News from bottomonium spectral functions in thermal QCD

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FASTSUM

Lattice QCD

- From thermal field theory

$$G(\tau_n) = \int_0^\infty K(\tau_n, \omega)\rho(\omega)\,\mathrm{d}\omega$$

- Thermal Kernel plays the role of a distribution function.

$$K(\tau_n, \omega) = \frac{\cosh(\omega[\tau - \beta/2])}{\sinh({}^{\omega\beta}/_{2})}$$

- Spectral function contains all the interesting physics.

Lattice NRQCD

- An effective field model, where the Langrangian is in powers of $v_b = p/m$
- Cuts off relativistic modes for bottom quark
- Quark and anti-quarks fields decouple in this limit
- Time evolution is given by initial value problem.
- Performing the shift $\omega = 2M + \omega'$ allows us to re-write thermal Kernel as $K(\tau_n, \omega) = e^{-\omega'\tau}$

Lattice NRQCD

$$S(x + a_\tau e_\tau) = \left(1 - \frac{a_\tau H_0|_{\tau+a_\tau}}{2k}\right)^k U_\tau^\dagger(x) \left(1 - \frac{a_\tau H_0|_\tau}{2k}\right)^k (1 - a_\tau \delta H)\, S(x)$$

$$H_0 = -\frac{\Delta^{(2)}}{2m_b}, \qquad \text{with} \qquad \Delta^{(2n)} = \sum_{i=1}^{3} \left(\nabla_i^+ \nabla_i^-\right)^n \qquad (k = 1)$$

$$\begin{aligned}
\delta H = &- \frac{\left(\Delta^{(2)}\right)^2}{8m_b^3} + \frac{ig_0}{8m_b^2}\left(\nabla^\pm \cdot E - E \cdot \nabla^\pm\right) \\
&- \frac{g_0}{8m_b^2}\sigma \cdot \left(\nabla^\pm \times E - E \times \nabla^\pm\right) - \frac{g_0}{2m_b}\sigma \cdot B \\
&+ \frac{a_s^2 \Delta^{(4)}}{24 m_b} - \frac{a_\tau \left(\Delta^{(2)}\right)^2}{16 k m_b^2}
\end{aligned}$$

δH gives corrections to $\mathcal{O}(v^4), \mathcal{O}(a_s^2), \mathcal{O}(a_\tau)$

m_b set from spin averaged S-wave masses

Gen2 to Gen2L

Gen 2

M_π = 392 MeV

ξ = 3.5

N_τ	128*	40	36	32	28	24	20	16
T [MeV]	44	141	156	176	201	235	281	352
T/T_c	0.24	0.76	0.84	0.95	1.09	1.27	1.52	1.90
$N_{\rm cfg}$	139	501	501	1000	1001	1001	1000	1001

Gen 2 ensembles, lattice size $24^3 \times N_\tau$

Gen 2L

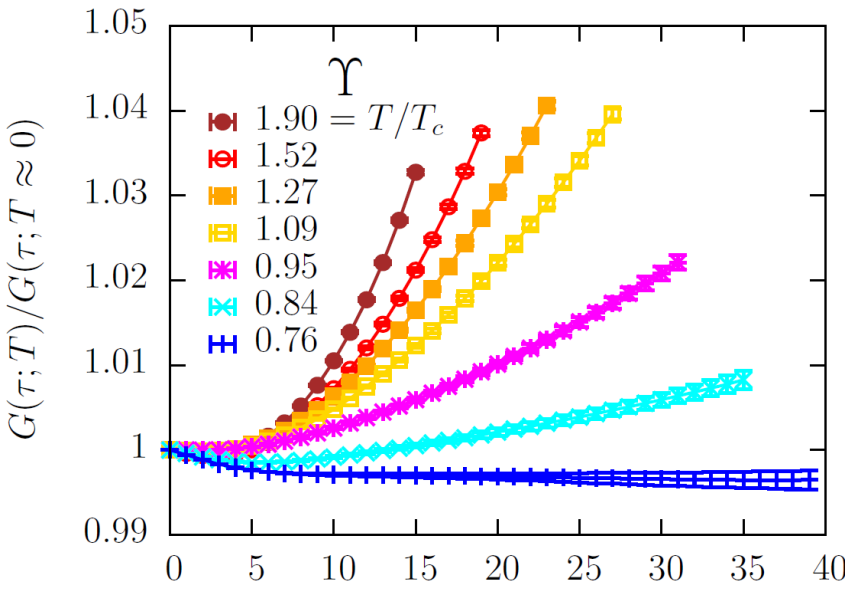
M_π = 236 MeV

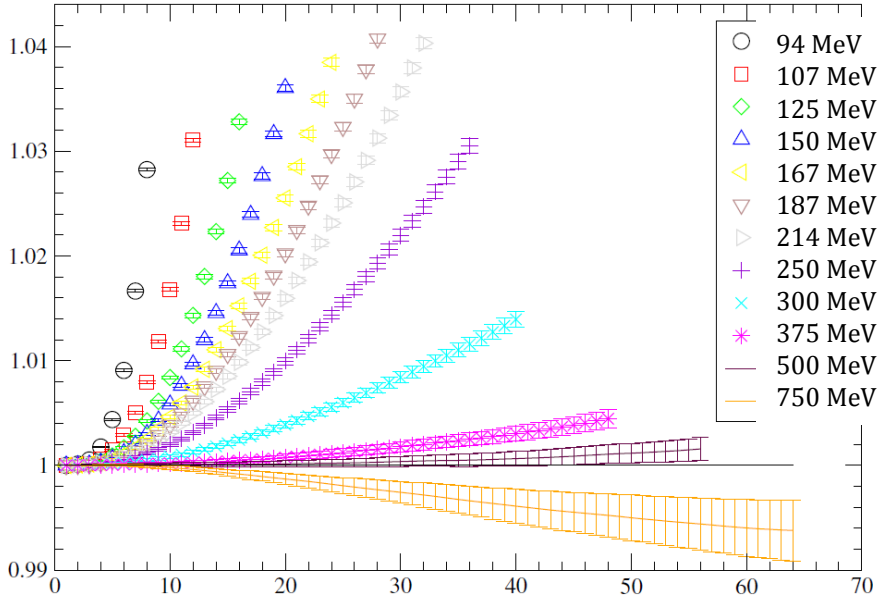
ξ = 3.45

N_τ	256*	128	64	56	48	40	36
T [MeV]	23	47	94	107	125	150	167
$N_{\rm cfg}$	750	300	500	500	500	500	500
N_τ	32	28	24	20	16	12	8
T [MeV]	187	214	250	300	375	500	750
$N_{\rm cfg}$	500	1000	1000	1000	1000	1000	1000

Gen 2L ensembles, lattice size $32^3 \times N_\tau$

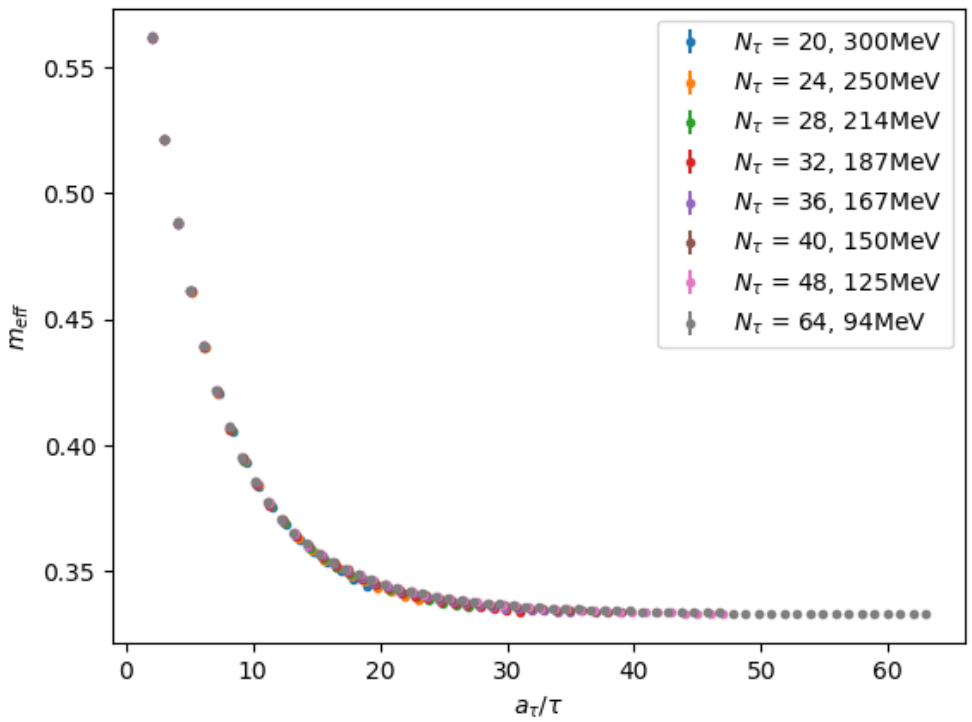
Thermal modification of Υ





- Consistent results for both generations
- Largest gap occurs corresponds to crossing T_c
- Very little enhancement

Effective mass of Υ

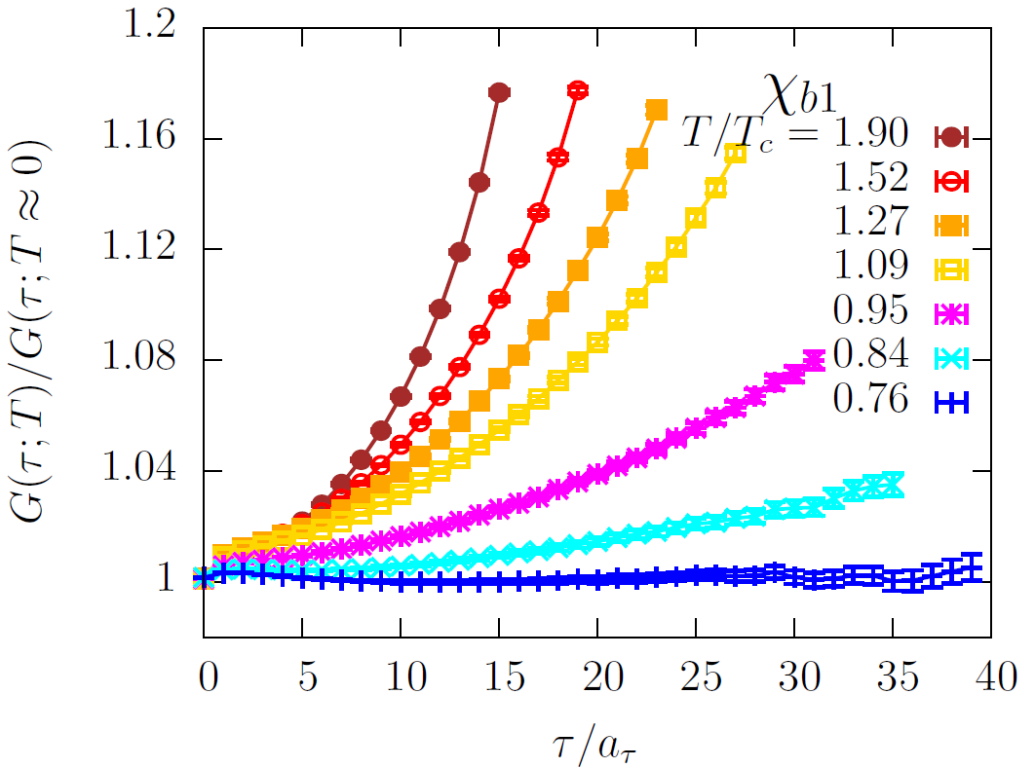


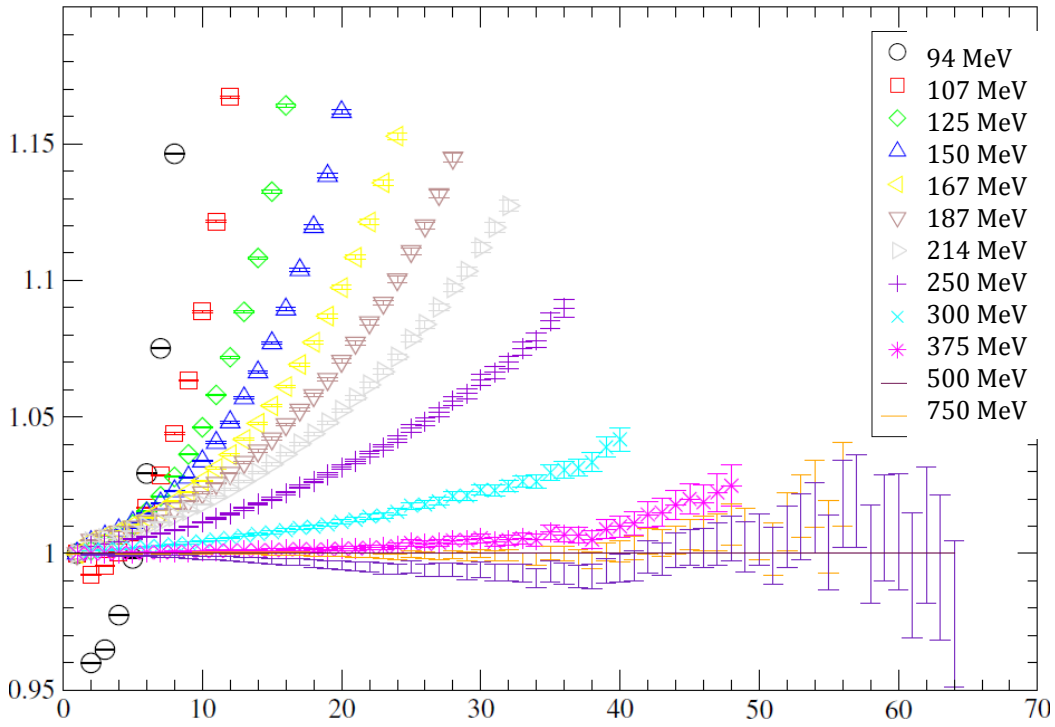
- The curve appear the same for all N_τ
- $a_\tau M_{eff}$ = 0.332
- Can now define energy shift M_0 = 7.463 GeV

$$M_{expt} = a_\tau \Delta E + M_0$$

$n^{S+1}L_J$	State	M_{expt} (MeV)
1^1S_0	η_b	9398.0(3.2)
2^1S_0	η_b'	9999(4)
1^3S_1	Υ	9460.30(26)
2^3S_1	Υ'	10023.26(31)
1^1P_1	h_b	9899.3(1.0)
1^3P_0	χ_{b0}	9859.44(52)
1^3P_1	χ_{b1}	9892.78(40)
1^3P_2	χ_{b2}	9912.21(40)

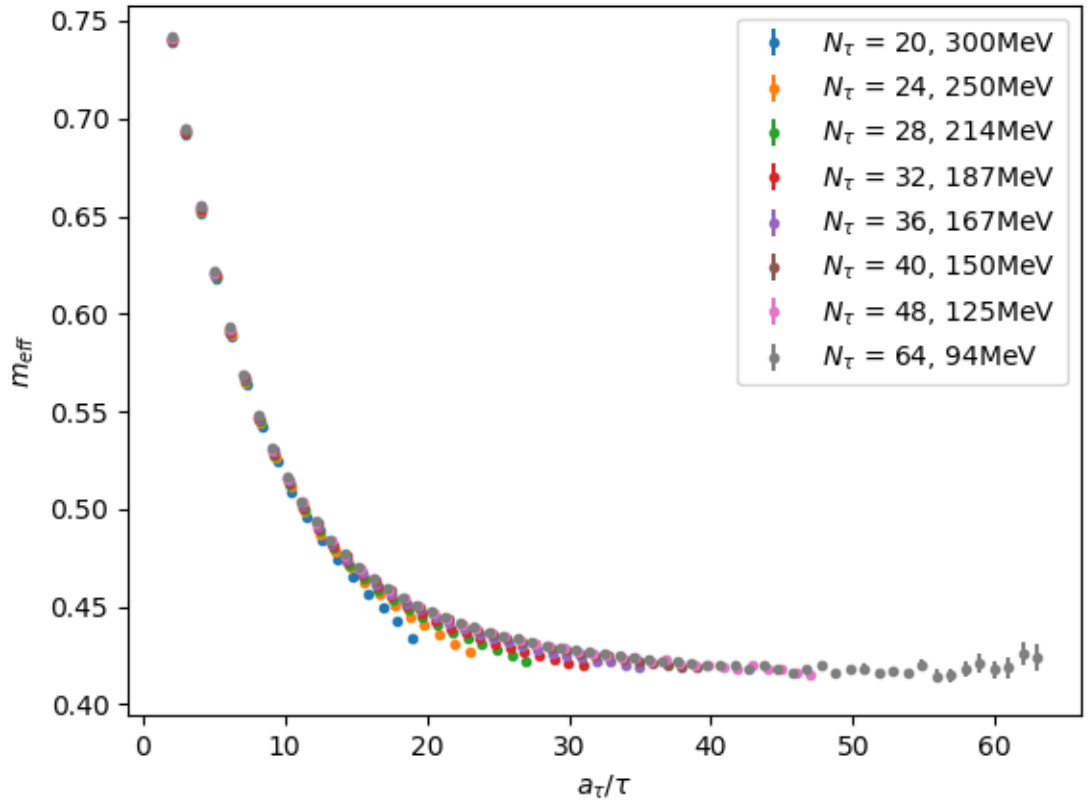
Thermal modification of χ_b





- Largest gap occurs at T_c again
- 20% percent enhancement at high temp
- Larger thermal effects expected

Effective mass of Y



- The curves spread for different N_τ
- Expected value is $a_\tau M_{eff} = 0.404$

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The ill-posed problem

Recall

$$G(\tau_n) = \int_0^\infty K(\tau_n, \omega)\rho(\omega)\,\mathrm{d}\omega$$

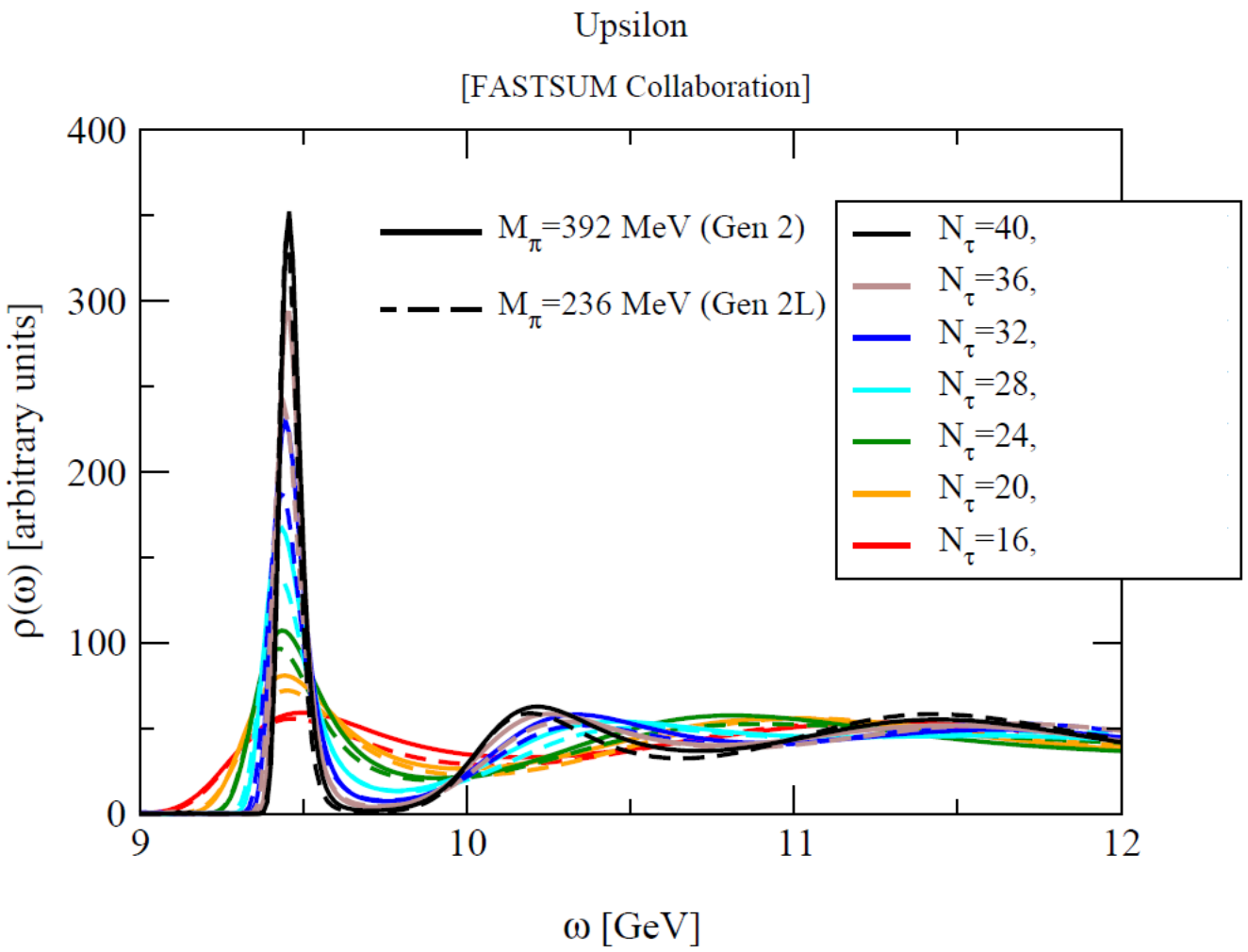
$G(\tau_n)$ ~ O(10) whilst $\rho(\omega)$ is in principle continuous, ~O(1000).

Furthermore, small errors in G would blow up in $\rho(\omega)$.

Can be treated as a 1D “image” reconstruction.

Requires numerical methods such as Maximum Entropy method (MEM) or Machine Learning.

MEM results



Strong agreement of ground-state energy between both generations.

Some melting of states at highest temperature

N_τ	128*	40	36	32	28	24	20	16
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Gen 2L ensembles, lattice size $32^3 \times N_\tau$

Machine Learning: Kernel Ridge Regression (KRR)

- Generalised form of linear regression

- Linear case: $y = \varphi(x)^T w$ → KRR case : $\mathbf{Y} = \mathbf{C}\left(X_i, X_j\right)^T \boldsymbol{\alpha}$

- **C** is our kernel function. Used to determine correlations between functions X_i and X_j.

- Parameter vector w → Parameter matrix $\boldsymbol{\alpha}$

KRR recipe

- Generate training data. (Spectral functions from realistic mock data. Correlators from Laplace transform)

- Generate kernel from training correlators.

$$\mathbf{C}(G_i, G_j) = \exp\left(-\frac{\sum_2^{N_\tau - 1}\{G_i(\tau_n) - G_j(\tau_n)\}^2}{\gamma^2}\right)$$

- Minimise Cost function

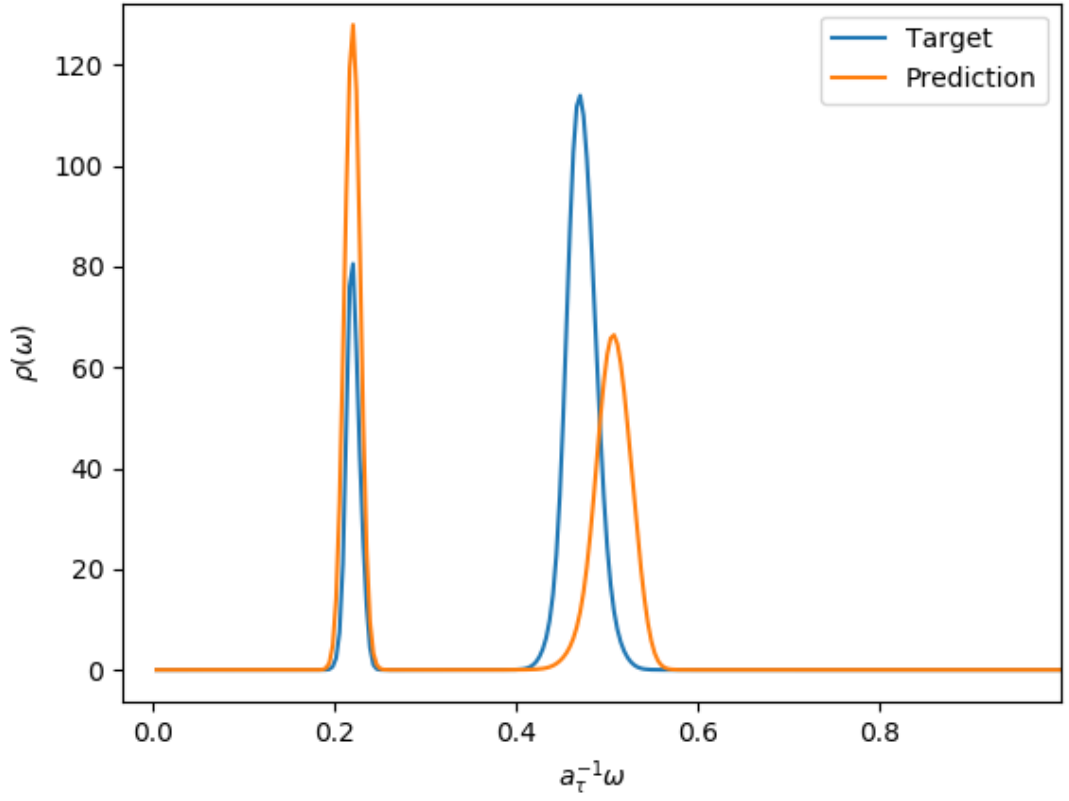
$$\mathrm{E}\left[\mathbf{C}(G_i, G_j), \boldsymbol{\alpha}\right] = \left(\mathbf{Y} - \mathbf{C}^{\mathrm{T}}\boldsymbol{\alpha}\right)^{\mathbf{2}} - \lambda\, \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\boldsymbol{\alpha}$$

$$\boldsymbol{\alpha} = (\mathbf{C} + \mathbf{I}\lambda)^{-\mathbf{1}}\,\mathbf{Y}$$

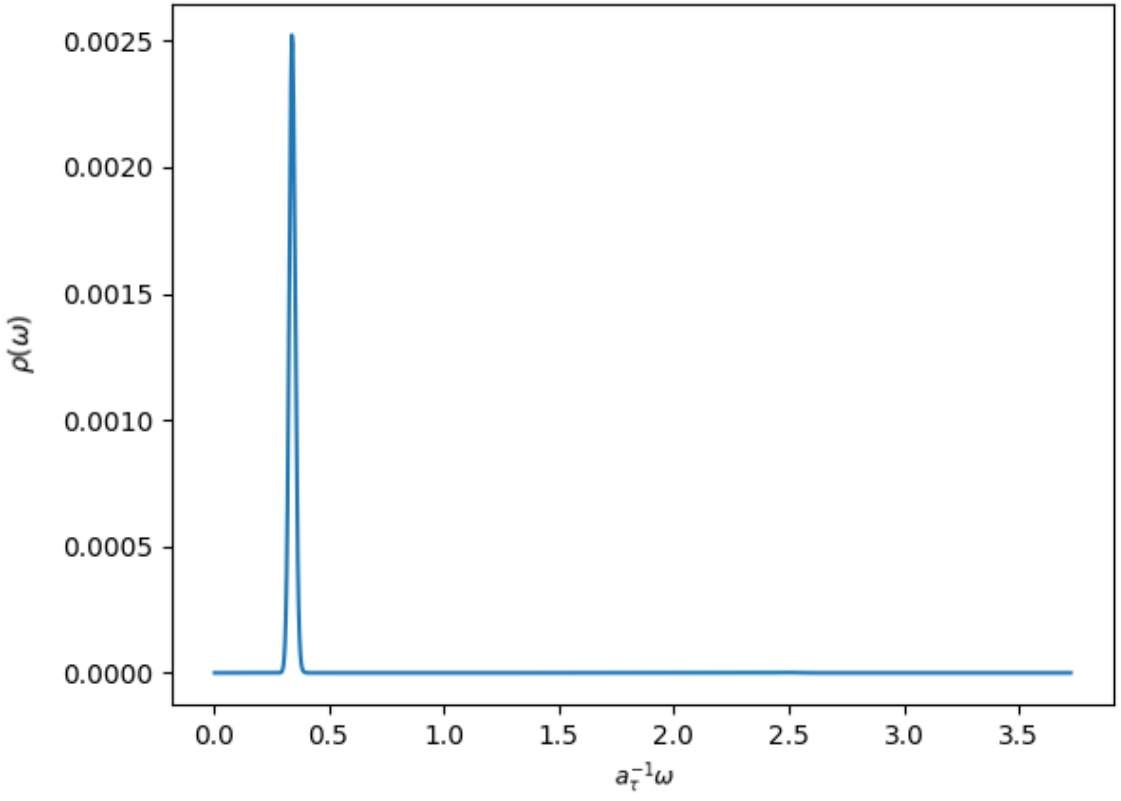
- Can now make predictions using real data using $\mathbf{Y} = \mathbf{C}(G_i, G_{real})^T\boldsymbol{\alpha}$.

Kernel Methods results (Preliminary)

Mock data



Υ Correlation Function



Summary

- We have seen thermal modification that is consistent with our previous generation of ensembles.
- Effective mass plots show consistent results with generation 2 (see arxiv 1402.6210)
- and expected value from experimental data
- MEM reconstruction for the Υ indicates agreement between generations
- KRR in its early stages

Thank you listening

Maximum Entropy Method (MEM)

Need to maximise $P(F|D)$

Bayes Theorem: $P(F|D)P(D) = P(D|F)P(F) \;=\; P(D \cap F)$

i.e. $$P(F|D) = \frac{P(D|F)P(F)}{P(D)}$$

But $P(D|F) \sim e^{-\chi^2}$ $\longrightarrow$ minimising $\chi^2 \neq$ maximising $P(F|D)$

$\longrightarrow$ *Maximum Likelihood Method* wrong??

$$P(F) \sim e^{\alpha S}$$

Shannon-Jaynes Entropy: $$S = \int_0^\infty \frac{d\omega}{2\pi}\left[\rho(\omega) - m(\omega) - \rho(\omega)\ln\frac{\rho(\omega)}{m(\omega)}\right]$$

Asakawa, Hatsuda, Nakahara, Prog.Part.Nucl.Phys. 46 (2001) 459