Zero modes of the Domain wall operator for 2+1 flavor lattices with $a^{-1} \approx 1$ GeV

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Contents

Introduction

- The large mass difference between η and η' is explained by the breaking of $\mathrm{U_A}(1)$ symmetry, which is originated from the non-trivial topological structure of QCD vacuum.
- A good sampling over topological sectors is important for η and η'. It's also important for general gauge field generation.
- We have seen some evidence showing that topology is not evolving well enough in our coarse $a^{-1} = 1$GeV DWF ensemble.
- Topology is usually measured after smearing the lattice. However, measurements are done before smearing. Are there ways to measure topological properties before smearing?

Lattices and calculation details

- We consider three ensembles in this study:

size	a^{-1}	m_π	total configurations	total eigenvectors	methods
12x32	1GeV	300MeV	70	3500	Lanczos
24x64	1GeV	140MeV	2	40	Ritz
24x64	2GeV	300MeV	2	60	Ritz

- We varied input mass for the fine ensemble and the 12x32 ensemble. We varied Ls for 12x32 ensemble. We calculated eigenvectors for Ls $= 8, 12, 16, 24, 48, 96, 192$.
- We are going to do more measurements for the physical coarse ensemble after July when we get our allocation.

Eigenvectors in the continuum

- In the continuum, the massless Dirac operator is $i\not{D}$. It has the following properties:
  - $i\not{D} = (i\not{D})^\dagger$, $\{i\not{D}, \gamma_5\} = 0$
- Therefor, for $i\not{D}\psi_\lambda = \lambda\psi_\lambda$, we have:
  - if $\lambda = 0$, $\gamma_5\psi_0 = \pm\psi_0$, $\langle\psi_0|\gamma_5|\psi_0\rangle = \pm 1$
  - if $\lambda \neq 0$, $\gamma_5\psi_\lambda = \psi_{-\lambda}$, $\langle\psi_\lambda|\gamma_5|\psi_\lambda\rangle = 0$, $\langle\psi_\lambda|\gamma_5|\psi_{-\lambda}\rangle = 1$
- The index theorem tells us that the number of zero (chiral) modes of $i\not{D}$ gives the topology of the field.
- $\gamma_5(\not{D} + m)$ is the Hermitian operator with mass. The properties are:
  - $\{\gamma_5(\not{D} + m), \gamma_5\} = 2m \Rightarrow \langle\psi^H_{\pm\sqrt{\lambda^2+m^2}}|\gamma_5|\psi^H_{\pm\sqrt{\lambda^2+m^2}}\rangle = \frac{m}{\lambda_H} = \frac{m}{\pm\sqrt{\lambda^2+m^2}}$.
  - Note that for all modes with $\lambda \ll m$, the eigenvectors of the Hermitian operator are chiral.
  - $(\gamma_5(\not{D} + m))^2 = \not{D}^\dagger\not{D} + m^2$, so the eigenvector space of $\gamma_5(\not{D} + m)$ is the same as $i\not{D}$ and $\lambda_H^2 = \lambda^2 + m^2$.

Eigenvectors in the continuum

- Given the eigenvectors of $i\not{D}$ with $\lambda \neq 0$, we find that the eigenvectors of $\gamma_5(\not{D} + m)$ are given as:

$$\psi^H_{\sqrt{\lambda^2+m^2}} = \frac{1}{\sqrt{2}}(e^{i\frac{\theta}{2}}\psi_\lambda + e^{-i\frac{\theta}{2}}\psi_{-\lambda}) \tag{1}$$

$$\psi^H_{-\sqrt{\lambda^2+m^2}} = \frac{1}{\sqrt{2}}(e^{i\frac{\theta}{2}}\psi_\lambda - e^{-i\frac{\theta}{2}}\psi_{-\lambda}) \tag{2}$$

  where θ is given by $e^{i\theta} = \frac{m+i\lambda}{\sqrt{m^2+\lambda^2}}$

- In addition, given $\gamma_5(\not{D} + m_1)$ and $\gamma_5(\not{D} + m_2)$, we have:

$$\langle \psi^H_{\sqrt{\lambda^2+m_1^2}} | \psi^H_{\sqrt{\lambda^2+m_2^2}}{}' \rangle = \cos(\theta_2 - \theta_1) \tag{3}$$

$$\langle \psi^H_{\sqrt{\lambda^2+m_1^2}} | \psi^H_{-\sqrt{\lambda^2+m_2^2}}{}' \rangle = i\sin(\theta_2 - \theta_1) \tag{4}$$

Eigenvectors for the lattice

- For domain wall Fermions, D_{DWF} (Shamir) is not Hermition. The Hermition operator is $D^H_{DWF} = \gamma_5 R_5 D_{DWF}$ and the mass term is contained in the matrix. The eigenvalue should have the following form [Liu, G. (2003)] for the low lying eigenvectors:

$$\Lambda^2 = n^2(\lambda^2 + (\delta m + m)^2) \tag{5}$$

- Here, Λ is the eigenvalue for the lattice, m is the input mass in domain wall Fermion. n, δm and λ are scale factor, residual mass and eigenvalue in the massless continuum, which have to be fitted.
- The anti-commutation relation is:

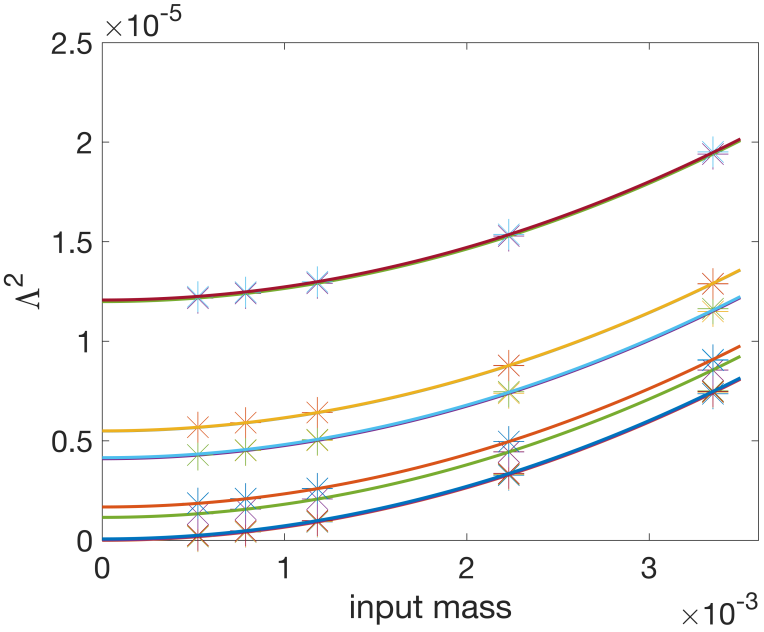
$$\{D_{DWF}, \Gamma_5\} = 2m_f Q^{(w)} + 2Q^{(mp)} \equiv 2Q \tag{6}$$

- where Γ_5 is 1 for $s >= L_s/2$ and -1 for $s < L_s/2$. This means that the chirality of the eigenvector follows:

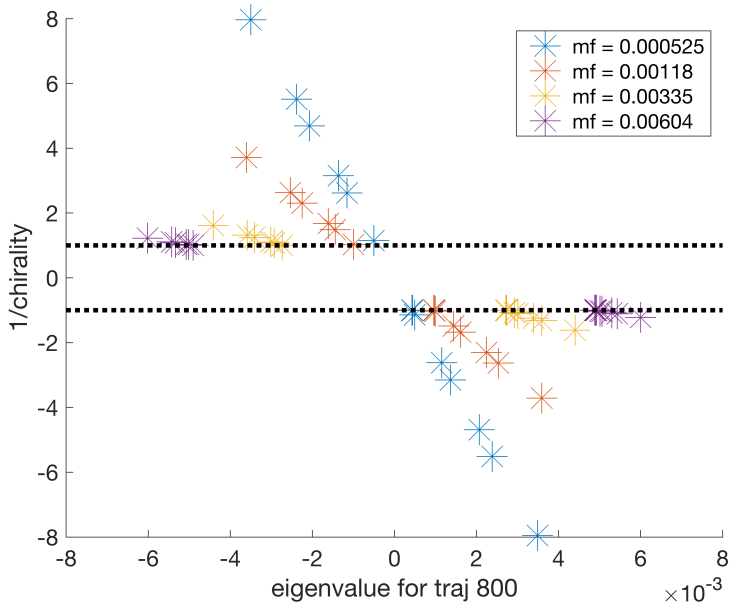
$$\langle\Psi_i|\Gamma_5|\Psi_i\rangle = \frac{\langle\Psi_i|Q|\Psi_i\rangle}{\Lambda_i} \Rightarrow \frac{1}{\langle\Psi_i|\Gamma_5|\Psi_i\rangle} \propto \Lambda_i \tag{7}$$

Eigenvectors for the fine lattice

- Here we show the results for the fine lattice with $a^{-1} = 2$ GeV on configuration 800. The left plot is a fit of eqn 5 and the right plot is a fit of eqn 7. The results confirm our understanding about the eigenvalues and chirality of the eigenvectors.



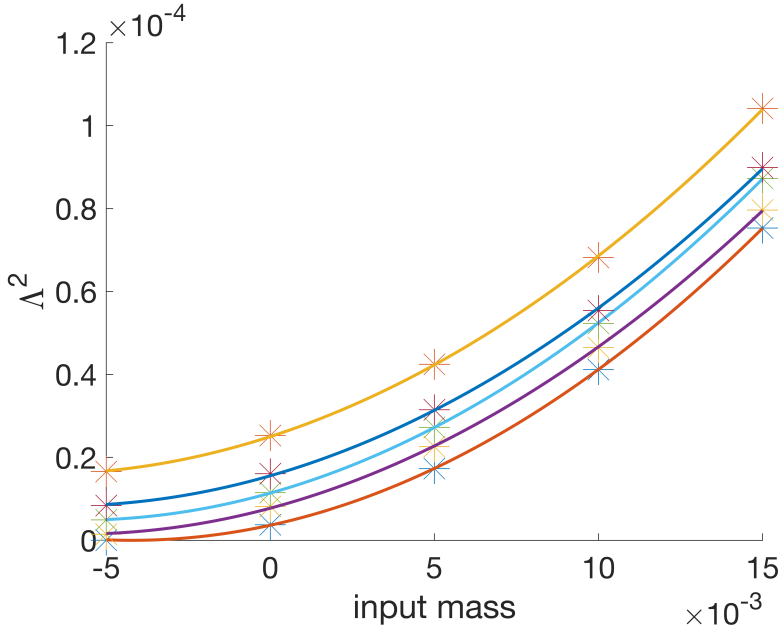
(a) Eigenvalue as a function of the input mass, Ls = 48



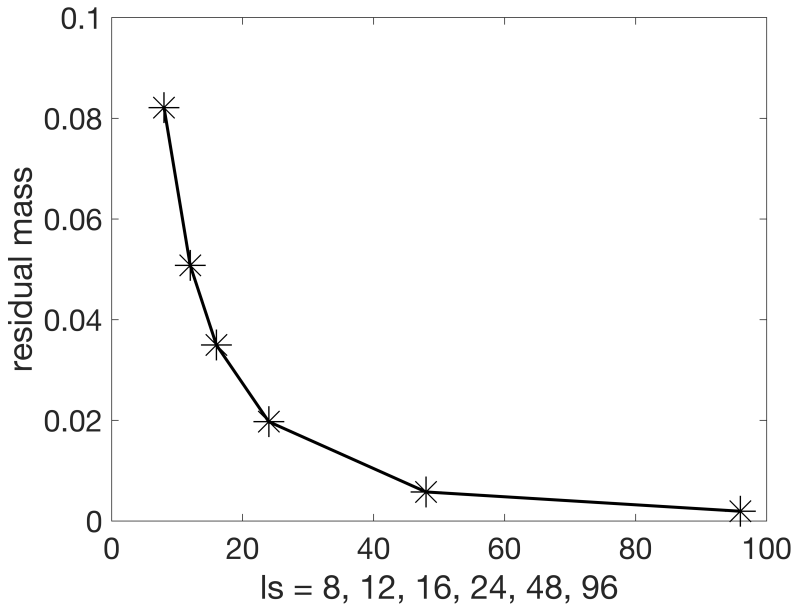
(b) $1/\langle\Psi|\Gamma_5|\Psi\rangle$ as a function of the eigenvalue, Ls = 48

Eigenvectors for the coarse lattice

- For the eigenvalues, we did similar study for the coarse lattices. We also obtained results for the residual mass. Results here are for the 12x32 ensemble. Again, it verifies our understanding about the eigenvalues.



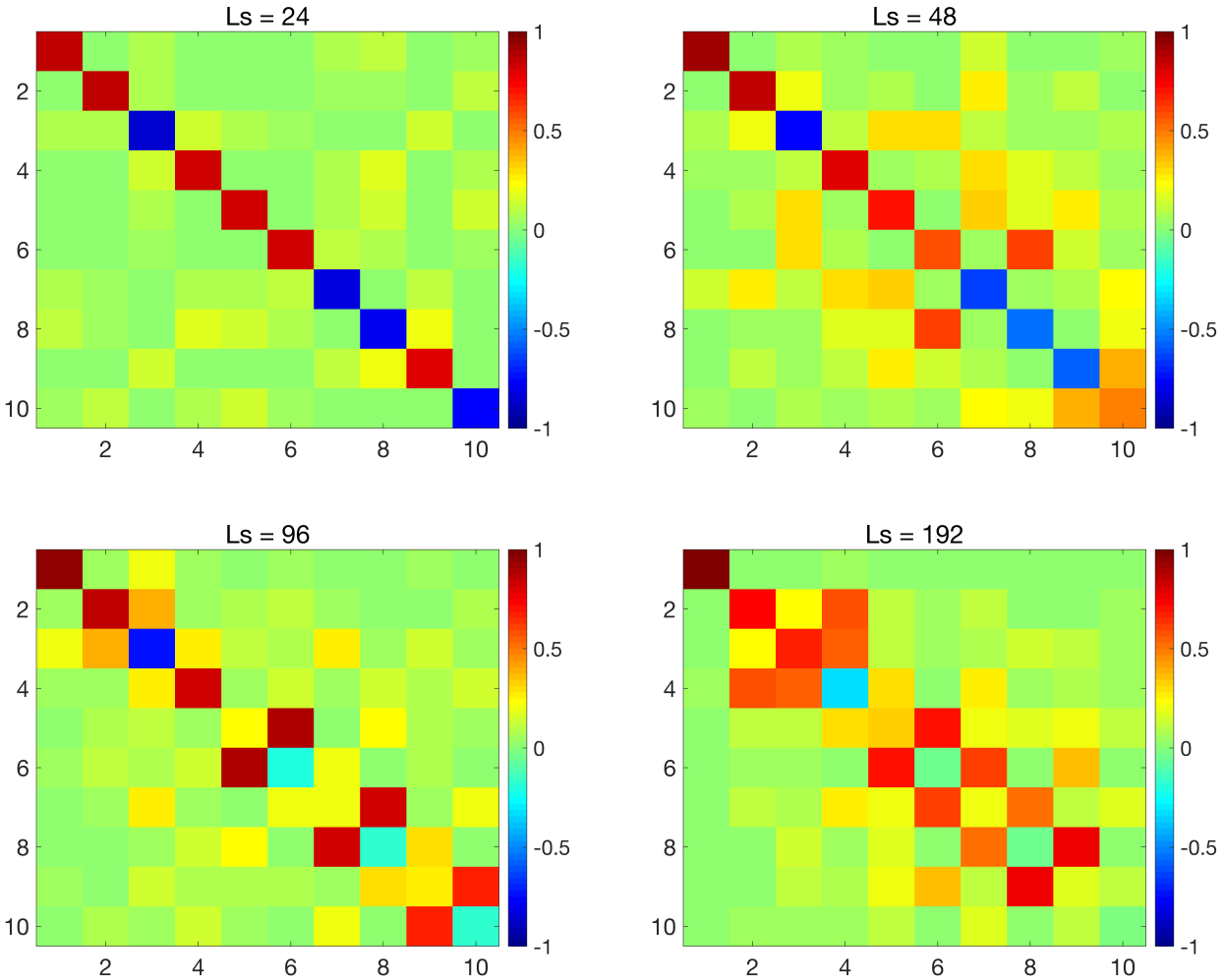
(a) Eigenvalue as a function of the input mass for ls = 48, config 700



(b) Residual mass for different Ls, config 700

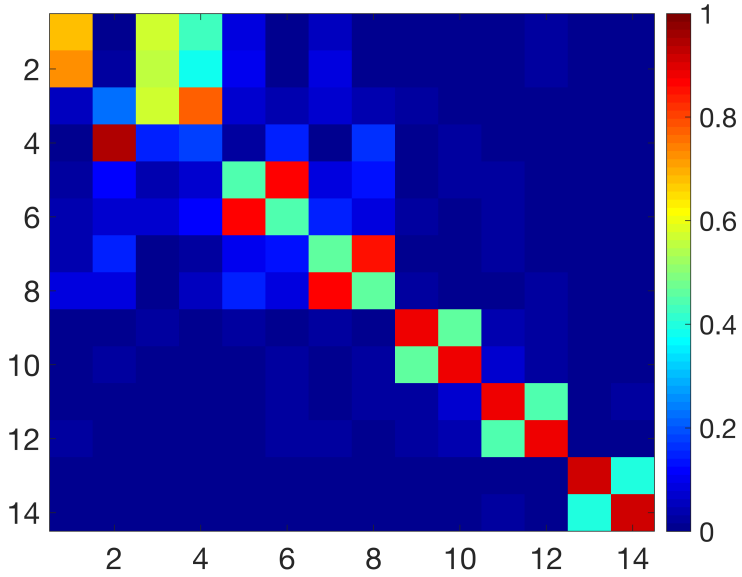
Results for different Ls

- We calculated eigenvectors for different Ls. Here we show chirality matrix $\langle \Psi_i | \Gamma_5 | \Psi_j \rangle$ for Ls = 24, 48, 96, 192, m = 0 on configuration 700, 12x32 ensemble.

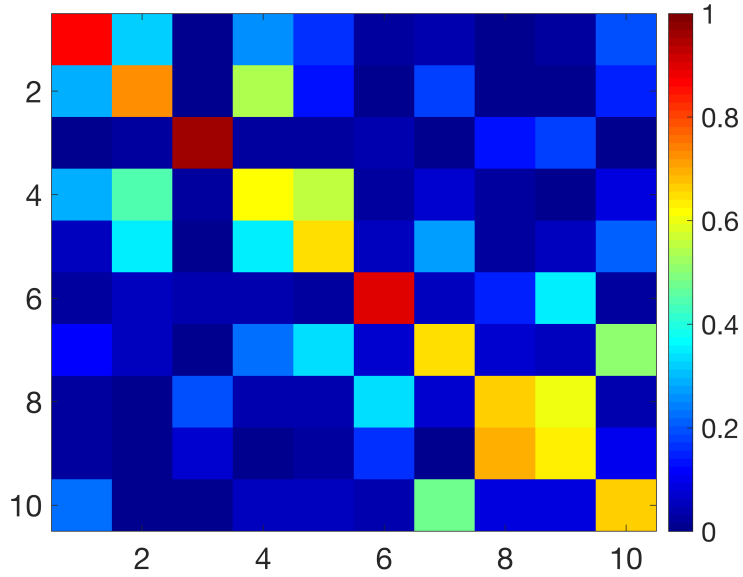


Mass effect

- When the input mass is changed, the eigenvectors are rotated for the non-zero modes.



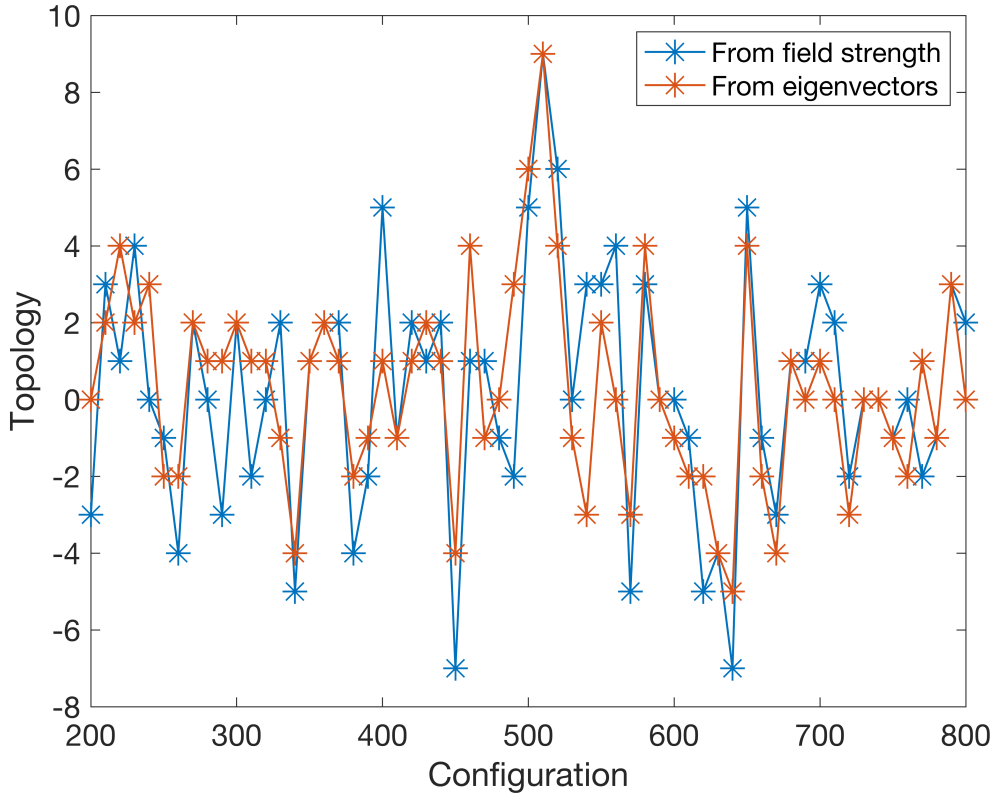
(a) Magnitude of inner product between eigenvectors with m = 0.000525 and m = 0.00604 for the fine lattice, Ls = 48, config 800



(b) Magnitude of inner product between eigenvectors with m = 0.01 and m = 0.02 for the 12x32 coarse lattice, Ls = 96, config 700

Topology from field strength and eigenvectors

- We compare topology calculated from field strength and topology calculated from eigenvectors for the 12x32 ensemble. For the results from field strength, we first ran Wilson flow with flow time t = 13.5. For the results from eigenvectors, we only consider chiral modes ($|\langle\Psi_i|\Gamma_5|\Psi_i\rangle|>= 0.9$) and round the chirality to ± 1.



Summary of the results about eigenvectors

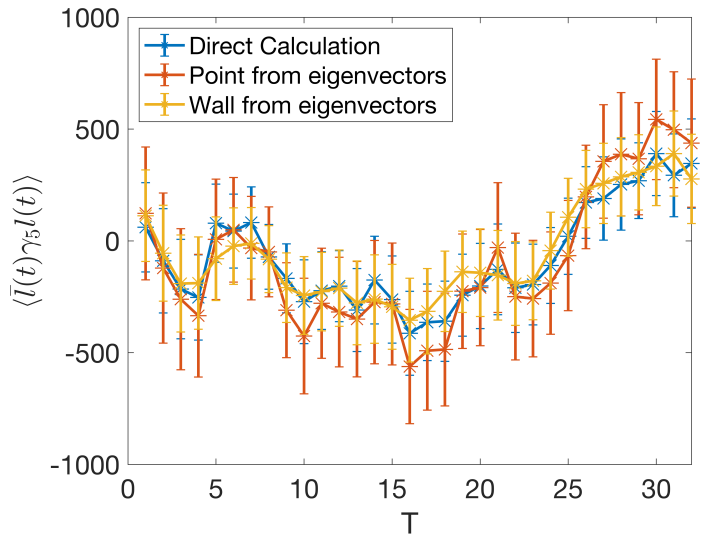
- **Eigenvalues**: The results fit the picture in the continuum. However, for the operator of the lattice, the effective mass is the sum of residual mass and input mass.
- **Chirality and mass effect**: The operator of the lattice contains a mass scale in it. The modes are chiral for eigenvalues much smaller than the mass. Eigenvectors of different masses are related by a rotation in the subspace. One needs very large Ls and 0 or even negative input mass to get results free of mass effect.
- **Topology**: The net topology is the difference between positive and negative chiral modes. The results are compared with the results from field strength. Two results generally agree with each other but some differences are found. We obtain topology without smearing and we get results for the individual zero-modes.

From eigenvectors to propagators

- Here we consider the wall to wall propagator $\langle \bar{l}(t)\gamma_5 l(t)\rangle$. It can also be calculated from eigenvectors. In the lattice, the formula is:

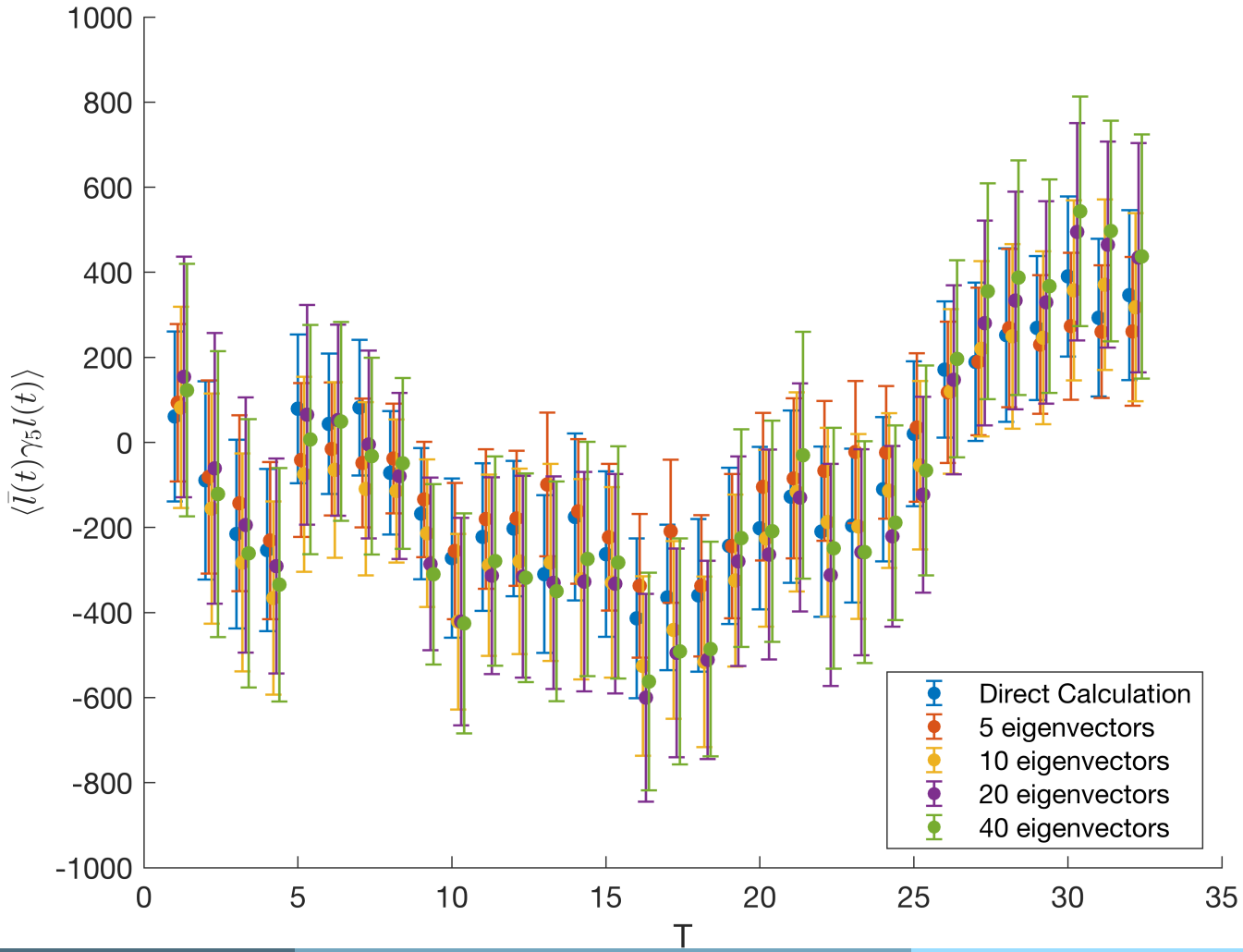
$$\sum_n \frac{\mathrm{Tr}(\sum_{\vec{x},\vec{x}'}(\Psi_n^\dagger(\vec{x},t)\Psi_n(\vec{x}',t)))}{\Lambda_n} \tag{8}$$

where the trace is over the fifth direction, spin and color indices. For point propagators, $\vec{x}$ should be the same as $\vec{x}'$. We consider 40 eigenvectors for config 200-800 with separation 10 (12x32 ensemble).



From eigenvectors to propagators

- It turns out that we can approximate the propagators with only the lowest 5 eigenvectors.



η and η' calculation

- With approximate SU(3) flavor symmetries:

$$|\eta\rangle = \frac{1}{\sqrt{6}}|\bar{u}\gamma_5 u + \bar{d}\gamma_5 d - 2\bar{s}\gamma_5 s\rangle \tag{9}$$

$$|\eta'\rangle = \frac{1}{\sqrt{3}}|\bar{u}\gamma_5 u + \bar{d}\gamma_5 d + 2\bar{s}\gamma_5 s\rangle \tag{10}$$

- With $|l\rangle = \frac{1}{\sqrt{2}}|\bar{u}\gamma_5 u + \bar{d}\gamma_5 d\rangle$ and $|s\rangle = |\bar{s}\gamma_5 s\rangle$, one can calculate the correlators for the light and strange state. Furthermore, the correlators consist of disconnected and connected parts [Norman Christ et al. PRL 2010]:

$$\begin{pmatrix} \langle l(t)l^\dagger(0)\rangle & \langle s(t)l^\dagger(0)\rangle \\ \langle l(t)s^\dagger(0)\rangle & \langle s(t)s^\dagger(0)\rangle \end{pmatrix} = \begin{pmatrix} C_{ll} - 2D_{ll} & -2\sqrt{2}D_{ls} \\ -\sqrt{2}D_{ls} & C_{ss} - D_{ss} \end{pmatrix} \tag{11}$$

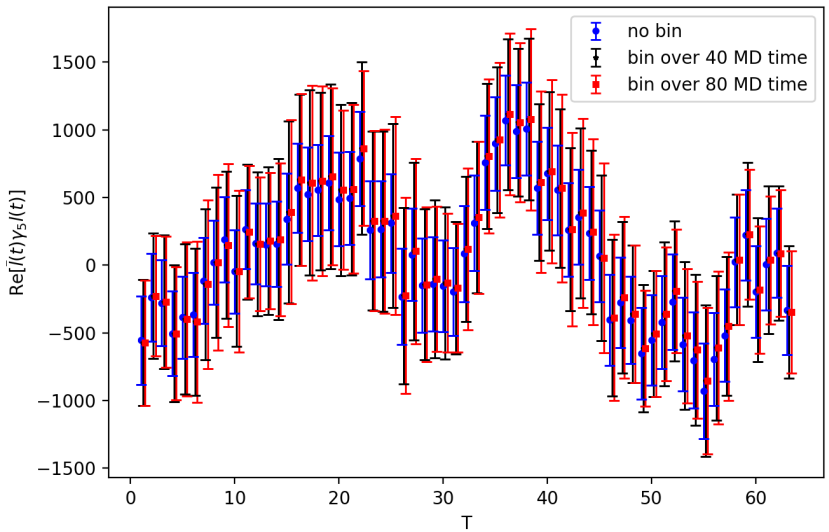
η and η' calculation

- Here we show our fitting results for η, η' and the mixing angle θ:

size	a^{-1}	m_π	m_η	$m_{\eta'}$	θ
24x64	1	0.140	0.523(25)	0.924(113)	-18.6(6.7)°
12x32	1	0.300	0.551(29)	1.17(24)	-11.3(5.0)°
Experiment	-	-	0.548	0.958	-10° $\sim$ -20°

The unit is GeV for masses and a^{-1}.

- We see that the results for η' have large errors. It's possible that there are issues with the topological properties of our coarse lattices.



Conclusion and outlook

- We calculated low lying eigenvectors for Hermitian Dirac operator on fine and coarse lattices and compared the results to the eigenvectors of the continuum. Particularly, we studied the eigenvalue and chirality properties. The topology of the lattice could be learned without smearing.
- We constructed propagators from the eigenvectors which are in good agreement with direct calculation.
- We calculated the mass of η and η'.
- We can further study the change in eigenvectors during the gauge evolution and study the effects of zero modes to η and η' in more details.