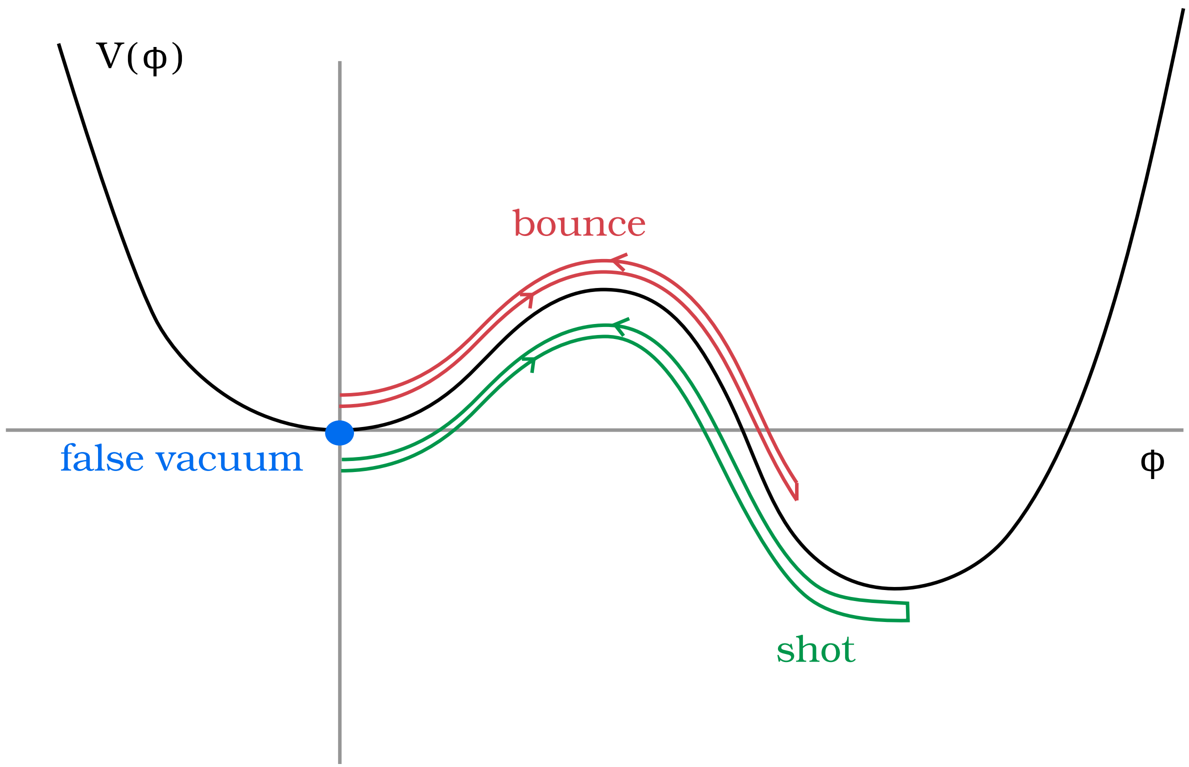
E.g. Airy function [Witten]: $Ai(\lambda) = \int dx e^{i\lambda\left(\frac{x^3}{3} - x\right)}, \quad \lambda > 0$ saddle points $x = \pm 1$



$$Ai(\lambda) = \int_{C_1+C_2} dx e^{i\lambda\left(\frac{x^3}{3} - x\right)}$$

Steepest descent paths for the Euclidean integral

Euclidean action has 3 saddle points [Schwarz et al]

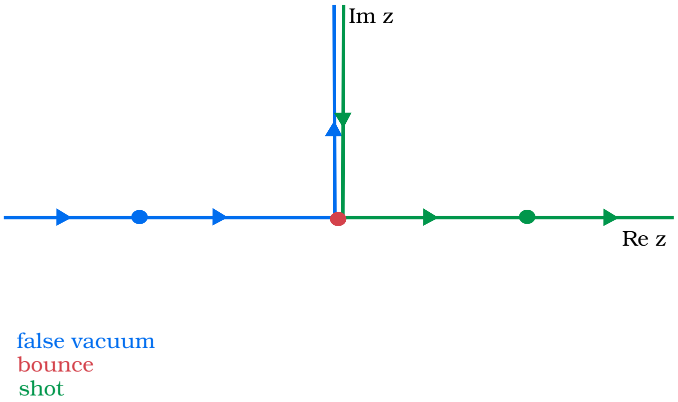


Again, for infinite T one has multi-bounce saddle-points

Shot has different boundary conditions for $\dot{\phi}$ at large t than false-vacuum/shot!

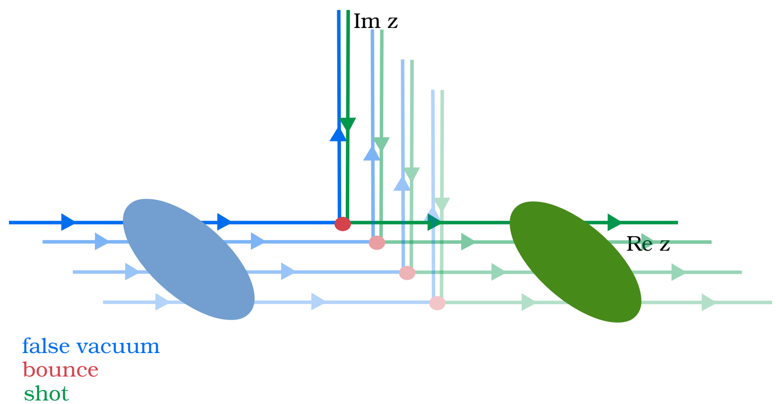
Steepest-descent flows for the Euclidean integral

Steepest-descent paths for 1-parameter family of functions passing through 1-bounce



Steepest-descent paths for the Euclidean integral

Steepest-descent paths for 1-parameter families passing through multi-bounces



Imaginary parts cancel from the vertical branches of flows: Z_E captures energy of true-vacuum state and is real.

False-vacuum dynamics just follows from steepest-descent flow connected to false vacuum [Schwarz et al]

$$\langle q|e^{-HT_E}|q\rangle = \mathcal{N}_0^{-2}\langle 0|e^{-HT_E}|0\rangle = Z_E[T_E] = Z_E^{\text{FV path}} + Z_E^{\text{shot path}} \in \mathbb{R}$$

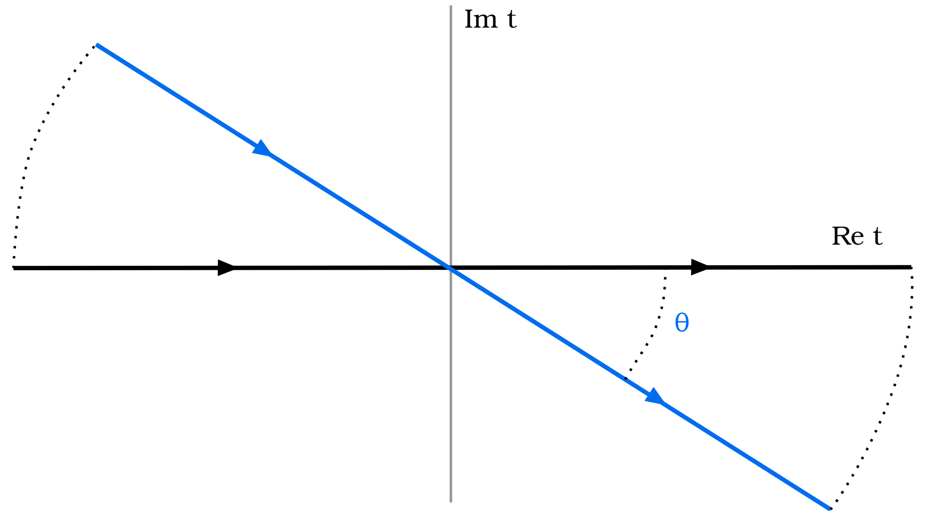
$$\mathcal{N}^{-2}\langle F|e^{-HT_E}|F\rangle = Z_E^{FV}[T_E] = Z_E^{\text{FV path}} \sim Z_E^{\text{Gaussian, FV}} + \exp\left(\frac{1}{2}Z_E^{\text{Gaussian,1-bounce}}\right)$$

Complexity can also be understood from relation to false-vacuum effective action [CT, A. Plascencia]

Bounce and fluctuation determinant for arbitrary rotations of the time contour

Choice of time-countours

We consider arbitrary rotations of the time contour away from Minkowski



$$S_\theta[\phi] = e^{-i\theta} \int d^4x \left[\frac{1}{2} \left(\frac{d\phi}{dt} \right)^2 \cdot e^{+2i\theta} - \frac{1}{2} \left(\vec{\nabla}\phi \right)^2 - V(\phi) \right].$$

Path integral

$$\langle q | e^{-ie^{-i\theta} H T_\theta} | q \rangle = Z_\theta[T_\theta] = \int [d\phi] e^{iS_\theta[\phi]}$$

What we need

Complex saddle-points for arbitrarily rotated time contours

Complex steepest descent flows passing through saddle points

[Cherman & Unsal, Turok, Tanizaki & Koike]

Definition of path integral measure along flows

Path integral of fluctuations along flows

Our work

The challenges

Expanding steepest descent flow around background:

$$\phi = \bar{\phi} + \Delta\phi$$

$$Z_\theta[T_\theta] = \int [d\phi] e^{iS_\theta[\phi]} \sim e^{iS_\theta[\bar{\phi}]} \int [d\Delta\phi] \, \exp\left\{ ie^{-i\theta} \int d^4x \left[-\frac{1}{2} \Delta\phi \, \mathfrak{M}_\theta \Delta\phi \right] \right\}$$

$\Delta\phi$ along flow is **complex**

$\mathfrak{M}_\theta$ is **not hermitian**! Are eigenfunctions orthogonal?

Steepest descent flow equations imply that $\Delta\phi$ cannot be expanded in terms of usual eigenfunctions of $\mathfrak{M}_\theta$, but rather "**flow eigenfunctions**"

$$\mathfrak{M}_\theta \chi_n = \kappa_n \chi_n^*$$

Is there any **connection to the usual determinant** of $\mathfrak{M}_\theta$?

How can one recover the **Euclidean result** for $\theta = \frac{\pi}{2}$?

Summary of results

Saddle points obtained by analytic continuation of Euclidean ones!

$$\bar{\phi}(t,\vec{x}) = \bar{\phi}_E(\tau = ie^{-i\theta}t, \vec{x})$$

Bounce action at saddle related to that of the Euclidean case, as follows from Cauchy theorem

$$iS_\theta[\bar{\phi}] = -S_E[\bar{\phi}_E]$$

Jacobian for integration along steepest descent flow is related to phase of $\mathfrak{M}_\theta$

$$\phi = \bar{\phi} + \Delta\phi, \quad \Delta\phi = \sqrt{-i}e^{i\theta/2}\sum_n g_n\chi_n, g_n \in \mathbb{R}$$

$$\mathfrak{M}_\theta\chi_n = \kappa_n\chi_n^*$$

$$[d\Delta\phi]_{\text{flow}} = \mathcal{J}\prod_n \frac{dg_n}{\sqrt{2\pi}}$$

$$\mathcal{J} = \left(\prod_n \sqrt{-i}e^{i\theta/2}\right) e^{-i/2\text{Arg}\,\det\mathfrak{M}_\theta}$$

OUR WORK

Integration over fluctuations on flow related to absolute value of determinant!

$$\prod_n \frac{dg_n}{\sqrt{2\pi}} \exp\left\{ie^{-i\theta}\int d^4x\left[-\frac{1}{2}\Delta\phi\,\mathfrak{M}_\theta\Delta\phi\right]\right\} = \prod \frac{1}{\sqrt{\kappa_n}} = |\det\mathfrak{M}_\theta|^{-1/2}$$

Summary of results

We can compute $\det \mathfrak{M}_\theta$ (full understanding in planar limit for the background)

Discrete eigenfunctions: Straightforward analytic continuations of Euclidean ones, with the **same real eigenvalues**

Continuum eigenfunctions: Can be constructed from Euclidean ones by a simultaneous analytic continuation in time and in the parameters characterizing asymptotic behaviour. Get **deformed, complex eigenvalues**

Despite non-hermiticity of $\mathfrak{M}_\theta$, eigenfunctions are **orthogonal** and **complete**

Continuum spectrum contribution contains generalization of **Coleman-Weinberg** potential

Final result compatible with **straightforward analytic continuation in length of time interval**:

$$\det \mathfrak{M}_\theta[T] = \det \mathfrak{M}_E[\mathcal{T}]|_{\mathcal{T}\to ie^{-i\theta}T}$$

Summary of results

Final Gaussian approximation near a saddle point:

$$Z_\theta^{\text{Gaussian},\bar\phi}[T] = e^{-S_E[\bar\phi_E]} \left(\frac{S_E}{2\pi}\right)^2 (Vie^{-i\theta}T)(\det{}'\mathfrak{M}_\theta)^{-1/2} \left(\prod_n \sqrt{-i}e^{i\theta/2}\right)$$

Classical action at saddle point

Zero mode integration

Jacobian + integration of remaining modes

$\prod_n \sqrt{-i}e^{i\theta/2}$ can be reabsorbed by volume-suppressed counterterm in potential. This gives

$$Z_\theta^{\text{Gaussian},\bar\phi}[T] = Z_E^{\text{Gaussian},\bar\phi}[\mathcal{T}]\Big|_{\mathcal{T}\to ie^{-i\theta}T}$$

Partition functions related by straightforward analytic continuation!

Decay rate from the optical theorem

As in Euclidean case, Z given by integration along steepest-descent flows passing through false vacuum and multi-bounces

$$\mathcal{N}^{-2}\langle F|e^{-ie^{-i\theta}HT_\theta}|F\rangle = Z_\theta^{\text{FV path}} \sim Z_\theta^{\text{Gaussian, FV}} + \exp\left(\frac{1}{2}Z_\theta^{\text{Gaussian, 1-bounce}}\right)$$

In terms of **scattering operator** for a **unit-norm vacuum state**

$$\langle F|e^{-ie^{-i\epsilon}HT_\theta}|F\rangle = 1 + i\mathcal{M}$$

For a stable vacuum there is no transition/bounce, so we have

$$\mathrm{Z}_\epsilon^{\text{Gaussian,FV}} = \mathcal{N}^{-2} \qquad i\mathcal{M} = \frac{1}{2}\frac{Z_\epsilon^{\text{Gaussian, 1-bounce}}}{Z_\epsilon^{\text{Gaussian,FV}}}$$

False-vacuum piece related to effective potential. For normalization constant $\mathcal{N}=1$

$$\mathrm{Z}_\epsilon^{\text{Gaussian,FV}} = e^{-iVV_{\text{CW}}(\phi_F)} = 1 \rightarrow V_{\text{CW}}(\phi_F) = 0$$

Effective potential at false vacuum fixed to zero by appropriate normalization!
(of field eigenstate $|\phi_F\rangle$ related to false vacuum)

Decay rate from the optical theorem

Total decay probability from optical theorem:

$$p = 2\mathrm{Im}\mathcal{M} = -2\mathrm{Re}(i\mathcal{M}) = -\mathrm{Re}\left(\frac{Z_\epsilon^{\text{Gaussian, 1-bounce}}}{Z_\epsilon^{\text{Gaussian,FV}}}\right)$$

Decay rate from probability per unit time and volume, using:

- Presence of one discrete negative mode around the bounce for arbitrary θ
- Ratio of determinants becomes T-independent and so matches Euclidean result

$$\gamma = \frac{p}{VT} = e^{-S_E[\bar{\phi}_E]}\left(\frac{S_E}{2\pi}\right)^2 \left|\frac{\det'(-\partial_\tau^2 - \vec{\nabla}^2 + V''(\phi_B))}{\det'(-\partial_\tau^2 - \vec{\nabla}^2 + V''(\phi_F))}\right|^{-1/2}$$

Real time tunneling calculation reproduces Euclidean result

Conclusions

Tunneling rates can be consistently calculated for any arbitrarily rotated time contour. We have a good understanding for QM and scalar fields in QFT in planar limit

The vacuum decay rate can be obtained from the optical-theorem for real-time amplitudes. The result reproduces the classic Euclidean calculation

Vacuum energy in flat space can be thought as being fixed to zero by demanding a proper normalization of the field/position eigenstate associated with the false vacuum