Properties of the η and η' mesons

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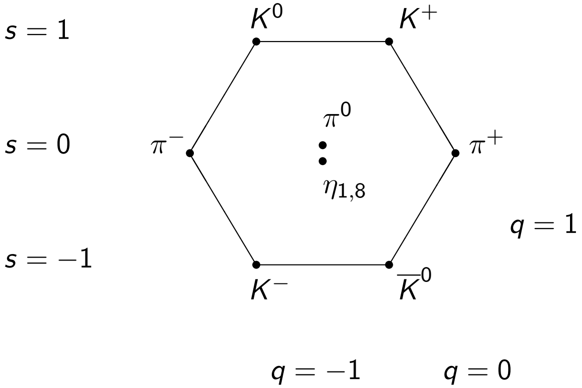
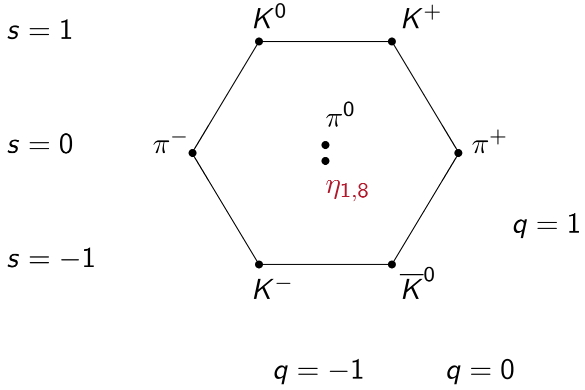
Outline

Quark model: "periodic table of pseudoscalar mesons"



$$SU(3)_L \times SU(3)_R \to SU(3)_V$$

Quark model: "periodic table of pseudoscalar mesons"



$$SU(3)_L \times SU(3)_R \to SU(3)_V$$

The QCD vacuum: Axial symmetry breaking

- η_1 becomes heavy compared to the octet mesons due to anomalous breaking of $U_A(1)$ axial symmetry:

$$\partial_\mu \widehat{A}^{a\mu} = (\overline{\psi}\gamma_5 \widehat{\{M, t^a\}}\psi) + \sqrt{2N_f}\delta^{a0}\widehat{q}_t, \quad a = 0, \ldots, 8$$

- Tightly connected to topological susceptibility χ: Large mass of η_1 explained by Witten and Veneziano:

$$\frac{f_\pi^2 m_{\eta_1}^2}{N_f} = \chi\Big|_{N_f=0}$$

(in the t'Hooft limit of $N_c \to \infty$)

η/η' mixing

- Simple $SU(3)$ flavour symmetry fails: Predicted singlet and octet particles

$$\eta_1 = \frac{u\bar{u} + d\bar{d} + s\bar{s}}{\sqrt{3}}, \qquad \eta_8 = \frac{u\bar{u} + d\bar{d} - 2s\bar{s}}{\sqrt{6}}$$

- In reality, η and η' are not flavour eigenstates:

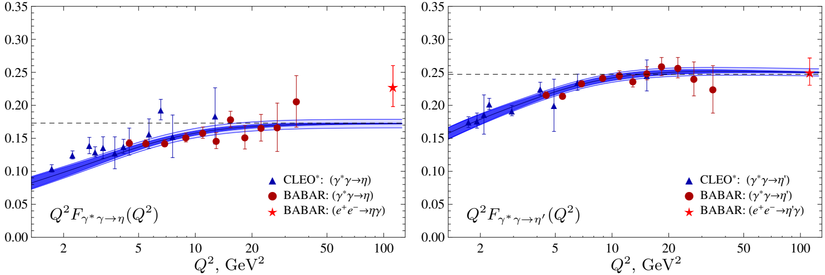
$$U(\theta,\theta') \begin{pmatrix} \mathcal{O}_{88} & \mathcal{O}_{81} \\ \mathcal{O}_{18} & \mathcal{O}_{11} \end{pmatrix} U^T(\theta,\theta') = \begin{pmatrix} \mathcal{O}_{\eta\eta} & 0 \\ 0 & \mathcal{O}_{\eta'\eta'} \end{pmatrix}$$

with mixing angles θ and θ',

$$\eta = \cos\theta\eta_8 - \sin\theta\eta_1, \qquad \eta' = \sin\theta'\eta_8 + \cos\theta'\eta_1$$

  - In a flavour symmetric world: $\eta = \eta_8 = \pi^0$

Phenomenology

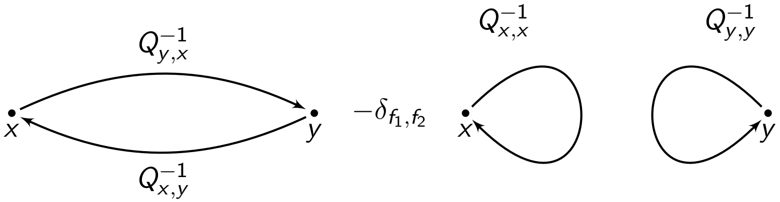


- η transition form factor $\gamma^*\gamma \to \eta$: Interest in large Q^2, cf. [Agaev et al.,1409.4311]

$$\lim_{Q^2\to\infty} Q^2 F_{\gamma^*\gamma\to\eta}(Q^2) = \sqrt{\frac{2}{3}}\left[f_\eta^8 + 2\sqrt{2} f_\eta^1(\mu_0)\left(1 - 2\frac{2N_f}{\pi\beta_0}\alpha_s(\mu_0)\right)\right]$$

- axialvector decay constants have never been determined at the physical point, but see, e.g., lattice determinations
  - from relating them to the pseudoscalar current using ChPT, [ETMC,1710.07986]
  - for two ensembles at a single lattice spacing [RQCD,1406.5449]

Stochastic estimation of disconnected loops

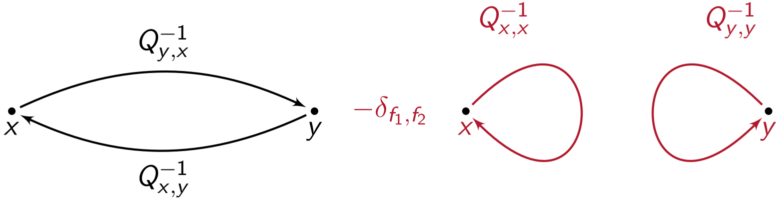


Wick contractions of mesons:

$$\langle q_{f_1}(y)\gamma_5\bar{q}_{f_2}(y)\bar{q}_{f_1}(x)\gamma_5 q_{f_2}(x)\rangle = q_{f_1}(y)\gamma_5\bar{q}_{f_2}(y)\bar{q}_{f_1}(x)\gamma_5 q_{f_2}(x)$$

$$= Q^{-1}_{f_1,f_1}(y,x)Q^{-1}_{f_2,f_2}(x,y) - \delta_{f_1,f_2}Q^{-1}_{f_1,f_2}(y,y)Q^{-1}_{f_1,f_2}(x,x)$$

Stochastic estimation of disconnected loops



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$\rightarrow$ disconnected loops arise

Stochastic Estimation of Disconnected loops

Stochastic estimation is required for the inversion of Q: The N_{stoch} linear systems $Q \mid s_i\rangle = \mid \eta_i\rangle$ are solved on random sources $\eta_{ix\alpha a} \in (\mathbb{Z}_2 + i\mathbb{Z}_2)/\sqrt{2}$

$$Q^{-1} = \frac{1}{N_{stoch}} \sum_{i}^{N_{stoch}} \mid s_i\rangle\langle\eta_i \mid + \mathcal{O}\left(\frac{1}{\sqrt{N_{stoch}}}\right)$$

→ extra stochastic noise in addition to the gauge noise.

Time dilution [Bernardson et al.,1993; Viehoff et al.,1998; O'Cais et al.,2005]

- put random sources at every 4th time slice
- set source at (open) boundaries to zero

Hopping parameter expansion [Thron et al.,1998; Michael et al., 2000; Bali et al.2005]

- use locality of the Wilson Dirac operator and expand in small κ
- using two and four applications for the pseudoscalar and axialvector loops, respectively

Matrix correlators

- Construct N bases from n biquark fields:

$$b_i(t, \vec{p}) = \sum_{j=0}^{n-1} B_{ij} \sum_{\vec{x}} e^{-i\vec{p}\vec{x}} \left(\bar{q}_j \gamma_5 q_j\right)(x),$$

where $B \in \mathbb{R}^{N\times n}$ matrix that defines a basis and subscripts are superindices defining flavour and smearing.

- matrix correlator

$$C(t) = \frac{1}{N_t} \sum_{t'=0}^{N_t-1} \begin{pmatrix} \langle b_1(t'+t) \mid b_1(t')\rangle & \cdots & \langle b_1(t'+t) \mid b_N(t')\rangle \\ \vdots & \ddots & \vdots \\ \langle b_N(t'+t) \mid b_1(t')\rangle & \cdots & \langle b_N(t'+t) \mid b_N(t')\rangle \end{pmatrix}.$$

- Note: C is a real, positive semidefinite and symmetric matrix.

Fitting to matrices

- Usually: solve GEVP to diagonalize C
- Instead: decompose correlator in terms of orthogonal (mass) eigenstates:

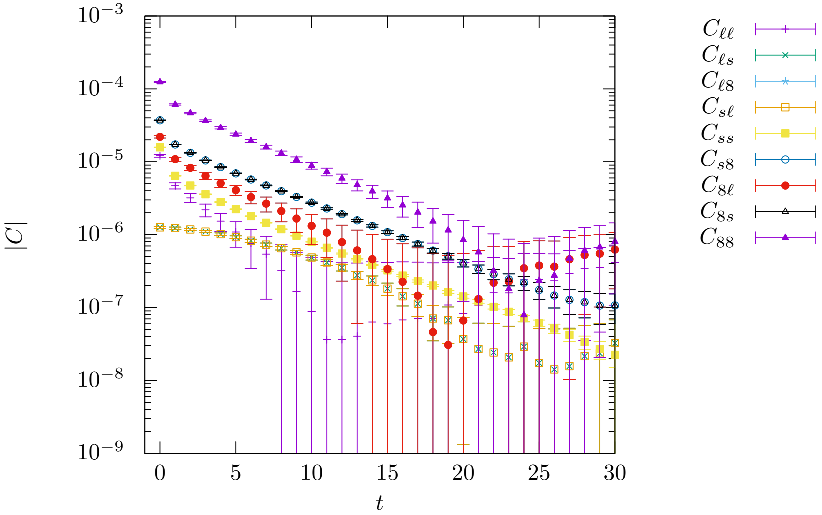
$$C(t)_{ij} = \sum_n \frac{1}{2E_n} \langle b_i \mid n \rangle \langle n \mid b_j \rangle \exp(-E_n t)$$
$$= \left[Z D(t) Z^T \right]_{ij},$$

where

$$Z_{in} = \frac{1}{\sqrt{2E_n}} \langle b_i \mid n \rangle, \qquad \text{and} \qquad D(t) = \mathrm{diag}_{n=0}(\exp(-E_n t))$$

- Note that Z does *not* depend on t.
- Need to truncate the sum: $Z : N \times \infty \to Z : N \times N_{\mathrm{st}}$
- There is no need for Z to be quadratic, may fit more or fewer states N_{st} than available bases N.

Example correlator



Improving the signal

Derivative trick

Replacing correlators with their derivatives removes any constant shifts in the correlator and decreases autocorrelations (similar to [Takashi,hep-lat/0701005], [Feng et al,0909.3255]):

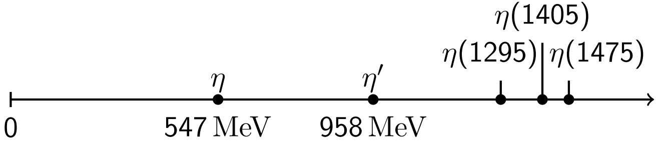
$$\partial_t C(t) = -ZED(t)Z^T$$

Combine data with dispersion relation

Signals typically better at $p^2 > 0$: Combine several correlators in a joint fit, using the dispersion relation:

$$E(p) = \sqrt{m^2 + p^2}, \qquad p_i = \frac{2\pi k_i}{L}$$

Excited states



- η' signal deteriorates at short Euclidean times
- plethora of states nearby
- coverage of excited states depends on the (octet or singlet) nature of these states and the overlap with the chosen basis
- need to include these states in the fit $\rightarrow$ need at least a 3x3 matrix
- typical fit windows $t \in [0.35, 1]\,\mathrm{fm}$
- fit functions contain multi-exponentials:

$$C_{ij}(t) = \sum_{n=0}^{N-1} Z_{in} D_{nn} Z_{jn} = \sum_{n=0}^{N-1} Z_{in} Z_{jn} \exp(-E_n t)$$

Generalized effective masses

- Excited states become visible when looking at

$$(\partial_t C(t))C^{-1} = -ZEZ^{-1} + \mathcal{O}(\exp(-(E_{N_{st}} - E_{N_{st}-1})t))$$

- leading term ZEZ^{-1} does not depend on time
- generalization of simple effective masses (for single correlators):

$$(\partial_t C(t))C^{-1} = \partial_t \log(C(t))$$

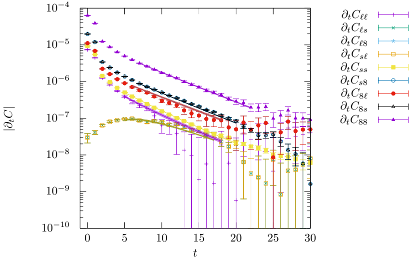
- can be included in a (joint) fit and helps constraining amplitudes (Z)
- leading term unchanged when using higher derivatives

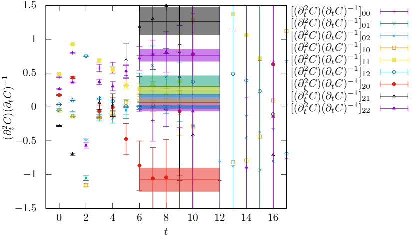
$$(\partial_t^2 C(t))(\partial_t C)^{-1} = -ZEZ^{-1} + \mathcal{O}(\exp(-(E_{N_{st}} - E_{N_{st}-1})t))$$

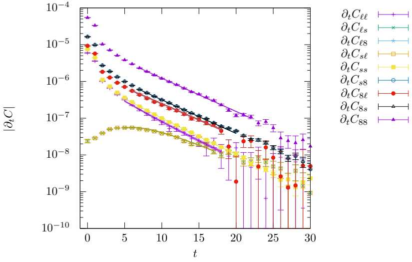
Fitting

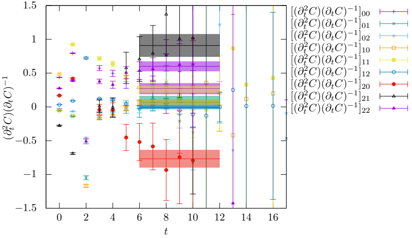
$$\partial_t C = -ZEDZ^T$$

$$(\partial_t^2 C)(\partial_t C)^{-1} = -ZEZ^{-1}$$

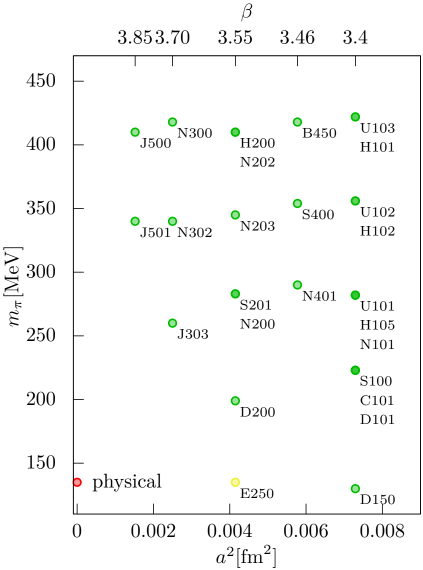


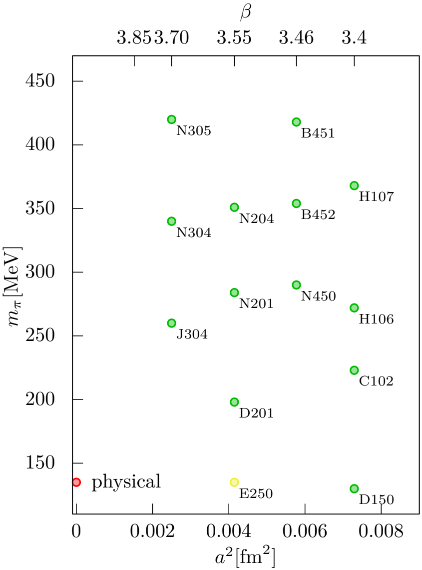






CLS Ensembles: overview





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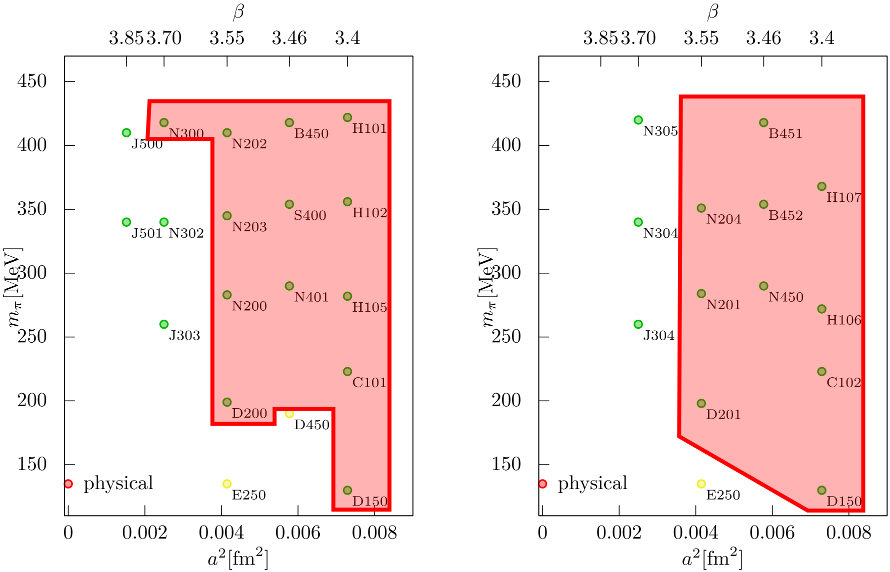
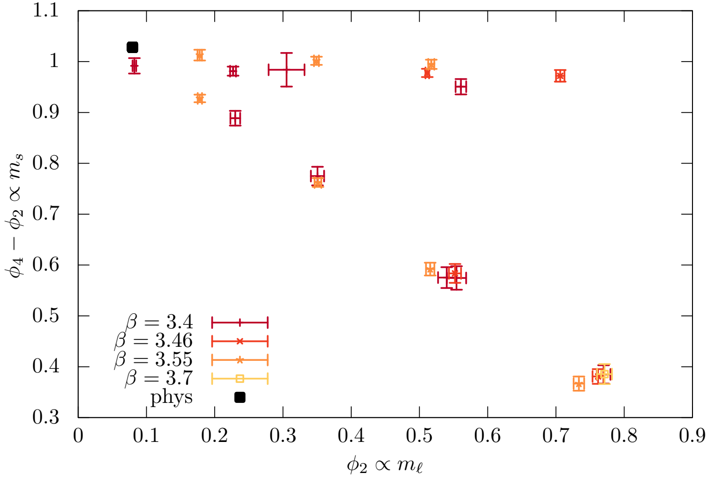
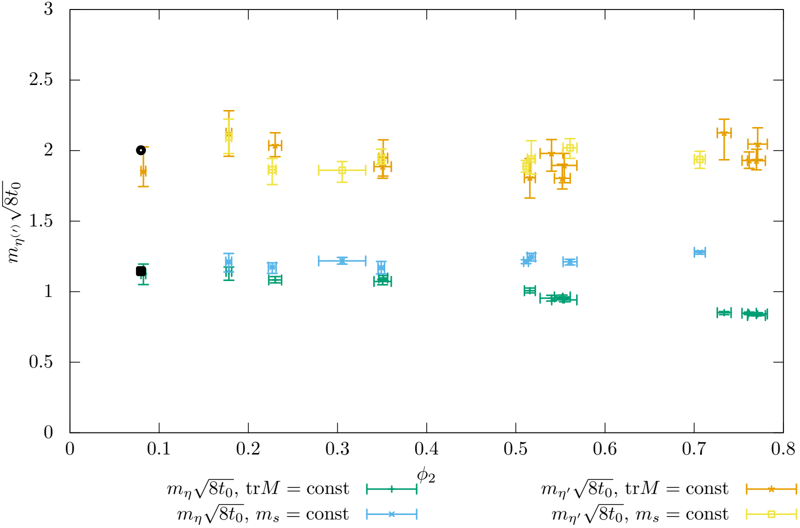


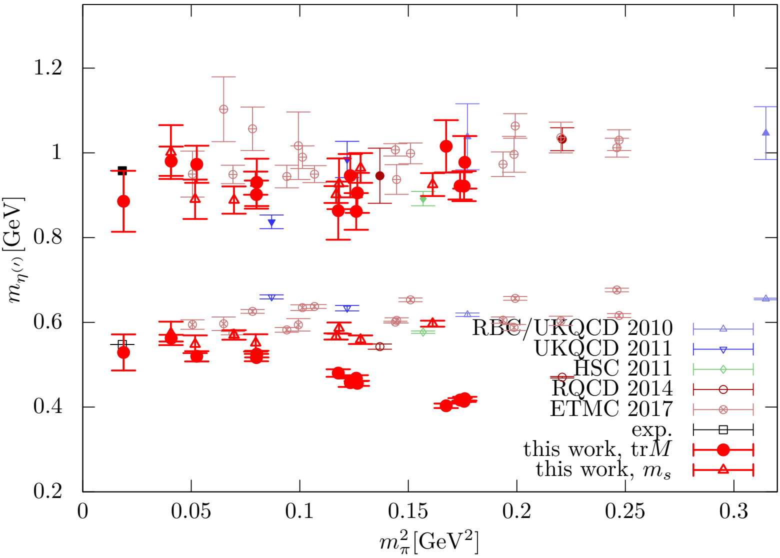
CLS Ensembles: mass plane



Masses


3
2.5
2
1.5
1
0.5
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
ϕ_2
$m_{\eta^{(\prime)}}\sqrt{8t_0}$
$m_\eta\sqrt{8t_0}$, tr$M$ = const
$m_\eta\sqrt{8t_0}$, m_s = const
$m_{\eta'}\sqrt{8t_0}$, tr$M$ = const
$m_{\eta'}\sqrt{8t_0}$, m_s = const

Masses



1.2
1
0.8
0.6
0.4
0.2
$m_{\eta^{(\prime)}}\,[\mathrm{GeV}]$
0
0.05
0.1
0.15
0.2
0.25
0.3
$m_\pi^2\,[\mathrm{GeV}^2]$
RBC/UKQCD 2010
UKQCD 2011
HSC 2011
RQCD 2014
ETMC 2017
exp.
this work, tr$M$
this work, m_s

LO ChPT

- LO ChPT [Gasser and Leutwyler,1984]:

$$M_+^2 = m_{\eta'}^2 + m_\eta^2 = M_0^2 + \tilde{M}_{\eta_8}^2$$
$$M_-^2 = m_{\eta'}^2 - m_\eta^2 = \frac{M_0^2 - \tilde{M}_{\eta_8}^2}{\cos 2\delta},$$

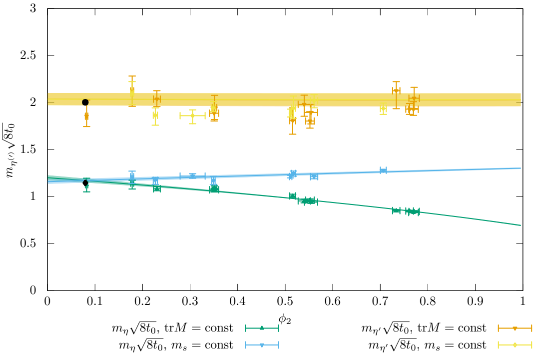
where

$$\tilde{M}_{\eta_8}^2 = \frac{1}{3}(4m_K^2 - m_\pi^2), \qquad \tan 2\delta = -\frac{4}{3}\sqrt{2}\gamma\frac{m_K^2 - m_\pi^2}{M_0^2 - \tilde{M}_{\eta_8}^2}$$

- work with dimensionless combinations of masses with $8t_0$
- both masses described by two LECs: M_0^2 and γ.
- parametrize lattice spacing effects:

$$m_{\eta^{(\prime)}}\left(1 + c_0^{(\prime)}a^2 + c_1^{(\prime)}a^2(2m_K^2 + m_\pi^2) + c_2^{(\prime)}a^2(m_K^2 - m_\pi^2)\right)$$

Masses



- lattice discretization effects: only some terms resolvable
- results extrapolate to physical points within errors:

$$M_\eta = 559(10)\,\mathrm{MeV} \qquad M_{\eta'} = 973(30)\,\mathrm{MeV}$$

- LO LECs:

$$M_0 = 970(32)\,\mathrm{MeV} \qquad \gamma = 0.32(34)$$

Decay constants: definitions

- LO ChPT predicts decay constants to be constant
- defined via local axialvector currents:

$$\sqrt{2}\langle 0 \mid \overline{\psi}\gamma_\mu\gamma_5 t^a \psi \mid \mathcal{M}\rangle = i f^a_{\mathcal{M}} p_\mu,$$

  where $\mathcal{M} = \eta, \eta', \ldots$ is a pseudoscalar state
  $\rightarrow$ 4 independent decay constants ($\mathcal{M} = \eta, \eta'$ and $a = 0, 8$)
- Again: can combine correlators at different momenta to improve the signal.

Decay constants: Renormalization and improvement

- Renormalization and improvement different for singlet ($a = 0$) and non-singlet ($a = 8$) currents:

$$f_{\mathcal{M}}^{a} = Z_A^a (1 + \tilde{b}_A a \,\mathrm{tr}\, M + b_A^a a m_a) \frac{1}{E_{\mathcal{M}}} \langle 0 \mid A_\mu^a + c_A^a \partial_\mu P^a \mid \mathcal{M} \rangle,$$

where m_a is a combination of (valence) quark masses.

- non-singlet renormalization Z_A^8 well-known nonperturbatively from, e.g., Schrödinger functional, [ALPHA,1604.05827;1808.09236]
- same for improvement coefficients [Korcyl,1607.07090;ALPHA,1502.04999]
- assume $b_A^0 = b_A^8 = b_A$ and $c_A^0 = c_A^8 = c_A$, $\tilde{b}_A = 0$

Renormalization of the Singlet Axialvector Current

- singlet renormalization factor unknown for our action
- but: difference to non-singlet Z_A known from lattice perturbation theory [Constantinou et al.,1610.06744] and starts at $\mathcal{O}(a_s^2)$ ($a_s = \alpha_s/(4\pi)$)

$$Z_A^0(a\mu) - Z_A^8(a\mu) = -\frac{1}{16} a_s^2(a^{-1}) c_F N_f \left(6 \log(a^2\mu^2) - 11.3716\right)$$

- anomalous dimension of singlet axialvector current vanishes to $\mathcal{O}(a_s)$ $\rightarrow Z_A^0(\mu = \infty)$ is finite
- Take perturbative Z_A^0 at $\mu = a^{-1}$ (no logs) and evolve to $\mu = \infty$, defining a scale-independent renormalization [Zöller, 1304.2232]

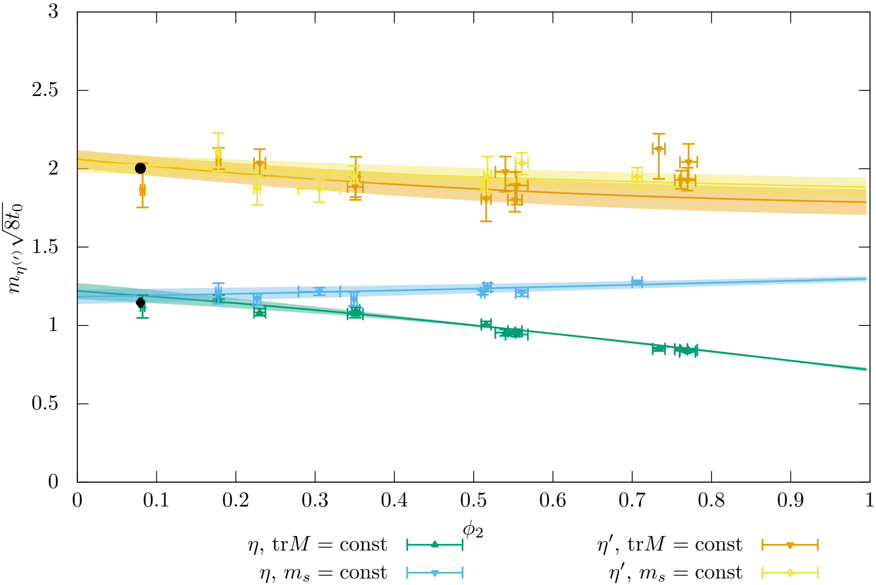
$$Z_A^{0\prime} = \left[1 - \frac{\gamma_{A1}^s}{\beta_0} a_s(\mu) + \frac{\gamma_{A1}^s}{2\beta_0}\left(\frac{\gamma_{A1}^s}{\beta_0} + \frac{\beta_1}{\beta_0} - \frac{\gamma_{A2}^s}{\gamma_{A1}^s}\right) a_s^2(\mu) + \cdots\right] Z_A^0(\mu).$$

- final results can be converted to low scales, using finite conversion factors $c(\mu) = Z_A^0(\mu)/Z_A^{0\prime}$

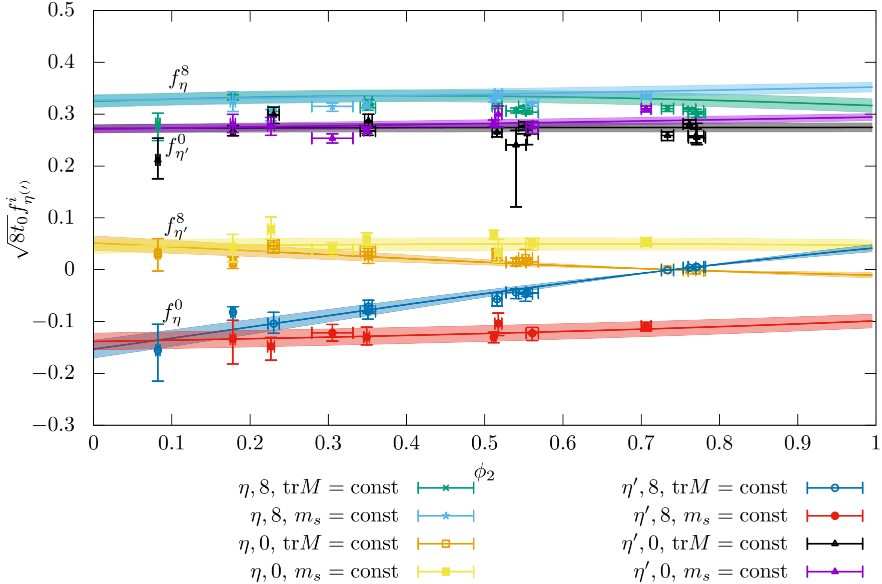
NLO ChPT

- Large-N_c NNLO chiral expansion available [Bickert et al.,1612.05473]
- truncate at NLO and obtain (complicated) functions of
  - ϕ_2 and ϕ_4, parametrizing the quark mass dependence
  - LECs $L_5, L_8, \Lambda_1, \Lambda_2, M_0^2$ (different from $\mathrm{SU}(3)$-ChPT LECs)
  - LECs $L_4, L_7, \ldots$ and chiral logs appear only at NNLO
- LECs connect decay constants with masses $\rightarrow$ joint fit
- visible discretization effects in octet decay constants

NLO masses: results



NLO decay constants: results



Large-N_c NLO fits

- vary different mass cuts, parametrization of a^2 effects
  $\rightarrow$ fits give consistent results with $1 < \chi^2/N_{\mathrm{df}} < 2$ (shown fit: 1.5)
- masses:

$$m_\eta = 568(20)\,\mathrm{MeV} \qquad m_{\eta'} = 965(29)\,\mathrm{MeV}$$

- decay constants:

$$\begin{aligned} f_\eta^8 &= 157(6)\,\mathrm{MeV} & f_{\eta'}^8 &= -65(8)\,\mathrm{MeV} & & \\ f_\eta^0 &= 22(6)\,\mathrm{MeV} & f_{\eta'}^0 &= 130(4)\,\mathrm{MeV} & & @\mu = \infty \\ f_\eta^0 &= 21(6)\,\mathrm{MeV} & f_{\eta'}^0 &= 121(4)\,\mathrm{MeV} & & @\mu = 1\,\mathrm{GeV} \end{aligned}$$

- LECs:

$$\begin{aligned} F &= 102(5)\,\mathrm{MeV} & M_0 &= 784(50)\,\mathrm{MeV} \\ L_5 &= 1.43(6) \times 10^{-3} & L_8 &= 1.18(12) \times 10^{-3} \\ \Lambda_1 &= -0.362(42) & \tilde{\Lambda} &= \Lambda_1 - 2\Lambda_2 = 0.07(25) \\ & & &\rightarrow \Lambda_2 \approx \frac{1}{2}\Lambda_1 \end{aligned}$$

Gluonic matrix elements

- Axial Ward Identities:

$$\partial_\mu \widehat{A}^{a\mu} = (\overline{\psi}\gamma_5 \widehat{\{M, t^a\}}\psi) + \sqrt{2N_f}\delta^{a0}\widehat{q}_t,$$

- $\langle 0 \mid \hat{q}_t \mid \eta^{(\prime)} \rangle$ contributes to the (singlet) decay constants
- two possible definitions
  - direct (gluonic):

$$\langle 0 \mid \hat{q} \mid \eta^{(\prime)} \rangle = B_G \langle 0 \mid q \mid \eta^{(\prime)} \rangle + Z_{GA} \langle 0 \mid \partial_\mu A_\mu \mid \eta^{(\prime)} \rangle$$

  - fermionic (via AWI):

$$\langle 0 \mid \hat{q} \mid \eta^{(\prime)} \rangle = \frac{1}{\sqrt{2N_f}} \left(\langle 0 \mid \partial_\mu \hat{A}^{0,\mu} \mid \eta^{(\prime)} \rangle - \langle 0 \mid \gamma_5 \widehat{\{M, t^0\}} \mid \eta^{(\prime)} \rangle \right)$$

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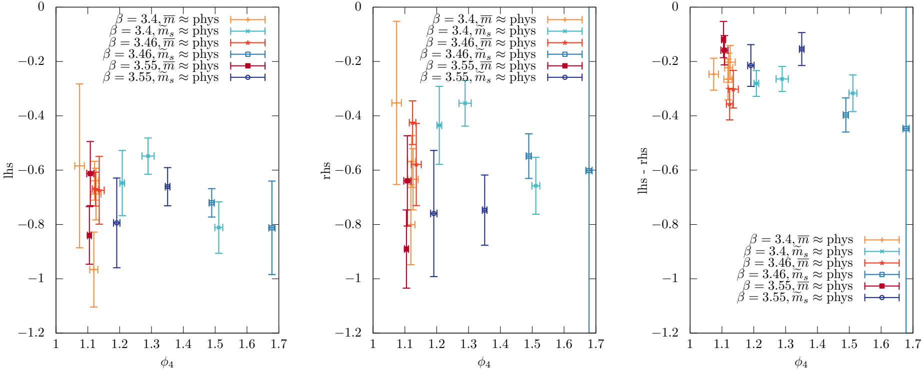
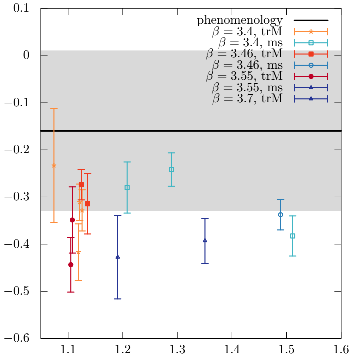
Test of the singlet AWI

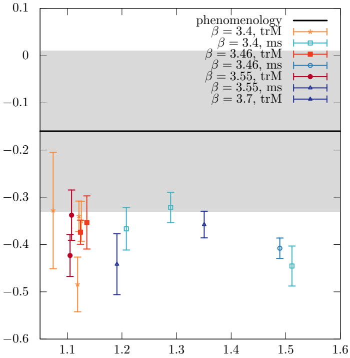


$$(8t_0)^{3/2}\left(\langle 0|\partial_\mu \widehat{A}^{0\mu}|\eta'\rangle - \langle 0|(\overline{\psi}\gamma_5\widehat{\{M,t^0\}}\psi + \sqrt{2N_f}\widehat{q}_t)|\eta'\rangle\right) = 0 + \mathcal{O}(a)$$

Gluonic matrix elements



gluonic



fermionic

In line with phenomenological estimate [Cheng et al.,0811.2577]:

$$\langle 0|\hat{q}_t|\eta'\rangle = -0.054(57)\,\mathrm{GeV}^3 = -0.16(17)(8t_0)^{-3/2}$$

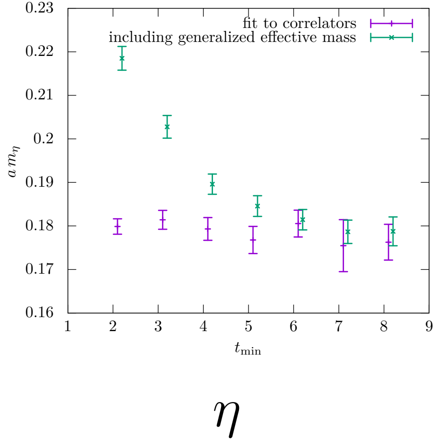
Conclusions

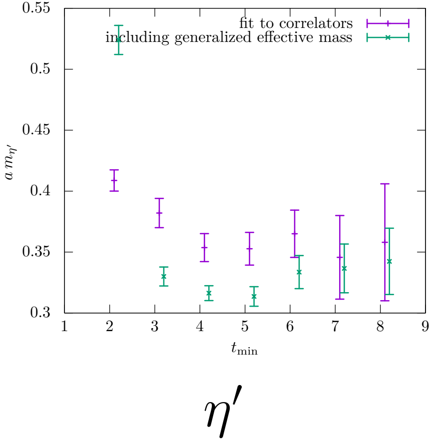
- continuum extrapolation of η and η' masses and decay constants in a combined Large-N_c ChPT NLO fit
- determination of decay constants directly from axialvector currents within a few percent accuracy
- anomalous matrix elements accessible but large discretization effects visible

Backup slides

Excited states analysis

example: D200 ($\beta = 3.55$, $m_\pi \approx 200\,\mathrm{MeV}$, $m_K \approx 480\,\mathrm{MeV}$)





Decay constants in the flavour basis

- relation to octet-singlet basis:

$$f^{\ell}_{\eta^{(\prime)}} = \sqrt{\frac{1}{3}} f^{8}_{\eta^{(\prime)}} + \sqrt{\frac{2}{3}} f^{0}_{\eta^{(\prime)}}, \qquad f^{s}_{\eta^{(\prime)}} = -\sqrt{\frac{2}{3}} f^{8}_{\eta^{(\prime)}} + \sqrt{\frac{1}{3}} f^{0}_{\eta^{(\prime)}}$$

- Reparametrization:

$$\begin{pmatrix} f^{\ell}_{\eta} & f^{s}_{\eta} \\ f^{\ell}_{\eta'} & f^{s}_{\eta'} \end{pmatrix} = \begin{pmatrix} f_{\ell} \cos\phi_{\ell} & -f_{s} \sin\phi_{s} \\ f_{\ell} \sin\phi_{\ell} & f_{s} \cos\phi_{s} \end{pmatrix}$$

- results @ $\mu = \infty$

$$f_{l} = 111(8)\,\mathrm{MeV} = 0.85(7) f_{\pi}, \qquad f_{s} = 173(9) = 1.33(5) f_{\pi}\,\mathrm{MeV}$$

$$\phi_{l} = 11.5(9)^{\circ}, \qquad \phi_{s} = 42(1.4)^{\circ}$$

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- results @ $\mu = 1\,\mathrm{GeV}$

$$f_l = 123(8)\,\mathrm{MeV} = 0.95(8) f_{\pi}, \qquad f_s = 169(9) = 1.3(5) f_{\pi}\,\mathrm{MeV}$$
$$\phi_l = 30(1)^{\circ}, \qquad \phi_s = 43(2)^{\circ}$$