Kaons on the lattice

Nicolas Garron

Birmingham, 3^{rd} December 2018,
New physics in kaon and beam-dump experiments

Introduction

Kaons are ideally suited for Lattice QCD

- Mesons: simpler and numerically cleaner than baryons
- Strange-light system: not too many different scales (compared eg. to charm and bottom)
- Can use the same discretisations for everything:
  light, strange, valence and sea quarks

Introduction

Kaons are ideally suited for Lattice QCD

- Mesons: simpler and numerically cleaner than baryons
- Strange-light system: not too many different scales (compared eg. to charm and bottom)
- Can use the same discretisations for everything:
  light, strange, valence and sea quarks

What's new ?

- Can now reach "physical" dynamical quark masses with various discretisations (Including Chiral fermions !)
- Inclusion of EM corrections in progress

Introduction

Kaons are ideally suited for Lattice QCD

- Mesons: simpler and numerically cleaner than baryons
- Strange-light system: not too many different scales (compared eg. to charm and bottom)
- Can use the same discretisations for everything:
  light, strange, valence and sea quarks

What's new ?

- Can now reach "physical" dynamical quark masses with various discretisations (Including Chiral fermions !)
- Inclusion of EM corrections in progress

Different lattice collaborations use $N_f = (2), 2+1, 2+1+1$ dynamical flavours with different discretisations

Outline

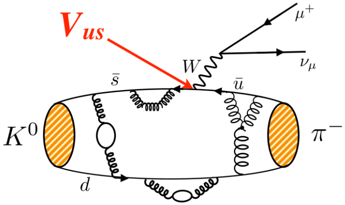
- Kl_3 and V_{us}
- Rare kaon decay $K \to \pi\nu\bar{\nu}$
- $K \to \pi\pi$ decay
- Kaon Mixing with and beyond the Standard Model
- Other perspective

K_{l3}

K_{l3} semileptonic form factor

Diagram from

[Aida X. El-Khadra @ Lattice2018]



Obtain $|V_{us} f_+(0)|$ from the experimental rate

$$\Gamma_{K\to\pi l\nu} = C_K^2 \frac{G_F^2 m_K^5}{192\pi^2} \, I \, S_{EW} \, \left[1 + 2\Delta_{SU(2)} + 2\Delta_{EM}\right] |V_{us} \, f_+(0)|^2$$

I is the phase space integral

$\Delta_{SU(2)}$ is the isospospin breaking correction

S_{EW} is the short distance electroweak correction

Δ_{EM} is the long distance electromagnetic correction

and $f_+(0)$ is the form factor we compute on the lattice

K_{l3} semileptonic form factor

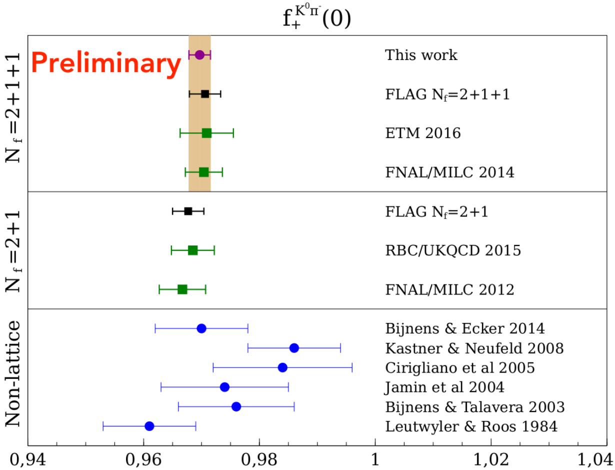
Obtain $|V_{us} f_+(0)|$ from the experimental rate

$$\Gamma_{K\to\pi l\nu} = C_K^2 \frac{G_F^2 m_K^5}{192\pi^2} \, I \, S_{EW} \, \left[1 + 2\Delta_{SU(2)} + 2\Delta_{EM}\right] |V_{us} \, f_+(0)|^2$$

$\Rightarrow$ determine $f_+(0)$ from the lattice to constraint V_{us}

K_{l3} semileptonic form factor II.

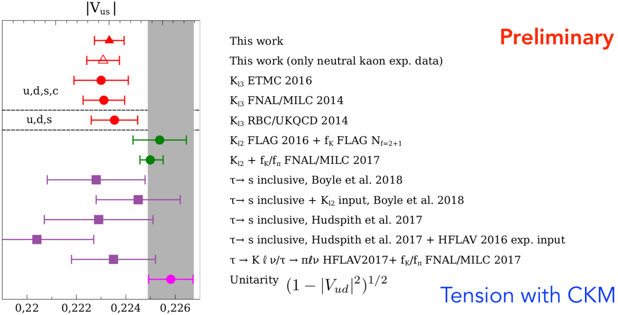
Talk from [Aida X. El-Khadra @ Lattice2018]



Preliminary results from Fermilab-MILC

K_{l3} semileptonic form factor II.

Talk from [Aida X. El-Khadra @ Lattice2018]



Tension with CKM unitarity: 2.2σ

Tensions with leptonic determinations:

- $\Gamma^{\text{exp}}_{K_{\ell 2}} + f_{K^\pm}$: 1.7σ
- $\Gamma^{\text{exp}}_{K_{\ell 2}} + f_{K^\pm}/f_{\pi^\pm} + |V_{ud}|$: 2.3σ

Preliminary results from Fermilab-MILC

K_{l3} semileptonic form factor

- Example of well-known quantity on the lattice
- Computed by many collaborations
- Allows for precision phenomenology
- All the effects/systematic erros have to be well under control
- Preliminary results from Fermilab-MILC find

$$\begin{aligned}\Delta_u &= |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 - 1 \\ &= -0.00151(38)_{f_+(0)}(35)_{f_K/f_\pi}(36)_{exp}(27)_{EM}\end{aligned}$$

Rare Kaon decay

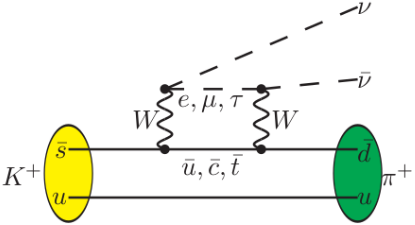
Rare kaon decay

Relevant for NA62

- $K \to \pi\nu\bar{\nu}$ or $K \to \pi l^+ l^-$
- FCNF, highly suppressed in the SM ($Br \sim 10^{-10}$), sensitivity to New-Physics
- $K \to \pi\nu\bar{\nu}$ is dominated by short-distance top-quark contribution
- But long-distance contribution from the charm is estimated to be of the same order as the SM uncertainty $(6 - 8\%)$
  [Isidori, Mescia, Smith '05, Buras, Buttazzo, Girrbach-Noe '15]
- Lattice exploratory studies of these long-distance contributions
  [Christ, Feng, Portelli, Sachrajda '16, Bai, Christ, Feng, Lawson, Portelli, Sachrajda '17]

Rare kaon decay

From [Xu Feng @Lattice 2017]



- Second order Weak interaction process
- Insertion of 2 Hamiltonian: $\Delta S = 1$ and $\Delta S = 0$
- Non-standard computation, requires new techniques to be developed
- Proof of concept and feasibility but no physical result yet

$K \to \pi\pi$ and CP violation

Background: Kaon decays and CP violation

- First discovery of CP violation was made in kaon system in 1964 (Christenson, Cronin, Fitch and Turlay)
- Noble prize in 1980 (Cronin and Fitch)
- Direct CP violation discovered in kaon decays at CERN and Fermilab

  [NAxx, KTeV '90-99] ... (Long story, controversies, drama, etc)
- Finally, very nice measurements, numbers from NA48 and KTeV:

$$\begin{cases} \textit{Indirect} \quad |\varepsilon| = (2.228 \pm 0.011) \times 10^{-3} \\ \textit{Direct} \quad Re\left(\frac{\varepsilon'}{\varepsilon}\right) = (1.66 \pm 0.23) \times 10^{-3} \\ \qquad\qquad\qquad\;\; = (1.65 \pm 0.26) \times 10^{-3} \quad \text{[PDG2018]} \end{cases}$$

Background: Kaon decays and CP violation

- Although very small effects, both direct and indirect CP violation are well established (experimentally) in $K \to \pi\pi$
- Expect sensitivity to New Physics
- Nice framework to test the Standard Model and constrain BSM theories

Background: Kaon decays and CP violation

- Although very small effects, both direct and indirect CP violation are well established (experimentally) in $K \to \pi\pi$
- Expect sensitivity to New Physics
- Nice framework to test the Standard Model and constrain BSM theories

What about the theoretical side ?

- ε and neutral kaon mixing "under control"

  SM and BSM contributions know with *decent* precision

- ε' and $K \to \pi\pi$: first "complete" computation only in 2015

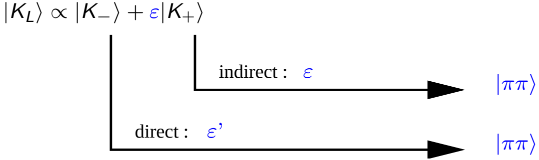
  Uncertainty on ε'/ε: Experiment $\sim 2 \times 10^{-4}$ (14%) vs Theory $(5-7) \times 10^{-4}$

Background: Kaon decays and CP violation

Flavour eigenstates $\begin{pmatrix} K^0 = \bar{s}\gamma_5 d \\ \overline{K}^0 = \bar{d}\gamma_5 s \end{pmatrix} \neq$ CP eigenstates $|K^0_\pm\rangle = \frac{1}{\sqrt{2}}\{|K^0\rangle \mp |\overline{K}^0\rangle\}$

They are mixed in the physical eigenstates $\begin{cases} |K_L\rangle & \sim & |K^0_-\rangle + \varepsilon|K^0_+\rangle \\ |K_S\rangle & \sim & |K^0_+\rangle + \varepsilon|K^0_-\rangle \end{cases}$

Direct and indirect CP violation in $K \to \pi\pi$



$K \to \pi\pi$ amplitudes

Two isospin channels: $\Delta I = 1/2$ and $\Delta I = 3/2$

$$K \to (\pi\pi)_{\mathrm{I}=0,2}$$

Corresponding amplitudes defined as

$$A[K \to (\pi\pi)_{\mathrm{I}}] = A_{\mathrm{I}} \exp(i\delta_{\mathrm{I}}) \qquad \text{/w } \mathrm{I} = 0, 2 \qquad \delta = \text{strong phases}$$

$K \to \pi\pi$ amplitudes

Two isospin channels: $\Delta I = 1/2$ and $\Delta I = 3/2$

$$K \to (\pi\pi)_{\mathrm{I}=0,2}$$

Corresponding amplitudes defined as

$$A[K \to (\pi\pi)_{\mathrm{I}}] = A_{\mathrm{I}} \exp(i\delta_{\mathrm{I}}) \qquad /\text{w } \mathrm{I} = 0, 2 \qquad \delta = \text{strong phases}$$

$\Rightarrow$ Need to compute the complex amplitudes A_0 and A_2

$\Delta I = 1/2$ rule

- Experimentally we find

$$\omega = \frac{\mathrm{Re} A_2}{\mathrm{Re} A_o} \sim 1/22$$

$\Delta I = 1/2$ rule

- Experimentally we find

$$\omega = \frac{\mathrm{Re}A_2}{\mathrm{Re}A_o} \sim 1/22$$

- Whereas "naive" theoretical estimate gives $1/2$

  $\Rightarrow$ Very long-standing puzzle, see e.g. [Gaillard & Lee '74, Altarelli & Maiani '74]

$\Delta I = 1/2$ rule

- Experimentally we find

$$\omega = \frac{\mathrm{Re}A_2}{\mathrm{Re}A_o} \sim 1/22$$

- Whereas "naive" theoretical estimate gives $1/2$

  $\Rightarrow$ Very long-standing puzzle, see e.g. [Gaillard & Lee '74, Altarelli & Maiani '74]

- Can it be explained by large non-perturbative QCD effects ?

- Still not yet completely understood

  Important progress have been made, in particular by RBC-UKQCD

- Note that the for the estimate of ϵ'/ϵ the experimental value of ω is used

We can derive the approximate formulae (see eg [De Rafael @ TASI'94]) (in the isospin limit)

$$\varepsilon' = \frac{i\omega \exp(i\delta_2 - \delta_0)}{\sqrt{2}} \left[\frac{\mathrm{Im}A_2}{\mathrm{Re}A_2} - \frac{\mathrm{Im}A_0}{\mathrm{Re}A_0} \right]$$

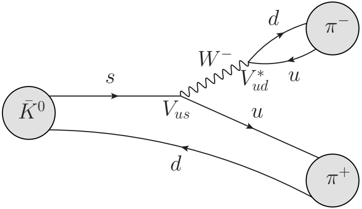
$$\varepsilon = e^{i\phi_\varepsilon} \left[\frac{\mathrm{Im}\langle \bar{K}^0 | H_{\mathrm{eff}}^{\Delta S=2} | K^0 \rangle}{\Delta m_K} + \frac{\mathrm{Im}A_0}{\mathrm{Re}A_0} \right]$$

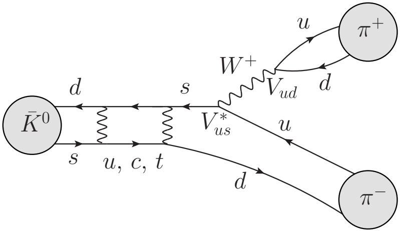
$\Rightarrow$ Related to $K^0 - \bar{K}^0$ mixing

$K \to \pi\pi$ amplitudes and $K - \bar{K}$ mixing

CP violation related to $\Delta S = 1$ and $\Delta S = 2$ processes

- Kaon decay $\Delta S = 1 : K \to \pi\pi$
- Neutral Kaon mixing $\Delta S = 2 : K \leftrightarrow \bar{K}$



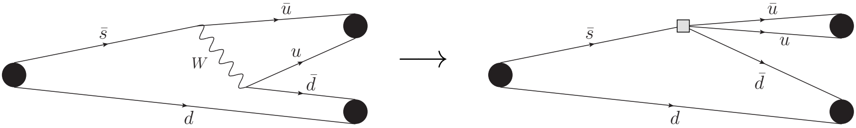


Figures from [Lellouch@ Les Houches'09]

$K \to \pi\pi$ Overview

Overview of the computation

Operator Product expansion

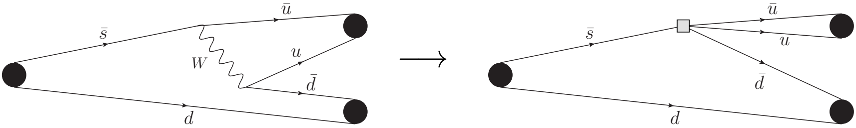


Describe $K \to (\pi\pi)_{\mathrm{I}=0,2}$ with an effective Hamiltonian [Ciuchini et al' 94, Buchalla, Buras, Lautenbacher '96]

$$H^{\Delta s=1} = \frac{G_F}{\sqrt{2}} \Big\{ \sum_{i=1}^{10} \big(V_{ud} V_{us}^* z_i(\mu) - V_{td} V_{ts}^* y_i(\mu) \big) Q_i(\mu) \Big\}$$

Overview of the computation

Operator Product expansion



Describe $K \to (\pi\pi)_{\mathrm{I}=0,2}$ with an effective Hamiltonian [Ciuchini et al' 94, Buchalla, Buras, Lautenbacher '96]

$$H^{\Delta s=1} = \frac{G_F}{\sqrt{2}} \Big\{ \sum_{i=1}^{10} \big(V_{ud} V_{us}^* z_i(\mu) - V_{td} V_{ts}^* y_i(\mu) \big) Q_i(\mu) \Big\}$$

Amplitude given by $A \propto \langle \pi\pi | H^{\Delta s=1} | K \rangle$

Short distance effects factorized in the Wilson coefficients y_i, z_i

Long distance effects factorized in the matrix elements

$\langle \pi\pi | Q_i(\mu) | K \rangle \longrightarrow$ task for the Lattice

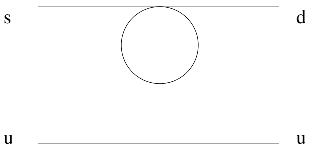
Isospin channels

10 four-quark operators, actually reduces to 7 in four-dimention

Only 3 of these operators contribute to the $\Delta I = 3/2$ channel

- A tree-level operator
- 2 electroweak penguins

No disconnect graphs contribute to the $\Delta I = 3/2$ channel



$\Rightarrow A_2$ is much simpler than A_0

$K \to \pi\pi$ Lattice results

$K \to (\pi\pi)_{I=2}$ Results

- First computation (2012): Physical kinematic, Near physical pion mass

  But only one coarse lattice spacing

  IDSDR $32^3 \times 64$, with $a^{-1} \sim 1.37\ \mathrm{GeV} \Rightarrow a \sim 0.14\ \mathrm{fm}$, $L \sim 4.6\ \mathrm{fm}$

$K \to (\pi\pi)_{I=2}$ Results

- First computation (2012): Physical kinematic, Near physical pion mass

  But only one coarse lattice spacing

  IDSDR $32^3 \times 64$, with $a^{-1} \sim 1.37\ \mathrm{GeV} \Rightarrow a \sim 0.14\ \mathrm{fm}$, $L \sim 4.6\ \mathrm{fm}$

- Latest computation (2015)

  Two lattice spacing, $n_f = 2+1$, large volume at the physical point

  New discretisation of the Domain-Wall fermion forumlation:
  Möbius Fermions [Brower, Neff, Orginos '12]

  - $48^3 \times 96$, with $a^{-1} \sim 1.73\ \mathrm{GeV} \Rightarrow a \sim 0.11\ \mathrm{fm}$, $L \sim 5.5\ \mathrm{fm}$
  - $64^3 \times 128$ with $a^{-1} \sim 2.36\ \mathrm{GeV} \Rightarrow a \sim 0.084\ \mathrm{fm}$, $L \sim 5.4\ \mathrm{fm}$
  - $am_{res} \sim 10^{-4}$

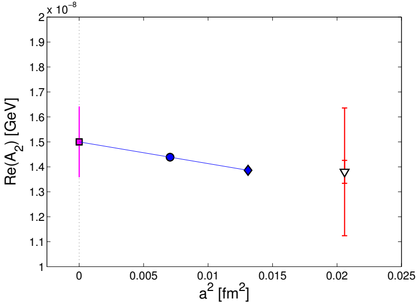
$K \to (\pi\pi)_{I=2}$ 2015 Results

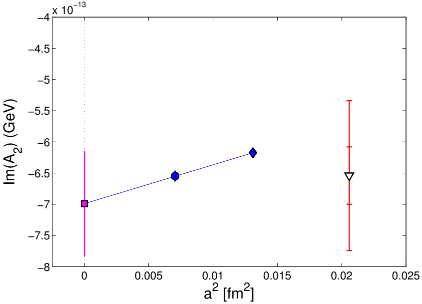
2012 [Blum, Boyle, Christ, N.G.,Goode, Izubuchi, Jung, Kelly, Lehner, Lightman, Liu, Lytle, Mawhinney, Sachrajda, Soni, Sturm, *PRL'12, PRD'12*]

$\mathrm{Re}\,A_2 = 1.381(46)_{\rm stat}(258)_{\rm syst}\,10^{-8}\,\mathrm{GeV}$ $\quad$ $\mathrm{Im}\,A_2 = -6.54(46)_{\rm stat}(120)_{\rm syst}10^{-13}\,\mathrm{GeV}$

2015 [Blum, Boyle, Christ, Frison, N.G., Janowski, Jung, Kelly, Lehner, Lytle, Mawhinney, Sachrajda, Soni, Hin, Zhang, *PRD'15*]

$\mathrm{Re}\,A_2 = 1.50(4)_{\rm stat}(14)_{\rm syst}\,10^{-8}\,\mathrm{GeV}$ $\quad$ $\mathrm{Im}\,A_2 = -6.99(20)_{\rm stat}(84)_{\rm syst}10^{-13}\,\mathrm{GeV}$





A_0, 2015

- First complete computation of the matrix elements $\langle \pi\pi | Q_i K \rangle$ (both isospin channel) with physical kinematics and quark masses

  [Bai, Blum, Boyle, Christ, Frison, N.G., Izubuchi, Jung, Kelly, Lehner, Mawhinney, Sachrajda, Soni, Zhang PRL'15]

- Pion mass $m_\pi = 143.1(2.0)\ \mathrm{MeV}$, single lattice spacing $a \sim 0.14\ \mathrm{fm}$

  Kaon mass $m_K = 490.6(2.4)\ \mathrm{MeV}$

- Physical kinematics achieved with G-Parity boundary conditions

  [Kim, Christ, '03 and '09]

- Requires algorithmic development, dedicated generation of gauge configurations, ...

- See talk by C.Kelly and proceeding from Lattice'14

A_0, 2015

- First complete computation of the matrix elements $\langle \pi\pi | Q_i K \rangle$ (both isospin channel) with physical kinematics and quark masses

  [Bai, Blum, Boyle, Christ, Frison, N.G., Izubuchi, Jung, Kelly, Lehner, Mawhinney, Sachrajda, Soni, Zhang PRL'15]

- Pion mass $m_\pi = 143.1(2.0)\ \mathrm{MeV}$, single lattice spacing $a \sim 0.14\ \mathrm{fm}$

  Kaon mass $m_K = 490.6(2.4)\ \mathrm{MeV}$

- Physical kinematics achieved with G-Parity boundary conditions

  [Kim, Christ, '03 and '09]

- Requires algorithmic development, dedicated generation of gauge configurations, ...

- See talk by C.Kelly and proceeding from Lattice'14

Another computation, [Ishizuka, Ishikawa, Ukawa, Yoshié '15] with Wilson fermions at threshold (unphysical kinematics)

A_0, 2015 update

Renormalisation at $\mu \sim 1.5\ \mathrm{GeV}$, combine with the Wilson coefficients

i	Re(A_0)(GeV)	Im(A_0)(GeV)
1	$1.02(0.20)(0.07) \times 10^{-7}$	0
2	$3.63(0.91)(0.28) \times 10^{-7}$	0
3	$-1.19(1.58)(1.12) \times 10^{-10}$	$1.54(2.04)(1.45) \times 10^{-12}$
4	$-1.86(0.63)(0.33) \times 10^{-9}$	$1.82(0.62)(0.32) \times 10^{-11}$
5	$-8.72(2.17)(1.80) \times 10^{-10}$	$1.57(0.39)(0.32) \times 10^{-12}$
6	$3.33(0.85)(0.22) \times 10^{-9}$	$-3.57(0.91)(0.24) \times 10^{-11}$
7	$2.40(0.41)(0.00) \times 10^{-11}$	$8.55(1.45)(0.00) \times 10^{-14}$
8	$-1.33(0.04)(0.00) \times 10^{-10}$	$-1.71(0.05)(0.00) \times 10^{-12}$
9	$-7.12(1.90)(0.46) \times 10^{-12}$	$-2.43(0.65)(0.16) \times 10^{-12}$
10	$7.57(2.72)(0.71) \times 10^{-12}$	$-4.74(1.70)(0.44) \times 10^{-13}$
Tot	$4.66(0.96)(0.27) \times 10^{-7}$	$-1.90(1.19)(0.32) \times 10^{-11}$
Exp	$3.3201(18) \times 10^{-7}$	-

A_0, 2015 update

Renormalisation at $\mu \sim 1.5\ \mathrm{GeV}$, combine with the Wilson coefficients

i	Re(A_0)(GeV)	Im(A_0)(GeV)
1	$1.02(0.20)(0.07) \times 10^{-7}$	0
2	$3.63(0.91)(0.28) \times 10^{-7}$	0
3	$-1.19(1.58)(1.12) \times 10^{-10}$	$1.54(2.04)(1.45) \times 10^{-12}$
4	$-1.86(0.63)(0.33) \times 10^{-9}$	$1.82(0.62)(0.32) \times 10^{-11}$
5	$-8.72(2.17)(1.80) \times 10^{-10}$	$1.57(0.39)(0.32) \times 10^{-12}$
6	$3.33(0.85)(0.22) \times 10^{-9}$	$-3.57(0.91)(0.24) \times 10^{-11}$
7	$2.40(0.41)(0.00) \times 10^{-11}$	$8.55(1.45)(0.00) \times 10^{-14}$
8	$-1.33(0.04)(0.00) \times 10^{-10}$	$-1.71(0.05)(0.00) \times 10^{-12}$
9	$-7.12(1.90)(0.46) \times 10^{-12}$	$-2.43(0.65)(0.16) \times 10^{-12}$
10	$7.57(2.72)(0.71) \times 10^{-12}$	$-4.74(1.70)(0.44) \times 10^{-13}$
Tot	$4.66(0.96)(0.27) \times 10^{-7}$	$-1.90(1.19)(0.32) \times 10^{-11}$
Exp	$3.3201(18) \times 10^{-7}$	-

A_0, 2015 update

Renormalisation at $\mu \sim 1.5\ \mathrm{GeV}$, combine with the Wilson coefficients

i	Re(A_0)(GeV)	Im(A_0)(GeV)
1	$1.02(0.20)(0.07) \times 10^{-7}$	0
2	$3.63(0.91)(0.28) \times 10^{-7}$	0
3	$-1.19(1.58)(1.12) \times 10^{-10}$	$1.54(2.04)(1.45) \times 10^{-12}$
4	$-1.86(0.63)(0.33) \times 10^{-9}$	$1.82(0.62)(0.32) \times 10^{-11}$
5	$-8.72(2.17)(1.80) \times 10^{-10}$	$1.57(0.39)(0.32) \times 10^{-12}$
6	$3.33(0.85)(0.22) \times 10^{-9}$	$-3.57(0.91)(0.24) \times 10^{-11}$
7	$2.40(0.41)(0.00) \times 10^{-11}$	$8.55(1.45)(0.00) \times 10^{-14}$
8	$-1.33(0.04)(0.00) \times 10^{-10}$	$-1.71(0.05)(0.00) \times 10^{-12}$
9	$-7.12(1.90)(0.46) \times 10^{-12}$	$-2.43(0.65)(0.16) \times 10^{-12}$
10	$7.57(2.72)(0.71) \times 10^{-12}$	$-4.74(1.70)(0.44) \times 10^{-13}$
Tot	$4.66(0.96)(0.27) \times 10^{-7}$	$-1.90(1.19)(0.32) \times 10^{-11}$
Exp	$3.3201(18) \times 10^{-7}$	-

Standard Model "prediction" for ε'/ε

ε'/ε can be computed from

$$Re(\varepsilon'/\varepsilon) = Re\left\{\frac{i\omega\exp(i\delta_2-\delta_0)}{\sqrt{2}\varepsilon}\left[\frac{\mathrm{Im}(A_2)}{\mathrm{Re}A_2}-\frac{\mathrm{Im}A_0}{\mathrm{Re}A_0}\right]\right\}$$

Combining our new value of $\mathrm{Im}A_0$ and δ_0 with

- our continuum value for $\mathrm{Im}A_2$
- the experimental value for $\mathrm{ReA_0}$, $\mathrm{ReA_2}$ and their ratio ω

we find

$$Re(\varepsilon'/\varepsilon) = 1.38(5.15)(4.43)\times 10^{-4}$$

Standard Model "prediction" for ε'/ε

we find

$$Re(\varepsilon'/\varepsilon) = 1.38(5.15)(4.43) \times 10^{-4}$$

The experimental value (average) is $Re(\varepsilon'/\varepsilon) = 16.6(2.3) \times 10^{-4}$

- Agreement only approximate $\sim 2.1\sigma$,
- Our error is $\sim$ 3 times larger than the experimental one
- But can be systematically reduced

Standard Model "prediction" for ε'/ε

- The experimental value (average) is $Re(\varepsilon'/\varepsilon) = 16.6(2.3) \times 10^{-4}$
- Our result is $Re(\varepsilon'/\varepsilon) = 1.38(5.15)(4.43) \times 10^{-4}$
- [Buras, Gorbahn, Jağer, Jamin '15] combine our results for the matrix elements in a different way and find $Re(\varepsilon'/\varepsilon) = 1.9(4.5) \times 10^{-4}$, ie $\sim 2.9\sigma$
- Another analysis [Kitahara, Nierste, Tremper '16] using new RGE for the Wilson coefficients and our results for the matrix elements finds $1.06(5.07) \times 10^{-4}$, which is $\sim 2.8\sigma$

Standard Model "prediction" for ε'/ε

- The experimental value (average) is $Re(\varepsilon'/\varepsilon) = 16.6(2.3) \times 10^{-4}$
- Our result is $Re(\varepsilon'/\varepsilon) = 1.38(5.15)(4.43) \times 10^{-4}$
- [Buras, Gorbahn, Jağer, Jamin '15] combine our results for the matrix elements in a different way and find $Re(\varepsilon'/\varepsilon) = 1.9(4.5) \times 10^{-4}$, ie $\sim 2.9\sigma$
- Another analysis [Kitahara, Nierste, Tremper '16] using new RGE for the Wilson coefficients and our results for the matrix elements finds $1.06(5.07) \times 10^{-4}$, which is $\sim 2.8\sigma$
- Another improvement on the Wilson coefficient on the way
  [Cerdà-Sevilla, Gorbahn, Jäger, Kokulu @ Kaon 2016]

ε'/ε Theory vs Theory vs Experiment

Recent updates

- [Gisbert & Pich Rept.Prog.Phys December 2017, QCD'18] claim that *long-distance re-scattering [effect] of the final pions in* $K \to \pi\pi$ were neglected

  After corrections

  $$\mathrm{Re}(\varepsilon'/\varepsilon) = 15 \pm 7 \times 10^{-4}$$

  in complete agreement with the SM

ε'/ε Theory vs Theory vs Experiment

Recent updates

- [Gisbert & Pich Rept.Prog.Phys December 2017, QCD'18] claim that *long-distance re-scattering [effect] of the final pions in* $K \to \pi\pi$ were neglected

  After corrections
  $$\mathrm{Re}(\varepsilon'/\varepsilon) = 15 \pm 7 \times 10^{-4}$$
  in complete agreement with the SM

- Phase shift puzzle ?

The phase shift puzzle

See [C.Kelly and T. Wang @Lattice2018] 2015 results

- For $(\pi\pi)_{I=2}$ we find $\delta_2 = -11.0(0.3)^o$
- For $(\pi\pi)_{I=0}$ we find $\delta_0 = 23.8(5.2)^o$

δ_0 differs from the dispersive approach see e.g. [Colangelo, Gasser, Leutwyler '01, Colangelo, Passemar, Stoffer '15]

$$\delta_2 = -11.4(?) \text{ and } \delta_0 = 35.0(?)$$

$\Rightarrow$ Is there a issue there ?

The phase shift puzzle

See [C.Kelly and T. Wang @Lattice2018] 2015 results

- For $(\pi\pi)_{I=2}$ we find $\delta_2 = -11.0(0.3)^o$
- For $(\pi\pi)_{I=0}$ we find $\delta_0 = 23.8(5.2)^o$

δ_0 differs from the dispersive approach see e.g. [Colangelo, Gasser, Leutwyler '01, Colangelo, Passemar, Stoffer '15]

$$\delta_2 = -11.4(?) \text{ and } \delta_0 = 35.0(?)$$

$\Rightarrow$ Is there a issue there ?

New analysis (RBC-UKQCD 2018) $\delta_2 = -11.3(0.1)$ and $\delta_0 \sim 31 - 34(??)$

The phase shift puzzle

See [C.Kelly and T. Wang @Lattice2018] 2015 results

- For $(\pi\pi)_{I=2}$ we find $\delta_2 = -11.0(0.3)^o$
- For $(\pi\pi)_{I=0}$ we find $\delta_0 = 23.8(5.2)^o$

δ_0 differs from the dispersive approach see e.g. [Colangelo, Gasser, Leutwyler '01, Colangelo, Passemar, Stoffer '15]

$$\delta_2 = -11.4(?) \text{ and } \delta_0 = 35.0(?)$$

$\Rightarrow$ Is there a issue there ?

New analysis (RBC-UKQCD 2018) $\delta_2 = -11.3(0.1)$ and $\delta_0 \sim 31 - 34(??)$

This change is due to the presence of a close excited state

The effect on the matrix elements is currently under investigation

Neutral kaon mixing

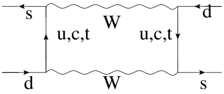
Based on work done in collaboration with [Boyle, Hudspith, Lytle]

and now also with [Kettle, Soni, Tsang]

Neutral kaon mixing in the SM

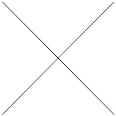
Indirect CP violation related to neutral kaon oscillations

in the SM this occurs though box diagrams with W exchange



$\mu \ll \overrightarrow{M_W}$

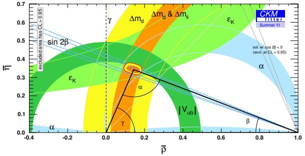
OPE



Factorise the non-perturbative contribution into

$$\langle \overline{K}^0 | \mathcal{O}_{LL}^{\Delta S=2}(\mu) | K^0 \rangle = \frac{8}{3} F_K^2 M_K^2 B_K(\mu) \qquad \text{w/ } \mathcal{O}_{LL}^{\Delta S=2} = (\bar{s}\gamma_\mu(1-\gamma_5)d)(\bar{s}\gamma^\mu(1-\gamma_5)d)$$

Related to ε via CKM parameters, schematically $\varepsilon \propto V_{\mathrm{CKM}} \times C(\mu) \times B_K(\mu)$



and beyond

In the SM, neutral kaon mixing occurs through W-exchanges $\rightarrow$ $(V-A)$

$$O_1^{\Delta s=2} = (\bar{s}\,(V-A)\,d)\,(\bar{s}\,(V-A)\,d)$$

Beyond the SM, other Dirac structure appear in the generic Hamiltonian

$$H^{\Delta s=2} = \sum_{i=1}^{5} C_i(\mu)\, O_i^{\Delta s=2}(\mu)\,.$$

We express them in terms of Lorentz matrices Vector, Axial, Scalar, Pseudo-scalar, Tensor

$$\begin{array}{l} (V-A)\times(V+A) \\ (S-P)\times(S+P) \\ (S-P)\times(S-P) \\ TT\times TT \end{array}$$

On the lattice, we compute $\langle \bar{K}^0 | O_i^{\Delta s=2} | K^0 \rangle$

B_K SM kaon mixing - Results

FLAG 2013 quotes an error of 1.3% dominated by the perturbative matching

Most recent determinations, in $\overline{\mathrm{MS}}$ at 3 GeV, $B_K^{\overline{\mathrm{MS}}}(3\mathrm{GeV})$

Collaboration	N_f	Discretisation	Result
RBC-UKQCD	$2+1$	Domain-Wall	$0.5293(17)_{stat+syst}(106)_{PT}$
SWME	$2+1$	Staggered	$0.518(3)_{stat}(26)_{syst}$
ETM	$2+1+1$	Twisted Mass	$0.506(17)_{stat+syst}(3)_{PT}$

B_K SM kaon mixing - Results

FLAG 2013 quotes an error of 1.3% dominated by the perturbative matching

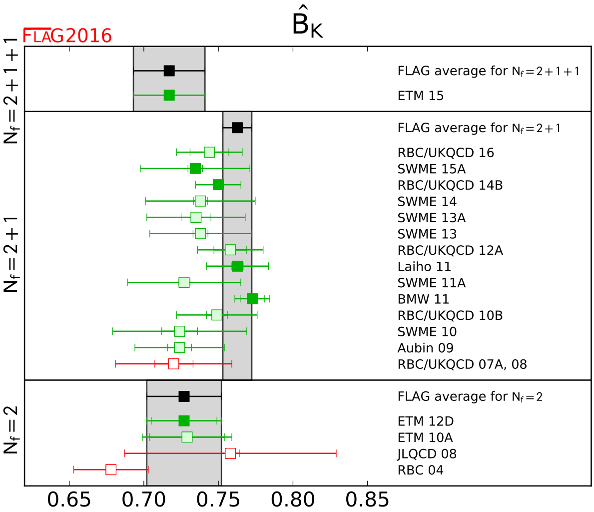
Most recent determinations, in $\overline{\mathrm{MS}}$ at 3 GeV, $B_K^{\overline{\mathrm{MS}}}(3\mathrm{GeV})$

Collaboration	N_f	Discretisation	Result
RBC-UKQCD	$2+1$	Domain-Wall	$0.5293(17)_{stat+syst}(106)_{PT}$
SWME	$2+1$	Staggered	$0.518(3)_{stat}(26)_{syst}$
ETM	$2+1+1$	Twisted Mass	$0.506(17)_{stat+syst}(3)_{PT}$

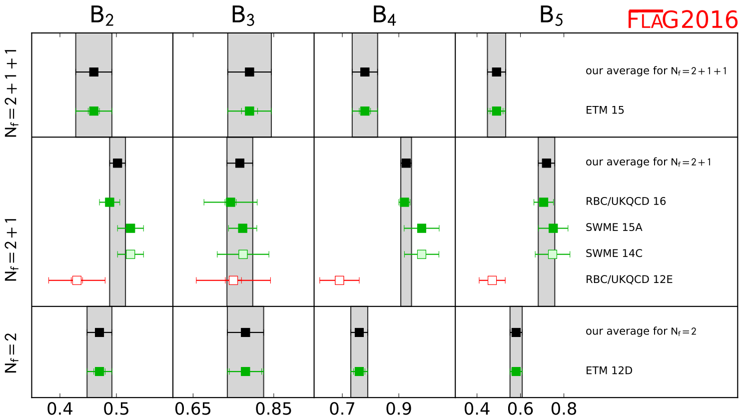
Note that the conversion Lattice $\rightarrow \overline{\mathrm{MS}}$ is only permformed at 1-loop in PT

But 2-loop on the way see [Jäger & Kvedaraite @ Lattice 2018]

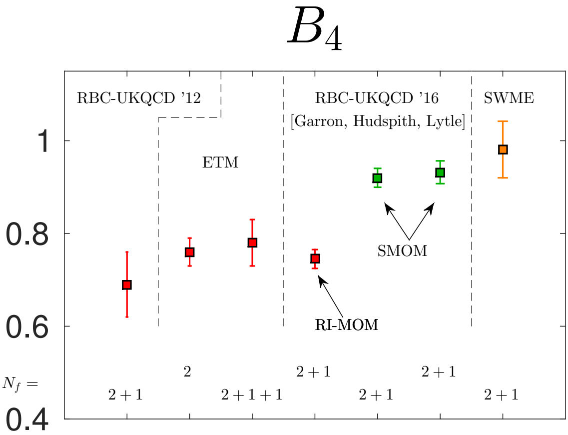
B_K SM kaon mixing - Results



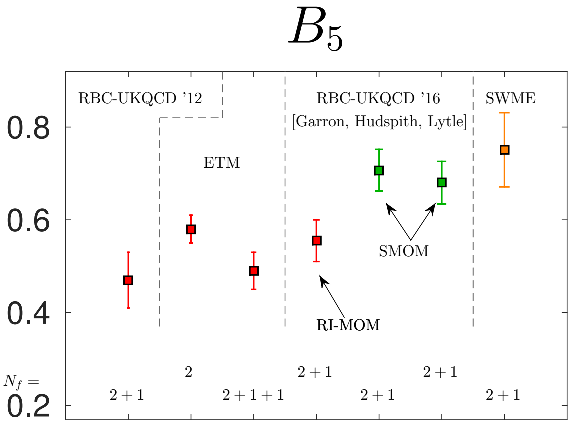
BSM kaon mixing - Results



BSM kaon mixing - Results



BSM kaon mixing - Results



Other perspectives

- QCD+QED: Huge effort (BMWc, ETMc, QCDSF, RBC-UKQCD, ...)

  Applications to decay amplitudes, K_{l2}, K_{l3} ...

  See e.g. [Sachrajda @ Lattice2018]

  and to $K \to \pi\pi$, see [Christ & Feng @ Lattice2017]

Other perspectives

- Improving the interface Lattice/Phenomenology

  Schematically

$$\text{experimental value} \sim \sum_i \underbrace{C_i(\mu)}_{PT} \times \underbrace{\langle O_i(\mu) \rangle}_{Lattice}$$

  - Matching Lattice/Pheno: $Lattice \overset{NPR}{\rightarrow}$ intermediate renorm. scheme $\overset{PT}{\rightarrow} \overline{\mathrm{MS}}$
  - Matching to $N_f = 3$ requires PT to be under control at $\mu \sim m_c$

Other perspectives

- Improving the interface Lattice/Phenomenology

  Schematically

$$\text{experimental value} \sim \sum_i \underbrace{C_i(\mu)}_{PT} \times \underbrace{\langle O_i(\mu)\rangle}_{Lattice}$$

  - Matching Lattice/Pheno: $Lattice \overset{NPR}{\rightarrow}$ intermediate renorm. scheme $\overset{PT}{\rightarrow} \overline{\mathrm{MS}}$
  - Matching to $N_f = 3$ requires PT to be under control at $\mu \sim m_c$

  Several improvement in progress

  - Higher order in PT, see Jäger & Kvedaraite @ Lattice 2018 for B_K
  - Better(?) NPR schemes, [Cahill, NG, Gorbahn, Gracey, Rakow, . . .]
  - Non-perturbative computation of the Wilson coefficient [Bruno @ Lattice 2017]
  - Renormalisation in position space [Tomi @ Lattice 2018]
  - . . .

Conclusions & Outlook

Lattice community is very active in the Kaon area

- Some observables are known with very good precisions and provide important checks of the SM and conatraints on BSM theories (ex: V_{us}
  - Dynamical fermions $N_f = 2, 2+1, 2+1+1$ flavours
  - Physical quark masses, several lattice spacings, large volume etc.
  - Several discretisation, including chiral fermions
  - Huge effort to incorporate QED effects
- New quantities, non-standard
  - New Last 5-8 years have seen tremendous progress in $K \to \pi\pi$ decays and $K - \bar{K}$ mixing
  - Progress toward long-distance contribution to $K \to \pi\nu\bar{\nu}$
- Improving the connection Lattice / Phenomenology

The RBC & UKQCD collaborations

BNL and BNL/RBRC

Yasumichi Aoki (KEK)
Mattia Bruno
Taku Izubuchi
Yong-Chull Jang
Chulwoo Jung
Christoph Lehner
Meifeng Lin
Aaron Meyer
Hiroshi Ohki
Shigemi Ohta (KEK)
Amarjit Soni

UC Boulder

Oliver Witzel

Columbia University

Ziyuan Bai
Norman Christ
Duo Guo
Christopher Kelly
Bob Mawhinney
Masaaki Tomii
Jiqun Tu
Bigeng Wang
Tianle Wang
Evan Wickenden
Yidi Zhao

University of Connecticut

Tom Blum
Dan Hoying (BNL)
Luchang Jin (RBRC)
Cheng Tu

Edinburgh University

Peter Boyle
Guido Cossu
Luigi Del Debbio
Tadeusz Janowski
Richard Kenway
Julia Kettle
Fionn O'haigan
Brian Pendleton
Antonin Portelli
Tobias Tsang
Azusa Yamaguchi

KEK

Julien Frison

University of Liverpool

Nicolas Garron

MIT

David Murphy

Peking University

Xu Feng

University of Southampton

Jonathan Flynn
Vera Guelpers
James Harrison
Andreas Juettner
James Richings
Chris Sachrajda

Stony Brook University

Jun-Sik Yoo
Sergey Syritsyn (RBRC)

York University (Toronto)

Renwick Hudspith

Backup

Definitions of ε and ε'

$$
\begin{aligned}
\varepsilon &= \frac{A\left[K_L \to (\pi\pi)_0\right]}{A\left[K_S \to (\pi\pi)_0\right]} \\
\varepsilon' &= \frac{1}{\sqrt{2}}\left(\frac{A\left[K_L \to (\pi\pi)_2\right] - \varepsilon \times A\left[K_S \to (\pi\pi)_2\right]}{A\left[K_S \to (\pi\pi)_0\right]}\right)
\end{aligned}
$$

Definitions of ε and ε'

$$
\begin{aligned}
\varepsilon &= \frac{A\left[K_L \to (\pi\pi)_0\right]}{A\left[K_S \to (\pi\pi)_0\right]} \\
\varepsilon' &= \frac{1}{\sqrt{2}}\left(\frac{A\left[K_L \to (\pi\pi)_2\right] - \varepsilon \times A\left[K_S \to (\pi\pi)_2\right]}{A\left[K_S \to (\pi\pi)_0\right]}\right)
\end{aligned}
$$

Or in terms of ε'/ε

$$
\frac{\varepsilon'}{\varepsilon} = \frac{1}{\sqrt{2}}\left(\frac{A\left[K_L \to (\pi\pi)_2\right]}{A\left[K_L \to (\pi\pi)_0\right]} - \frac{A\left[K_S \to (\pi\pi)_2\right]}{A\left[K_S \to (\pi\pi)_0\right]}\right)
$$

Non Perturbative Renormalisation (NPR)

A few words on the renormalisation

First step: remove the divergences

Non-perturbative Renormalisation à la Rome-Southampton [Martinelli et al '95]

$$Q_i^{lat}(a) \to Q_i^{MOM}(\mu, a) = Z(\mu, a)_{ij} Q_j^{lat}(a)$$

and take the continuum limit

$$Q_i^{MOM}(\mu, 0) = \lim_{a^2 \to 0} Q_i^{MOM}(\mu, a)$$

Second step: Matching to $\overline{\mathrm{MS}}$, done in perturbation theory [Sturm et al., Lehner and Sturm, Gorbahn and Jäger, Gracey, ...]

$$Q_i^{MOM}(\mu, 0) \to Q_i^{\overline{\mathrm{MS}}}(\mu) = (1 + r_1 \alpha_S(\mu) + r_2 \alpha_S(\mu)^2 + \ldots)_{ij} Q_j^{MOM}(\mu, 0)$$

The Rome Southampon method [Martinelli et al '95]

Consider a quark bilinear $O_\Gamma = \bar{\psi}_2 \Gamma \psi_1$

Define

$$\Pi(x_2, x_1) = \langle \psi_2(x_2) O_\Gamma(0) \bar{\psi}_1(x_1) \rangle = \langle S_2(x_2, 0) \Gamma S_1(0, x_1) \rangle$$

In Fourier space $S(p) = \sum_x S(x, 0) e^{ip.x}$

$$\Pi(p_2, p_1) = \langle S_2(p_2) \Gamma S_1(p_1)^\dagger) \rangle$$

The Rome Southampon method [Martinelli et al '95]

Consider a quark bilinear $O_\Gamma = \bar{\psi}_2 \Gamma \psi_1$

Define

$$\Pi(x_2, x_1) = \langle \psi_2(x_2) O_\Gamma(0) \bar{\psi}_1(x_1) \rangle = \langle S_2(x_2, 0) \Gamma S_1(0, x_1) \rangle$$

In Fourier space $S(p) = \sum_x S(x, 0) e^{ip.x}$

$$\Pi(p_2, p_1) = \langle S_2(p_2) \Gamma S_1(p_1)^\dagger) \rangle$$

Amputated Green function

$$\Lambda(p_2, p_1) = \langle S_2(p_2)^{-1} \rangle \langle S_2(p_2) \Gamma S_1(p_1)^\dagger) \rangle \langle (S_2(p_1)^{\dagger\,-1}) \rangle$$

The Rome Southampon method [Martinelli et al '95]

Consider a quark bilinear $O_\Gamma = \bar\psi_2 \Gamma \psi_1$

Define

$$\Pi(x_2, x_1) = \langle \psi_2(x_2) O_\Gamma(0) \bar\psi_1(x_1) \rangle = \langle S_2(x_2, 0) \Gamma S_1(0, x_1) \rangle$$

In Fourier space $S(p) = \sum_x S(x,0) e^{ip.x}$

$$\Pi(p_2, p_1) = \langle S_2(p_2) \Gamma S_1(p_1)^\dagger) \rangle$$

Amputated Green function

$$\Lambda(p_2, p_1) = \langle S_2(p_2)^{-1} \rangle \langle S_2(p_2) \Gamma S_1(p_1)^\dagger) \rangle \langle (S_2(p_1)^{\dagger\,-1}) \rangle$$

Rome Southampton original scheme (RI-MOM), $p_1 = p_2 = p$ and $\mu = \sqrt{p^2}$

$$Z(\mu, a) \times \lim_{m \to 0} \mathrm{Tr}(\Gamma \Lambda(p, p))_{\mu^2 = p^2} = \mathrm{Tree}$$

The Rome Southampon method [Martinelli et al '95]

Remarks

- Can be generalised to the 4q-operators mixing case

The Rome Southampon method [Martinelli et al '95]

Remarks

- Can be generalised to the 4q-operators mixing case
- Non-perturbative off-shell and massless scheme(s)
- Requires gauge fixing (unlike Schrödinger Functional)

The Rome Southampon method [Martinelli et al '95]

Remarks

- Can be generalised to the 4q-operators mixing case
- Non-perturbative off-shell and massless scheme(s)
- Requires gauge fixing (unlike Schrödinger Functional)

Note that the choice of projector and kinematics is not unique

- In particular, SMOM scheme

$$p_1 \neq p_2 \text{ and } p_1^2 = p_2^2 = (p_1 - p_2)^2$$

- Can use $\not{q}$ as projector

The Rome Southampon method [Martinelli et al '95]

Remarks

- Can be generalised to the 4q-operators mixing case
- Non-perturbative off-shell and massless scheme(s)
- Requires gauge fixing (unlike Schrödinger Functional)

Note that the choice of projector and kinematics is not unique

- In particular, SMOM scheme

$$p_1 \neq p_2 \text{ and } p_1^2 = p_2^2 = (p_1 - p_2)^2$$

- Can use $\not{q}$ as projector
- In principle the results should agree after conversion to $\overline{\mathrm{MS}}$, and extrapolation to the continuum limit

Renormalisation basis of the $\Delta F = 2$ operators

As for BSM neutral meson mixing one needs to renormalise 5 operators ,

$$
\begin{array}{ll}
(27,1) & O_1^{\Delta S=2} \;=\; \gamma_\mu \times \gamma_\mu + \gamma_\mu\gamma_5 \times \gamma_\mu\gamma_5 \\[1em]
(8,8) & \left\{ \begin{array}{lcl} O_2^{\Delta s=2} & = & \gamma_\mu \times \gamma_\mu - \gamma_\mu\gamma_5 \times \gamma_\mu\gamma_5 \\ O_3^{\Delta s=2} & = & 1 \times 1 - \gamma_5 \times \gamma_5 \end{array} \right. \\[1em]
(6,\bar{6}) & \left\{ \begin{array}{lcl} O_4^{\Delta s=2} & = & 1 \times 1 + \gamma_5 \times \gamma_5 \\ O_5^{\Delta s=2} & = & \sigma_{\mu\nu} \times \sigma_{\mu\nu} \end{array} \right.
\end{array}
$$

So the renormalisation matrix has the form

$$
\mathcal{Z}^{\Delta S=2} = \begin{pmatrix} \mathcal{Z}_{11} & & & & \\ & \mathcal{Z}_{22} & \mathcal{Z}_{23} & & \\ & \mathcal{Z}_{32} & \mathcal{Z}_{33} & & \\ & & & \mathcal{Z}_{44} & \mathcal{Z}_{45} \\ & & & \mathcal{Z}_{54} & \mathcal{Z}_{55} \end{pmatrix}
$$

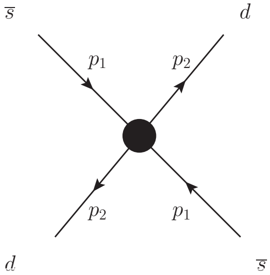
More details on NPR

- Setup is the similar to RBC-UKQCD

  In particular we follow [Arthur & Boyle '10]

- We implement momentum sources [Gockeler et al '98] to achieve high stat. accuracy

- Non exceptional kinematic with symmetric point $p_1^2 = p_2^2 = (p_2 - p_1)^2$



  to suppress IR contaminations [Sturm et al', RBC-UKQCD '09 '10]

Choice of SMOM scheme

- Orientation of the momenta kept fixed

$$p_1 = \frac{2\pi}{L}[n\,,0\,,n\,,0] \qquad p_2 = \frac{2\pi}{L}[0\,,n\,,n\,,0]$$

$\Rightarrow$ Well defined continuum limit

Choice of SMOM scheme

- Orientation of the momenta kept fixed

$$p_1 = \frac{2\pi}{L}[n\ ,0\ ,n\ ,0] \qquad p_2 = \frac{2\pi}{L}[0\ ,n\ ,n\ ,0]$$

$\Rightarrow$ Well defined continuum limit

- We chose γ_μ projectors, for example

$$P^{(\gamma_\mu)} \quad \leftrightarrow \quad \gamma_\mu \times \gamma_\mu + \gamma_\mu\gamma_5 \times \gamma_\mu\gamma_5$$

$\Rightarrow$ Z factor of a four quark operator O in the scheme (γ_μ, γ_μ) defined by

$$\lim_{m\to 0} \frac{Z_O^{(\gamma_\mu,\gamma_\mu)}}{Z_V^2} \left. \frac{P^{(\gamma_\mu)}\left\{\Lambda_O\right\}}{\left(P^{(\gamma_\mu)}\left\{\Lambda_V\right\}\right)^2} \right|_{\mu^2=p^2} = Tree$$

- Note that this defines an off-shell massless scheme

Step-scaling

- Rome-Southampton method requires a *windows*

$$\Lambda_{QCD}^2 \ll \mu^2 \ll (\pi/a)^2$$

- And our lattice spacings are $a^{-1} \sim 2.2, 1.7, 1.3 GeV$

Step-scaling

- Rome-Southampton method requires a *windows*

$$\Lambda_{QCD}^2 \ll \mu^2 \ll (\pi/a)^2$$

- And our lattice spacings are $a^{-1} \sim 2.2, 1.7, 1.3 GeV$

- we follow [Arthur & Boyle '10] and [Arthur, Boyle, NG, Kelly, Lytle '11] and define

$$\sigma(\mu_2,\mu_1) = \lim_{a^2\to 0}\lim_{m\to 0}\left[(P\Lambda(\mu_2,a))^{-1}P\Lambda(\mu_1,a)\right] = \lim_{a^2\to 0} Z(\mu_2,a)Z(\mu_1,a)^{-1}$$

Step-scaling

- Rome-Southampton method requires a *windows*

$$\Lambda^2_{QCD} \ll \mu^2 \ll (\pi/a)^2$$

- And our lattice spacings are $a^{-1} \sim 2.2, 1.7, 1.3 GeV$

- we follow [Arthur & Boyle '10] and [Arthur, Boyle, NG, Kelly, Lytle '11] and define

$$\sigma(\mu_2, \mu_1) = \lim_{a^2 \to 0} \lim_{m \to 0} \left[(P\Lambda(\mu_2, a))^{-1} P\Lambda(\mu_1, a) \right] = \lim_{a^2 \to 0} Z(\mu_2, a) Z(\mu_1, a)^{-1}$$

- We use 3 lattice spacings to compute $\sigma(2\text{ GeV}, 1.5\text{ GeV})$ but only the two finest to compute $\sigma(3\text{ GeV}, 2\text{ GeV})$ and get

$$Z(3\text{ GeV}, a) = \sigma(3\text{ GeV}, 2\text{ GeV})\,\sigma(2\text{ GeV}, 1.5\text{ GeV})\,Z(1.5\text{ GeV}, a)$$

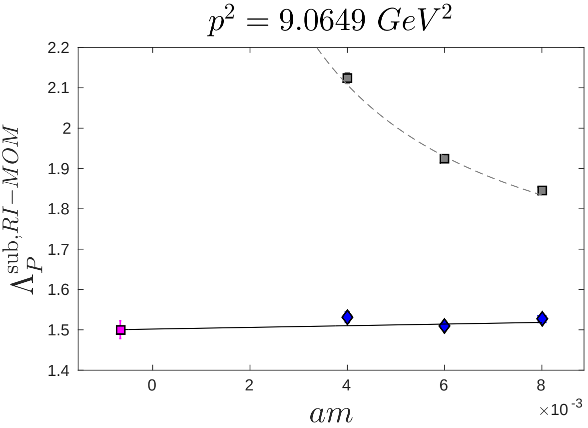
Pole subtraction

- The Green functions might suffer from IR poles, $\sim 1/p^2$, or $\sim 1/m_\pi^2$ which can pollute the signal
- In principle these poles are suppressed at high μ but they appear to be quite important at $\mu \sim 3$ GeV for some quantities which allow for pion exchanges
- The traditional way is to "subtract " these contamination by hand

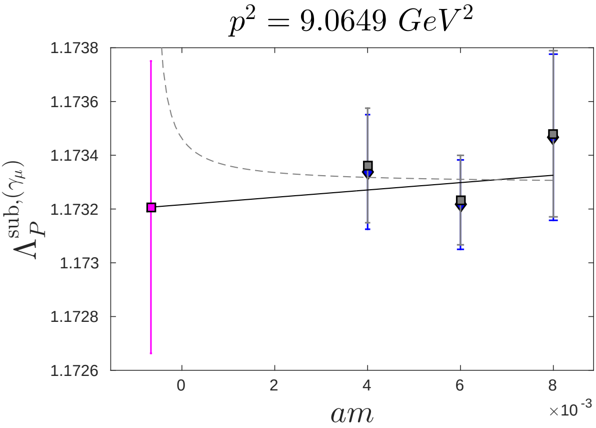
Pole subtraction

- The Green functions might suffer from IR poles, $\sim 1/p^2$, or $\sim 1/m_\pi^2$ which can pollute the signal
- In principle these poles are suppressed at high μ but they appear to be quite important at $\mu \sim 3$ GeV for some quantities which allow for pion exchanges
- The traditional way is to "subtract " these contamination by hand
- However these contaminations are highly suppressed in a SMOM scheme, with non-exceptional kinematics
- We argue that this pion pole subtractions is not-well under control and that schemes with exceptional kinematics should be discarded

Pole subtraction



Pole subtraction



Better MOM schemes ?

More MOM schemes

Renormalisation scale is μ, given by the choice of kinematics

- Original RI-MOM scheme

$$p_1 = p_2 \text{ and } \mu^2 \equiv p_1^2 = p_2^2$$

  But this lead to "exceptional kinematics' and bad IR poles

More MOM schemes

Renormalisation scale is μ, given by the choice of kinematics

- Original RI-MOM scheme

$$p_1 = p_2 \text{ and } \mu^2 \equiv p_1^2 = p_2^2$$

But this lead to "exceptional kinematics' and bad IR poles

- then RI-SMOM scheme

$$p_1 \neq p_2 \text{ and } \mu^2 \equiv p_1^2 = p_2^2 = (p_1 - p_2)^2$$

Much better IR behaviour [Sturm et al., Lehner and Sturm, Gorbahn and Jäger, Gracey, . . .]

More MOM schemes

Renormalisation scale is μ, given by the choice of kinematics

- Original RI-MOM scheme

$$p_1 = p_2 \text{ and } \mu^2 \equiv p_1^2 = p_2^2$$

But this lead to "exceptional kinematics' and bad IR poles

- then RI-SMOM scheme

$$p_1 \neq p_2 \text{ and } \mu^2 \equiv p_1^2 = p_2^2 = (p_1 - p_2)^2$$

Much better IR behaviour [Sturm et al., Lehner and Sturm, Gorbahn and Jäger, Gracey, . . .]

- We are now studying a generalisation (see also [Bell and Gracey])

$$p_1 \neq p_2 \text{ and } \mu^2 \equiv p_1^2 = p_2^2, \quad (p_1 - p_2)^2 = \omega\mu^2 \text{ where } \omega \in [0, 4]$$

Note that $\omega = 0 \leftrightarrow RI - MOM$ and $\omega = 1 \leftrightarrow RI - SMOM$

In collaboration with [...,Cahill, Gorbahn, Gracey, Perlt , Rakow, ...]