News from bottomonium spectral functions in thermal QCD

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FASTSUM
Lattice QCD

- From thermal field theory

\[ G(\tau_n) = \int_0^\infty K(\tau_n, \omega) \rho(\omega) \, d\omega \]

- Thermal Kernel plays the role of a distribution function.

\[ K(\tau_n, \omega) = \frac{\cosh(\omega[\tau - \beta/2])}{\sinh(\omega\beta/2)} \]

- Spectral function contains all the interesting physics.
Lattice NRQCD

- An effective field model, where the Langrangian is in powers of $v_b = p/m$

- Cuts off relativistic modes for bottom quark

- Quark and anti-quarks fields decouple in this limit

- Time evolution is given by initial value problem.

- Performing the shift $\omega = 2M + \omega'$ allows us to re-write thermal Kernel as $K(\tau_n, \omega) = e^{-\omega'\tau}$
Lattice NRQCD

\[ S(x + a_r e_r) = \left(1 - \frac{a_r H_0 |\tau + a_r|}{2k}\right)^k U_1(x) \left(1 - \frac{a_r H_0 |\tau|}{2k}\right)^k (1 - a_r \delta H) S(x) \]

\[ H_0 = -\frac{\Delta^{(2)}}{2m_b}, \quad \text{with} \quad \Delta^{(2n)} = \sum_{i=1}^{3} (\nabla_i^+ \nabla_i^-)^n \quad (k = 1) \]

\[ \delta H = -\frac{\left(\Delta^{(2)}\right)^2}{8m_b^2} + \frac{ig_b}{8m_b^2} (\nabla^\perp \cdot E - E \cdot \nabla^\perp) - \frac{g_b}{8m_b^2} \sigma \cdot (\nabla^\perp \times E - E \times \nabla^\perp) - \frac{g_b}{2m_b} \sigma \cdot B + \frac{a^2 \Delta^{(4)}}{24m_b} - \frac{a_r \left(\Delta^{(2)}\right)^2}{16km_b^2} \]

\( \delta H \) gives corrections to \( \mathcal{O}(v^4), \mathcal{O}(a^2_b), \mathcal{O}(a_r) \)

\( m_b \) set from spin averaged S-wave masses
**Gen2 to Gen2L**

**Gen 2**

\[ M_\pi = 392 \text{ MeV} \]

\[ \xi = 3.5 \]

<table>
<thead>
<tr>
<th>( N_\tau )</th>
<th>128*</th>
<th>40</th>
<th>36</th>
<th>32</th>
<th>28</th>
<th>24</th>
<th>20</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T ) [MeV]</td>
<td>44</td>
<td>141</td>
<td>156</td>
<td>176</td>
<td>201</td>
<td>235</td>
<td>281</td>
<td>352</td>
</tr>
<tr>
<td>( T/T_c )</td>
<td>0.24</td>
<td>0.76</td>
<td>0.84</td>
<td>0.95</td>
<td>1.09</td>
<td>1.27</td>
<td>1.52</td>
<td>1.90</td>
</tr>
<tr>
<td>( N_{\text{cfg}} )</td>
<td>139</td>
<td>501</td>
<td>501</td>
<td>1000</td>
<td>1001</td>
<td>1001</td>
<td>1000</td>
<td>1001</td>
</tr>
</tbody>
</table>

Gen 2 ensembles, lattice size \( 24^3 \times N_\tau \)

**Gen 2L**

\[ M_\pi = 236 \text{ MeV} \]

\[ \xi = 3.45 \]

<table>
<thead>
<tr>
<th>( N_\tau )</th>
<th>256*</th>
<th>128</th>
<th>64</th>
<th>56</th>
<th>48</th>
<th>40</th>
<th>36</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T ) [MeV]</td>
<td>23</td>
<td>47</td>
<td>94</td>
<td>107</td>
<td>125</td>
<td>150</td>
<td>167</td>
</tr>
<tr>
<td>( N_{\text{cfg}} )</td>
<td>750</td>
<td>300</td>
<td>500</td>
<td>500</td>
<td>500</td>
<td>500</td>
<td>500</td>
</tr>
<tr>
<td>( N_\tau )</td>
<td>32</td>
<td>28</td>
<td>24</td>
<td>20</td>
<td>16</td>
<td>12</td>
<td>8</td>
</tr>
<tr>
<td>( T ) [MeV]</td>
<td>187</td>
<td>214</td>
<td>250</td>
<td>300</td>
<td>375</td>
<td>500</td>
<td>750</td>
</tr>
<tr>
<td>( N_{\text{cfg}} )</td>
<td>500</td>
<td>1000</td>
<td>1000</td>
<td>1000</td>
<td>1000</td>
<td>1000</td>
<td>1000</td>
</tr>
</tbody>
</table>

Gen 2L ensembles, lattice size \( 32^3 \times N_\tau \)
Thermal modification of $\gamma$

- Consistent results for both generations
- Largest gap occurs corresponds to crossing $T_c$
- Very little enhancement
Effective mass of $\Upsilon$

- The curve appear the same for all $N_\tau$
- $a_\tau M_{\text{eff}} = 0.332$
- Can now define energy shift $M_0 = 7.463$ GeV

$$M_{\text{expt}} = a_\tau \Delta E + M_0$$

<table>
<thead>
<tr>
<th>$n^{S+1}L_J$</th>
<th>State</th>
<th>$M_{\text{expt}}$ (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1^1S_0$</td>
<td>$\eta$</td>
<td>9398.0(3.2)</td>
</tr>
<tr>
<td>$2^2S_0$</td>
<td>$\eta'$</td>
<td>9990(4)</td>
</tr>
<tr>
<td>$1^3S_1$</td>
<td>$\Upsilon$</td>
<td>9460.30(26)</td>
</tr>
<tr>
<td>$2^2S_1$</td>
<td>$\Upsilon'$</td>
<td>10023.26(31)</td>
</tr>
<tr>
<td>$1^1P_1$</td>
<td>$\eta_b$</td>
<td>9899.3(1.0)</td>
</tr>
<tr>
<td>$1^3P_0$</td>
<td>$\chi_{b0}$</td>
<td>9859.44(52)</td>
</tr>
<tr>
<td>$1^3P_1$</td>
<td>$\chi_{b1}$</td>
<td>9892.78(40)</td>
</tr>
<tr>
<td>$1^3P_2$</td>
<td>$\chi_{b2}$</td>
<td>9912.21(40)</td>
</tr>
</tbody>
</table>
Thermal modification of $\chi_b$

- Largest gap occurs at $T_c$ again
- 20% percent enhancement at high temp
- Larger thermal effects expected
Effective mass of $\Upsilon$

- The curves spread for different $N_\tau$
- Expected value is $a_\tau M_{\text{eff}} = 0.404$
The ill-posed problem

Recall

\[ G(\tau_n) = \int_0^{\infty} K(\tau_n, \omega) \rho(\omega) \, d\omega \]

\( G(\tau_n) \sim O(10) \) whilst \( \rho(\omega) \) is in principle continuous, \( \sim O(1000) \).

Furthermore, small errors in \( G \) would blow up in \( \rho(\omega) \).

Can be treated as a 1D “image” reconstruction.

Requires numerical methods such as Maximum Entropy method (MEM) or Machine Learning.
MEM results

Strong agreement of ground-state energy between both generations.

Some melting of states at highest temperature
Machine Learning: Kernel Ridge Regression (KRR)

- Generalised form of linear regression

- Linear case: $y = \varphi(x)^T w \rightarrow$ KRR case: $Y = C(X_i, X_j)^T \alpha$

- $C$ is our kernel function. Used to determine correlations between functions $X_i$ and $X_j$.

- Parameter vector $w \rightarrow$ Parameter matrix $\alpha$
KRR recipe

- Generate training data. (Spectral functions from realistic mock data. Correlators from Laplace transform)

- Generate kernel from training correlators.

\[
C(G_i, G_j) = \exp\left( -\frac{\sum_{n=1}^{N_T} (G_i(\tau_n) - G_j(\tau_n))^2}{\gamma^2} \right)
\]

- Minimise Cost function

\[
E[C(G_i, G_j), \alpha] = (Y - C^T \alpha)^2 - \lambda \alpha^T C^T \alpha
\]

\[
\alpha = (C + I\lambda)^{-1} Y
\]

- Can now make predictions using real data using \( Y = C(G_i, G_{real})^T \alpha \).
Kernel Methods results (Preliminary)

Mock data

$\rho(\omega)$ vs $a^{-1}\omega$

- **Target**
- **Prediction**

$Y$ Correlation Function

$\rho(\omega)$ vs $a^{-1}\omega$
Summary

- We have seen thermal modification that is consistent with our previous generation of ensembles.
- Effective mass plots show consistent results with generation 2 (see arxiv 1402.6210)
- and expected value from experimental data
- MEM reconstruction for the $\Upsilon$ indicates agreement between generations
- KRR in its early stages

Thank you listening
Maximum Entropy Method (MEM)

Need to maximise \( P(F|D) \)

Bayes Theorem: \( P(F|D)P(D) = P(D|F)P(F) = P(D \cap F) \)

i.e. \( P(F|D) = \frac{P(D|F)P(F)}{P(D)} \)

But \( P(D|F) \sim e^{-\chi^2} \rightarrow \) minimising \( \chi^2 \neq \) maximising \( P(F|D) \)

\( \rightarrow \) Maximum Likelihood Method wrong??

\( P(F) \sim e^{\alpha S} \)

Shannon-Jaynes Entropy:
\[
S = \int_0^\infty \frac{d\omega}{2\pi} \left[ \rho(\omega) - m(\omega) - \rho(\omega)\ln\frac{\rho(\omega)}{m(\omega)} \right]
\]

Asakawa, Hatsuda, Nakahara, Prog.Part.Nucl.Phys. 46 (2001) 459