Dirac eigenvalue spectrum of Nf =2+1 QCD towards the chiral limit using HISQ action

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Introduction

- ✔ **What is Dirac spectrum?**

$$\rho(\lambda, m) = \frac{1}{V} \sum_k \langle \delta(\lambda - \lambda_k) \rangle \, , D\phi = i\lambda\phi$$

- ✔ **Why are we interested in the Dirac spectrum?**

$$\boxed{\mathrm{SU(2)}_{\mathrm{L}} \times \mathrm{SU(2)}_{\mathrm{R}}} \qquad \boxed{\mathrm{U(1)}_{\mathrm{A}}}$$

- • **A probe of the spontaneous chiral symmetry breaking through the Banks-Casher relation**

- • **Whether the spectral density have a gap in the near-zero mode will signal the effective $\mathrm{U(1)}_{\mathrm{A}}$ restoration above $\mathrm{T_C}$**

Banks and Casher (1980)

Cohen (1996)

Aoki et.al (2013)

Chiral & $U(1)_A$ symmetry and Dirac spectrum

Bazavov et al., Phys.Rev. D86 (2012) 094503

The order parameter of chiral phase transition:

$$\langle \bar{\psi}\psi \rangle \xrightarrow{V \to \infty} \frac{N_f}{4} \int_0^\infty d\lambda \frac{2m\rho(\lambda, m)}{\lambda^2 + m^2}$$

An indicator of the $U(1)_A$ symmetry breaking/restoration:

$$\chi_\pi - \chi_\delta \xrightarrow{V \to \infty} \frac{N_f}{4} \int_0^\infty d\lambda \frac{4m^2\rho(\lambda, m)}{(\lambda^2 + m^2)^2}$$

How can we get the Dirac spectrum?

- Commonly used method: Lanczos algorithm to calculate the individual low-lying eigenvalues

- **Here we utilized the Chebyshev filtering technique combined with a stochastic estimate of the mode number**

$$\bar{n}[s,t] \approx \frac{1}{N_r} \sum_{r=1}^{N_r} \sum_{j=0}^{p} g_j^p \gamma_j \langle \xi_r^{\dagger} T_j(A) \xi_r \rangle$$

T_j : **Chebyshev polynomial**

γ_j : **coefficient**

p : **polynomial order**

$$\rho(\lambda, \delta) = \frac{1}{V} \frac{\bar{n}[\lambda - \delta/2, \lambda + \delta/2]}{\delta} \quad (\lambda \geq \delta/2)$$

Giusti,Luscher, arXiv: 0812.3638
Di Napoli et al., arXiv: 1308.4275
Fodor et al., arXiv: 1605.08091
Cossu et al., arXiv: 1601.00744

Chebyshev filtering

Stochastic counting of eigenvalues of a **Hermitian matrix A** in a given [s, t] within [-1,1]:

$$n[s,t] \simeq \frac{1}{N_r} \sum_{r=1}^{N_r} \xi_r^\dagger h(A) \xi_r$$

Chebyshev approximation:

$$h(A) \simeq \sum_{j=0}^{p} g_j^p \gamma_j T_j(A)$$

h(x)

1.0

s t
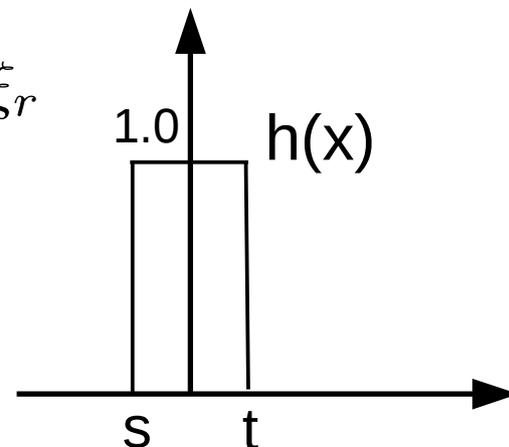
Chebyshev polynomial:

$$T_0(A) = 1, T_1(A) = A, T_j(A) = 2AT_{j-1}(A) - T_{j-2}(A) \quad (j \geq 2)$$

Rescale the D⁺D to have the eigenvalues of A are in the range [-1,1]

$$A = \frac{D^\dagger D - \frac{(\lambda_{max}^{D^\dagger D} + \lambda_{min}^{D^\dagger D})}{2}\mathbf{1}}{\frac{(\lambda_{max}^{D^\dagger D} + \lambda_{min}^{D^\dagger D})}{2}\mathbf{1}}$$

Giusti,Luscher, arXiv: 0812.3638
Di Napoli et al., arXiv: 1308.4275
Fodor et al., arXiv: 1605.08091
Cossu et al., arXiv: 1601.00744

$\lambda_{min} = 0$ is set. λ_{max} is estimated by the power method

Simulation setup

- **Actions**:

  Tree level improved gauge action

  Highly improved staggered quark action

- **Lattice size:**
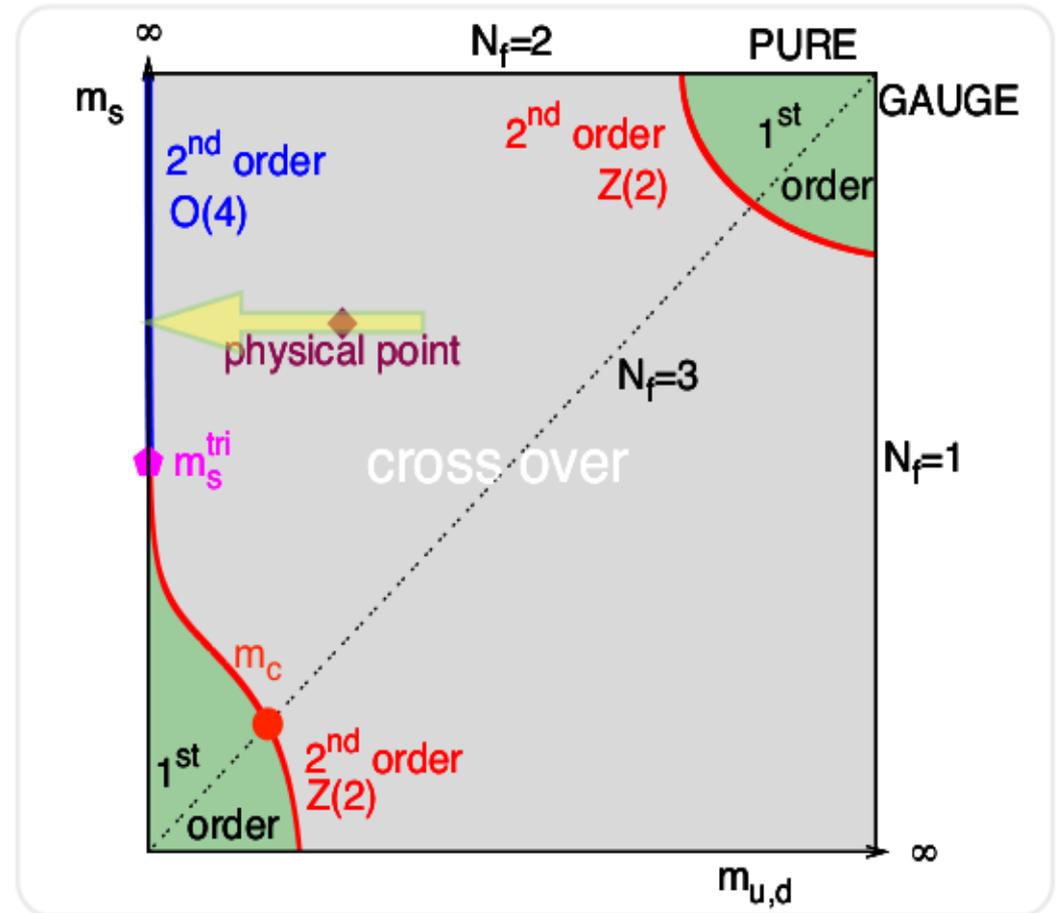
  $N_\tau = 8$, $N_\sigma = 32, 40, 56$

- **Quark mass:**

  m_s:set to its physical value

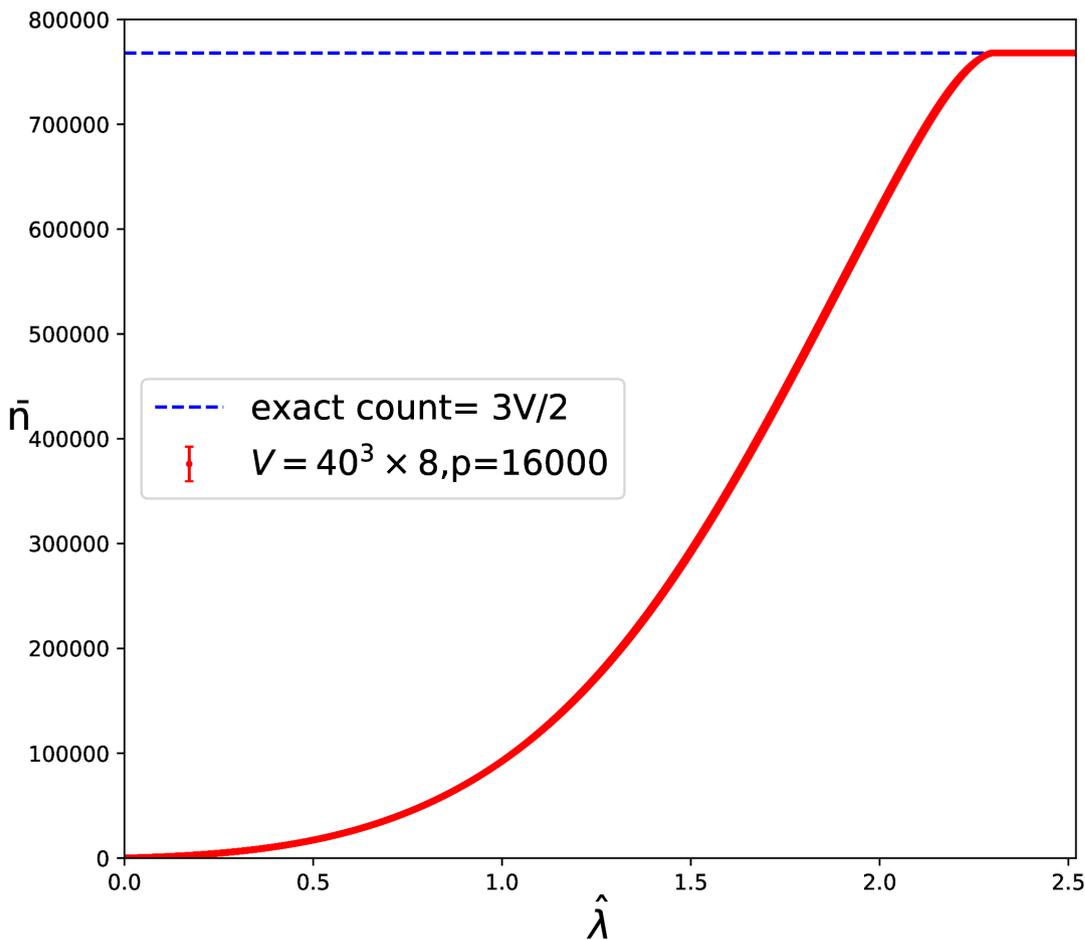 $m_l/m_s = 1/40, 1/80, 1/160$

  ($m_\pi = 110, 80, 55$ MeV)

- **Temperature:**

  $T = 137 \sim 166$ MeV

$$\bar{n}[s,t] \approx \frac{1}{N_r} \sum_{r=1}^{N_r} \sum_{j=0}^{p} g_j^p \gamma_j \langle \xi_r^\dagger T_j(A) \xi_r \rangle$$

$$A = \frac{D^\dagger D - \frac{(\lambda_{max}^{D^\dagger D} + \lambda_{min}^{D^\dagger D})}{2} 1}{\frac{(\lambda_{max}^{D^\dagger D} + \lambda_{min}^{D^\dagger D})}{2} 1}$$
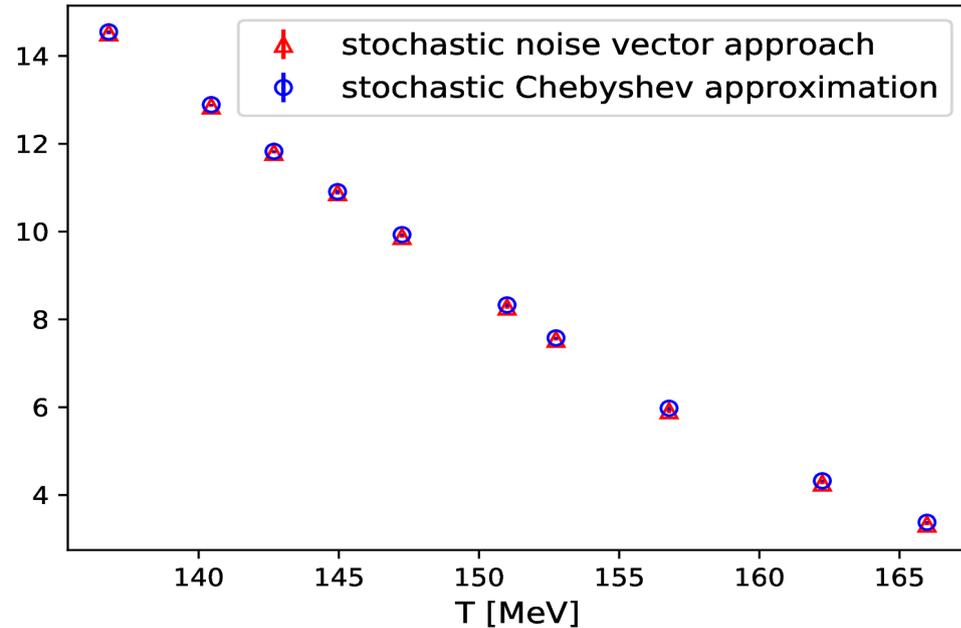
$$\sqrt{\lambda^{D^\dagger D}} = \sqrt{\lambda^A \frac{(\lambda_{max}^{D^\dagger D} + \lambda_{min}^{D^\dagger D})}{2} + \frac{(\lambda_{max}^{D^\dagger D} + \lambda_{min}^{D^\dagger D})}{2}}$$

Here λ is the positive eigenvalue of D (not of A)

The mode number converges to the exact count

Comparison of the chiral observables from eigenvalue spectrum and stochastic noise vector approach

$< \bar{\psi}\psi >_l /T^3, \quad m_l = m_s/40, \quad V = 40^3 \times 8$

- stochastic noise vector approach
- stochastic Chebyshev approximation

$(\chi_\pi - \chi_\delta)/T^2, \quad m_l = m_s/40, \quad V = 40^3 \times 8$

- stochastic noise vector approach
- stochastic Chebyshev approximation

$$\langle \bar{\psi}\psi \rangle = \frac{N_f}{4} \frac{1}{V} \langle \mathrm{Tr} M^{-1} \rangle$$

$$\langle \bar{\psi}\psi \rangle \xrightarrow{V \to \infty} \frac{N_f}{4} \int_0^\infty d\lambda \frac{2m\rho(\lambda, m)}{\lambda^2 + m^2}$$

$$\chi_\pi - \chi_\delta = \frac{N_f}{4} \frac{1}{mV} \langle \mathrm{Tr} M^{-1} \rangle + \frac{N_f}{4} \frac{1}{V} \langle \mathrm{Tr} M^{-2} \rangle$$
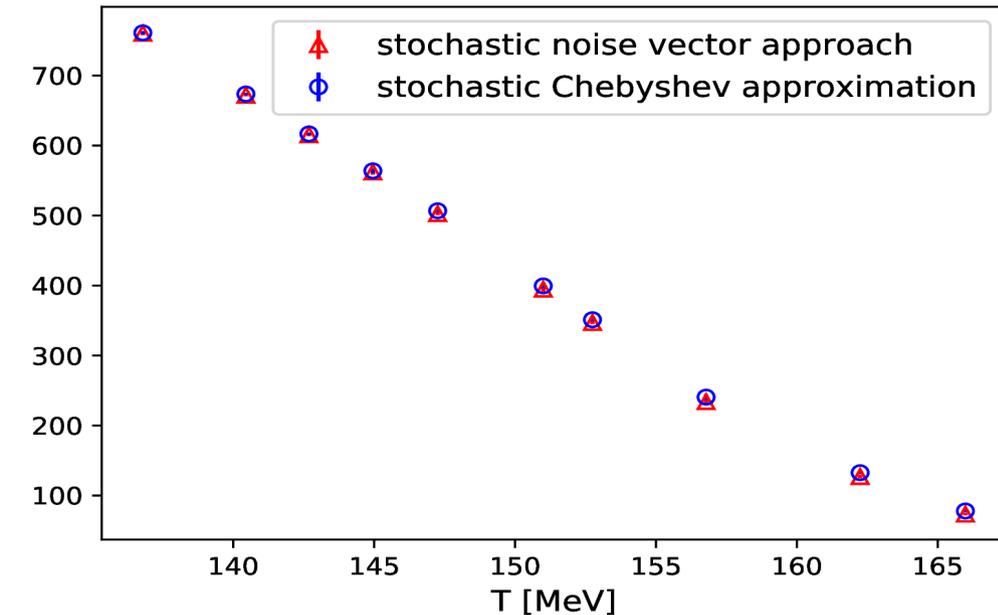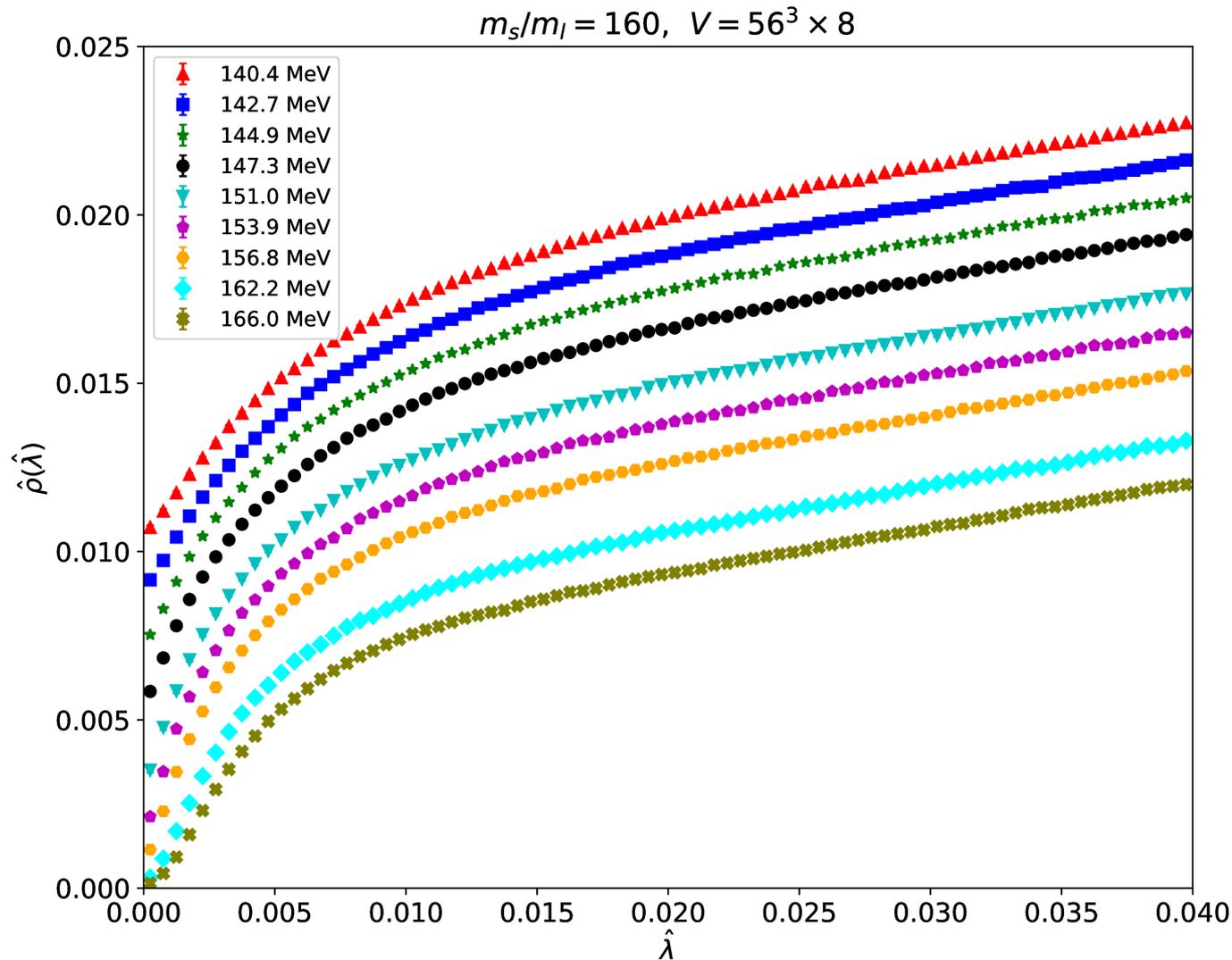
$$\chi_\pi - \chi_\delta \xrightarrow{V \to \infty} \frac{N_f}{4} \int_0^\infty d\lambda \frac{4m^2 \rho(\lambda, m)}{(\lambda^2 + m^2)^2}$$

Remarkable agreement between chiral observables evaluated from eigenvalue spectrum and stochastic noise vector approach

T dependence of low-lying eigenvalue spectrum

$m_s/m_l = 160, \quad V = 56^3 \times 8$

Legend:
- 140.4 MeV
- 142.7 MeV
- 144.9 MeV
- 147.3 MeV
- 151.0 MeV
- 153.9 MeV
- 156.8 MeV
- 162.2 MeV
- 166.0 MeV

Axis labels: $\hat{\rho}(\hat{\lambda})$ (vertical), $\hat{\lambda}$ (horizontal)

- Eigenvalue around zero is suppressed as T increases

- There is no clear evidence of a gap around zero up to T=166 MeV

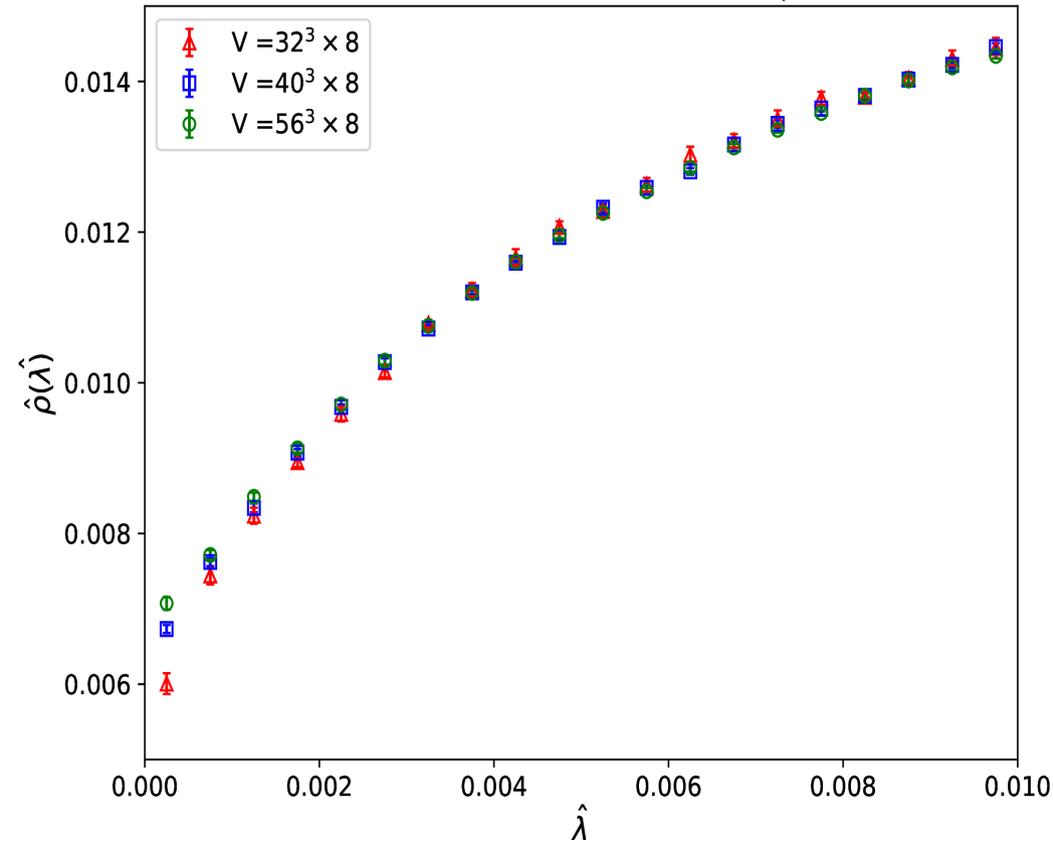(C.f. T_{pc} = 150.7(3) MeV)

Sea quark mass dependence of ρ(λ)
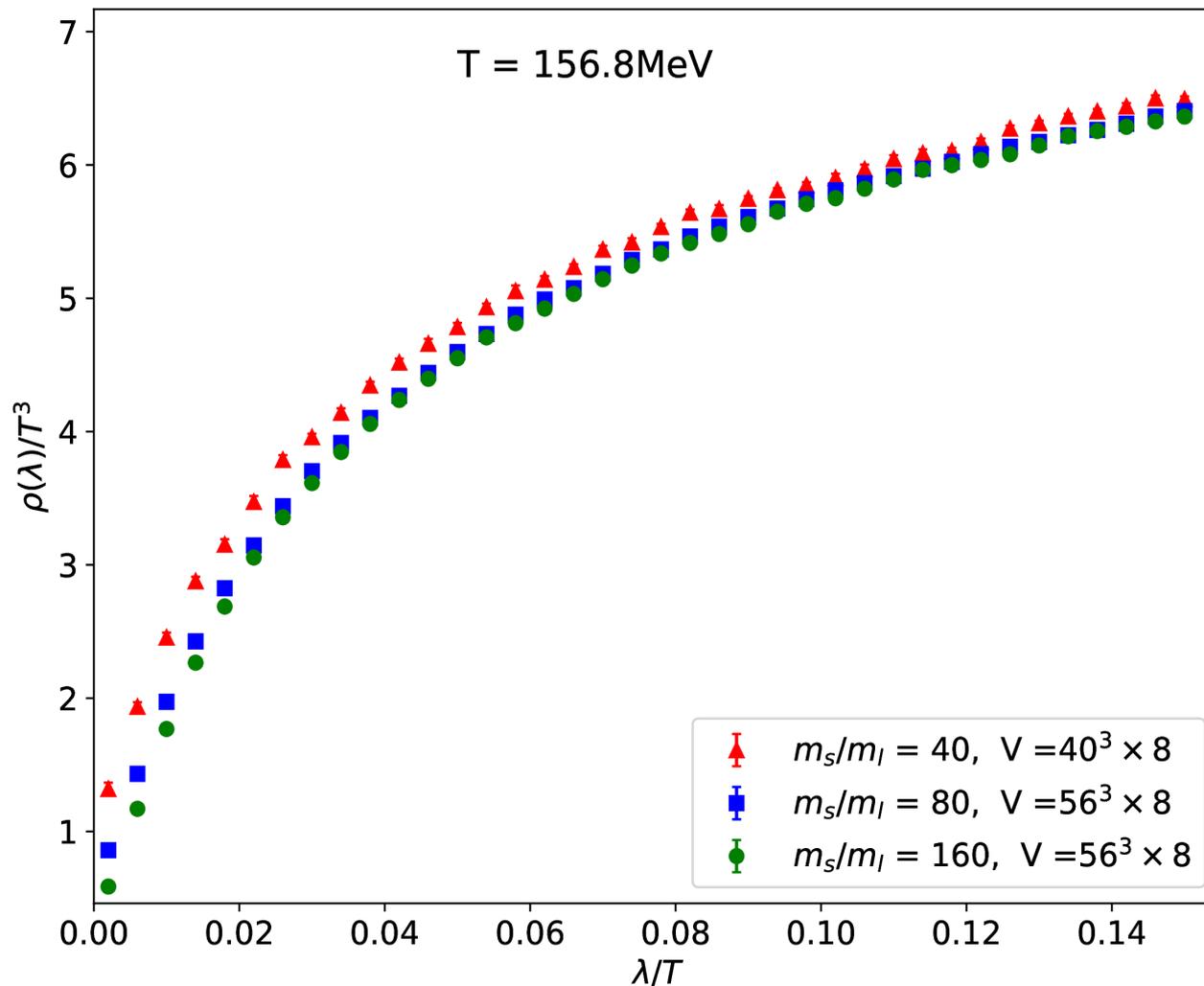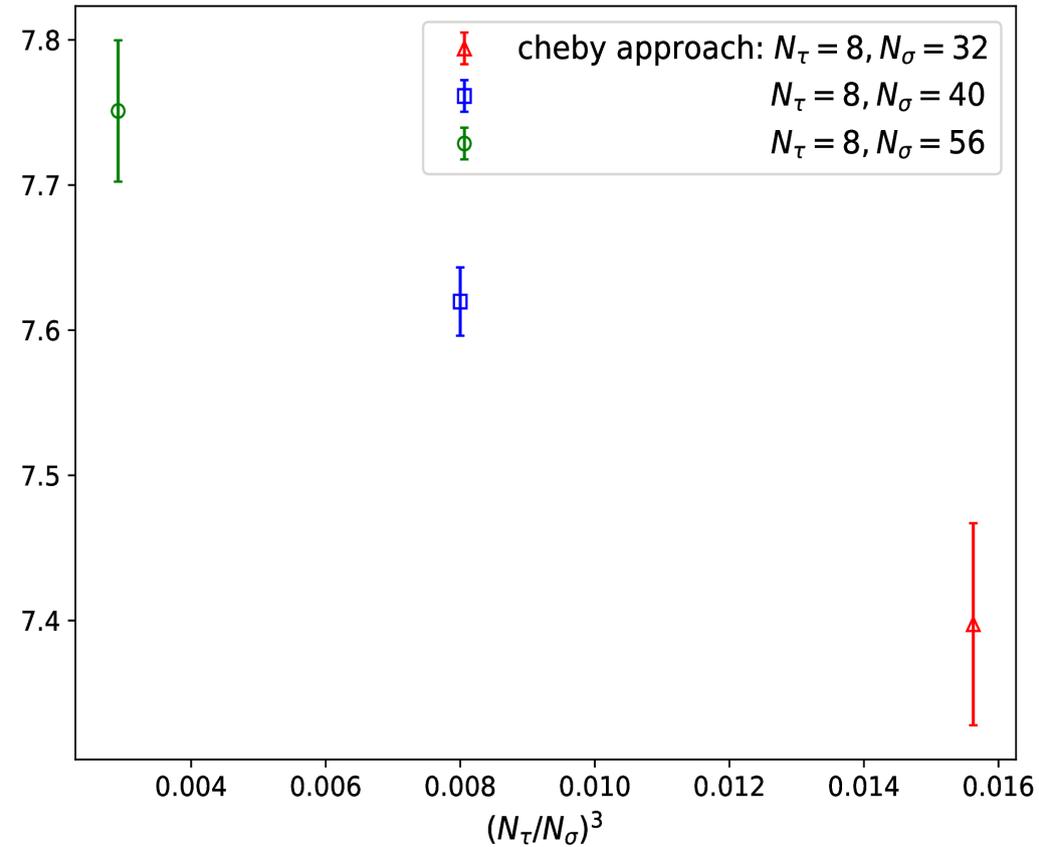

Small λ part: larger quark mass dependence
Large λ part: smaller quark mass dependence

Volume dependence of ρ(λ)

The near zero mode show larger volume dependence

The chiral condensate increases as the volume is increased

Short summary for now

- The stochastic chebyshev approximation method is very reliable

- Chiral observables can be reproduced well from Dirac spectrum

- Temperature, quark mass and volume dependence of spectrum density behave as we expected

Estimate of ρ(0,m)

Fit ρ(λ,m) to a cubic polynomial:
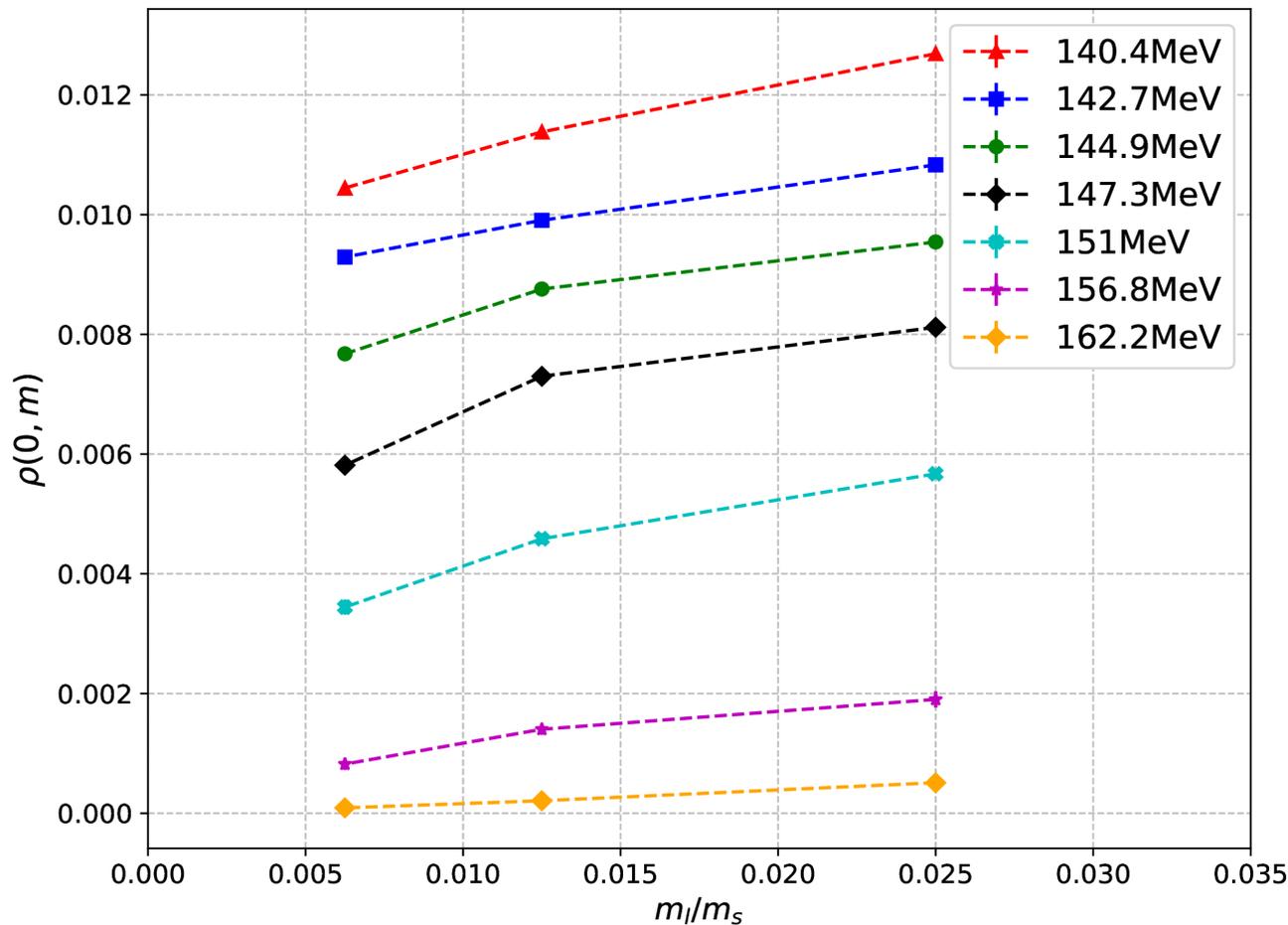$$\rho(\lambda,m) = C_0 + C_1\lambda + C_2\lambda^2 + C_3\lambda^3 \implies \rho(0,m) = C_0$$
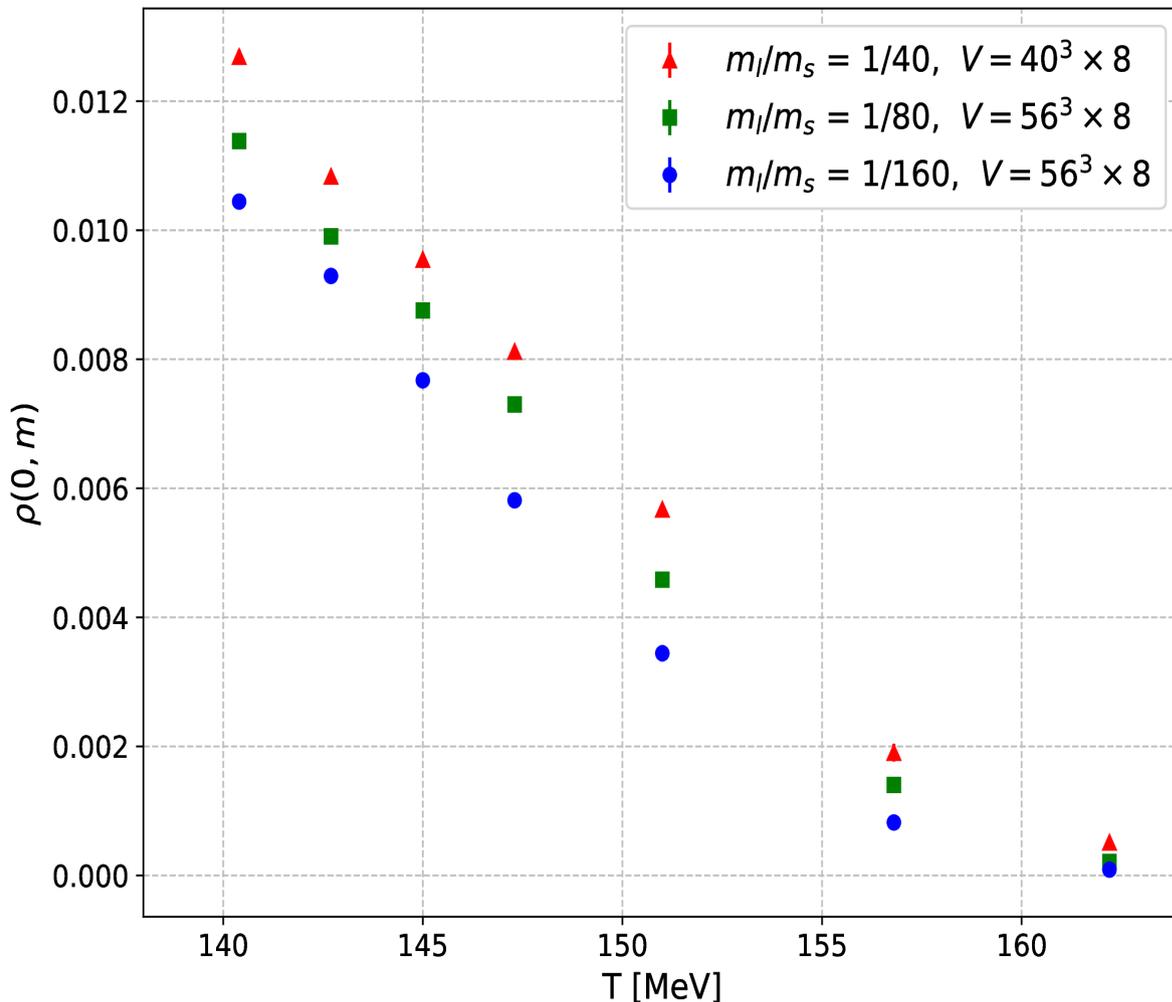
ρ(0,m) decrease as T increases and indicates a diminishing chiral condensate according to the Banks-Casher formula

Quark mass dependence of ρ(0,m)

$$T \geq T_c : \text{ If } \rho(0, m) \sim m \overset{m \to 0}{\implies} \boxed{\begin{array}{c} \langle \overline{\psi}\psi \rangle = 0 \\ \chi_\pi - \chi_\delta = \chi_{\text{disc}} \neq 0 \end{array}}$$

Bazavov et al., [HotQCD]
arXiv:1205.3535

T_{pc} estimated from the χ_M :

157.8(1) MeV (m_l/m_s=1/40)

153.7(3) MeV (m_l/m_s=1/80)

150.7(3) MeV (m_l/m_s=1/160)

H.-T. Ding et al., [HotQCD]
arXiv:1903.04801

ρ(0,m) has a linear dependence on the mass above T$_c$

U(1)$_A$ remains broken above T$_c$

Conclusions

- The eigenvalue filtering technique is very reliable to obtain the eigenvalue spectrum.

- Chiral observables can be reproduced well from Dirac spectrum.

- Spectral density at zero has a linear quark mass dependence above T_c.

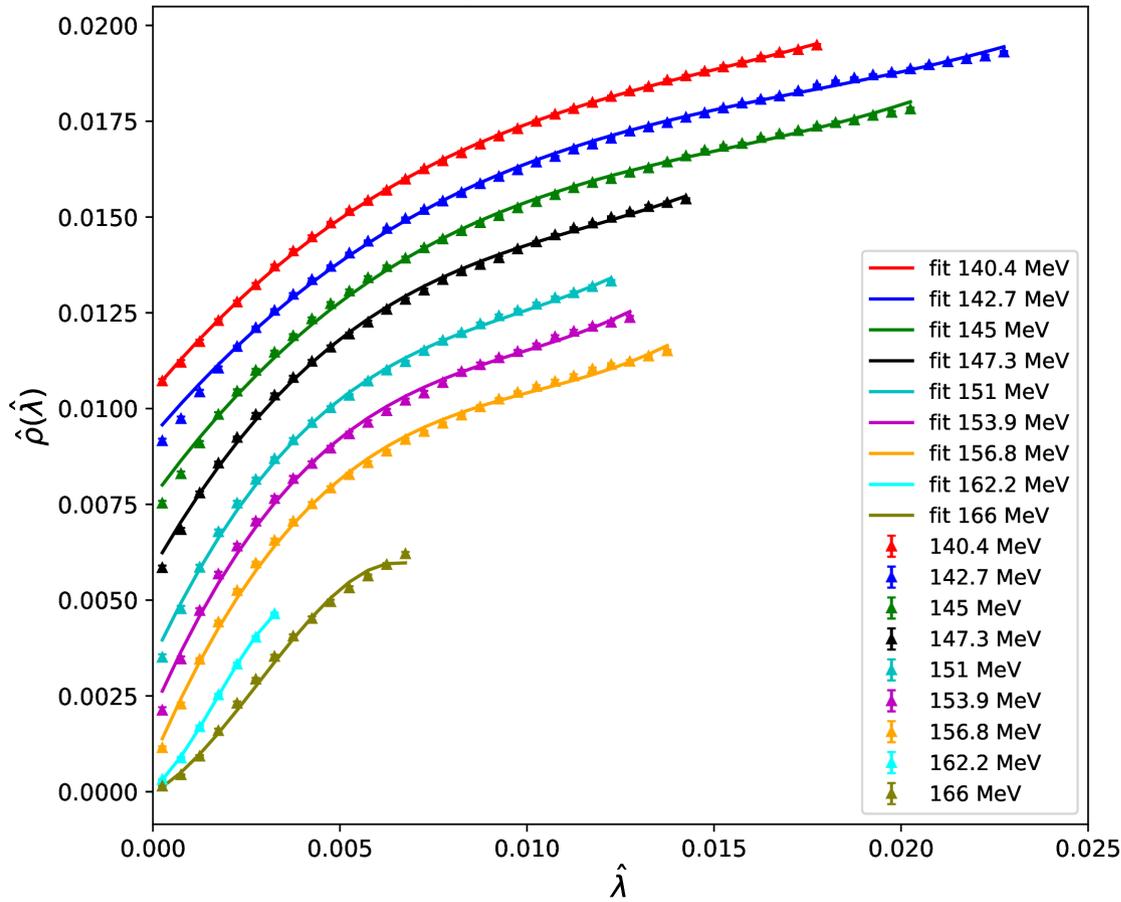- $U(1)_A$ symmetry remains broken above T_c in the chiral limit.

Outlook

- Higher temperature

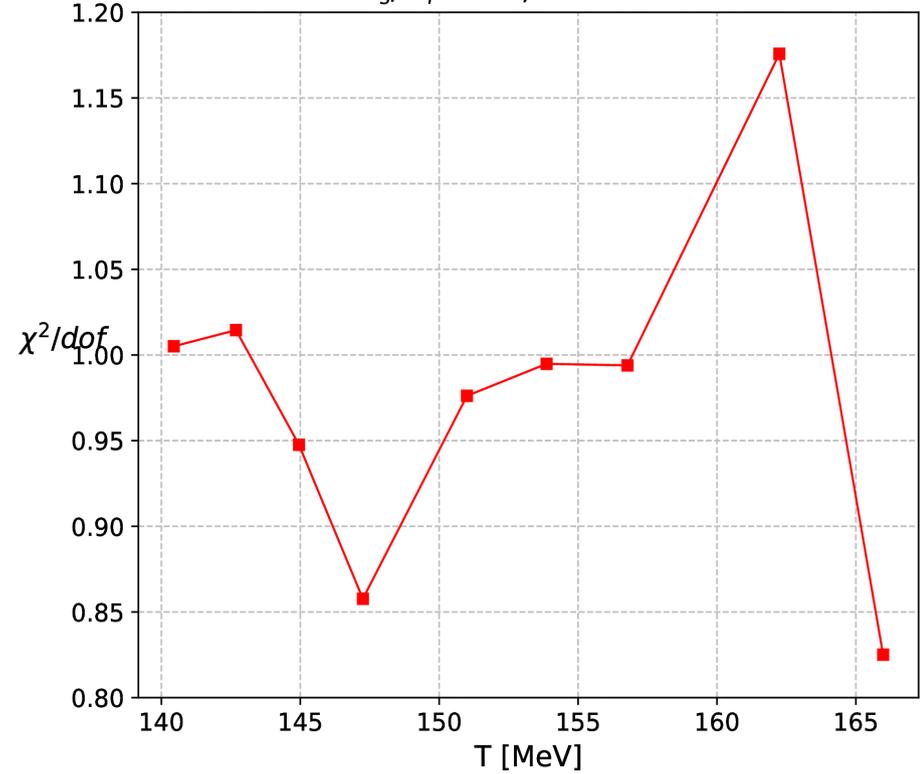- Cutoff dependence

- Comparison with Lanczos algorithm

Backup

Fit ρ(λ,m) to a cubic polynomial:
$\rho(\lambda,m) = C_0 + C_1\lambda + C_2\lambda^2 + C_3\lambda^3$

$m_s/m_l = 160, \quad V = 56^3 \times 8$

possible behaviors for ρ(λ,m)

Instanton

$T \gg T_c$: $\quad \rho(\lambda,m) = C_i m^2 \delta(\lambda) + C_\lambda \lambda + C_m m + O(\lambda m)$

$m \to 0, \lambda \to 0$: $\quad \rho(0,m) = C_m m$

Ansatz	$\langle \bar{\psi}\psi \rangle$	χ_π	χ_δ	$\chi_\pi - \chi_\delta$	χ_{disc}
$m^2 \delta(\lambda)$	m	1	-1	2	2
λ	$-2m\ln(m)$	$-2\ln(m)$	$-2\ln(m)$	2	0
m	πm	π	0	π	π

in the chiral limit

If ρ(0,m)~m $\Rightarrow$
$$\langle \bar{\psi}\psi \rangle = 0$$
$$\chi_\pi - \chi_\delta = \chi_{disc} \neq 0$$

$$\gamma_j^{[s,t]} = \begin{cases} \frac{1}{\pi} \big(\arccos(s) - \arccos(t) \big) & \text{for } j = 0 \\ \frac{2}{\pi} \frac{\sin(j \arccos(s)) - \sin(j \arccos(t))}{j} & \text{for } j > 0 \end{cases} \qquad (1)$$

damping factor :

$$g_j^p = \frac{(1 - \frac{j}{p+2}) \sin \alpha_p \cos(j\alpha_p) + \frac{1}{p+2} \cos \alpha_p \sin(j\alpha_p)}{\sin \alpha_p}, \alpha_p = \frac{\pi}{p+2}$$