On the Lefschetz thimbles structure of the Thirring model

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Outline

Introduction

The sign problem

- We would like to compute expectation values

$$\langle O \rangle \equiv \frac{1}{Z} \int D\psi \, O[\psi] e^{-S[\psi]}$$

  by importance sampling, sampling fields configurations from $P \propto e^{-S[\psi]}$.

- However, there are cases of physical interests where S is complex $\rightarrow$ sign problem

- Some interesting approaches to attack the sign problem are those in which one complexifies the field variables (these are the subject of today's and tomorrow's parallel talks).

Introduction

Lefschetz thimbles regularization

- One of such approaches is thimble regularization. Idea: complexify the degrees of freedom of the theory and deform the integration paths. Picard-Lefschetz theory: attached to each critical point p_σ exists a manifold $\mathcal{J}_\sigma$ s.t.

$$\int_{\mathcal{C}} dz^n\, O(z)\, e^{-S(z)} = \sum_\sigma n_\sigma e^{-iS^I_\sigma} \int_{\mathcal{J}_\sigma} dz^n\, O(z)\, e^{-S^R_\sigma}$$

- The thimble $\mathcal{J}_\sigma$ attached to a critical point p_σ is the union of the steepest ascent paths leaving the critical points

$$\frac{dz_i}{dt} = \frac{\partial \bar{S}}{\partial \bar{z}_i}\ , \text{ with i.c. } z_i(-\infty) = z_{\sigma,i}$$

Along the flow, the imaginary part of the action is constant.

- The tangent space at p_σ is spanned by the Takagi vectors, which can be found by diagonalizing the Hessian at the critical point

$$H(p_\sigma) v^{(i)} = \lambda^\sigma_i \bar{v}^{(i)}$$

Introduction

Lefschetz thimbles regularization

- A natural parametrization for a point on the thimble is $z \in \mathcal{J}_\sigma \leftrightarrow (\hat{n}, t)$, where $\hat{n}$ defines the direction along which the path leaves the critical point and t is the integration time.

- Using this parametrization, the thimbles decomposition of an expectation value $\langle O \rangle$ takes the form

$$\langle O \rangle = \frac{\sum_\sigma n_\sigma \int D\hat{n}\, 2 \sum_i \lambda_i^\sigma n_i^2 \int dt\, e^{-S_{eff}(\hat{n},t)}\, O\, e^{i\omega(\hat{n},t)}}{\sum_\sigma n_\sigma \int D\hat{n}\, 2 \sum_i \lambda_i^\sigma n_i^2 \int dt\, e^{-S_{eff}(\hat{n},t)}\, e^{i\omega(\hat{n},t)}}$$

where $V(\hat{n}, t)$ is the parallel transported basis, $S_{eff}(\hat{n}, t) = S_R(\hat{n}, t) - log\, |detV(\hat{n}, t)|$ and $\omega(\hat{n}, t) = arg(detV(\hat{n}, t)))$.

Introduction

Lefschetz thimbles regularization

- Observe that, when only one thimble contributes, one can rewrite $\langle O \rangle = \frac{\langle O e^{i\omega} \rangle_\sigma}{\langle e^{i\omega} \rangle_\sigma}$, having defined

$$\langle f \rangle_\sigma = \int D\hat{n} \, \frac{Z_{\hat{n}}}{Z_\sigma} \, f_{\hat{n}}$$

$$Z_\sigma = \int D\hat{n} \, Z_{\hat{n}}, \; Z_{\hat{n}} = \left(2 \sum_i \lambda_i^\sigma n_i^2\right) \int dt \, e^{-S_{eff}(\hat{n},t)}$$

$$f_{\hat{n}} = \frac{1}{Z_{\hat{n}}} \left(2 \sum_i \lambda_i^\sigma n_i^2\right) \int dt \, f(\hat{n}, t) e^{-S_{eff}(\hat{n},t)}$$

$\rightarrow$ importance sampling, $P_{acc}(\hat{n}' \leftarrow \hat{n}) = min\left(1, \frac{Z_{\hat{n}'}}{Z_{\hat{n}}}\right)$.

- Can be generalized to more than one thimble:

$$\langle O \rangle = \frac{\sum_\sigma n_\sigma Z_\sigma \langle O e^{i\omega} \rangle_\sigma}{\sum_\sigma n_\sigma Z_\sigma \langle e^{i\omega} \rangle_\sigma}$$

- Let's consider the $0+1$-dimensional Thirring model

$$S = \beta \sum (1 - cos(z_n)) + log\, detD$$

$$detD = \frac{1}{2^{L-1}}(cosh(L\hat{\mu} + i\sum z_n) + cosh(L\hat{m}))\ ,\ \hat{\mu} \equiv a\mu\ ,\ \hat{m} = asinh(am)$$

- It has been shown before that one thimble is not enough to capture the full content of the theory (i.e. JHEP1511(2015)078; but see also JHEP 1605(2016)053)

- We wanted to try collecting contributions from different thimbles with our approach

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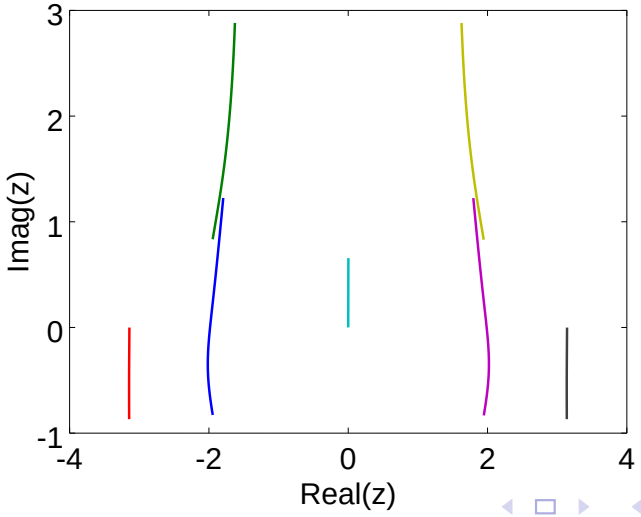
Critical points of the theory ($L = 2$, $\beta = 1$)

- To test if we are able to recover the results of the full theory, we first start from the simple case of $L = 2$, with a coupling $\beta = 1$; the critical points are determined by imposing

$$\frac{\partial S}{\partial z_n} = \beta sin(z_n) - i \frac{sinh(L\mu + i \sum z_n)}{cosh(L\mu + i \sum z_n) + cosh(Lm)} = 0$$

  The second term depends on the fields only through the sum $s \equiv \sum z_n$, then it must be $sin(z_n) = sin(z)\ \forall n$ and z_n can be either z or $\pi - z$.

- Critical points in the $n_- = 0$ sector for $\mu = 0 \ldots 2$



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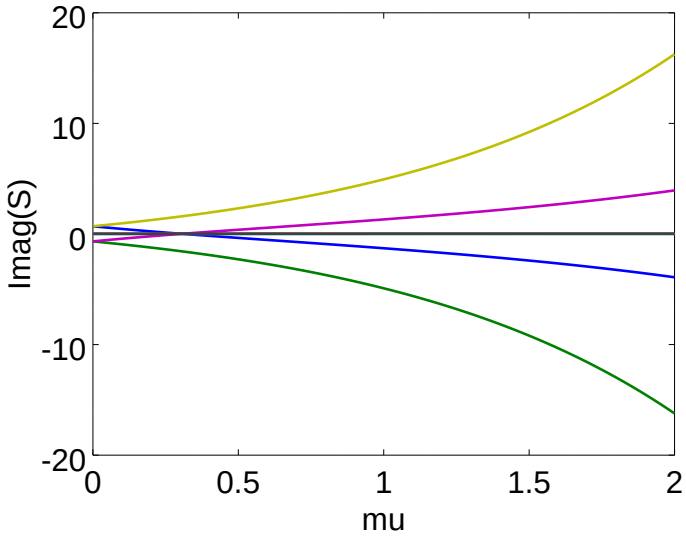
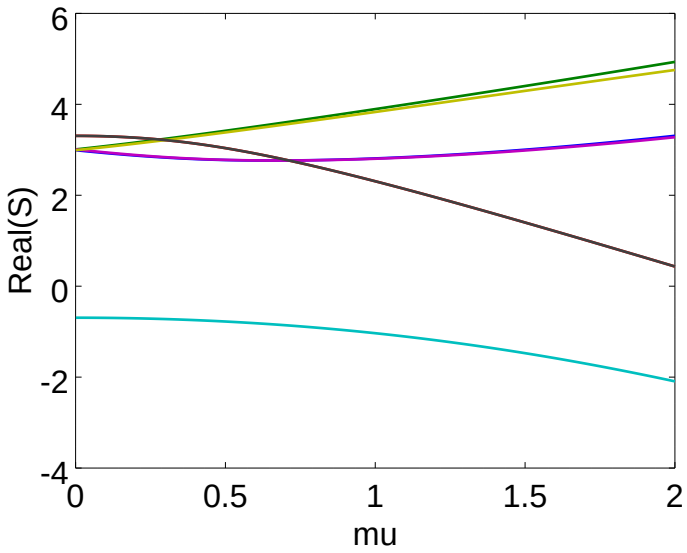
Stokes phenomenon ($L = 2$, $\beta = 1$)

- We look for changes in the intersection numbers after Stokes phenomena



(a) SI



(b) SR

Note: due to a simmetry, the two thimbles that enter the thimbles decomposition at $\mu \gtrsim 0.30$ give conjugate contributions.

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Results from MC ($L = 2$, $\beta = 1$)

- How to collect the contributions from more than one thimble?
- Last year we proposed to calculate the corrections to the semiclassical weight as $\langle \frac{Z^G_{\hat{n}}}{Z_{\hat{n}}} \rangle^{-1}$, it turns out this doesn't work very well for this theory.
- Nevertheless, we can resort to the method we proposed previously:

$$\langle O \rangle = \frac{n_0 Z_0 e^{-iS_0^I} \langle O e^{i\omega_0} \rangle_0 + n_{12} Z_{12} e^{-iS_{12}^I} \langle O e^{i\omega_{12}} \rangle_{12}}{n_0 Z_0 e^{-iS_0^I} \langle e^{i\omega_0} \rangle_0 + n_{12} Z_{12} e^{-iS_{12}^I} \langle e^{i\omega_{12}} \rangle_{12}}$$

$$\equiv \frac{\langle O e^{i\omega_0} \rangle_0 + \alpha \langle O e^{i\omega_{12}} \rangle_{12}}{\langle e^{i\omega_0} \rangle_0 + \alpha \langle e^{i\omega_{12}} \rangle_{12}} \ .$$

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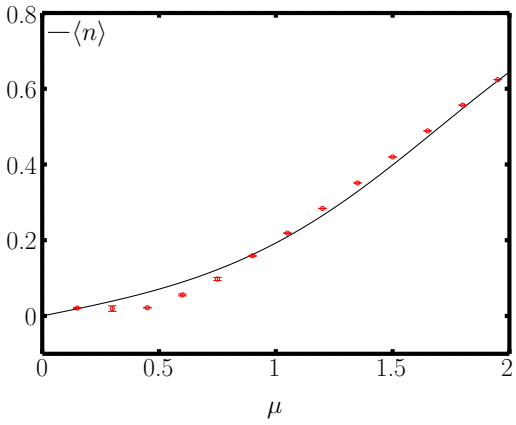
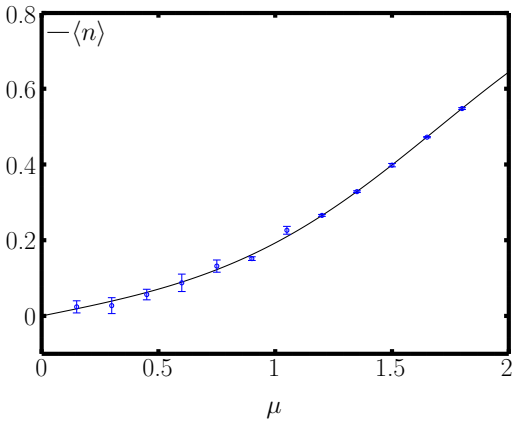
Results from MC ($L = 2$, $\beta = 1$)

- Fermion number density

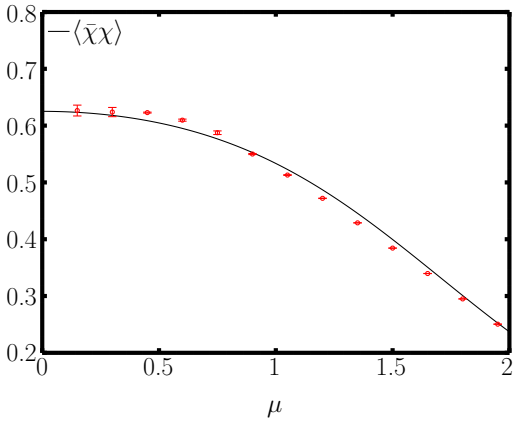


(c) Results from σ_0

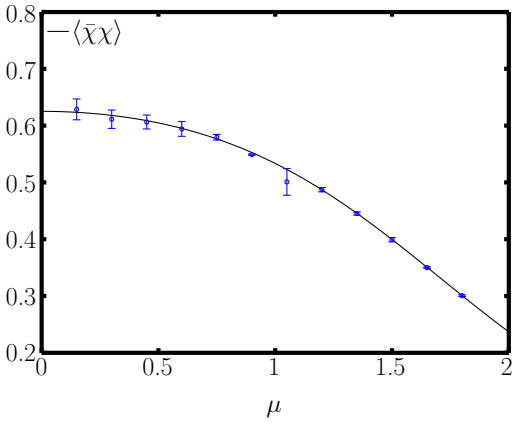


(d) Results from $\sigma_{0,i}$

- Condensate

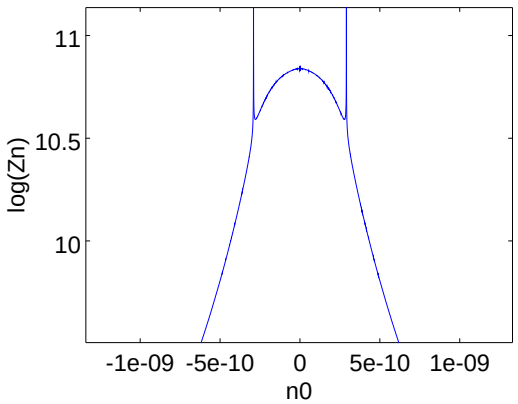


(e) Results from σ_0

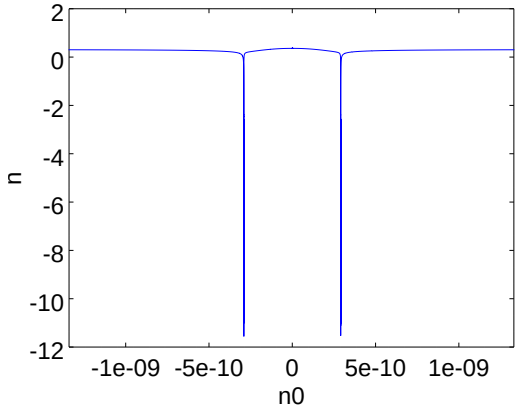


(f) Results from $\sigma_{0,i}$

- At least in this case, where we only have 2 degrees of freedom, one may also compute expectation values by direct numerical integration.



(g) $LogZ_n$ vs n_0



(h) n vs n_0

- Strong dependence of Z_n on n_0, that is the component of the initial displacement along the tangent space at the critical point associated to the larger Takagi value. Sharp peaks in some regions of n_0. These observations appear to hold also for higher L.

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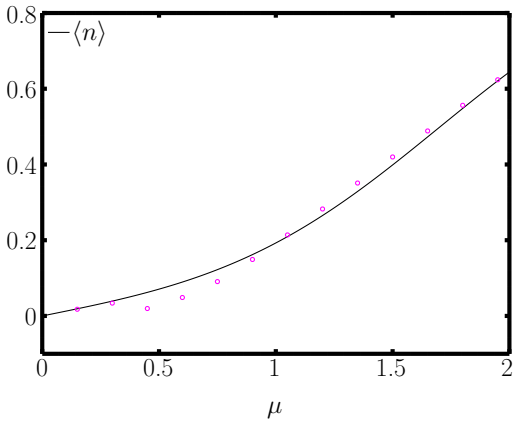
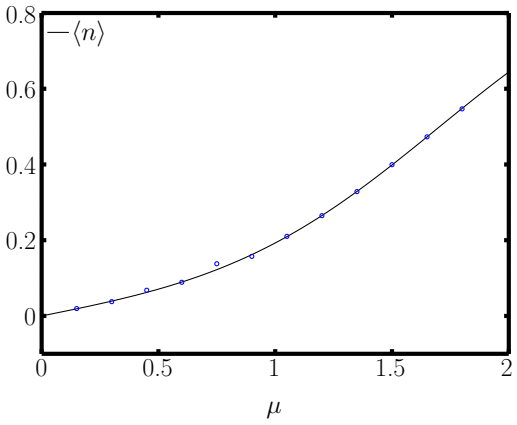
Numerical integration ($L = 2$, $\beta = 1$)

- Fermion number density

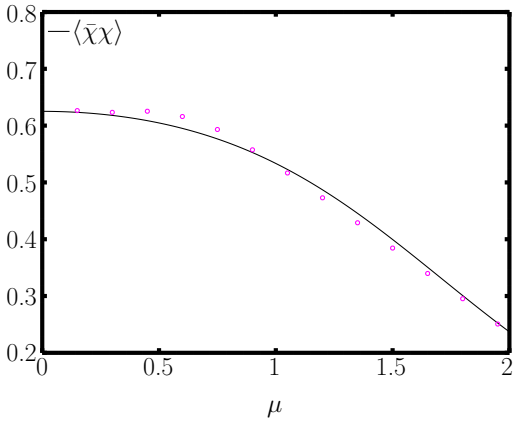


(i) Results from σ_0

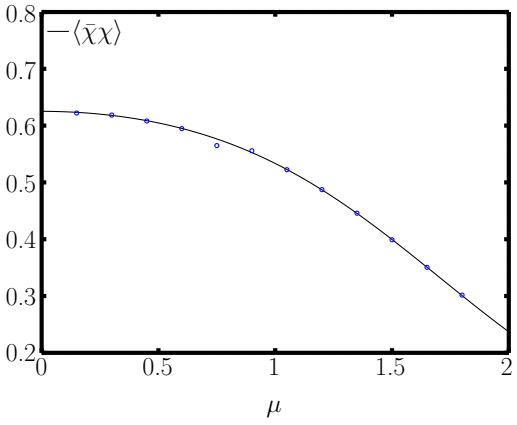


(j) Results from $\sigma_{0,i}$

- Condensate



(k) Results from σ_0



(l) Results from $\sigma_{0,i}$

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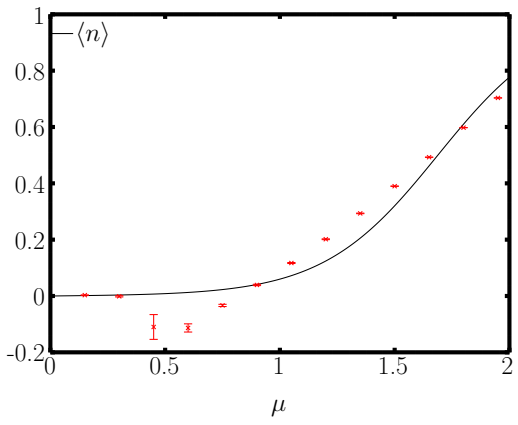
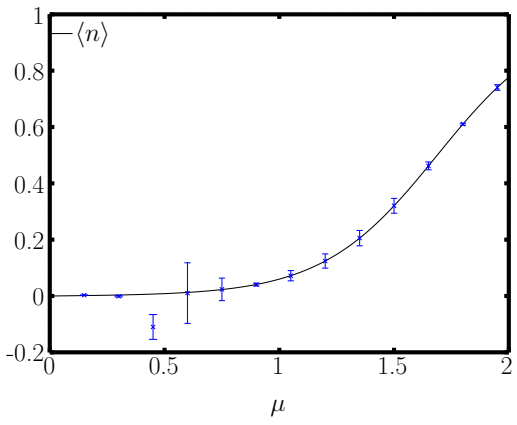
Results from MC ($L = 4$, $\beta = 1$)

- Fermion number density

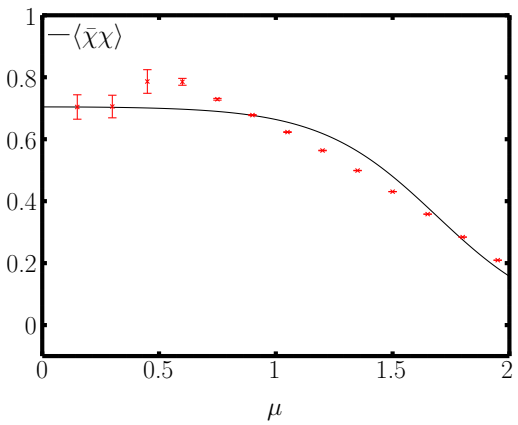


(m) Results from σ_0

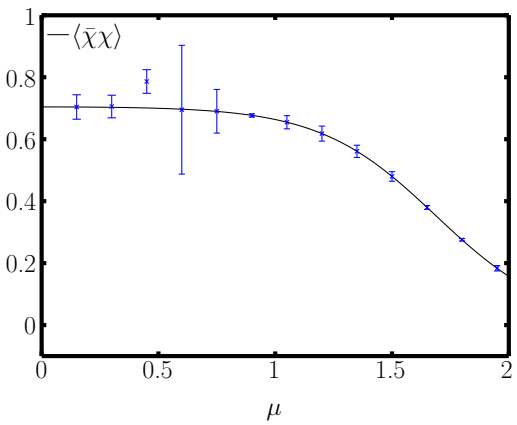


(n) Results from $\sigma_{0,i}$

- Condensate



(o) Results from σ_0



(p) Results from $\sigma_{0,i}$

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Reweighting and Taylor expansion for thimbles

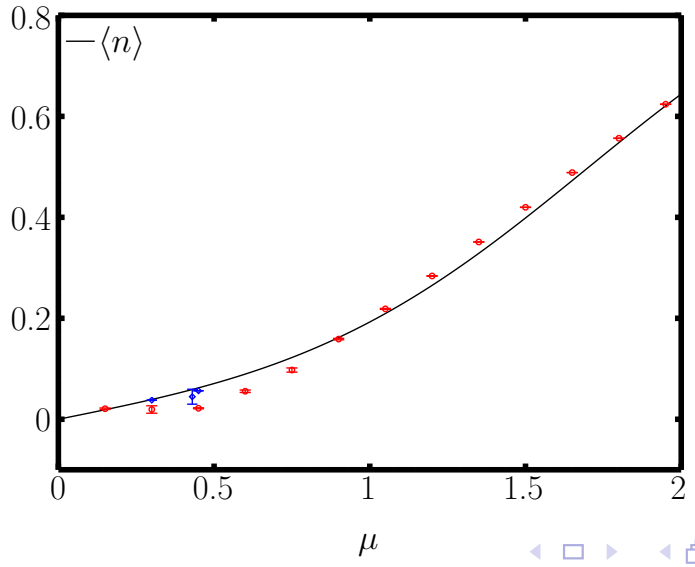
- Reweighting:

$$\int dx\, O[x]\, e^{-\beta \sum(1-cos(z_n))-S_F} =$$

$$= \int dx\, O[x] e^{-(\beta-\beta')\sum(1-cos(z_n))}\, e^{-\beta' \sum(1-cos(z_n))-S_F}$$

- Taylor expansion:

$$\langle O\rangle(\mu) = \langle O\rangle(\mu_0) + \left.\frac{\partial O}{\partial \mu}\right|_{\mu_0} (\mu-\mu_0) + \dots$$



Conclusions

Conclusions

- We have studied the $(0+1)$-dimensional Thirring model for $L = 2$ and $L = 4$ with a strong coupling $\beta = 1$.
- Discrepancies between the analytical solution and the results from one-thimble simulations seem to disappear after collecting the contribution from the sub-leading thimble.
- Very preliminary, we explored reweighting and Taylor expansion for thimbles.