Exploring the QCD phase diagram via reweighting from isospin chemical potential

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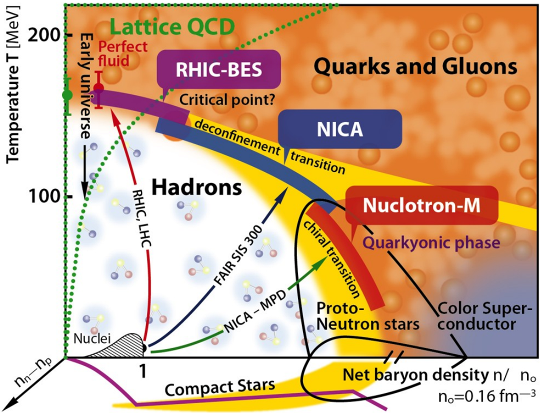
Emmy Noether-Programm
DFG Deutsche Forschungsgemeinschaft

STIFTUNG GIERSCH

Outline

- introduction
  - QCD with isospin asymmetry: phase diagram
  - (spontaneous) symmetry breaking, pion condensation
- dependence of phase boundary on chemical potentials
  - reweighting to $\mu_S > 0$
  - reweighting to $\mu_B > 0$
- decoupling of auxiliary quarks
  - reweighting in m_q
- summary & outlook

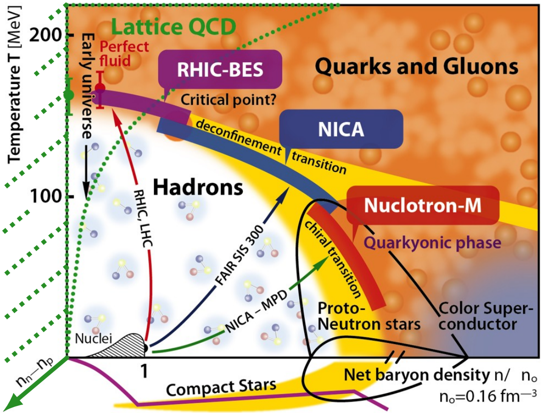
QCD phase diagram



(taken from NICA)

QCD phase diagram

QCD
with
isospin



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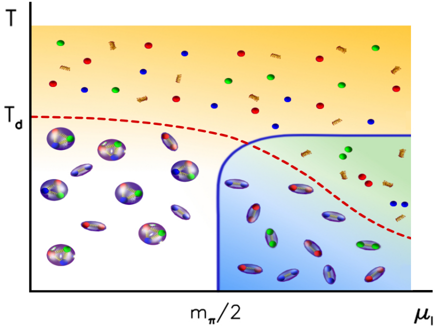
QCD isospin phase diagram

- baryon chemical potential $\mu_B = 0$
- **isospin chemical potential** $\mu_I = (\mu_u - \mu_d)/2$
- rich phase structure: [B. Brandt, G. Endrödi, S. Schmalzbauer '18]
  - vacuum (white)
  - quark-gluon plasma
  - pion condensate (BEC)
  - BCS phase
- dependence of BEC phase boundary on μ_B, μ_S?



Simulation Details

- QCD partition function for $N_f = 2+1$ rooted staggered quarks

$$\mathcal{Z} = \int \mathcal{D}[U] \, (\det \mathcal{M}_{ud} \mathcal{M}_s)^{1/4} \, e^{-S_G}$$

- quark matrices with $\eta_5 = (-1)^{n_t+n_x+n_y+n_z}$

$$\mathcal{M}_{ud} = \begin{pmatrix} \not{D}_{\mu_I} + m_{ud} & \lambda\eta_5 \\ -\lambda\eta_5 & \not{D}_{-\mu_I} + m_{ud} \end{pmatrix}, \qquad \mathcal{M}_s = \not{D}_0 + m_s$$

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$$\det \mathcal{M}_{ud} = \det\left(M^\dagger M + \lambda^2\right) \in \mathbb{R}_{>0} \qquad M = \not{D}_{\mu_I} + m_{ud}$$

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- first studies [Kogut, Sinclair '02] [de Forcrand, Stephanov, Wenger '07]
- in this work: stout-smeared quarks, physical pion masses, tree-level Symanzik improved gluons, 8^4 lattice

Pion condensation: symmetry breaking

- QCD with light quarks

$$\mathcal{M}_{ud} = \not{D} + m_{ud}$$

- chiral symmetry breaking pattern

$$\mathrm{SU}(2)_V$$

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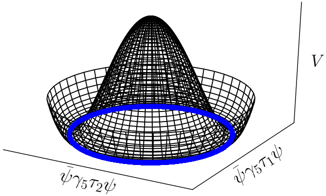

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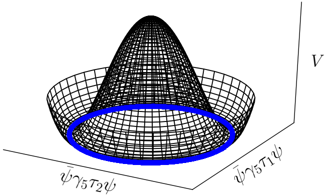

- **pion condensate** $\langle \bar{\psi}\gamma_5\tau_{1,2}\psi \rangle = 0$ (finite volume)
- zero-eigenvalues: accumulation, slowing down (Goldstone mode)

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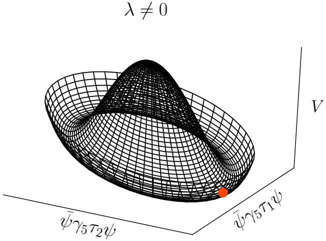
$$\mathrm{SU}(2)_V \rightarrow \mathrm{U}(1)_{\tau_3} \rightarrow \varnothing$$



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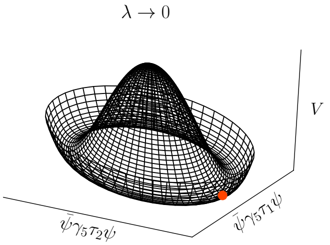

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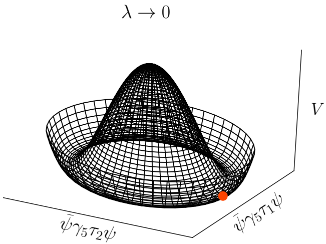

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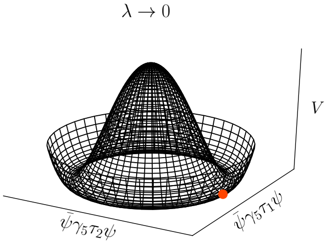

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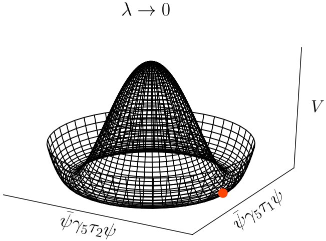

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  e.g. reweighting

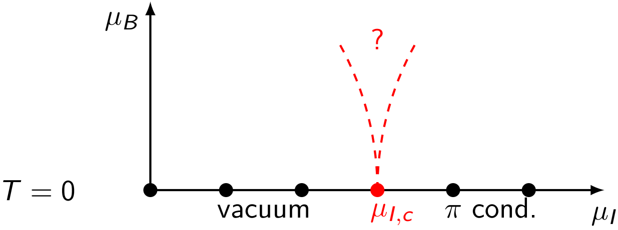
$$\langle O \rangle_0 = \frac{\langle OR_\lambda \rangle_\lambda}{\langle R_\lambda \rangle_\lambda}$$

Reweighting in μ: motivation

$T = 0$

- until now: $\mu_B, \mu_S = 0$

Reweighting in μ: motivation



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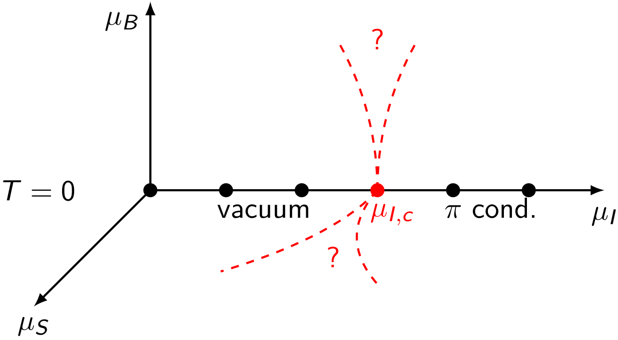

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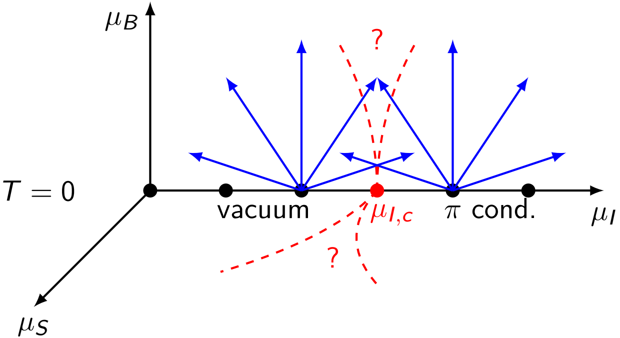

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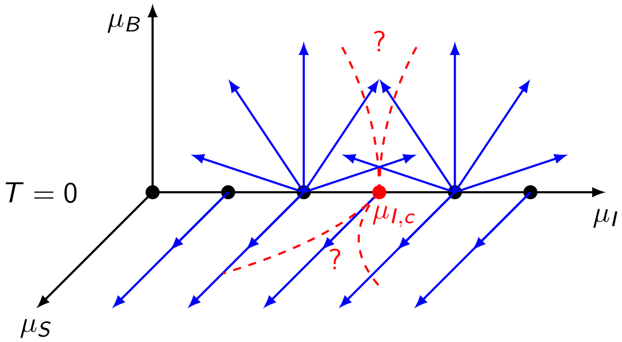

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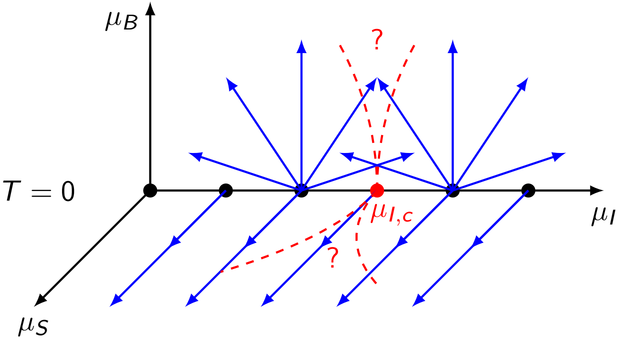

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- want to explore $\mu_B > 0$, $\mu_S > 0$: phase boundary
- reweight from pure μ_I into $\mu_I - \mu_B$ and $\mu_I - \mu_S$ plane
  - look for lines of constant observables
  - check overlap and sign problem

Reweighting in μ: determinant reduction

- for one quark $R = R_\lambda \left(\det M_{\mu'} / \det M_\mu\right)^{1/4}$
- scan over μ_S, μ_B, μ_I: need to compute $\det M_{\mu'}$ for many different μ'

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- **determinant reduction** [Toussaint '90] [Fodor, Katz '02]

$$\det M_\mu = e^{-3 V_s L_t \mu} \det \left(P - e^{L_t \mu}\right)$$

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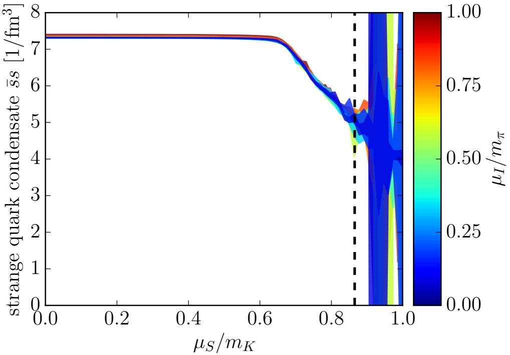
- **analytic μ-dependence**: calculate eigenvalues of P just once
- measure observables in target ensemble
  - $\mu_{u/d} = \mu_B \pm \mu_I$
  - pion condensate $\pi = \frac{T}{V}\frac{\partial \log \mathcal{Z}}{\partial \lambda}$ ↯ for $\lambda = 0$
  - chiral condensate $\bar{\psi}\psi = \frac{T}{V}\frac{\partial \log \mathcal{Z}}{\partial m_{ud}}$ via numerical derivative
  - benchmark with isospin density $n_I = \frac{T}{V}\frac{\partial \log \mathcal{Z}}{\partial \mu_I}$

Reweighting to $\mu_S > 0$: results



- no kaon cond. below $\mu_S < 0.865\ m_K$ [A. Mammarella, M. Mannarelli '15]
- precursor of transition due to finite size / temperature effects

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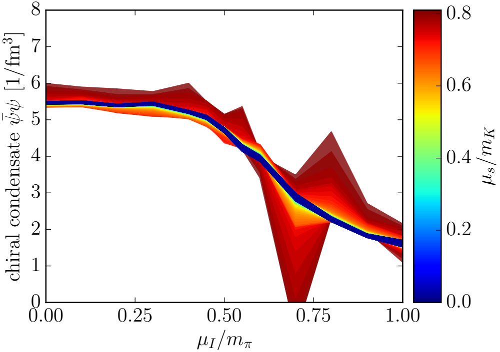
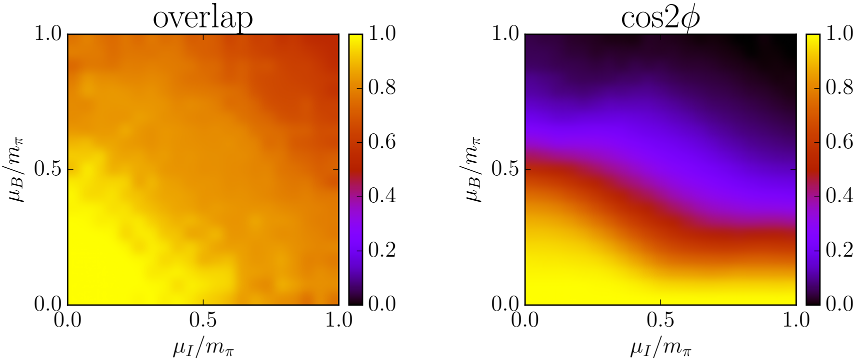


- no kaon cond. below $\mu_S < 0.865\ m_K$ [A. Mammarella, M. Mannarelli '15]
- precursor of transition due to finite size / temperature effects
- no visible effect on BEC phase boundary
  before sign problem gets too strong ($\mu_S \approx 0.7\ m_K$)

Reweighting to $\mu_B > 0$: results



- reasonable overlap, moderate sign problem

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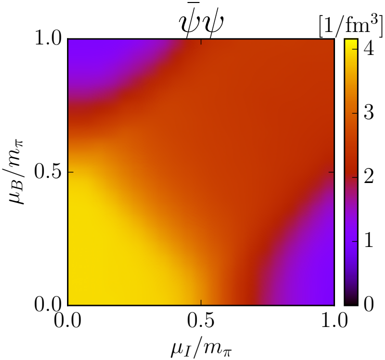
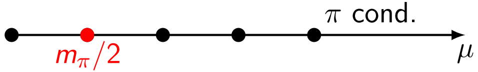


- reasonable overlap, moderate sign problem
- BEC phase boundary bends towards highter values of μ_I
- unexpected behavior below $\mu_{I,c} = m_\pi/2$ (no Silver Blaze for high μ_B)
- strong μ_B-dependence of temperature / finite size effects?

Complementary approach: decouple auxiliary quarks



$\mathcal{Z}_2(m_u, m_d; \mu, -\mu)$

- start from degenerate light quarks with isospin asymmetry

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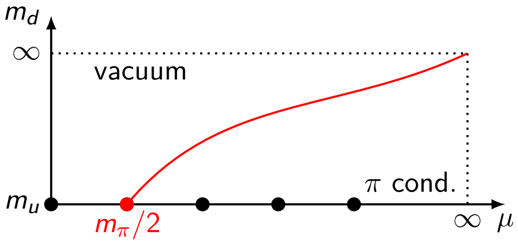

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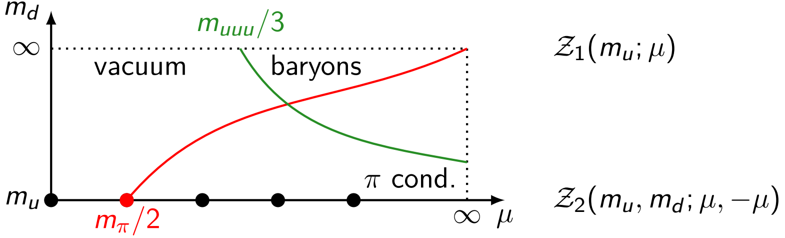

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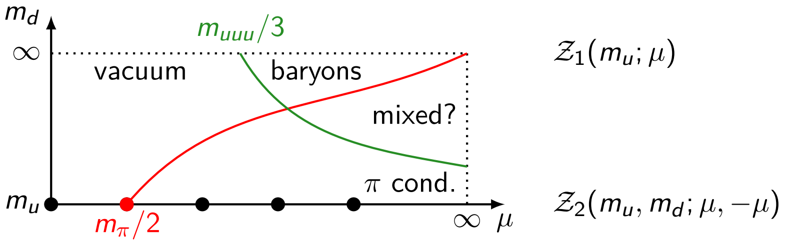

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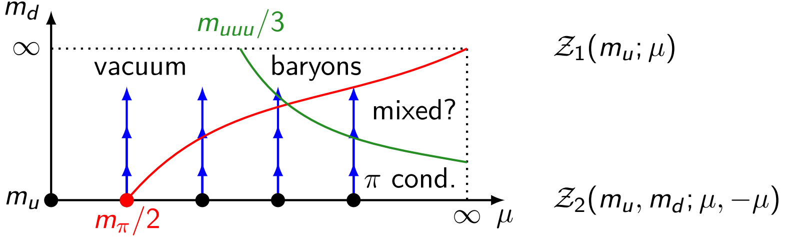

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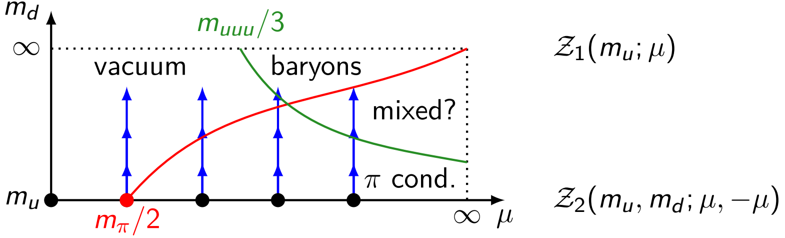

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- via reweighting in m_d: $R = R_\lambda \left[\det(\not{D}_{-\mu} + m_d) / \det(\not{D}_{-\mu} + m_u)\right]^{1/4}$
- want to have $N_f = 2 + 1$ theory after decoupling: start from 5-flavor QCD and repeat above procedure for **2 auxiliary quarks**

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  prohibits to formulate clear statement

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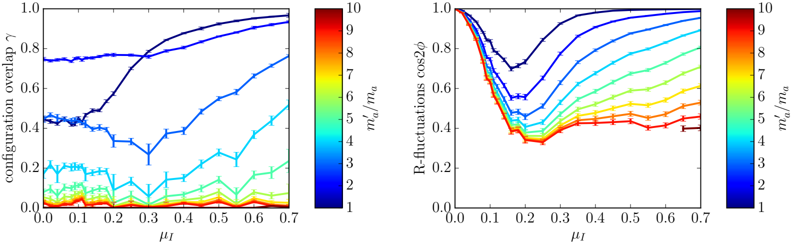

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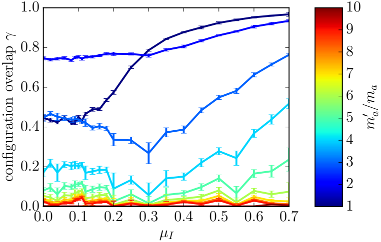

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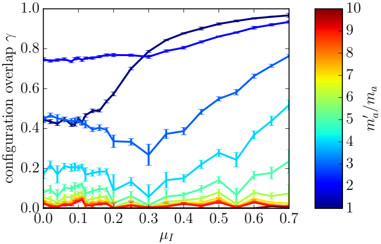

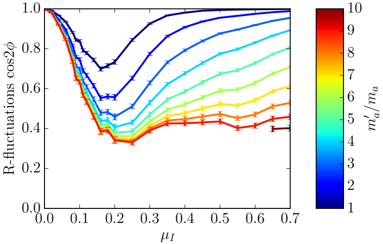

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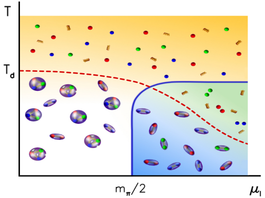
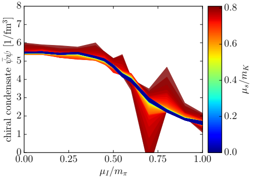
- optimally: multi-parameter-reweighting in m_a and β

Summary

- QCD with isospin chemical potential
- methods, underlying concepts



- dependence of phase boundary on μ_s, μ_B
- approach via decoupling of quarks



Outlook:

- study temperature / finite size effects

Thank you!

backup: reliability of reweighting

- for normalized $\sum_i R_i = 1$ and sorted $R_1 \geq \cdots \geq R_N$, estimate **overlap** γ as [F. Csikor et. al. '04] [C. Schmidt '04]

$$\sum_{i=1}^{N\gamma/2} R_i = 1 - \gamma/2.$$

$$\gamma = \begin{cases} 1 \;:\; R_1 = \cdots = R_N & \Rightarrow \text{ big overlap} \\ 0 \;:\; R_1 \gg \sum_{i=2}^{N} R_i & \Rightarrow \text{ only one configuration relevant } \lightning \end{cases}$$

- **sign problem**: the phase fluctuation of $R = |R|e^{i\phi}$ is

$$\cos(2\phi) = \Re \frac{R^2}{|R|^2}$$

$$\cos(2\phi) = \begin{cases} 1 \;:\; \text{no fluctuations} & \Rightarrow \text{ no sign problem} \\ 0 \;:\; \text{strong fluctuations} & \Rightarrow \text{ severe sign problem } \lightning \end{cases}$$