Exploring the QCD phase diagram via reweighting from isospin chemical potential

Sebastian Schmalzbauer
in collaboration with B. Brandt, F. Cuteri, G. Endrődi
Outline

• introduction
  • QCD with isospin asymmetry: phase diagram
  • (spontaneous) symmetry breaking, pion condensation
• dependence of phase boundary on chemical potentials
  • reweighting to $\mu_S > 0$
  • reweighting to $\mu_B > 0$
• decoupling of auxiliary quarks
  • reweighting in $m_q$
• summary & outlook
QCD phase diagram

(taken from NICA)
QCD phase diagram

QCD with isospin

(taken from NICA)
QCD isospin phase diagram

- baryon chemical potential $\mu_B = 0$
- isospin chemical potential $\mu_I = (\mu_u - \mu_d)/2$
- rich phase structure: [B. Brandt, G. Endrödi, S. Schmalzbauer ’18]
  - vacuum (white)
  - quark-gluon plasma
  - pion condensate (BEC)
  - BCS phase
- dependence of BEC phase boundary on $\mu_B, \mu_S$?
Simulation Details

- QCD partition function for $N_f = 2 + 1$ rooted staggered quarks

$$Z = \int \mathcal{D}[U] \ (\det \mathcal{M}_{ud}\mathcal{M}_s)^{1/4} e^{-S_G}$$

- quark matrices with $\eta_5 = (-1)^{n_t+n_x+n_y+n_z}$

$$\mathcal{M}_{ud} = \begin{pmatrix} \phi_{\mu I} + m_{ud} & \lambda \eta_5 \\ -\lambda \eta_5 & \phi_{-\mu I} + m_{ud} \end{pmatrix}, \quad \mathcal{M}_s = \phi_0 + m_s$$
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- no sign problem due to $\eta_5 \tau_1 \mathcal{M}_{ud} \tau_1 \eta_5 = \mathcal{M}_{ud}^\dagger$:

$$\det \mathcal{M}_{ud} = \det \left( \mathcal{M}^\dagger \mathcal{M} + \lambda^2 \right) \in \mathbb{R}_{>0} \quad \mathcal{M} = \hat{\phi}_{\mu I} + m_{ud}$$
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- first studies [Kogut, Sinclair ’02] [de Forcrand, Stephanov, Wenger ’07]

- in this work: stout-smeared quarks, physical pion masses, tree-level Symanzik improved gluons, 8$^4$ lattice
Pion condensation: symmetry breaking

- QCD with light quarks
  \[ M_{ud} = \mathcal{O} + m_{ud} \]
- chiral symmetry breaking pattern
  \[ SU(2)_V \]
Pion condensation: symmetry breaking

- QCD with light quarks

\[ \mathcal{M}_{ud} = \Phi + m_{ud} + \mu \gamma_0 \tau_3 \]

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\[ \text{SU}(2)_V \rightarrow \text{U}(1)_{\tau_3} \]
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- pion condensate \( \langle \bar{\psi} \gamma_5 \tau_{1,2} \psi \rangle = 0 \) (finite volume)

- zero-eigenvalues: accumulation, slowing down (Goldstone mode)
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- need to extrapolate \( \lambda \to 0 \) for physical results
  
  e.g. reweighting

\[ \langle O \rangle_0 = \frac{\langle OR_\lambda \rangle_\lambda}{\langle R_\lambda \rangle_\lambda} \]
Reweighting in $\mu$: motivation

$T = 0$

- until now: $\mu_B, \mu_S = 0$

$\mu_I$ cond.

$\mu_{I,c}$

vacuum
Reweighting in $\mu$: motivation

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- want to explore $\mu_B > 0, \mu_S > 0$: phase boundary
- reweight from pure $\mu_I$ into $\mu_I - \mu_B$ and $\mu_I - \mu_S$ plane
  - look for lines of constant observables
  - check overlap and sign problem
Reweighting in $\mu$: determinant reduction

- for one quark $R = R_\lambda \left( \frac{\det M_{\mu'}}{\det M_\mu} \right)^{1/4}$
- scan over $\mu_S, \mu_B, \mu_I$: need to compute $\det M_{\mu'}$ for many different $\mu'$
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- **determinant reduction** [Toussaint '90] [Fodor, Katz '02]

\[
\det M_{\mu} = e^{-3V_s L_t \mu} \det \left( P - e^{L_t \mu} \right)
\]

- **analytic $\mu$-dependence**: calculate eigenvalues of $P$ just once
Reweighting in $\mu$: determinant reduction

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- determinant reduction [Toussaint '90] [Fodor, Katz '02]

$$\det M_\mu = e^{-3V_sL_t\mu} \det \left( P - e^{L_t\mu} \right)$$

- analytic $\mu$-dependence: calculate eigenvalues of $P$ just once
- measure observables in target ensemble
  - $\mu_u/d = \mu_B \pm \mu_I$
  - pion condensate $\pi = \frac{T}{V} \frac{\partial \log Z}{\partial \lambda} \uparrow$ for $\lambda = 0$
  - chiral condensate $\bar{\psi}\psi = \frac{T}{V} \frac{\partial \log Z}{\partial m_{ud}}$ via numerical derivative
  - benchmark with isospin density $n_I = \frac{T}{V} \frac{\partial \log Z}{\partial \mu_I}$
Reweighting to $\mu_S > 0$: results

- no kaon cond. below $\mu_S < 0.865 \, m_K$ [A. Mammarella, M. Mannarelli '15]
- precursor of transition due to finite size / temperature effects
Reweighting to $\mu_S > 0$: results

- no kaon cond. below $\mu_S < 0.865 \, m_K$ [A. Mammarella, M. Mannarelli '15]
- precursor of transition due to finite size / temperature effects
- no visible effect on BEC phase boundary before sign problem gets too strong ($\mu_S \approx 0.7 \, m_K$)
Reweighting to $\mu_B > 0$: results

- reasonable overlap, moderate sign problem
Reweighting to $\mu_B > 0$: results

- reasonable overlap, moderate sign problem
- BEC phase boundary bends towards higher values of $\mu_I$
- unexpected behavior below $\mu_{I,c} = m_\pi/2$ (no Silver Blaze for high $\mu_B$)
- strong $\mu_B$-dependence of temperature / finite size effects?
Complementary approach: decouple auxiliary quarks

- start from degenerate light quarks with isospin asymmetry

\[ \pi \text{ cond. } Z_2(m_u, m_d; \mu, -\mu) \]

\[ m_\pi / 2 \]
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- start from degenerate light quarks with isospin asymmetry
- increase $m_d$: change in $m_\pi$ and $\mu_{I,C}$
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- $m_d \to \infty$: decoupling of down-quark: baryonic phase, sign problem

\[ Z_1(m_u; \mu) \]
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- increase $m_d$: change in $m_\pi$ and $\mu I, c$
- $m_d \to \infty$: **decoupling** of down-quark: baryonic phase, **sign problem**
- via reweighting in $m_d$: $R = R_\lambda \left[ \det(\mathcal{D}_{-\mu} + m_d) / \det(\mathcal{D}_{-\mu} + m_u) \right]^{1/4}$

\[ Z_1(m_u; \mu) \]
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- increase $m_d$: change in $m_\pi$ and $\mu I, c$
- $m_d \to \infty$: decoupling of down-quark: baryonic phase, sign problem
- via reweighting in $m_d$: $R = R_\lambda \left[ \det(\not{D}_-\mu + m_d)/\det(\not{D}_-\mu + m_u) \right]^{1/4}$
- want to have $N_f = 2 + 1$ theory after decoupling: start from 5-flavor QCD and repeat above procedure for 2 auxiliary quarks
Reweighting in $m_q$: observations

- adding quarks induces drastic changes: different $a, m_\pi, T, \bar{\psi}\psi, \ldots$
  - prohibits to formulate clear statement
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- adding quarks induces drastic changes: different $a$, $m_\pi$, $T$, $\bar{\psi}\psi$, ... prohibits to formulate clear statement
- start from higher quark masses to match $\bar{\psi}\psi$ values, lower $m_{ud}$ to physical values together with decoupling auxiliary quarks
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- see changes in the auxiliary quark sector, but we are more interested in the remaining quarks
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- see changes in the auxiliary quark sector, but we are more interested in the remaining quarks
- optimally: multi-parameter-reweighting in $m_a$ and $\beta$
Summary

- QCD with isospin chemical potential
- methods, underlying concepts

- dependence of phase boundary on $\mu_s, \mu_B$
- approach via decoupling of quarks

Outlook:
- study temperature / finite size effects
Thank you!
for normalized $\sum_i R_i = 1$ and sorted $R_1 \geq \cdots \geq R_N$, estimate overlap $\gamma$ as [F. Csikor et. al. '04] [C. Schmidt '04]

$$\frac{N\gamma}{2} \sum_{i=1}^{\gamma/2} R_i = 1 - \frac{\gamma}{2}.$$ 

$$\gamma = \begin{cases} 
1 : R_1 = \cdots = R_N & \Rightarrow \text{big overlap} \\
0 : R_1 \gg \sum_{i=2}^{N} R_i & \Rightarrow \text{only one configuration relevant} \\
\end{cases}$$

**sign problem**: the phase fluctuation of $R = |R|e^{i\phi}$ is

$$\cos(2\phi) = \Re \frac{R^2}{|R|^2}$$

$$\cos(2\phi) = \begin{cases} 
1 : \text{no fluctuations} & \Rightarrow \text{no sign problem} \\
0 : \text{strong fluctuations} & \Rightarrow \text{severe sign problem} \\
\end{cases}$$