Study of 2+1 flavor finite-temperature QCD using improved Wilson quarks at the physical point with the gradient flow

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Application of GF to thermodynamics of QCD

Gradient flow

Flowed operators are free from UV divergences and short-distance singularities.

Suzuki's general method based on GF to correctly calculate any renormalized observables on the lattice:

Define renormalized operators in the continuum, and evaluate their flowed ones on the lattice.

Effects of the flow can be removed by taking \( t \to 0 \).

Applicable also to observables whose base symmetry is broken on the lattice (Poincaré inv., chiral sym. ...).

We apply to QCD with Wilson-type quarks, to cope with the problems due to the chiral violation.

Our first study: 2+1 flavor QCD with heavy u,d quarks on a fine lattice

Taniguchi et al.(WHOT-QCD), Phys.Rev. D 95, 054502 (2017); D 96, 014509 (2017)

\[ m_{PS}/m_V \approx 0.63 \]
\[ a \approx 0.07 \text{fm} \]

The method seems to work well:

- **EoS from En.Mom.Tensor**
- **Chiral susceptibility** (disconnected)
- **Topological susceptibility**

- Gluonic and fermionic definitions agree with each other, already at \( a \neq 0 \) with Wilson-type quarks!
- Power-low behavior consistent with DIGA.
- EoS by GF agrees well with conventional integral method at \( T \leq 300 \text{ MeV} \) (\( N_t \geq 10 \)).
- Disagreement at \( T \geq 350 \text{ MeV} \) may be attributed to \( O((at)^2 = 1/Nt^2) \) artifacts at \( N_t \leq 8 \).
- Clear peak at \( T_{pc} \approx 190 \text{ MeV} \), as suggested by Polyakov loop etc.
- Peak higher with decreasing \( m_q \).
1. An extension of the study to 
   **2+1 flavor QCD with physical u,d,s quarks.**

2. Test of **2-loop matching coefficients** for EMT 
   recently calculated by 
   revisiting 2+1 flavor QCD with heavy u,d quarks.
(2+1)-flavor **phys.pt. QCD**

- **RG-improved Iwasaki gauge + NP O(α)-improved Wilson quarks**
- **T=0** configs. of PACS-CS (β=1.9, 32³×64, a≈0.09 fm) [Phys.Rev.D79, 034503 (2009)] 80 configs.
  All quark masses fine-tuned to the **phys.pt.** by reweighting [Phys.Rev.D81, 074503 (2010)] using mπ, mK, m/μ inputs.
- **T>0** by fixed-scale approach, (32³×Nt, Nt = 4, 5, ..., 16, 18): T≈122, 137 – 549 MeV.
  Odd Nt too, to have a finer T-resolution.
  Generated directly at the phys.pt. w/o reweighting [β=1.9, Kud=0.13779625, Ks=0.13663377].
- Gauge meas. every 5 tau, quark meas. every 50 tau.

**Where is Tpc for physical mq?** Expect Tpc^{phys} < 190 MeV.

**Lattice slightly coarser than the heavy QCD case (a≈0.07 fm).**

**Expect lattice artifacts of O((αT)²=1/Nt²) at Nt ≤ 8 (T≥274 MeV)**

<table>
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<th>T[MeV]</th>
<th>T/T_{pc} N_t t_{1/2} gauge confs. fermion confs.</th>
<th>0</th>
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<th>32</th>
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<td>129</td>
<td>17 9.03125</td>
<td>137</td>
<td>16 8 212 212</td>
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<td>146</td>
<td>15 7.03125 42 42</td>
<td>157</td>
<td>14 6.125 650 65</td>
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<td>13 5.28125 550 55</td>
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<td>12 4.5 610 61</td>
<td>199</td>
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<td>10 3.125 690 69</td>
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<td>244</td>
<td>9 2.53125 780 78</td>
<td>274</td>
<td>8 2 680 68</td>
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<td>5 0.78125 130 130</td>
<td>548</td>
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</table>
Gauge and Quark Flows

We adopt the simplest flow by Lüscher:

**Gauge flow:** standard Wilson flow

\[
\partial_t B_\mu(t, x) = D_\nu G_{\nu\mu}(t, x), \quad B_\mu(t = 0, x) = A_\mu(x)
\]

\[
G_{\mu\nu}(t, x) = \partial_\mu B_\nu(t, x) - \partial_\nu B_\mu(t, x) + [B_\mu(t, x), B_\nu(t, x)],
\]

\[
D_\nu G_{\nu\mu}(t, x) = \partial_\nu G_{\nu\mu}(t, x) + [B_\nu(t, x), G_{\nu\mu}(t, x)].
\]

**Quark flow:** as suggested by Lüscher

\[
\partial_t \chi_f(t, x) = \Delta \chi_f(t, x), \quad \chi_f(t = 0, x) = \psi_f(x),
\]

\[
\partial_t \bar{\chi}_f(t, x) = \bar{\chi}_f(t, x)\bar{\Delta}, \quad \bar{\chi}_f(t = 0, x) = \bar{\psi}_f(x),
\]

\[
\Delta \chi_f(t, x) \equiv D_\mu D_\mu \chi_f(t, x), \quad D_\mu \chi_f(t, x) \equiv [\partial_\mu + B_\mu(t, x)] \chi_f(t, x),
\]

\[
\bar{\chi}_f(t, x)\bar{\Delta} \equiv \bar{\chi}_f(t, x)\bar{D}_\mu \bar{D}_\mu, \quad \bar{\chi}_f(t, x)\bar{D}_\mu \equiv \bar{\chi}_f(t, x)\left[\bar{\partial}_\mu - B_\mu(t, x)\right]
\]

only gauge fields involved

**Quark field renormalization:**

\[
\chi_R(t, x) = Z_\chi \chi_0(t, x) \quad Z_\chi = \sqrt{\varphi(t)} \quad \varphi_f(t) = \frac{-6}{(4\pi)^2 t^2 \left< \bar{\chi}_f(t, x) \bar{D}_\mu \chi_f(t, x) \right>_0}.
\]

No more renormalizations needed for any composite op's.
Full QCD En.Mom.Tensor by GF

Measure operators which can mix with EMT at $t \neq 0$:

\[
\tilde{\mathcal{O}}_{1\mu\nu}(t, x) \equiv G_{\mu\rho}(t, x)G^{a}_{\nu\rho}(t, x),
\]
\[
\tilde{\mathcal{O}}_{2\mu\nu}(t, x) \equiv \delta_{\mu\nu}G^{a}_{\rho\sigma}(t, x)G^{a}_{\rho\sigma}(t, x),
\]

and combine them

\[
T_{\mu\nu}(x) = \lim_{t \to 0} \left\{ c_1(t) \left[ \tilde{\mathcal{O}}_{1\mu\nu}(t, x) - \frac{1}{4} \tilde{\mathcal{O}}_{2\mu\nu}(t, x) \right] \\
+ c_2(t) \left[ \tilde{\mathcal{O}}_{2\mu\nu}(t, x) - \langle \tilde{\mathcal{O}}_{2\mu\nu}(t, x) \rangle_0 \right] \\
+ c_3(t) \sum_{f=u,d,s} \left[ \tilde{\mathcal{O}}_{4\mu\nu}(t, x) - 2\tilde{\mathcal{O}}_{5\mu\nu}(t, x) - \langle \tilde{\mathcal{O}}_{5\mu\nu}(t, x) \rangle_0 \right] \\
+ c_4(t) \sum_{f=u,d,s} \left[ \tilde{\mathcal{O}}_{5\mu\nu}(t, x) - \langle \tilde{\mathcal{O}}_{5\mu\nu}(t, x) \rangle_0 \right] \\
+ \sum_{f=u,d,s} c_5^f(t) \left[ \tilde{\mathcal{O}}_{5\mu\nu}(t, x) - \langle \tilde{\mathcal{O}}_{5\mu\nu}(t, x) \rangle_0 \right] \right\},
\]

Physical EMT extracted by $t \to 0$ extrapolation

using matching coefficients by Makino-Suzuki calculated in 1-loop PT:

- to make $t \to 0$ smoother by removing known small-$t$ mixings & $t$-dep. in the continuum
- to match the renormalization schemes when the observable is scheme-dependent

Higher-order errors may affect at small $t \neq 0$.


We discuss it later.
We adopt the renormalization scale $\mu = \mu_0(t) \equiv \frac{1}{\sqrt{2\epsilon^\gamma t}}$ suggested by HKL, in the matching coefficients.

$=>$ wider linear windows $=>$ reduction of uncertainties due to the $t \rightarrow 0$ extrapolation on coarse lattices

$[ t \text{ with } \mu = \mu_0 ] > [ t \text{ with conventional } \mu = 1/\sqrt{(8t)} ]$

$=> \mu_0$ extends the perturbative region towards larger $t$

$\triangleright$ EoS: $(e+p)/T^4$

$\epsilon = -\langle T_{00} \rangle$, $p = \frac{1}{3} \sum \langle T_{ii} \rangle$

$\checkmark$ Similar to the heavy QCD case. The method works well.

(cf) With conventional $\mu = 1/\sqrt{(8t)}$, $t/a^2 > 1.5$ out of pert. region at $a \approx 0.09$fm.
(2+1)-flavor \textit{phys.pt.} QCD with GF

\begin{itemize}
  \item \textbf{EoS:} \((e-3p)/T^4\)
  \item \(\epsilon = -\langle T_{00} \rangle, \quad p = \frac{1}{3} \sum_i \langle T_{ii} \rangle\)
\end{itemize}

Similar to the heavy QCD case. The method seems to work well.

\(T \approx 122\text{-}137\text{MeV} \) in the transition region ??

Need more statistics/data at low \(T\) 's

Borsany et al., JHEP 1011, 077 (2010), KS, cont. lim.
(2+1)-flavor **phys.pt.** QCD with GF

➤ chiral cond. (VEV-subtracted, $\mu=2\text{GeV}$)

- $u, d$

- heavy QCD

*Lighter quarks show a sharper transition/crossover*
(2+1)-flavor \textbf{phys.pt.} QCD with GF

\textbf{chiral suscep.} (disconnected, \( \mu = 2 \text{GeV} \))

- \( u, d \)

\[ \chi_{\bar{u}u, \bar{d}d} \]

\[ \chi_{\bar{s}s} \]

- Similar to the heavy QCD case. The method seems to work well.
- \( T_{pc}^{\text{phys}} < 157 \text{ MeV} \) \((T \approx 122-137 \text{ MeV} \text{ in the transition region ??)}\)
- Need more statistics/data at low \( T \)'s
1. An extension of the study to
   2+1 flavor QCD with physical u,d,s quarks.

2. Test of 2-loop matching coefficients for EMT
   recently calculated by

   revisiting 2+1 flavor QCD with heavy u,d quarks.
2-loop coefficients for EMT

\[ c_1(t) = \frac{1}{g^2} \left\{ 1 + \frac{g^2}{(4\pi)^2} \left[ -\frac{7}{3} C_A + \frac{3}{2} T_F - \beta_0 L(\mu, t) \right] \right. \\
\left. + \frac{g^4}{(4\pi)^4} \left[ -\frac{14482}{405} - \frac{16546}{135} \ln 2 + \frac{1187}{10} \ln 3 \right] + C_A T_F \left( \frac{59}{9} \text{Li}_2 \left( \frac{1}{4} \right) + \frac{10873}{810} + \frac{73}{54} \pi^2 \right) \\
- \frac{2773}{135} \ln 2 + \frac{302}{45} \ln 3 \right) + C_F T_F \\
\left( -\frac{256}{9} \text{Li}_2 \left( \frac{1}{4} \right) + \frac{2587}{108} - \frac{7}{9} \pi^2 \right) \\
- \frac{106}{9} \ln 2 - \frac{161}{18} \ln 3 \right] + O(g^6) \right\} , \]

etc. with \( L(\mu, t) \equiv \ln \left( \frac{2\mu^2 t}{\pi} \right) + \gamma_E \)

Removing known small-\( t \) properties further, we may expect a milder \( t \)-dep. at small \( t \).

Fig. 1: Equation (3.1) as a function of \( \frac{t}{T^2} \) for \( \frac{T}{T_c} = 1 \).

In each panel, the order of perturbation theory and the choice of the renormalization scale are indicated. The errors are statistical only. The extrapolation of the continuum limit (the gray band) to \( t = 0 \) is plotted by the black circle (obtained by the fit range (3.2)), the white circle (obtained by the fit range (3.3)), and the white triangle (obtained by the fit range (3.4)).

first test in quenched QCD


☑ Results of EoS with 1- and 2-loop coefficients are consistent with each other.

☑ With 2-loop coefficients, \( t \)-dep. is milder.

☑ Thus, 2-loop coefficients reduce systematic errors from the \( t \to 0 \) extrapolation.
(2+1)-flavor QCD with 2-loop coefficients

➤ coefficients for full QCD EMT

Harlander et al. used the equation of motion (EoM) in the continuum

\[ 0 = \mathcal{O}_{4,\mu\nu}(x) + 2 \mathcal{O}_{5,\mu\nu}(x) \]

\[ \mathcal{O}_{4,\mu\nu}(x) \equiv \delta_{\mu\nu} \bar{\psi}_f(x) \not\!D \psi_f(x), \]

\[ \mathcal{O}_{5,\mu\nu}(x) \equiv \delta_{\mu\nu} m_{f,0} \bar{\psi}_f(x) \psi_f(x) \]

to reduce the number of independent operators/coefficients, assuming that the EMT operators are isolated.

This should be OK when we take the continuum limit.

However, EoM gets corrections at $a \neq 0$ on the lattice.

$\Rightarrow$ May introduce another source of errors.

(Note 1) EoM not used in the quenched coefficients.

(Note 2) EoM affects the trace-part of EMT only.
(2+1)-flavor heavy QCD with 2-loop coefficients

To test the effects of 2-loop coefficients in full QCD, we have revisited the case of QCD with heavy u,d quarks.

We adopt $\mu_0$ as the ren. scale. (The difference from $1/\sqrt{(8t)}$ was small in EoS on this fine lattice.)

$$\frac{(e+p)}{T^4}$$ in which EoM not used. $\leftarrow$ trace-less combination of EMT

- $m_{PS}/m_V \approx 0.63$
- $a \approx 0.07$fm

- 1-loop and 2-loop results are completely consistent with each other.
- 1-loop results sufficiently flat in $t$ to extract the $t\to 0$ limit on this fine lattice for this operator.

=> no apparent improvements with 2-loop coeff's. in this case.
(2+1)-flavor heavy QCD with 2-loop coefficients

\begin{equation}
\frac{(e-3p)}{T^4}
\end{equation}
in which EoM is used in the 2-loop HKL coefficients.

- 1-loop (w/o EoM) and 2-loop (w/ EoM) results are well consistent with each other.
- 1-loop results sufficiently flat in t
  \Rightarrow no apparent improvements with 2-loop coeff's. in this case.
To identify the effects of EoM, we compare

- 1-loop Makino-Suzuki coefficients w/o EoM
- 1-loop HKL coefficients w/ EoM

They are consistent with each other.

\[ \frac{(e-3p)}{T^4} \] in which EoM is used in the HKL coefficients.
Summary

2+1 flavor **phys. pt. QCD thermodynamics with GF**

- slightly coarser lattice \(a \approx 0.09\) fm, \(32^3 \times Nt (Nt=4,5,\ldots,16,18)\): \(T \approx 122, 237-549\) MeV

- Similar to the heavy QCD case. The method seems to work well here too.
- Choosing \(\mu_0\) as the renormalization scale helps on coarse lattices.
- \(T_{pc}^{phys} < 157\) MeV \(\leq\) Need more statistics/data at low-\(T\)'s.
  Simulations at \(Nt=15, 17, 20\) are coming.

2+1 flv. **heavy QCD revisited with 2-loop coefficients by HKL**

- Works well for EoS.
  But, unlike the qQCD case, no particular benefits so far for EoS.
  \(\leq\) t-dep. well flat in our case.
  They may help on coarse lattices / other observables.

- Effect of EoM looks small in EoS.

We are applying 2-loop coefficients to chiral cond./suscept. and also to the phys. pt.
BACKUP
Suzuki's method

General method to correctly calculate any renormalized observables making use of the finiteness of GF

- Finiteness of GF
  - Can evaluate flowed operators non-perturbatively on the lattice.

- Define renormalized operators in the continuum, and evaluate their flowed ones on the lattice in the cont. lim.. Effects of the flow can be removed by taking $t \to 0$.

- Applicable to any observables related to symmetries violated on the lattice (translational inv., rotational inv., chiral sym. etc.).

- The order of $a \to 0$ and $t \to 0$ may be interchanged when linear window is identified.

Taniguchi et al. (WHOT-QCD), Phys.Rev. D 96, 014509 (2017)
At $a\neq 0$, additional mixing with unwanted operators:

$$T_{\mu\nu}(t, x, a) = T_{\mu\nu}(t, x) + \frac{a^2}{t} + \sum B_{f\mu\nu}(am_f)^2 + C_{\mu\nu}(aT)^2 + D_{\mu\nu}(a\Lambda_{\text{QCD}})^2$$

$$+ a^2 S'_{\mu\nu}(x) + \mathcal{O}(a^4),$$

Singular term at $t \approx 0$ due to mixing with $D=4$ ops. => should be handled properly in the $t\rightarrow 0$ extrapolation.

We remove the contamination of singular terms by identifying "linear windows" in which the linear term looks dominating.

Note: in the case $a\rightarrow 0$ is taken first, one have to remove singular data at small $t$ by hand to get a reliable $a\rightarrow 0$ value.

The method seems to work when we are close to the continuum limit.