Conformal invariance versus Weyl invariance

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Motivation

In the UV masses should be unimportant → CFT

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Intuitive idea of scale invariance, no preferred length

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Intuitive idea of scale invariance, no preferred length

Three realizations of this symmetry

Scale invariance (in flat spacetime) $\underline{x^\mu \to \lambda\, x^\mu}$

Conformal invariance (in flat spacetime) Conformal group

Conformal invariance in curved space,
Weyl Symmetry $\underline{g_{\mu\nu} \to \Omega(x)^2\, g_{\mu\nu}}$

Motivation

Scale
invariance

Conformal
invariance

Weyl
invariance

Motivation

Scale invariance

Conformal invariance

Weyl invariance

Rosenfeld energy momentum tensor

(on-shell) $$\eta^{\mu\nu}T_{\mu\nu} = \underbrace{\partial_\mu V^\mu}_{\text{Virial}}$$

Conserved scale current

$$j^\mu = x^\lambda T^\mu_\lambda - V^\mu$$

[Callan, Coleman & Jackiw]

Motivation

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Conformal invariance

Weyl invariance

The virial is a total derivative

$$V^{\mu} = \partial_{\nu} \sigma^{\mu\nu}$$

Conserved current

$$K^{\mu\nu} = \left(2x^{\nu} x_{\rho} - x^{2} \delta^{\nu}_{\rho}\right) T^{\rho\mu} - 2x^{\nu} V^{\mu} + 2\sigma^{\mu\nu}$$

The EM tensor can be improved

$$\Theta^{\mu\nu} = T^{\mu\nu} + \frac{1}{2} \partial_{\lambda} \partial_{\rho} X^{\lambda\rho\mu\nu} \searrow X = X(\sigma)$$

Motivation

Scale invariance ⟷ **Conformal invariance**

Conditions for scale inv. to become conformal invariants [Nakayama]

d=2 scale inv. implies conformal invariance for local theories [Zamolodchikov] [Polchinski]

Counterexample 2d theory of elasticity [Riva & Cardy]

d=4 not known example of scale inv. but not conformal inv. [Coleman & Jackiw]

Weyl invariance

Motivation

Scale invariance ↔ **Conformal invariance**

These symmetries only make sense in flat spacetimes

What happens when gravity is present?

Weyl invariance

Motivation

Scale invariance

Conformal invariance

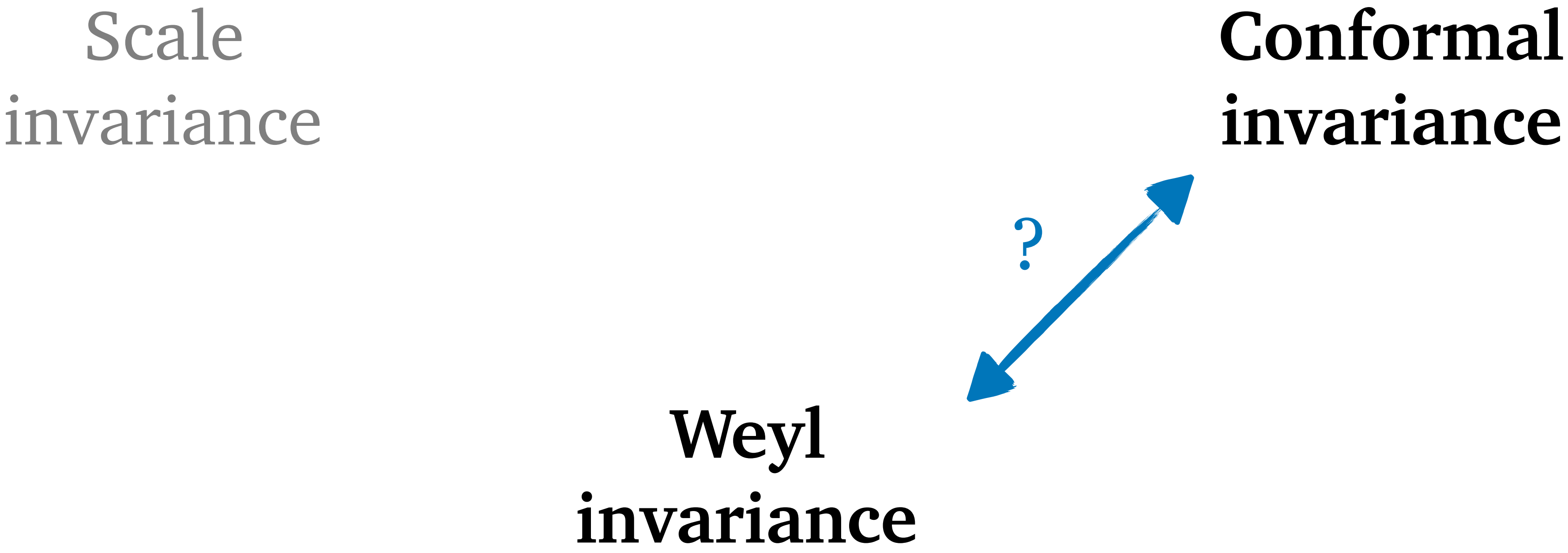
Weyl invariance

We can study this symmetry in the linear limit $g_{\mu\nu} = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}$

$$g_{\mu\nu} = \Omega(x)^2 g_{\mu\nu} \quad \rightarrow \quad \underline{\delta h_{\mu\nu} = \frac{2\omega(x)}{\kappa}\eta_{\mu\nu}}$$

Linearized symmetry LWeyl

Motivation



Scale invariance

Conformal invariance

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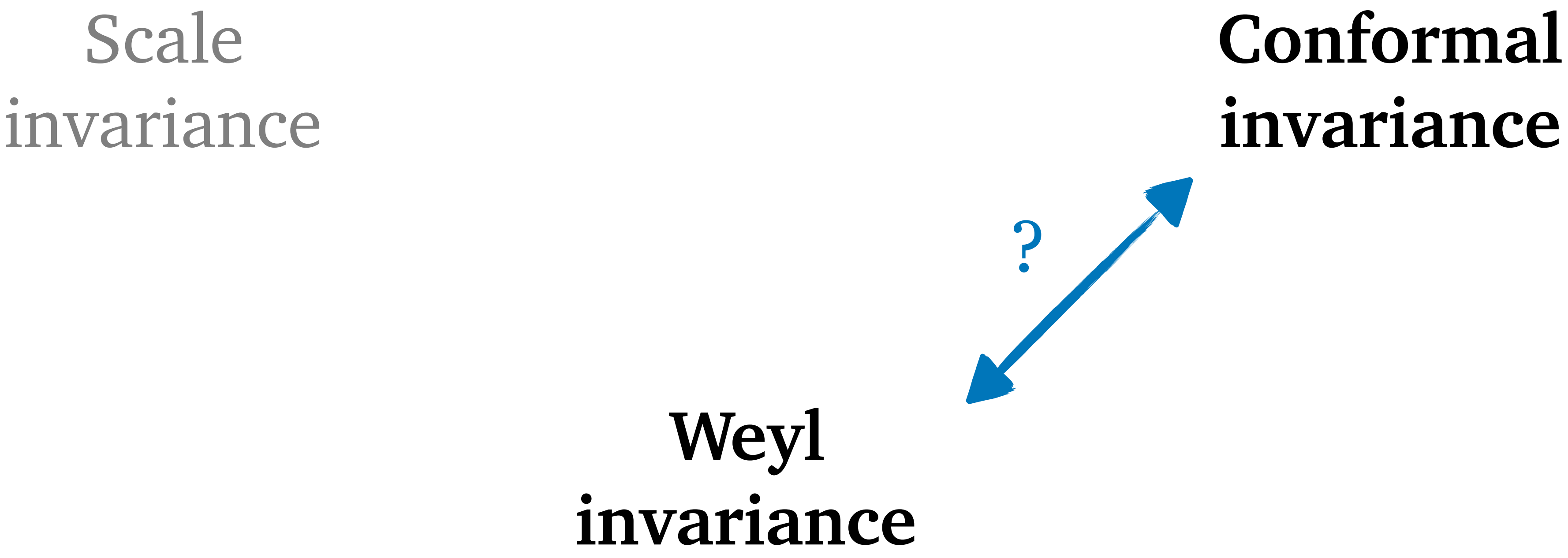
Weyl invariance

Motivation



Scale invariance

Conformal invariance

?

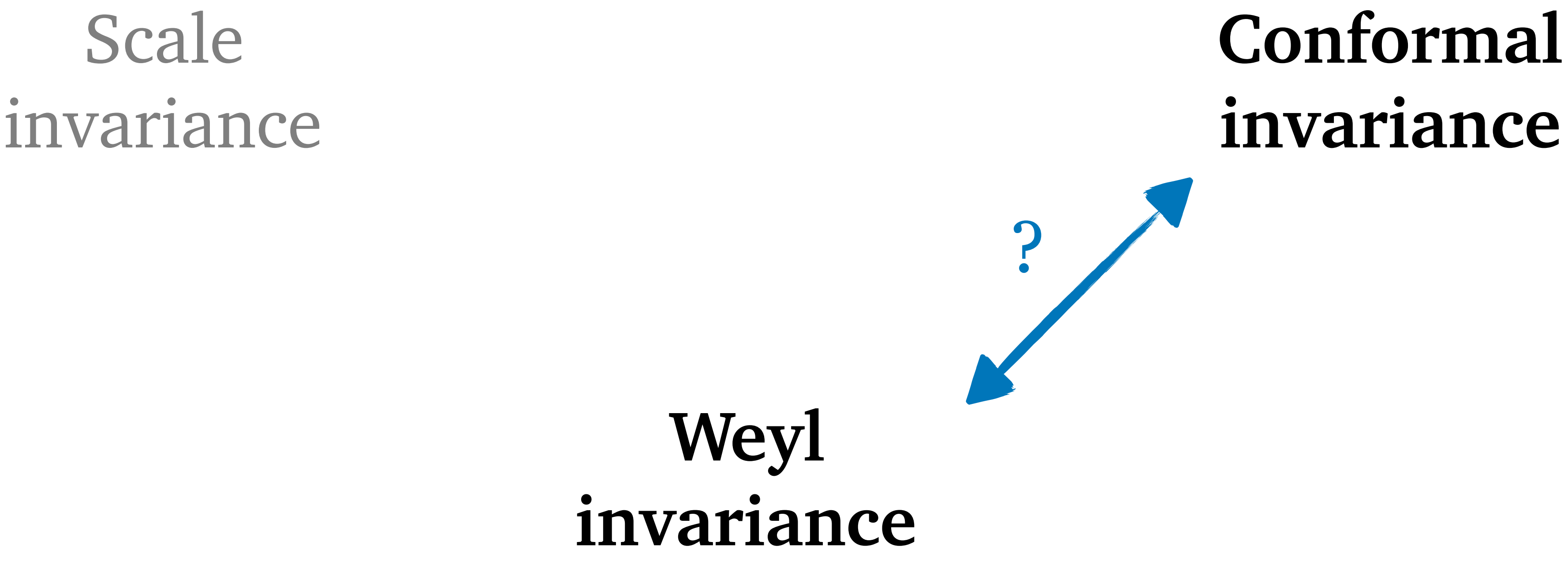
Weyl invariance

The scalar field conformal coupling to gravity is an easy example

Improvement of the EM tensor ↔ Conformal and Weyl invariant coupling of scalar field to gravity

$$\int d^4x \left\{ \frac{1}{6}\phi^2 R + (\partial_\mu \phi)^2 \right\}$$

Motivation



Scale
invariance

**Conformal
invariance**

?

**Weyl
invariance**

Our aim is to explore the difference between this symmetries for the spin 2 field

We study operators that come from the weak field expansion of linear and quadratic theories of gravity

Symmetries of the low energy spin 2 action

We study the gravitational field as a fluctuation around flat spacetime

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa\, \underline{h_{\mu\nu}}$$

Symmetric tensor representing
the graviton

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Symmetric tensor representing the graviton

Up to quadratic order in the fluctuations

Dimension 4 operators

Dimension 6 operators

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$$\left.\begin{aligned}
\mathscr{D}_1 &= \frac{1}{4}\partial_\mu h_{\nu\rho}\partial^\mu h^{\nu\rho} \\
\mathscr{D}_2 &= -\frac{1}{2}\partial^\lambda h_{\lambda\rho}\partial_\sigma h^{\rho\sigma} \\
\mathscr{D}_3 &= \frac{1}{2}\partial^\lambda h \partial^\sigma h_{\lambda\sigma} \\
\mathscr{D}_4 &= -\frac{1}{4}\partial_\mu h \partial^\mu h
\end{aligned}\right) \quad S_{lin.} = \int d^4x \sum_{i=1}^{4} \alpha_i \mathscr{D}_i$$

Dimension 6 operators

$$\left.\begin{aligned}
\mathscr{O}_1 &= h_{\alpha\beta}\left(\partial^\alpha\partial^\beta\partial^\gamma\partial^\delta\right) h_{\gamma\delta} \\
\mathscr{O}_2 &= h_{\alpha\beta}\left(\partial^\alpha\partial^\beta\eta^{\gamma\delta}\Box\right) h_{\gamma\delta} \\
\mathscr{O}_3 &= h_{\alpha\beta}\left(\partial^\alpha\partial^\gamma\eta^{\beta\delta}\Box\right) h_{\gamma\delta} \\
\mathscr{O}_4 &= h_{\alpha\beta}\left(\eta^{\alpha\gamma}\eta^{\beta\delta}\Box^2\right) h_{\gamma\beta} \\
\mathscr{O}_5 &= h_{\alpha\beta}\left(\eta^{\alpha\beta}\eta^{\gamma\delta}\Box^2\right) h_{\gamma\delta}
\end{aligned}\right) \quad S_{quad.} = \kappa^2 \int d^4x \sum_{i=1}^{4} g_i \mathscr{O}_i$$

[Alvarez, Blas, Garriga & Verdaguer]

Symmetries of the low energy spin 2 action

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$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa \underline{h_{\mu\nu}}$$

Symmetric tensor representing the graviton

Up to quadratic order in the fluctuations

Dimension 4 operators

Come from linearization of theories linear in curvature

$$\int d^4x \left(\sqrt{g}R\right)^{O(\kappa^2)}$$

Dimension 6 operators

Come from linearization of theories quadratic in curvature

$$\int d^4x \left[\sqrt{g}(\alpha R^2 + \beta R^2_{\mu\nu} + \gamma R^2_{\mu\nu\rho\sigma})\right]^{O(\kappa^2)}$$

Dimension 4 operators

LDiff	LTDiff	LWeyl	Conformal

Dimension 4 operators

LDiff LTDiff LWeyl Conformal

Invariance under $\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$

Conditions $\alpha_1 = \alpha_2, \quad \alpha_2 = \alpha_3, \quad \alpha_3 = \alpha_4$

Dimension 4 operators

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Invariance under $\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \quad \partial_\mu \xi^\mu = 0$

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Conditions $\alpha_1 = \alpha_2$

Invariance under $\delta h_{\mu\nu} = \dfrac{2\omega(x)}{\kappa} \eta_{\mu\nu}$

Conditions $\alpha_1 + \alpha_3 = n\alpha_4, \quad 2\alpha_2 = n\alpha_3$

Dimension 4 operators

LDiff LTDiff LWeyl Conformal

Only theory with LDiff and linear in curvature

$$S_{EH} = \int d^4x \left(\sqrt{g}R\right)^{O(\kappa^2)}$$

Dimension 4 operators

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LDiff and LWeyl symmetries are incompatible in n=4

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$$S_{EH} = \int d^4x \left(\sqrt{g}R\right)^{O(\kappa^2)}$$

LDiff and LWeyl symmetries are incompatible in n=4

LTDiff and LWeyl compatible giving

$$S_{WTDiff} = \int d^n x \left[g^{1/n} \left(R + \frac{(n-1)(n-2)}{4n^2} \frac{(\nabla g)^2}{g^2} \right) \right]^{O(\kappa^2)}$$

Description of Unimodular Gravity $S_{EH}[g^{-1/n} g_{\mu\nu}]$

Dimension 4 operators

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We can infer the theories that are invariant under this symmetries with the lowest order

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We need to see if we can write the EM tensor as $T = \partial_\mu \partial_\nu \sigma^{\mu\nu}$

In this case

$$\sigma^{\mu\nu} = \left(3 - \frac{n}{2}\right) \left\{ \frac{\alpha_1}{8} \eta^{\mu\nu} h^{\alpha\beta} h_{\alpha\beta} - \frac{\alpha_2}{4} h^{\rho\mu} h^{\nu}_{\rho} + \frac{\alpha_4}{4} h^{\mu\nu} h - \frac{\alpha_5}{8} \eta^{\mu\nu} h^2 \right\}$$

No constraints on the coupling constants

Dimension 4 operators

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At this order, LWeyl is more restrictive than Conformal invariance

Dimension 6 operators

LDiff LTDiff LWeyl Conformal

Invariance under $\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$

Conditions $2g_1 + g_2 + g_3 = 0, \quad g_2 + 2g_5 = 0, \quad g_3 + 2g_4 = 0$

Dimension 6 operators

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Dimension 6 operators

LDiff LTDiff LWeyl Conformal

Theories LDiff and quadratic in curvature

$$\int d^4x \left[\sqrt{g}(\alpha R^2 + \beta R^2_{\mu\nu} + \gamma R^2_{\mu\nu\rho\sigma})\right]^{O(\kappa^2)}$$

Dimension 6 operators

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$$\int d^4x \left[\sqrt{g}(\alpha R^2 + \beta R^2_{\mu\nu} + \gamma R^2_{\mu\nu\rho\sigma})\right]^{O(\kappa^2)}$$

LDiff and LWeyl symmetries can be compatible

$$\int d^n x \sqrt{g} \left\{ \left(\alpha - g_2 \frac{(n-1)(n-2)}{2(n-3)} \right) E_4 + g_2 \frac{(n-1)(n-2)}{2(n-3)} W_n \right\}$$

Dimension 6 operators

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Gauss-Bonet density

Weyl tensor squared

$$E_4 \equiv R^2_{\alpha\beta\gamma\delta} - 4\,R^2_{\alpha\beta} + R^2$$

$$W_n \equiv R^2_{\alpha\beta\gamma\delta} - \frac{4}{n-2}R^2_{\alpha\beta} + \frac{2}{(n-1)(n-2)}R^2$$

(total derivative)

Dimension 6 operators

LDiff **LTDiff** **LWeyl** Conformal

LTDiff and LWeyl compatible giving

$$S_{WTDiff} = \int d^n x \sqrt{\tilde{g}} \left\{ \alpha \tilde{R}_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma} + \beta \tilde{R}_{\mu\nu} \tilde{R}^{\mu\nu} + \gamma \tilde{R}^2 \right\}$$

where $\tilde{g}_{\mu\nu} = g^{-1/n} g_{\mu\nu}$

Dimension 6 operators

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where $\tilde{g}_{\mu\nu} = g^{-1/n} g_{\mu\nu}$

We need to see if we can write the EM tensor as $T = \partial_\mu \partial_\nu \sigma^{\mu\nu}$

In this case

$$\sigma^{\mu\nu} = \left(4 - \frac{n}{2}\right) \left\{ \frac{g_1}{2} \left(h^{\mu\alpha} \partial_\alpha \partial^\beta h^\nu_\beta + h^{\nu\alpha} \partial_\alpha \partial^\beta h^\mu_\beta \right) + \frac{g_2}{2} \left(h^{\mu\nu} \square h + h \square h^{\mu\nu} \right) + \frac{g_3}{2} \left(h^{\mu\lambda} \square h^\nu_\lambda + h^{\nu\lambda} \square h^\mu_\lambda \right) \right.$$
$$\left. + g_4 \eta^{\mu\nu} h_{\alpha\beta} \square h^{\alpha\beta} + g_5 \eta^{\mu\nu} h \square h \right\}$$

Dimension 6 operators

We cannot construct an scale invariant theory with dimension 4 and dimension 6 operators altogether

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For example the following Lagrangian is not even scale invariant

$$L' = \alpha\phi\Box\phi + \frac{\beta}{M^2}\phi\Box^2\phi = \alpha L_1 + \beta L_2$$

As we have

$$T = -\alpha\underline{\left(1-\frac{n}{2}\right)}L_1 - \frac{\beta}{M^2}\underline{\left(2-\frac{n}{2}\right)}L_2 \neq \partial_\mu V^\mu$$

Different factors

And the eom

$$\alpha\Box\phi + \frac{\beta}{M^2}\Box^2\phi = 0$$

Summary & Conclusions

	Weyl	Conformal
Dim. 4	Very restrictive	Off-shell, invariant in n=6
	No WDiff invariant theories	On-shell, no restriction on the coupling constants
	WTDiff invariant theories (UG description)	Improvement in any n
Dim. 6	Very restrictive	Off-shell, invariant in n=8
	Only one WDiff invariant theory	On-shell, no restriction on the coupling constants
	WTDiff invariant theories	Improvement in any n

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What about global Weyl invariance?

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Different Ward identities for Conformal and Weyl. Also, sensitive to the dynamical character of gravity?

Thank you!

Dimension 4 operators

LDiff LTDiff LWeyl **Conformal**

In this case

$$\sigma^{\mu\nu} = \left(3 - \frac{n}{2}\right)\left\{\frac{\alpha_1}{8}\eta^{\mu\nu}h^{\alpha\beta}h_{\alpha\beta} - \frac{\alpha_2}{4}h^{\rho\mu}h^{\nu}_{\rho} + \frac{\alpha_4}{4}h^{\mu\nu}h - \frac{\alpha_5}{8}\eta^{\mu\nu}h^2\right\}$$

We can find an improvement for any dimension

$$\Theta^{\mu\nu} = T^{\mu\nu} + \frac{1}{2}\partial_\lambda\partial_\rho X^{\lambda\rho\mu\nu} \longrightarrow \Theta = 0$$

$$X^{\lambda\rho\mu\nu} = g^{\lambda\rho}\sigma_+^{\mu\nu} - g^{\lambda\mu}\sigma_+^{\rho\nu} - g^{\lambda\nu}\sigma_+^{\mu\rho} + g^{\mu\nu}\sigma_+^{\lambda\rho} - \frac{1}{3}g^{\lambda\rho}g^{\mu\nu}\sigma^{\alpha}_{+\alpha} + \frac{1}{3}g^{\lambda\mu}g^{\rho\nu}\sigma^{\alpha}_{+\alpha}$$

Backup

$$0 = \delta Z = \int \mathscr{D} g_{\mu\nu} \mathscr{D} \phi \int d(vol)\, \omega(x) \left[-2 g^{\mu\nu} \frac{\delta S}{\delta g^{\mu\nu}} - \frac{n-2}{2} \phi \frac{\delta S}{\delta \phi} \right]$$

Unimodular Gravity (UG)

$$\begin{tikzcd}
S_{UG}[G] \arrow[r, "G_{\mu\nu} \;=\; g^{1/n}\, g_{\mu\nu}\;^{*}"] \arrow[d, "\delta G_{\mu\nu}"'] & S_{WTDiff}[g^{1/n}\, g_{\mu\nu}] \arrow[d, "\delta g_{\mu\nu}"] \\
\boxed{R[G]_{\mu\nu} - \frac{1}{n} R[G] G_{\mu\nu} = 0} \arrow[r, "G_{\mu\nu} \;=\; g^{1/n}\, g_{\mu\nu}\;^{*}"] \arrow[d, "\nabla_\nu R[G] = 2\nabla^\mu R_{\mu\nu}[G]"'] & \boxed{R[g]_{\mu\nu} - \frac{1}{n} R[g] g_{\mu\nu} = f(g)} \arrow[d, "\nabla_\nu R[g] = 2\nabla^\mu R_{\mu\nu}[g]"] \\
\boxed{R[G]_{\mu\nu} - \frac{1}{2}(R[G] - 2\lambda) G_{\mu\nu} = 0} \arrow[r, "G_{\mu\nu} \;=\; g^{1/n}\, g_{\mu\nu}\;^{*}"] & \boxed{R[g]_{\mu\nu} - \frac{1}{2}(R[g] g_{\mu\nu} - 2\lambda) = h(g)}
\end{tikzcd}$$

WTDiff reduces to UG in the gauge $g = 1$

* Non invertible

Motivation

Renormalization procedure → Introduction of an energy scale?

Flat spacetime

Breaking of scale invariance summarised in β-functions

Some known theories with vanishing β-functions ($\mathcal{N} = 2$ Supersymmetry)

↓

Curved spacetime (background)

Anomaly in the trace of the energy momentum tensor

Conformal anomaly

Duff, Capper, Christensen, Fulling, Desser, Isham, Englert, Truffin, Gastman, Berends, Vilkovisky, Fradkin, Tseytlin,...

$$T^{\mu}_{\mu} = a\left(W_4^2 + \frac{2}{3}\Box R\right) + b\,E_4 + c\,F^a_{\mu\nu}F_a^{\mu\nu}$$

Backup

Weyl operator in the Ward identity

$$0 = \delta Z = \int \mathscr{D} g_{\mu\nu} \mathscr{D}\phi \int d(vol)\, \omega(x) \left[-2g^{\mu\nu} \frac{\delta S}{\delta g^{\mu\nu}} - \frac{n-2}{2} \phi \frac{\delta S}{\delta \phi} \right]$$