# The energy-momentum tensor in lattice QCD and the Equation of State

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# PLAN OF THE TALK

- Introduction
- Renormalization of the energy-momentum tensor
- The QCD EoS
- Conclusions

# Introduction

- The energy-momentum tensor $T_{\mu\nu}$ is a very important quantity for a QFT: it is connected to the translation, rotation and dilatation symmetries.

- The expectation values of its matrix elements are directly related to physical quantities like pressure, entropy and energy density, transport coefficients.

- Non-perturbative physics: $T_{\mu\nu}$ on the lattice

S. Caracciolo et al., NPB 375 (1992) 195; Ann. Phys. 197 (1990) 119

| Continuum | Lattice |
|---|---|
| $T_{\mu\nu}$ | $T^R_{\mu\nu} = Z\, T^{[6]}_{\mu\nu} + z\, T^{[3]}_{\mu\nu} + \beta\, T^{[1]}_{\mu\nu}$ |
| two-index symmetric tensor: {10} rep of SO(4) | {10} ➔ {6} + {3} + {1} of H(4) |

- Z and z: renormalization constants approaching 1 in the continuum limit

  Using Ward Identity: defined in Perturbation Theory and computed at 1-loop

How defining and computing Z and z non-perturbatively?

- EoS is usually obtained from the direct measurement of the trace anomaly

G. Boyd et al., Nucl. Phys. B469 (1996) 419

# QFT in a moving frame

rest frame

$$T_{\mu\nu} = \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

$$w = e + p = Ts$$

$$\gamma(\vec{v}) = \frac{1}{\sqrt{1-\vec{v}^2}}$$

moving frame

$$T_{ij} = \gamma^2 w\, v_i v_j + p\,\delta_{ij}$$

$$T_{00} = \gamma^2 w - p$$

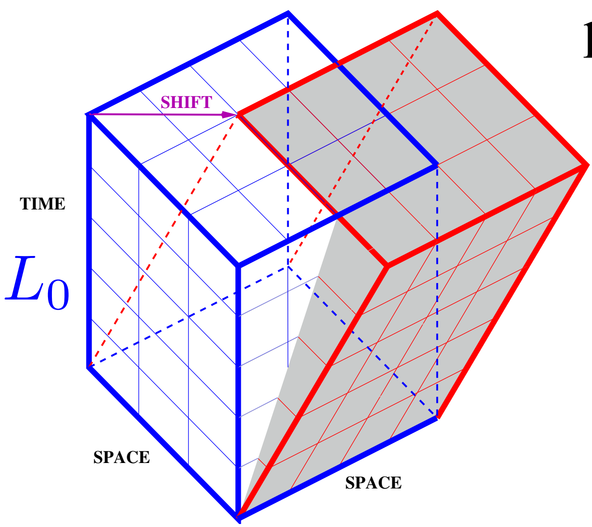
$$T_{0k} = \gamma^2 w v_k$$

**Landau, Lifschiz, vol 6, "Fluid mechanics"**

$$\frac{s}{T^3} = \frac{T_{0k}}{T^4\gamma^2 v_k} = \frac{T_{00}+T_{kk}}{T^4\gamma^2(1+v_k^2)}$$

A thermal quantum field theory in a moving reference frame:

**L. Giusti and H. Meyer, PRL 2011, JHEP 2011 and 2013**




That corresponds to introducing a shift $\xi$ when closing the periodic boundary conditions along the temporal direction

- Non-perturbative definition and calculation of the renormalization constants $Z$ and $z$ in SU(3) YM

- Equation of State of SU(3) YM with high accuracy in the range 0 – 230 $T_c$

**L. Giusti and M. Pepe, PRL 2014, PRD 2015, PLB 2017**

$$\frac{s}{T^3} = \frac{L_0^4(1+\xi^2)^3}{\xi_k} Z\langle T_{0k}\rangle_\xi = \frac{L_0^4(1+\xi^2)^3}{(1-\xi^2)} z\langle T_{00}-T_{kk}\rangle_\xi$$

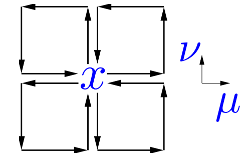
entropy density is the primary observable

# Lattice QCD in a moving frame

- The action: $$S = S_G + S_F$$

$$S_G = \frac{1}{g_0^2}\sum_{x,\mu\nu} \mathrm{Re}\;\mathrm{Tr}\big[1 - U_{\mu\nu}(x)\big] \qquad S_F = \sum_x \overline{\psi}(x)(D_W + M_0 + D_{sw})\psi(x) \qquad F_{\mu\nu}(x) = \frac{1}{8}[Q_{\mu\nu}(x) - Q_{\nu\mu}(x)]$$

$$D_W = \frac{1}{2}\sum_\mu \{\gamma_\mu(\nabla^*_\mu + \nabla_\mu) - \nabla^*_\mu\nabla_\mu\} \qquad D_{sw} = \frac{c_{sw}}{4}\sum_{\mu\nu}\sigma_{\mu\nu}F_{\mu\nu}(x) \qquad Q_{\mu\nu}(x) = \sum$$




$$U_\mu(x_0 + L_0, \mathbf{x}) = U_\mu(x_0, \mathbf{x} - L_0\xi) \qquad \overline{\psi}(x_0 + L_0, \mathbf{x}) = -\overline{\psi}(x_0, \mathbf{x} - L_0\xi) \qquad \psi(x_0 + L_0, \mathbf{x}) = -\psi(x_0, \mathbf{x} - L_0\xi)$$

$$U_\mu(x_0, \mathbf{x} + L_k\mathbf{k}) = U_\mu(x_0, \mathbf{x}) \qquad \overline{\psi}(x_0, \mathbf{x} + L_k\mathbf{k}) = \overline{\psi}(x_0, \mathbf{x}) \qquad \psi(x_0, \mathbf{x} + L_k\mathbf{k}) = \psi(x_0, \mathbf{x})$$

- The energy-momentum tensor:

$$T^G_{\mu\nu}(x) = \frac{1}{g_0^2}F^a_{\mu\rho}(x)F^a_{\nu\rho}(x) - \delta_{\mu\nu}\mathcal{L}^G(x) \qquad \mathcal{L}^G(x) = \frac{1}{4g_0^2}F^a_{\rho\sigma}(x)F^a_{\rho\sigma}(x) \qquad F^a_{\mu\nu}(x) = 2\,\mathrm{Tr}[F_{\mu\nu}(x)T^a]$$

$$T^F_{\mu\nu}(x) = \frac{1}{2}\Big\{\overline{\psi}(x)\gamma_\mu\overset{\leftrightarrow}{\nabla}_\nu\psi(x) + \overline{\psi}(x)\gamma_\nu\overset{\leftrightarrow}{\nabla}_\mu\psi(x)\Big\} - \delta_{\mu\nu}\mathcal{L}^F(x) \qquad \mathcal{L}^F(x) = \sum_\mu \overline{\psi}(x)\Big\{\gamma_\mu\overset{\leftrightarrow}{\nabla}_\mu + M_0\Big\}\psi(x)$$

$$\{10\} \rightarrow \{6\} + \{3\} + \{1\}$$

$$T^R_{\mu\nu} = T^{R,[6]}_{\mu\nu} + T^{R,[3]}_{\mu\nu} + T^{R,[1]}_{\mu\nu} = \begin{pmatrix} (T^R_{00} - T^R) & T^R_{01} & T^R_{02} & T^R_{03} \\ T^R_{01} & (T^R_{11} - T^R) & T^R_{12} & T^R_{13} \\ T^R_{02} & T^R_{12} & (T^R_{22} - T^R) & T^R_{23} \\ T^R_{03} & T^R_{13} & T^R_{23} & (T^R_{33} - T^R) \end{pmatrix} + T^R\mathbb{1}$$

[6] = green, [3] = red, [1] = blue

$$T^R = T^R_{\mu\mu}$$

# Lattice QCD: renormalization conditions

- Using Ward identities related to translation symmetry in YM

L. Giusti and M. Pepe, PLB 2017

$$\langle T_{0k}^{R,[6]}\rangle_\xi = Z(g_0^2)\langle T_{0k}^{[6]}\rangle_\xi = -\frac{\partial}{\partial \xi_k} f[\xi] = \frac{1}{L_0 V}\frac{\partial}{\partial \xi_k}\log \mathcal{Z}[\xi]$$

$$\langle T_{0k}^{R,[6]}\rangle_\xi = Z(g_0^2)\langle T_{0k}^{[6]}\rangle_\xi = \frac{\xi_k}{1-\xi_k^2} z(g_0^2)\langle T_{00}^{[3]} - T_{kk}^{[3]}\rangle_\xi = \frac{\xi_k}{1-\xi_k^2}\langle T_{00}^{R,[3]} - T_{kk}^{R,[3]}\rangle_\xi$$

- In QCD non-trivial mixing between the gluonic and the fermionic components

$$T_{\mu\nu}^{R,[6]} = Z_G(g_0^2)\, T_{\mu\nu}^{G,[6]} + Z_F(g_0^2)\, T_{\mu\nu}^{F,[6]} \qquad T_{\mu\nu}^{R,[3]} = z_G(g_0^2)\, T_{\mu\nu}^{G,[3]} + z_F(g_0^2)\, T_{\mu\nu}^{F,[3]}$$

- We need a second parameter to disentangle the mixing: couple fermions to a constant Abelian field $\lambda_\mu = e^{i\theta_\mu}$

$$U_\mu(x) \to \lambda_\mu U_\mu(x)$$

by a change of variables ⇒ fermionic phases at the boundaries.
It is related to the conserved flavor-singlet vector current $\mathbf{V}_\mu$: $\quad i\langle V_\mu\rangle_{\xi,\theta} = \frac{\partial}{\partial \theta_\mu} f[\xi,\theta]$

- In the thermodynamic limit we have

$$f[L_0;\xi,\theta_\mu] = f[\frac{L_0}{\gamma};0,\gamma\widetilde{\theta}_0] \qquad \gamma = \frac{1}{\sqrt{1+\xi^2}} \qquad \widetilde{\theta}_0 = \theta_0 + \xi_k\theta_k$$

generalizes to systems with fermions the equation $f[L_0;\xi] = f[\frac{L_0}{\gamma};0]$

note that $\mu_{\mathcal{I}} = i\gamma\widetilde{\theta}_0$ can be interpreted as an imaginary chemical potential

# Lattice QCD: renormalization conditions

- [6] representation: $\theta_\mu \to \theta_0 = \theta_A$ and $\theta_B$

$$\langle T^{R,[6]}_{0k}\rangle_{\xi,\theta_A} = Z_G(g_0^2)\, \langle T^{G,[6]}_{\mu\nu}\rangle_{\xi,\theta_A} + Z_F(g_0^2)\, \langle T^{F,[6]}_{\mu\nu}\rangle_{\xi,\theta_A} = -\frac{\partial}{\partial \xi_k} f[\xi,\theta_A] = \frac{1}{L_0 V}\frac{\partial}{\partial \xi_k}\log \mathcal{Z}[\xi,\theta_A]$$

$$\langle T^{R,[6]}_{0k}\rangle_{\xi,\theta_B} = Z_G(g_0^2)\, \langle T^{G,[6]}_{\mu\nu}\rangle_{\xi,\theta_B} + Z_F(g_0^2)\, \langle T^{F,[6]}_{\mu\nu}\rangle_{\xi,\theta_B} = -\frac{\partial}{\partial \xi_k} f[\xi,\theta_B] = \frac{1}{L_0 V}\frac{\partial}{\partial \xi_k}\log \mathcal{Z}[\xi,\theta_B]$$

! $\langle T^{F,[6]}_{\mu\nu}\rangle_{\xi,\theta}$ has a different dependence on $\theta_\mu$ w.r.t. $\langle T^{G,[6]}_{\mu\nu}\rangle_{\xi,\theta}$

direct

indirect: via quark interaction

the r.h.s. can be measured directly or one can use that

$$\frac{\partial}{\partial \xi_k} f[\xi,\theta_B] = \frac{\partial}{\partial \xi_k} f[\xi,\theta_A] + i\int_{\theta_A}^{\theta_B} \frac{\partial}{\partial \xi_k}\langle V_0\rangle_{\xi,\theta}\, d\theta$$

- [3] representation: $\theta_\mu \to \theta_0 = \theta_A$ and $\theta_B$

$$Z_G(g_0^2)\, \langle T^{G,[6]}_{\mu\nu}\rangle_{\xi,\theta} + Z_F(g_0^2)\, \langle T^{F,[6]}_{\mu\nu}\rangle_{\xi,\theta} = \frac{\xi_k}{1-\xi_k^2}\Big\{ z_G(g_0^2)\langle T^{G,[3]}_{00} - T^{G,[3]}_{kk}\rangle_{\xi,\theta} + z_F(g_0^2)\langle T^{F,[3]}_{00} - T^{F,[3]}_{kk}\rangle_{\xi,\theta}\Big\}$$

# Perturbation Theory at 1-loop order

- $Z_G$, $Z_F$, $z_G$, $z_F$ computed at 1-loop in Perturbation Theory in the framework of shifted b.c. : plaquette gauge action + Wilson fermions + clover term

⬇

- <u>check</u>: full agreement with 1-loop data in the literature obtained with other methods.

S. Caracciolo et al., PLB 1991, NPB 1996

S. Capitani and GC Rossi, NPB 1995

- <u>news n.1</u>: $Z_G$, $Z_F$, $z_G$, $z_F$ computed at 1-loop with Wilson clover fermions ⇒ 1-loop improved definition in their non-perturbative measurement   in progress

- <u>news n.2</u>: O(a) improvement coefficients of $T_{\mu\nu}$ for degenerate massive quarks

$$T^{G,[6]}_{0k,imp} = (1 + b^{[6]}_G m_q) T^{G,[6]}_{0k} \qquad (T^{G,[3]}_{00} - T^{G,[3]}_{kk})_{imp} = (1 + b^{[3]}_G m_q)(T^{G,[3]}_{00} - T^{G,[3]}_{kk})$$

$$T^{F,[6]}_{0k,imp} = (1 + b^{[6]}_F m_q)\left[T^{F,[6]}_{0k} + c^{[6]}_T \delta T^{F,[6]}_{0k}\right] \qquad (T^{F,[3]}_{00} - T^{F,[3]}_{kk})_{imp} = (1 + b^{[3]}_F m_q)\left[(T^{F,[3]}_{00} - T^{F,[3]}_{kk}) + c^{[3]}_T \delta T^{F,[3]}_{0k}\right]$$

$$\delta T^{F,[k]}_{0k} \to \overline{\psi}(x)\left[\sigma_{0\rho}F_{k\rho} + \sigma_{k\rho}F_{0\rho}\right]\psi(x) \qquad m_q = m_0 - m_{cr}$$

$$b^{[k]}_G = b^{[k],(1)}_G g_0^2 + \ldots \qquad b^{[k]}_F = 1 + b^{[k],(1)}_F g_0^2 + \ldots \qquad c^{[k]}_G = c^{[k],(1)}_T g_0^2 + \ldots$$

# The QCD Equation of State in a moving frame

- QCD EoS: up to T~ 1-2 GeV using staggered quarks

  up to T~ 600 MeV using Wilson quarks

**Budapest-Wuppertal Collab., 2014**
**A. Bazavov, P. Petreczky, J. Weber, PRD 2018**
**HOT-QCD Collab., PRD 2014**
**MILC Collab., PRD 2014**
**WHOT-QCD Collab., PRD 2018**
**tmFT Collab., PRD 2015**

integral method: pressure by integrating the trace anomaly,

challenging: in the same system two different physical scales, T=0 and T finite

**Engels et al., NPB 1982**

- shifted b.c.: extrapolation to the continuum limit in an independent way at every T

  fix T and quark masses: choose $L_0$ and then run at the proper bare parameters

- $s/T^3$ is the primary observable that is measured

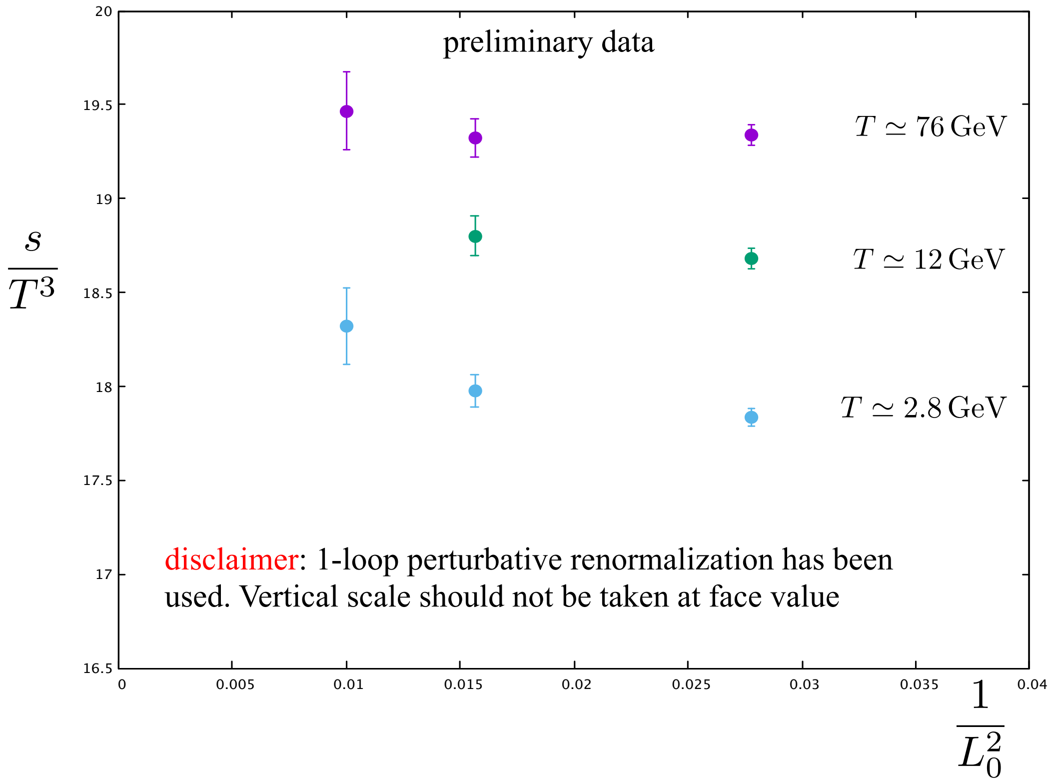
$$\frac{s}{T^3} = \frac{L_0^4(1+\xi^2)^3}{\xi_k}\left[Z_G\langle T_{0k}^{G,[6]}\rangle_\xi + Z_F\langle T_{0k}^{F,[6]}\rangle_\xi\right]$$

s = entropy density; e = energy density; p = pressure; T = temperature

$$p(T) - p(T_0) = \int_{T_0}^{T} s(T)dT \quad ; \qquad e(T) = Ts(T) - p(T) \quad \text{or} \quad e(T) - e(T_0) = \int_{s_0}^{s} T(s)ds$$

# The numerical study

- AIM: calculation of the QCD EoS up to 80 GeV, the Electro-Weak scale
- 8 physical temperatures in the range 2.5 GeV – 80 GeV
- at each temperature 3 values of the temporal extension $L_0$= 6, 8 and 10 and spatial size L=288; the shift is (1,0,0) (YM: small lattice artefacts).
- we consider 3 flavors of massless Wilson clover quarks with the Wilson SU(3) gauge action




small lattice artefacts

final accuracy about 1%

work in progress for the non-perturbative calculation of the renormalization constants

# Conclusions

- The framework of shifted boundary conditions confirms to be a simple, robust and numerically efficient method to investigate the thermal features of a QFT.

- We have provided a non-perturbative definition of the renormalization constants of the energy-momentum tensor $T_{\mu\nu}$ of lattice QCD

- That definition can be also implemented in practice for the non-perturbative calculation of the renormalization constants by Monte Carlo simulations: numerical work is in progress.

- Perturbative calculations at 1-loop: check of the framework, improved definition of the renormalization constants, calculation of the improvement coefficients of $T_{\mu\nu}$

- Results of the QCD EoS in the unexplored range of temperatures: 2.5 – 80 GeV

Perturbative approaches fail to give a good description of SU(3) Yang-Mills theory at high T: what about QCD?