Schwinger-Keldysh Formalism for Lattice Gauge Theories

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Nonequilibrium and Nonperturbative

Nonequilibrium processes in particle physics
(ex. the creation and evolution of QGP)
$\longrightarrow$ Nonperturbative framework

• The standard method
$\Rightarrow$ The imaginary-time formalism of lattice QCD

[Taniguchi et al.;1901.01666, Asakawa et al.;hep-lat/0011040]

• Analytic continuation is impossible
$\Rightarrow$ We need to infer spectral functions.
(cf. ansatzes, MEM etc.)

The Schwinger-Keldysh formalism

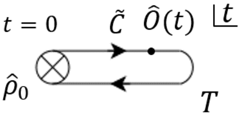
Real-time Green's functions directly

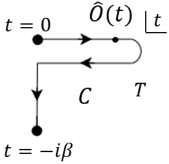
- Formulation for lattice scalar field
- Study of a sign problem with generalized Lefschetz thimble method

[Alexandru et al.;1605.08040, 1704.06404]

$\Longrightarrow$ Schwinger-Keldysh formalism for Lattice QCD

Schwinger-Keldysh formalism





An observable for nonequilibrium processes:

$$\begin{aligned}\langle \hat{O}(t)\rangle &\equiv \mathrm{Tr}\{\hat{\rho}_0 \hat{O}(t)\} \\ &= \mathrm{Tr}\{\hat{\rho}_0\ \mathrm{e}^{i\hat{H}T}\mathrm{e}^{-i\hat{H}(T-t)}\hat{O}_S\ \mathrm{e}^{-i\hat{H}t}\} \\ &= \int \mathscr{D}\phi\ \langle\phi_0|\hat{\rho}_0|\phi_0'\rangle\ O(t)\ \mathrm{e}^{i\int_{\tilde{C}} dt \mathscr{L}[\phi,\partial\phi]} \\ &\to \int \mathscr{D}\phi\ O(t)\ \mathrm{e}^{i\int_{C} dt \mathscr{L}[\phi,\partial\phi]}\Big|_{\phi(0)=\pm\phi(-i\beta)}\end{aligned}$$

$$(\hat{\rho}_0 = \mathrm{e}^{-\beta\hat{H}}/Z)$$

Strategies for lattice formulation of SK formalism (1)

- Continuum theories

$$\langle \hat{O}(t) \rangle_\beta = \mathrm{Tr}\left\{ \mathrm{e}^{-\beta \hat{H}} \ \mathrm{e}^{i\hat{H}T} \ \mathrm{e}^{-i\hat{H}(T-t)} \hat{O}_S \ \mathrm{e}^{-i\hat{H}t} \right\} / Z, \quad (\hat{\rho}_0 = \mathrm{e}^{-\beta \hat{H}}/Z)$$

- Lattice theories

$$Z = \mathrm{Tr}\{\hat{\rho}_0\} = \mathrm{Tr}\{(\hat{T}_{+1})^{N_\beta}\}, \quad \text{where } \hat{T}_{+1} = \mathrm{e}^{-a_0 \hat{H}_{\mathrm{latt}}}.$$

$$\langle \hat{O}(n_t) \rangle_\beta \equiv \mathrm{Tr}\left\{ (\hat{T}_{+1})^{N_\beta} (\hat{T}_{-i})^{N_T} (\hat{T}_{+i})^{N_T - n_t} \hat{O}_S (\hat{T}_{+i})^{n_t} \right\} / Z$$

- The ideal method

$$\hat{T}_{+1} = \mathrm{e}^{-a_0 \hat{H}_{\mathrm{latt}}} \longrightarrow \hat{T}_{\pm i} \equiv \mathrm{e}^{\mp i a_0 \hat{H}_{\mathrm{latt}}}$$

- A typical imaginary-time TM

$$\hat{T}_{+1} = \mathrm{e}^{-a_0 \sum_{\boldsymbol{n}} \frac{V(\hat{\phi}(x))}{2}} \ \mathrm{e}^{-\sum_{\boldsymbol{n}} \frac{a_0}{2} \hat{\Pi}^2(x)} \ \mathrm{e}^{-a_0 \sum_{\boldsymbol{n}} \frac{V(\hat{\phi}(x))}{2}}$$

Simple replacement $a_0 \to i a_0 \implies$ unitary, but $[\hat{T}_{+1}, \hat{T}_{\pm i}] \neq 0$.

Strategies for lattice formulation of SK formalism (2)

Our conditions for real-time TMs

- Natural extentions of the imaginary-time TMs
- Unitarity condition

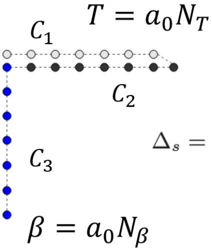
$$\hat{T}_{+i}\hat{T}_{-i} = 1, \quad \hat{T}_{\pm i} = \hat{A}\hat{U}_{\pm i}\hat{A}^{-1}, \quad \hat{U}^{\dagger}_{\pm i} = \hat{U}_{\mp i} = \hat{U}^{-1}_{\pm i}.$$

- Commutativity condition

$$[\hat{T}_{+1}, \hat{T}_{\pm i}] = O(a).$$

A partition function and action
of the lattice QCD in the SK formalism.

$$Z = \mathrm{Tr}\left\{(\hat{T}_{+1})^{N_\beta}(\hat{T}_{-i})^{N_T}(\hat{T}_{+i})^{N_T}\right\}$$
$$= \int [\mathcal{D}\phi] \mathrm{e}^{-S_{\mathrm{SK/latt}}}$$



$$\Delta_s = \begin{cases} i & (s \in C_1), \\ -i & (s \in C_2), \\ 1 & (s \in C_3). \end{cases}$$

TMs and SK formalism for lattice QCD

•Imaginary-time TM of Gauge bosons $\quad (U_\tau = U_i(\tau, \boldsymbol{n}))$

$$T^G_{+1}(U_\tau, U_{\tau+1}) = \mathrm{e}^{-\frac{a}{2}(V(U_\tau)+V(U_{\tau+1}))}$$
$$\times \prod_{i,\boldsymbol{n}} e^{-6N/ \quad g_0^2 a} \sum_R d_R \left(L_R(1/g_0^2 a)\right) \times \mathrm{Tr}_R(U_\tau U^\dagger_{\tau+1})$$

$V(U_\tau) = \sum_{k,l,\boldsymbol{n}}[1-(U_{kl}+U^\dagger_{kl})/2]/g_0^2 a, \quad U_{kl}$: plaquettes

•Imaginary-time TM of Wilson fermions

$$T^F_{+1} = A_{\tau+1}\ (1-\ \mathcal{H}_{\tau+1}/2)\ \frac{1}{1+\mathcal{H}_\tau/2} A_\tau^{-1},\ \mathrm{w}/\ A_\tau = B_\tau^{-1/2}[(2+aD_{3W})\gamma_0]_\tau$$

$\mathcal{H}_\tau = \gamma_0 a_0 D_{3W,\tau}\ \frac{1}{2+a_0 D_{3W,\tau}},\ B_\tau = \delta_{\boldsymbol{n},\boldsymbol{n}'} + a_0(m_0\ \delta_{\boldsymbol{n},\boldsymbol{n}'} + \sum_k \frac{\nabla_k \nabla_k^\dagger}{2})$

•TM for SK formalism for lattice QCD $\quad \hat{T}_{+1} = \hat{T}^G_{+1} \otimes \hat{T}^F_{+1}$

$$Z^{\mathrm{QCD}}_{\mathrm{latt}}[\beta] = \mathrm{Tr}\{(\hat{T}_{+1})^{N_\beta}\}$$
$$= \int \mathscr{D}U \mathscr{D}\psi \mathscr{D}\overline{\psi} \mathrm{e}^{-(S_G[U]+S_F[\psi,\overline{\psi},\mu])}.$$

TMs and SK formalism for lattice QCD

• Real-time TM of Gauge bosons $(U_s = U_i(s, \boldsymbol{n}))$

$$T^G_{\Delta}(U_s, U_{s+1}) = \mathrm{e}^{-\frac{\Delta_s a}{2}(V(U_s)+V(U_{s+1}))} \times \prod_{i,\boldsymbol{n}} e^{-6N/\Delta_s g_0^2 a} \sum_R d_R \left(L_R(1/g_0^2 a)\right)^{\Delta_s} \mathrm{Tr}_R(U_s U^\dagger_{s+1})$$

$V(U_s) = \sum_{k,l,\boldsymbol{n}} [1 - (U_{kl} + U^\dagger_{kl})/2]/g_0^2 a$, U_{kl}: plaquettes

• Real-time TM of Wilson fermions

$$T^F_{\Delta} = A_{s+1}\ (1 - \Delta_{s+1}\mathcal{H}_{s+1}/2)\ \frac{1}{1+\Delta_{s+1}\mathcal{H}_s/2} A_s^{-1},\ \text{w/}\ A_s = B_s^{-1/2}[(2 + aD_{3W})\gamma_0]_s$$

$\mathcal{H}_s = \gamma_0 a_0 D_{3W,s}\ \frac{1}{2+a_0 D_{3W,s}}$, $B_s = \delta_{\boldsymbol{n},\boldsymbol{n}'} + a_0(m_0\ \delta_{\boldsymbol{n},\boldsymbol{n}'} + \sum_k \frac{\nabla_k \nabla_k^\dagger}{2})$

• TM for SK formalism for lattice QCD $\hat{T}_{\Delta} \equiv \hat{T}^G_{\Delta} \otimes \hat{T}^F_{\Delta}$

$$Z^{\mathrm{QCD}}_{\mathrm{SK/latt}}[\beta, T] = \mathrm{Tr}\{(\hat{T}_{+1})^{N_\beta} (\hat{T}_{-i})^{N_T} (\hat{T}_{+i})^{N_T}\} \quad (\alpha_{s\Delta_s} = [\Delta_s D_{3W,s} + B_s^{1/2} A_s]/2)$$

$$= \int \mathscr{D}U \mathscr{D}\psi \mathscr{D}\overline{\psi} \mathrm{e}^{-(S_{G/SK}[U] + S_{F/SK}[\psi,\overline{\psi},\mu])} \times \prod_{s=1}^{2N_T} \{\det\ \alpha_{s\Delta_s}\}^{-1}.$$

The action for gauge fields and fermionic fields

• $U_0(s,\boldsymbol{n}) = 1$ except $s = 2N_T + N_\beta - 1$ (Polyakov loops)

SK action for the lattice gauge bosons

$$S_{G/SK} = \sum_s \left\{ \frac{\Delta_s a}{2} (V(U_s) + V(U_{s+1})) + K_{\Delta_s}(U_s, U_{s+1}) \right\}$$

$$\mathrm{e}^{-K_{\Delta_s}(U_s,U_{s+1})} = \begin{cases} \exp\left[-\sum_{\boldsymbol{n},i} \frac{1}{g_0^2 a} \mathrm{Tr}[2 - U_s U_{s+1}^\dagger - U_{s+1} U_s^\dagger] \right], \ (s \in C_3) \\ \prod_{\boldsymbol{n},i} \left(\mathrm{e}^{-\frac{2N}{g_0^2 \Delta_s a}} \sum_R d_R \left(L_R(1/g_0^2 a) \right)^{\Delta_s} \mathrm{Tr}_R(U_s U_{s+1}^\dagger) \right), \\ \qquad (s \in C_1 \cup C_2) \end{cases}$$

SK action for the Wilson fermions

$$S_{F/SK} = \sum_{n,s,s'} \overline{\psi}(s,\boldsymbol{n}) \left[-\left(\frac{1-\gamma_0}{2} \right) \nabla_0 - \left(\frac{1+\gamma_0}{2} \right) \nabla_0^\dagger + a_0 D_{3W} V_{s,s'} \right] \psi(s',\boldsymbol{n})$$

$$V_{s,s'} = \begin{pmatrix} \frac{1+\Delta_s}{2}\delta_{s,s'} - \frac{1-\Delta_s}{2}\delta_{s,s'+1} & 0 \\ 0 & \frac{1+\Delta_{s-1}}{2}\delta_{s,s'} - \frac{1-\Delta_s}{2}\delta_{s+1,s'} \end{pmatrix}.$$

- Response to time-dependent external sources

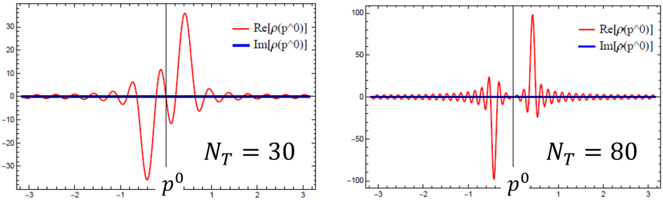
ex. EM field($U(1)$ link field) $\quad V_\mu(t,\boldsymbol{n}) = \mathrm{e}^{ia_0 A_\mu(t,\boldsymbol{n})}$

$$\Delta \langle \hat{J}_k(t,\boldsymbol{n})\rangle_\beta = ia_0 \sum_{t'=0}^{t} a^d \sum_{\boldsymbol{n}'} \langle [\hat{J}(t,\boldsymbol{n}), \hat{J}(t',\boldsymbol{n}')]\rangle_\beta A_0(t',\boldsymbol{n}'),$$

$$\sigma = \frac{1}{d}\sum_{t=0}^{T}\sum_{s=t}^{-N_\beta - t}\sum_{\boldsymbol{n}} \langle \hat{J}_k(t,\boldsymbol{n})\hat{J}_k(s,\boldsymbol{n})\rangle_\beta = \frac{1}{d} N_\beta \sum_{t=0}^{T} G_{kk}^{\mathrm{Kubo}}(n,n')$$

- Spectral function

$\rho(p;N_T,N_\beta,a_0) := \sum_{t=0}^{N_T} \mathrm{e}^{ip^0(t-N_T/2)} \langle [\hat{\phi}(t), \hat{\phi}(N_T/2)]\rangle$.



$a_0 = 1,\ N_\beta = 8,\ \hat{m}_0 = 0,\ \hat{p}_1 = 0.4267$

Summary and future works

Summary

- We have constructed the real-time TMs for the link fields and the fermionic fields, and formulated the SK formalism for lattice QCD.
- We have investigated linear response theory in this formalism, and defined the spectral function nonperturbatively.

Future works

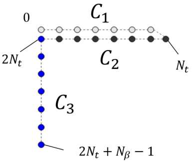
- Renormalizability of the real-time Green's functions
- Construction of energy momentum tensor in the SK formalism (cf. gradient flow method [Suzuki;1304.0533])
- Applications of the Lefschez thimble and the complex Langevin methods

Appendix

Exact expressions of scalar Green's functions in LSK

$$G_C(s,s') = \begin{pmatrix} \Delta_F(t,t') & \Delta(t,\tau') \\ \Delta^T(\tau,t') & \Delta_\beta(\tau,\tau') \end{pmatrix},$$

$$\begin{array}{l} s \in C_1 \cap C_2 \cap C_3, \\ t \in C_1 \cap C_2,\ \tau \in C_3 \end{array}, \quad t = \begin{cases} \tilde{t} & (t \in C_1) \\ 2N_T - \tilde{t} & (t \in C_2) \end{cases}$$



$$\begin{aligned} \Delta_F(t,t') = A(E') \Big(& \mathrm{e}^{-iE\tilde{t}} \left[\alpha_+^2 \mathrm{e}^{iE\tilde{t}'} + \alpha_+\alpha_- \mathrm{e}^{-iE\tilde{t}'} + \alpha_+\alpha_- \mathrm{e}^{-NE} \mathrm{e}^{iE\tilde{t}'} + \alpha_-^2 \mathrm{e}^{-NE} \mathrm{e}^{iE\tilde{t}'} \right] \\ & + \mathrm{e}^{iE\tilde{t}} \left[\alpha_-^2 \mathrm{e}^{-iE\tilde{t}'} + \alpha_+\alpha_- \mathrm{e}^{iE\tilde{t}'} + \alpha_+\alpha_- \mathrm{e}^{-NE} \mathrm{e}^{iE\tilde{t}'} + \alpha_+^2 \mathrm{e}^{-NE} \mathrm{e}^{-iE\tilde{t}'} \right] \theta_C(t-t') \\ & + (t \leftrightarrow t') \Big) \end{aligned}$$

$$\Delta_\beta(\tau,\tau') = A(E') \left\{ \mathrm{e}^{-E'|\tau-\tau'|} + \mathrm{e}^{-NE'} \mathrm{e}^{E'|\tau-\tau'|} \right\}$$

$$\begin{aligned} \Delta(t,\tau') = A(E') \Big\{ & \mathrm{e}^{-iE\tilde{t}} \left[\alpha_+ \mathrm{e}^{-E'(N_\beta-\tau')} + \alpha_- \mathrm{e}^{-E'\tau'} \right] \\ & + \mathrm{e}^{iE\tilde{t}} \left[\alpha_- \mathrm{e}^{-E'(N_\beta-\tau')} + \alpha_+ \mathrm{e}^{-iE'\tau'} \right] \Big\} \end{aligned}$$

$$A(E') = \frac{1}{2\sinh\ E'} \frac{1}{1-\mathrm{e}^{-NE'}},$$

$$\alpha_\pm = \frac{1}{2}\left(1 \pm \frac{\sinh\ E'}{\sin\ E}\right),\ \cos\ E = 1 - \frac{w^2}{2},\ \cosh\ E' = 1 + \frac{w^2}{2}$$

Free Green's functions and the spectral function

- We get analytic solutions of free Green's functions from the action.
- Definition of spectral functions on a lattice with finite lattice size

$$\rho(p; N_T, N_\beta, a_0) := \sum_{n_t=0}^{N_T} \mathrm{e}^{ip^0(n_t - N_T/2)} \left\langle [\hat{\phi}(n_t), \hat{\phi}(N_T/2)] \right\rangle$$

$$= \frac{1}{2\sin\ E} \sum_{n'_t=-N_T/2}^{N_T/2} (\cos(p^0 - E)n'_t - \cos(p^0 + E)n'_t)$$

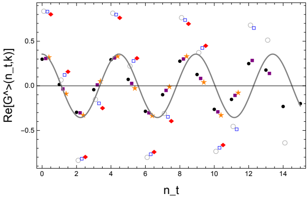
$$\xrightarrow{N_T \to \infty} 2\pi\ \mathrm{sign}\ (p^0)\delta(4\sin^2\ \frac{p^0}{2} - \sum_{i=1}^{d} 4\sin^2\ \frac{p^i}{2} - m_0^2)$$

$$\xrightarrow{a \to 0} 2\pi\ \mathrm{sign}\ (p^0)\delta((p^0)^2 - \vec{p}^2 - m_0^2)$$

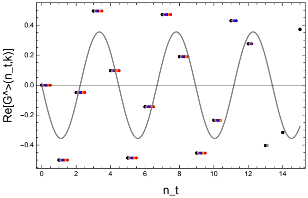
$\cosh\ E = 1 + \frac{w^2}{2},\ w^2 = \sum_{i=1}^{d} 4\sin^2\ \frac{p^i}{2} + m_0^2$

Time translation properties of $G^>$ in equilibrium

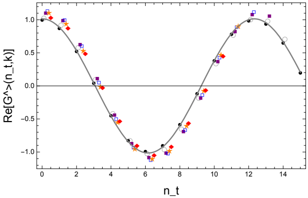
$G^>(n_t, n_t') = \langle \hat{\phi}(n_t) \hat{\phi}(n_t') \rangle$ (Various reference times)



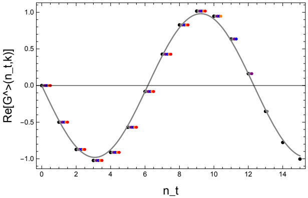
(a) $\hat{m}_0 = 1,\ p_1 = 0.99$ (Re)



(b) $\hat{m}_0 = 1,\ p_1 = 0.99$ (Im)



(a) $\hat{m}_0 = 0.1,\ p_1 = 0.5$ (Re)



(b) $\hat{m}_0 = 0.1,\ p_1 = 0.5$ (Im)

The action and TMs for the scalar field

SK action for the lattice scalar boson [Alexandru et al.;1704.06404]

$$S_{SC/SK}$$
$$= \sum_{s,\boldsymbol{n}} a^4 \left[\frac{(\phi_{s+1,\boldsymbol{n}} - \phi_{s,\boldsymbol{n}})^2}{2\Delta_s a^2} + \frac{\Delta_s}{2} \sum_{\boldsymbol{k}} \left(\frac{(\phi_{s+1,\boldsymbol{n}+\hat{\boldsymbol{k}}} - \phi_{s+1,\boldsymbol{n}})^2}{2a^2} + \frac{(\phi_{s,\boldsymbol{n}+\boldsymbol{k}} - \phi_{s,\boldsymbol{n}})^2}{2a^2} \right) - \frac{\Delta_s}{2} \left(V(\phi_{s,\boldsymbol{n}}) + V(\phi_{s+1,\boldsymbol{n}}) \right) \right]$$

•Description in TMs

$$Z = \mathrm{Tr} \left\{ (\hat{T}^{SC}_{+1})^{N_\beta} (\hat{T}^{SC}_{-i})^{N_T} (\hat{T}^{SC}_{+i})^{N_T} \right\} = \int \mathcal{D}[\phi] \mathrm{e}^{-S_{SC/SK}}$$

$$\hat{T}^{SC}_{\Delta} \equiv \mathrm{e}^{-a\Delta_s \frac{V(\hat{\phi}(x))}{2}} \mathrm{e}^{-\frac{a\Delta_s}{2} \hat{\Pi}^2(x)} \mathrm{e}^{-a\Delta_s \frac{V(\hat{\phi}(x))}{2}}$$

$$T^{SC}_{+i} T^{SC}_{-i} = 1, \qquad \left(T^{SC}_{+i}\right)^\dagger = T^{SC}_{-i}, \qquad \left[T^{SC}_{+1}, T^{SC}_{\pm i}\right] \neq 0$$

The action for gauge fields

SK action for the lattice gauge boson

$$S_{G/SK} = \sum_s \left\{ \frac{\Delta_s a}{2}(V(U_s) + V(U_{s+1})) + K_{\Delta_s}(U_s, U_{s+1}) \right\}$$

$$V(U_s) = \textstyle\sum_{k,l,\boldsymbol{n}}[1 - (U_{kl}(n) + U_{kl}^\dagger)/2]/g_0^2 a$$

$$\mathrm{e}^{-K_{\Delta_s}(U_s,U_{s+1})} = \begin{cases} \exp\left[-\sum_{\boldsymbol{n},i} \frac{1}{g_0^2 a} \mathrm{Tr}[2 - U_s U_{s+1}^\dagger - U_{s+1} U_s^\dagger]\right], & (s \in C_3) \\ \prod_{\boldsymbol{n},i} \left(\mathrm{e}^{-\frac{2N}{g_0^2 \Delta_s a}} \sum_R d_R \left(L_R(1/g_0^2 a)\right)^{\Delta_s} \mathrm{Tr}_R(U_s U_{s+1}^\dagger) \right), & (s \in C_1 \cup C_2) \end{cases}$$

Caracter expansion ($U \in$SU(N), R, d_R: a label of irreducible representation and its dimension)

$$\exp\left[\frac{1}{g_0^2 a}\mathrm{Tr}(U + U^\dagger)\right] = \sum_R d_R L_R(1/g_0^2 a)\mathrm{Tr}(U),$$

$$L_R(1/g_0^2 a) \equiv \frac{1}{d_R}\int dU \mathrm{Tr}(U_R)\ \exp\ \left[\frac{1}{g_0^2 a}\mathrm{Tr}(U + U^\dagger)\right]$$

The action for fermionic fields

SK action for the Wilson fermions

$$S_{F/SK} = \sum_{n,n'} \overline{\psi}(n) \left[-\left(\frac{1-\gamma_0}{2}\right) \delta_{s+1,s'} \delta_{\boldsymbol{n},\boldsymbol{n}'} - \left(\frac{1+\gamma_0}{2}\right) \delta_{s,s'+1} \delta_{\mathbf{n},\mathbf{n}'} + 1\dot{\delta}_{s,s'} \delta_{\boldsymbol{n},\boldsymbol{n}'} + aV_{s,s'} D_{3W} \right] \psi(n')$$

$$D_{3W,\tau} = \sum_k \left(\frac{\nabla_k \nabla_k^\dagger}{2} + \gamma_k \frac{\nabla_k - \nabla_k^\dagger}{2} \right) + m_0 \delta_{\boldsymbol{n},\boldsymbol{n}'},$$

$$\nabla_k = \delta_{\boldsymbol{n}+\hat{\boldsymbol{k}},\boldsymbol{n}'} U_k(n) - \delta_{\boldsymbol{n},\boldsymbol{n}'}, \quad \nabla_k^\dagger = \delta_{\boldsymbol{n},\boldsymbol{n}'+\hat{\boldsymbol{k}}} U_k^\dagger(n) - \delta_{\boldsymbol{n},\boldsymbol{n}'},$$

$$\frac{\nabla_k \nabla_k^\dagger}{2} \equiv -\frac{\nabla_k + \nabla_k^\dagger}{2}$$

$$V_{s,s'} = \begin{pmatrix} \frac{1+\Delta_s}{2}\delta_{s,s'} - \frac{1-\Delta_s}{2}\delta_{s,s'+1} & 0 \\ 0 & \frac{1+\Delta_{s-1}}{2}\delta_{s,s'} - \frac{1-\Delta_s}{2}\delta_{s+1,s'} \end{pmatrix}.$$