Chiral Ward Identities for Dirac Eigenmodes with Staggered Fermions

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Motivation

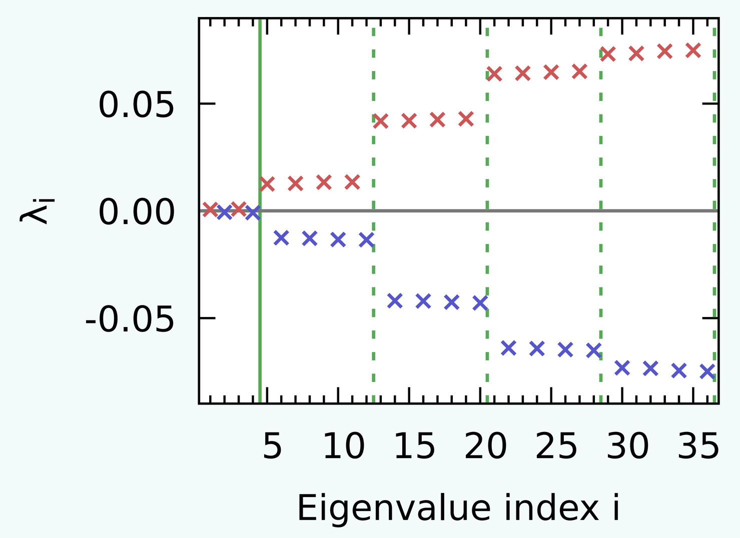
1. We want to identify zero modes of staggered fermions and understand chiral behaviors of Dirac eigenmodges using staggered fermions.
2. We want to apply Lanczos algorithm developed here to the distillation method which will be used to calculate $\pi - \pi$ scattering phase shifts.
3. We want to apply Lanczos algorithm developed here to obtain all-to-all propagators for staggered quarks.

Eigenvalues for staggered fermions

- $D_s^\dagger = -D_s$: eigenvalues are purely imaginary or zero.

$$D_s |f_\lambda(x)\rangle = i\lambda |f_\lambda(x)\rangle \tag{1}$$

- $\Gamma_\varepsilon D_s = -D_s \Gamma_\varepsilon$ for $\Gamma_\varepsilon \equiv [\gamma_5 \otimes \xi_5]$ is : nonzero eigenvalues pair as $\pm$.
- SU(4) taste symmetry induces four-fold degeneracy.



- 20^4 quenched lattice ($N_f = 0$)
- $a \cong 0.077\,\mathrm{fm}$
- (topological charge) $Q_t = -1$
- HYP staggered fermions
- Implicitly restarted Lanczos with Chebyshev polynomial acceleration.

Chirality for staggered fermions

- Chirality operators for staggered fermions :

$$\Gamma_5 \equiv [\gamma_5 \otimes \mathbb{1}] , \qquad \Gamma_\varepsilon \equiv [\gamma_5 \otimes \xi_5] , \tag{2}$$

where

$$\langle f | (\mathcal{O}_1 \otimes \mathcal{O}_2) | f \rangle \equiv \int d^4x \, f^\dagger(x_A) \overline{(\mathcal{O}_1 \otimes \mathcal{O}_2)}_{AB} U(x_A, x_B) f(x_B) \tag{3}$$

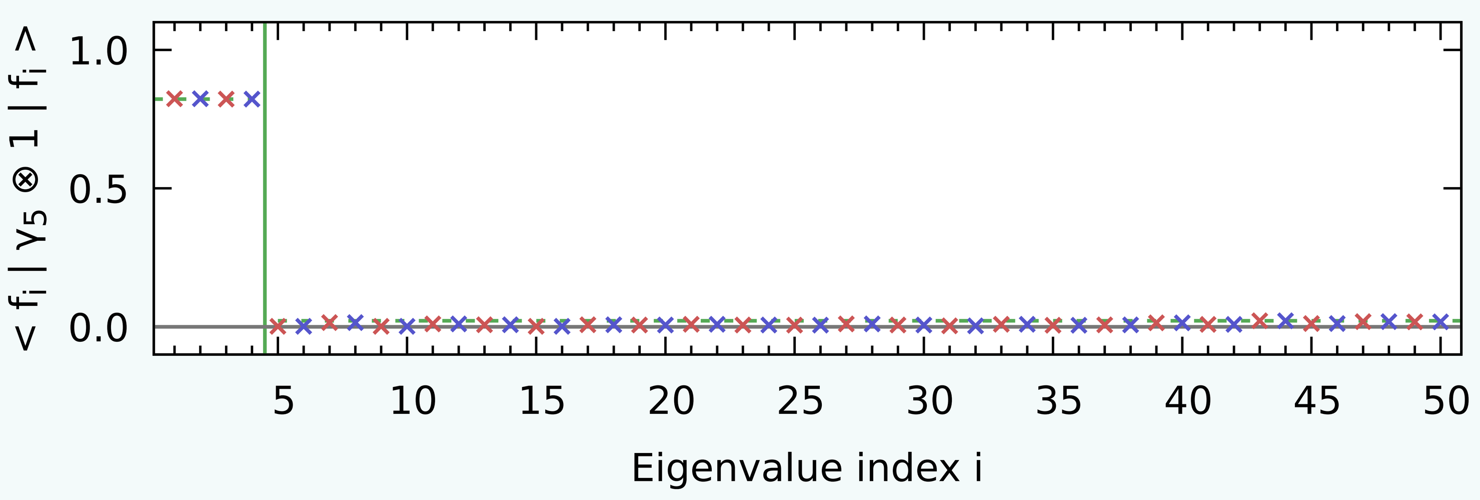
with

$$\overline{(\mathcal{O}_1 \otimes \mathcal{O}_2)}_{AB} = \frac{1}{4} \mathrm{Tr}\,(\gamma_A^\dagger \mathcal{O}_1 \gamma_B \mathcal{O}_2^\dagger) , \tag{4}$$

$$U(x_A, x_B) = \mathbb{P}_{\mathrm{SU}(3)} \left[\sum_{p \in \mathcal{C}} V(x_A, x_{p_1}) V(x_{p_1}, x_{p_2}) V(x_{p_2}, x_{p_3}) V(x_{p_3}, x_B) \right] \tag{5}$$

- $x_A = 2x + A$, $x_B = 2x + B$ where A, B are the hypercubic vectors with $A_\mu, B_\mu \in \{0, 1\}$
- $V(x, y)$: (HYP-smeared) gluon link from x to y
- $\mathbb{P}_{\mathrm{SU}(3)}$: SU(3) projection
- $\mathcal{C}$: complete set of the shortest paths from A to B

- (topological charge) $Q_t = -1$



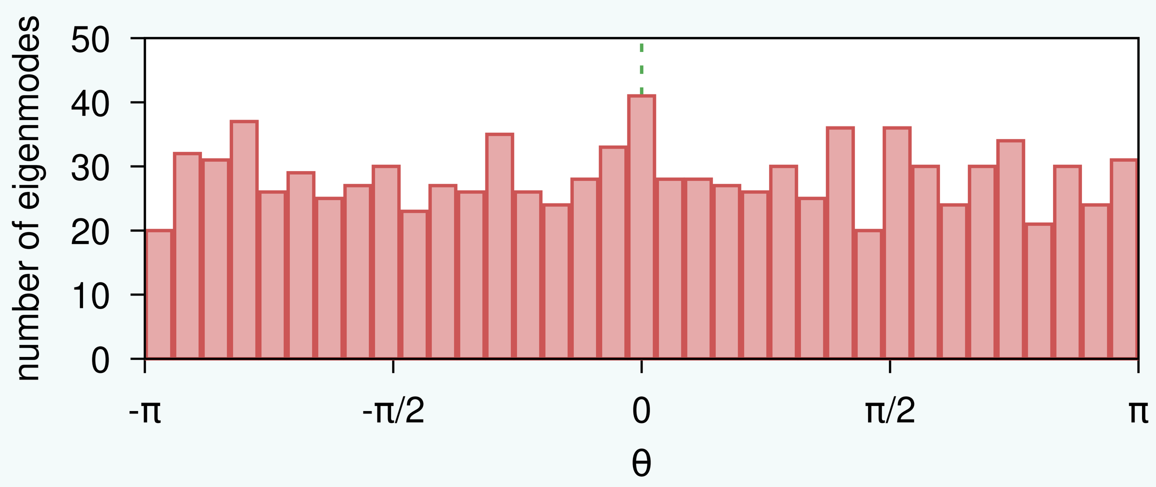
Γ_ε transformation

- Γ_ε transforms $|f_{\pm\lambda}\rangle$ to $|f_{\mp\lambda}\rangle$ with a random phase θ :

$$\Gamma_\varepsilon \, |f_{+\lambda}\rangle = e^{+i\theta} \, |f_{-\lambda}\rangle , \tag{6}$$

$$\Gamma_\varepsilon \, |f_{-\lambda}\rangle = e^{-i\theta} \, |f_{+\lambda}\rangle . \tag{7}$$

- Distribution of θ



Chiral Ward identity

- Ward identity for the chirality of staggered fermions :

$$e^{+i\theta} [\gamma_5 \otimes \mathbb{1}] \, |f^s_{-\lambda}\rangle = [\mathbb{1} \otimes \xi_5] \, |f^s_{+\lambda}\rangle , \tag{8}$$

$$e^{-i\theta} [\gamma_5 \otimes \mathbb{1}] \, |f^s_{+\lambda}\rangle = [\mathbb{1} \otimes \xi_5] \, |f^s_{-\lambda}\rangle . \tag{9}$$

- Define

$$\Gamma_5(\alpha, \beta) \equiv \langle f_\alpha | [\gamma_5 \otimes \mathbb{1}] | f_\beta \rangle , \qquad \Xi_5(\alpha, \beta) \equiv \langle f_\alpha | [\mathbb{1} \otimes \xi_5] | f_\beta \rangle . \tag{10}$$

- Another form of Chiral Ward identity :

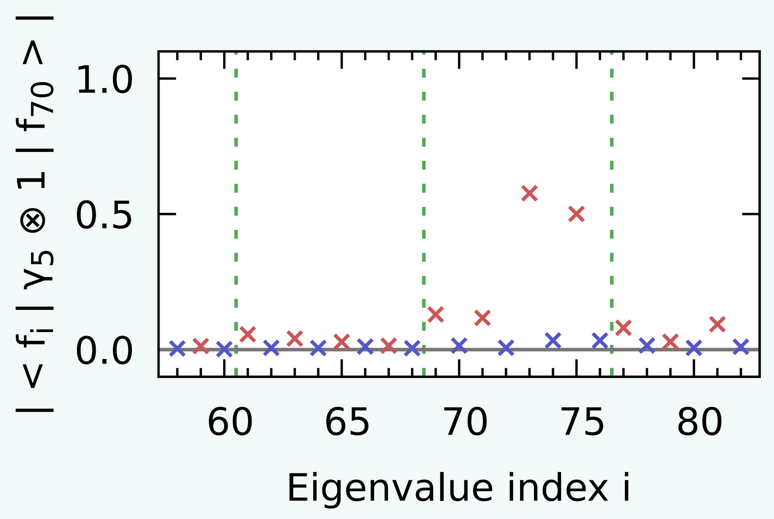
$$\begin{aligned} & |\Gamma_5(\alpha,\beta)| = |\Xi_5(-\alpha,\beta)| = |\Xi_5(\alpha,-\beta)| = |\Gamma_5(-\alpha,-\beta)| \\ & = |\Gamma_5(\beta,\alpha)| = |\Xi_5(-\beta,\alpha)| = |\Xi_5(\beta,-\alpha)| = |\Gamma_5(-\beta,-\alpha)| . \end{aligned} \tag{11}$$

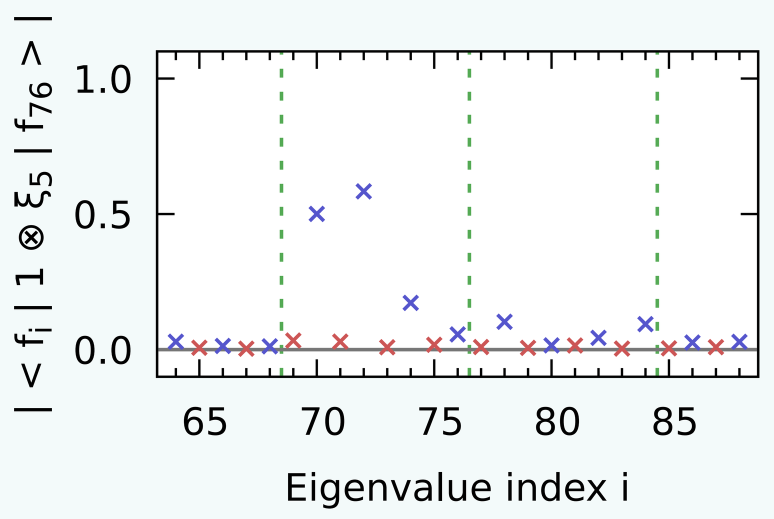
- Note that $\lambda_{2n} = -\lambda_{2n-1}$. ($\lambda_{2n-1} \leftrightarrow +\alpha$ and $\lambda_{2n} \leftrightarrow -\alpha$)
- For diagonal elements ($\alpha = \beta$),

parameter	value	parameter	value
$\lvert\Gamma_5(\lambda_1, \lambda_1)\rvert$	0.8238257	$\lvert\Xi_5(\lambda_2, \lambda_1)\rvert$	0.8238257
$\lvert\Xi_5(\lambda_1, \lambda_2)\rvert$	0.8238257	$\lvert\Gamma_5(\lambda_2, \lambda_2)\rvert$	0.8238257

- For off-diagonal elements ($\alpha \neq \beta$),

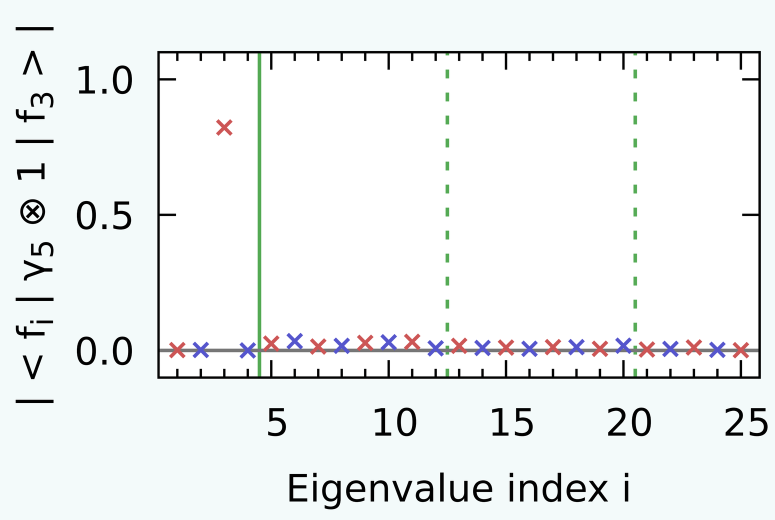
parameter	value	parameter	value
$\lvert\Gamma_5(\lambda_{75}, \lambda_{70})\rvert$	0.5008622	$\lvert\Gamma_5(\lambda_{70}, \lambda_{75})\rvert$	0.5008622
$\lvert\Xi_5(\lambda_{69}, \lambda_{75})\rvert$	0.5008622	$\lvert\Xi_5(\lambda_{75}, \lambda_{69})\rvert$	0.5008622
$\lvert\Xi_5(\lambda_{70}, \lambda_{76})\rvert$	0.5008622	$\lvert\Xi_5(\lambda_{76}, \lambda_{70})\rvert$	0.5008622
$\lvert\Gamma_5(\lambda_{69}, \lambda_{76})\rvert$	0.5008622	$\lvert\Gamma_5(\lambda_{76}, \lambda_{69})\rvert$	0.5008622

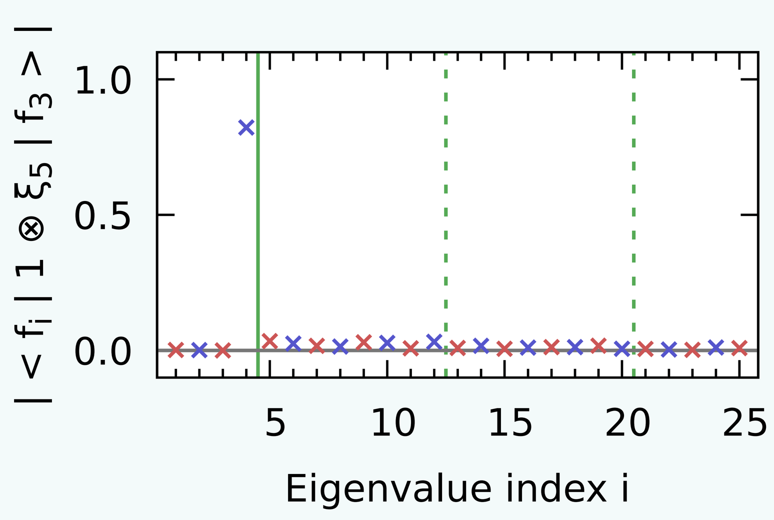




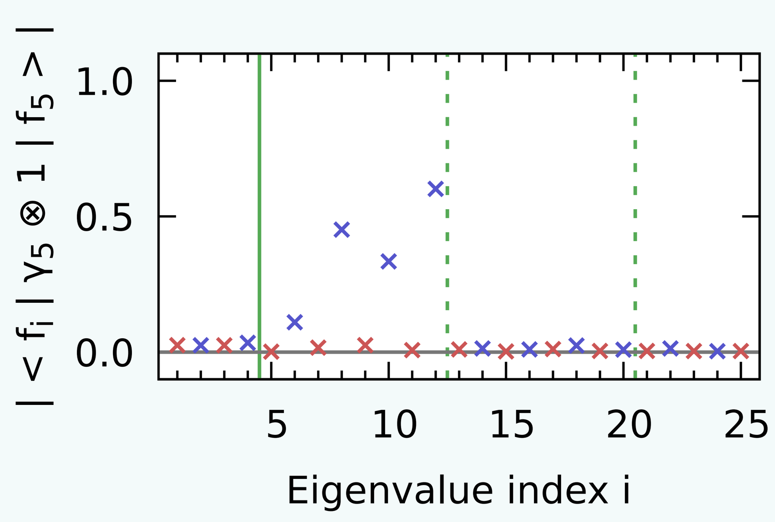
Leakage of chirality

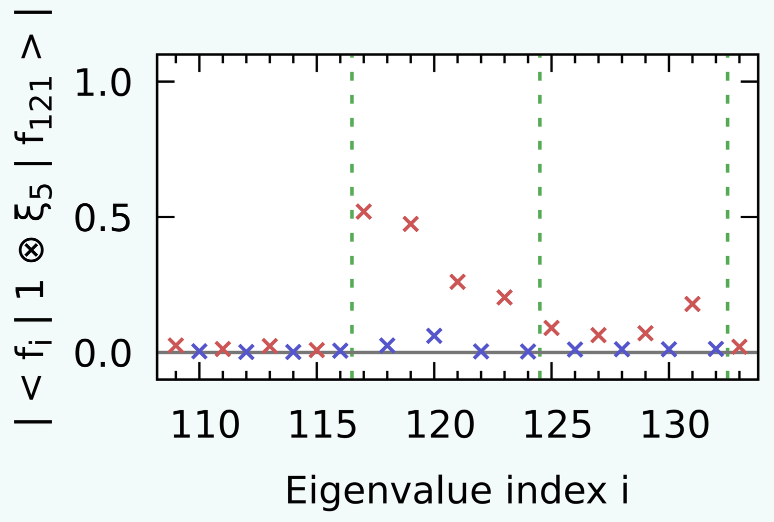
- A zero mode also can be identified by its leakage pattern through $\Xi_5 \equiv [\mathbb{1} \otimes \xi_5]$.





- For non-zero modes, chiralities *leak* into its eight-folded degenerate pairs (not applicable for higher modes such as $i > 500$).





Conclusion

- We introduced chiral Ward identities for staggered fermions. Chiral properties of staggered Dirac eigenmodes are related through Ward identities.
- We can identify zero modes of staggered fermions not only by measuring the $[\gamma_5 \otimes \mathbb{1}]$ chirality, but also with the leakage pattern of chiralities $[\gamma_5 \otimes \mathbb{1}]$ and $[\mathbb{1} \otimes \xi_5]$.

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