Development of Hamiltonian Effective Field Theory (HEFT) of Two Body Rest System within Partial Waves Mixing

Yan Li[1] D. B. Leinweber[2] A. W. Thomas[2] Jia-jun Wu[1]

[1]School of Physical Sciences, University of Chinese Academy of Sciences (UCAS), Beijing 100049, China

[2]Special Research Center for the Subatomic Structure of Matter (CSSM), Department of Physics, University of Adelaide, Adelaide 5005, Australia

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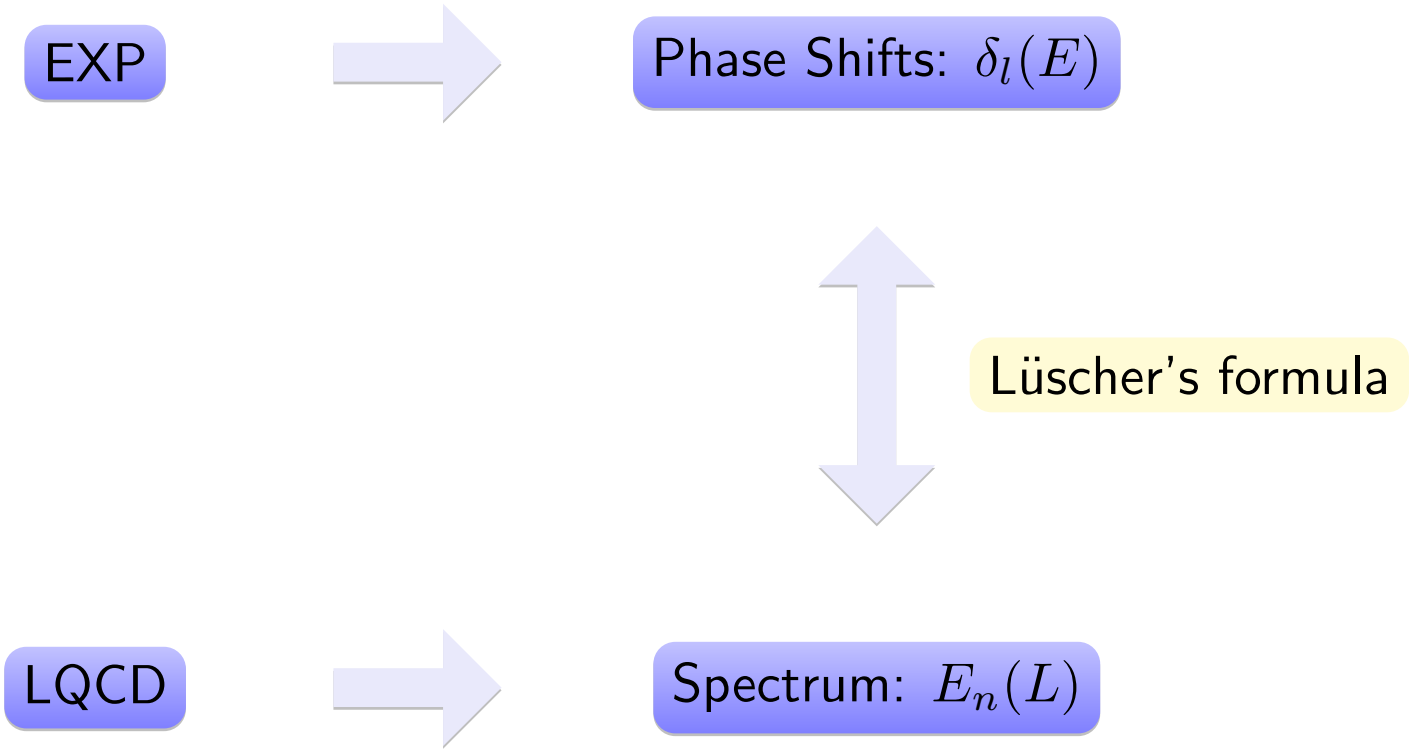
Background and Motivation



NPB 354 (1991) 531-578
Martin Lüscher.

Phase Shifts:

$$H = -\frac{1}{2\mu}\nabla^2 + V(r)$$

Spectrum:

$$H_L = -\frac{1}{2\mu}\nabla^2\big|_L + \sum_{\mathbf{n}\in\mathbb{Z}^3} V(|\mathbf{r}+\mathbf{n}L|)$$

Lüscher's formula is potential independent.

Phase Shifts:

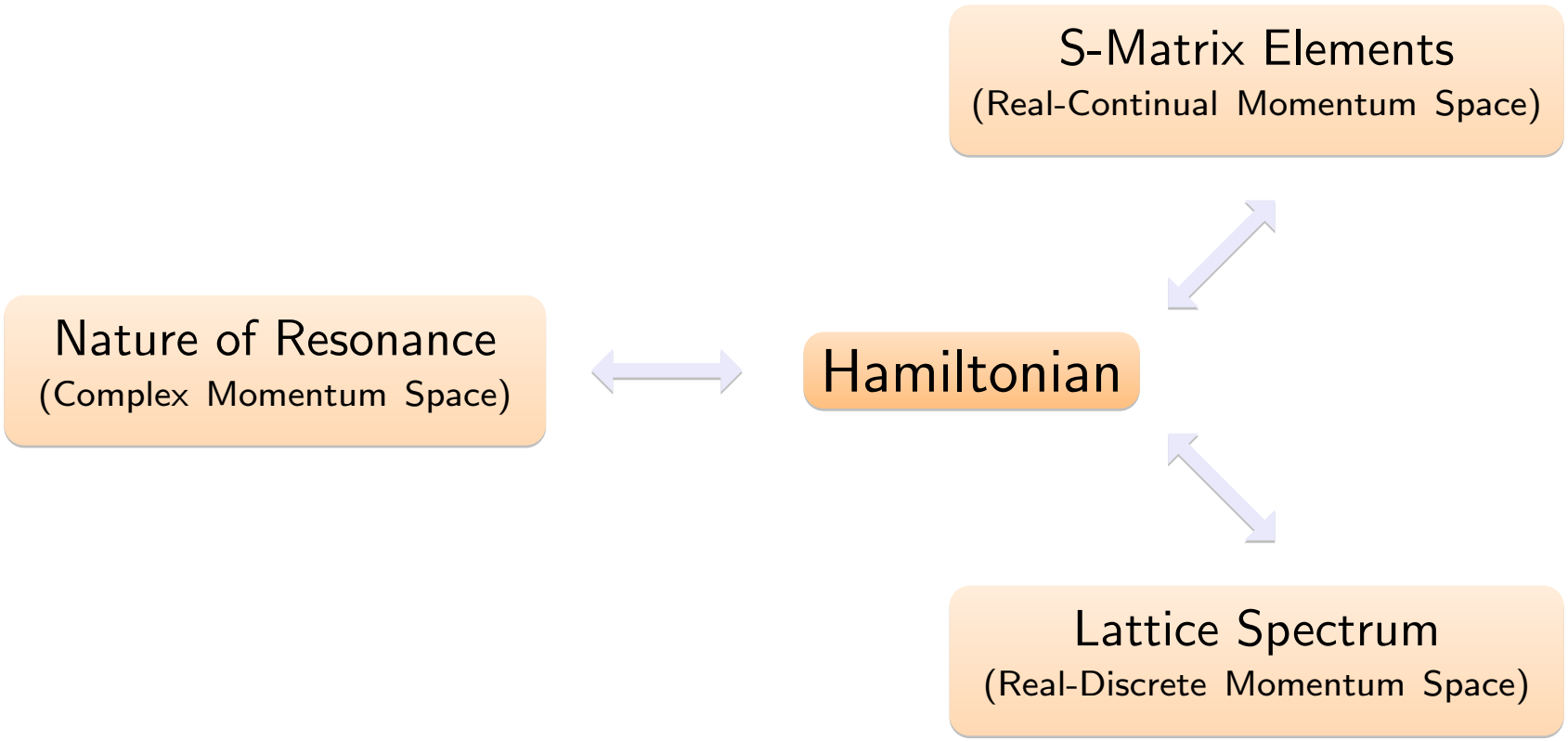
$$H = -\frac{1}{2\mu}\nabla^2 + V(r)$$

Spectrum:

$$H_L = -\frac{1}{2\mu}\nabla^2\big|_L + \sum_{\mathbf{n}\in\mathbb{Z}^3} V(|\mathbf{r} + \mathbf{n}L|)$$

Let the potential be the bridge between them.
Fit $V(r;\{par\})$ to one, predict the other.

HEFT


S-Matrix Elements
(Real-Continual Momentum Space)
Nature of Resonance
(Complex Momentum Space)
Hamiltonian
Lattice Spectrum
(Real-Discrete Momentum Space)

$$\hat{H}_L = \hat{H}_{0L} + \hat{V}_L$$
$$= \sum_{\mathbf{n}\in\mathbb{Z}^3} -\frac{1}{2\mu}\left(\frac{2\pi n}{L}\right)^2 |\mathbf{n}\rangle\langle\mathbf{n}| + \sum_{\mathbf{n}',\mathbf{n}\in\mathbb{Z}^3} V_L(\frac{2\pi\mathbf{n}'}{L}, \frac{2\pi\mathbf{n}}{L}) |\mathbf{n}'\rangle\langle\mathbf{n}|$$

$$\sum_{\mathbf{n}\in\mathbb{Z}^3} \rightarrow \sum_{\mathbf{n}\in\mathbb{Z}^3}^{|\mathbf{n}|^2\leq N_{\text{cut}}}$$

For $N_{\text{cut}} = 600$, we have 61565 states.

(For $L = 3$ fm, $N_{\text{cut}} \sim 10$ GeV)

$$\hat{H}_L = \hat{H}_{0L} + \hat{V}_L$$
$$= \sum_{\mathbf{n}\in\mathbb{Z}^3} -\frac{1}{2\mu}\left(\frac{2\pi n}{L}\right)^2 |\mathbf{n}\rangle\langle\mathbf{n}| + \sum_{\mathbf{n}',\mathbf{n}\in\mathbb{Z}^3} V_L(\frac{2\pi\mathbf{n}'}{L}, \frac{2\pi\mathbf{n}}{L}) |\mathbf{n}'\rangle\langle\mathbf{n}|$$

In the early works, for pure s- and p- waves,

$61565 \rightarrow$ about 600 .

$$
\begin{aligned}
\mathbf{0}^+ &= \mathbf{A}_1^+ \,, \\
\mathbf{1}^- &= \mathbf{T}_1^- \,, \\
\mathbf{2}^+ &= \mathbf{E}^+ \oplus \mathbf{T}_2^+ \,, \\
\mathbf{3}^- &= \mathbf{A}_2^- \oplus \mathbf{T}_1^- \oplus \mathbf{T}_2^- \,, \\
\mathbf{4}^+ &= \mathbf{A}_1^+ \oplus \mathbf{E}^+ \oplus \mathbf{T}_1^+ \oplus \mathbf{T}_2^+ \,,
\end{aligned}
$$

Irreducible representations

- The irreps of the rotation group will reduce into the irreps of the cubic group.
- Partial waves decoupling in the infinite volume can be mixed in the finite volume.

Partial Waves Mixing in HEFT

$$\hat{H}_L = \hat{H}_{0L} + \sum_{\mathbf{n}',\mathbf{n}\in\mathbb{Z}^3} V_L(\frac{2\pi\mathbf{n}'}{L}, \frac{2\pi\mathbf{n}}{L}) \left|\mathbf{n}'\right\rangle \left\langle\mathbf{n}\right|$$

$$\hat{H}_L = \hat{H}_{0L} + \sum_{N',N} \sum_{\Gamma,F',F} v_{\Gamma,F',F}(k_{N'}, k_N) \sum_{\alpha} \left|N';\Gamma,F',\alpha\right\rangle \left\langle N;\Gamma,F,\alpha\right|$$

$|N;\Gamma,F,\alpha\rangle$ denotes the α-th vector of the F-th occurrence of the irreps Γ in a given $N = |\mathbf{n}|^2$.

- Choose the basis $|N;\Gamma,F,\alpha\rangle$ with its states coupling as few as possible to the potential.
- Find the relation between $v_{\Gamma,F',F}$ and V_L.

Partial Waves Mixing in HEFT

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$$\hat{H}_L = \hat{H}_{0L} + \sum_{N',N} \sum_{\Gamma,F',F} v_{\Gamma,F',F}(k_{N'}, k_N) \sum_{\alpha} \left|N';\Gamma,F',\alpha\right\rangle \left\langle N;\Gamma,F,\alpha\right|$$

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$|\mathbf{n}\rangle$

$\downarrow \quad \sum_{\hat{\mathbf{n}}} Y_{lm}(\hat{\mathbf{n}})\,|\mathbf{n}\rangle$

$|N;l,m\rangle$

$N=\lvert\mathbf{n}\rvert^2$	0	1	2	3	$\cdots$
$\lvert\mathbf{n}\rangle$	1	6	12	8	$\cdots$
$\lvert N;l,m\rangle$	1	∞	∞	∞	$\cdots$

$l = 0,\ 1,\ 2,\ \cdots$

$m = -l,\ \cdots,\ +l$

$$
\begin{aligned}
N=0: &\quad (0,0,0) \\
N=1: &\quad (+1,0,0) \quad (0,+1,0) \quad (0,0,+1) \\
&\quad (-1,0,0) \quad (0,-1,0) \quad (0,0,-1) \\
&\cdots\cdots
\end{aligned}
$$

$|\mathbf{n}\rangle$

$\downarrow \quad \sum_{\hat{\mathbf{n}}} Y_{lm}(\hat{\mathbf{n}})\,|\mathbf{n}\rangle$

$|N;l,m\rangle$

$N=\lvert\mathbf{n}\rvert^2$	0	1	2	3	$\cdots$
$\lvert\mathbf{n}\rangle$	1	6	12	8	$\cdots$
$\lvert N;l,m\rangle$	1	∞	∞	∞	$\cdots$
with l_{cut}	1	$(l_{\text{cut}}+1)^2$	$(l_{\text{cut}}+1)^2$	$(l_{\text{cut}}+1)^2$	$\cdots$

$l = 0,\ 1,\ 2,\ \cdots,\ l_{\text{cut}}$

$m = -l,\ \cdots,\ +l$

$N=0:\quad (0,0,0)$

$N=1:\quad (+1,0,0)\quad (0,+1,0)\quad (0,0,+1)$

$\qquad\quad (-1,0,0)\quad (0,-1,0)\quad (0,0,-1)$

$\cdots\cdots$

$\sum_{l=0}^{l_{\text{cut}}}(2l+1) = (l_{\text{cut}}+1)^2 = 25$ for $l_{\text{cut}} = 4$

Dimension: big number $\rightarrow (l_{\text{cut}}+1)^2 \times N_{\text{cut}}+1$

$\sim 25\times 600$ for $l_{\text{cut}} = 4,\ N_{\text{cut}} = 600$

$|\mathbf{n}\rangle$

$\downarrow \quad \sum_{\hat{\mathbf{n}}} Y_{lm}(\hat{\mathbf{n}})\,|\mathbf{n}\rangle$

$|N;l,m\rangle$

$\downarrow \quad \sum_{m} [C_l]_{\Gamma,f,\alpha;m}\,|N;l,m\rangle$

$|N,l;\Gamma,f,\alpha\rangle$

$N=\lvert\mathbf{n}\rvert^2$	0	1	2	3	$\cdots$
$\lvert\mathbf{n}\rangle$	1	6	12	8	$\cdots$
$\lvert N;l,m\rangle$	1	∞	∞	∞	$\cdots$
with l_{cut}	1	$(l_{\text{cut}}+1)^2$	$(l_{\text{cut}}+1)^2$	$(l_{\text{cut}}+1)^2$	$\cdots$
$\lvert N,l;\Gamma,f,\alpha\rangle$	1	F_{cut}	F_{cut}	F_{cut}	$\cdots$

$$\mathbf{0}^+ = \mathbf{A}_1^+ \,, \qquad F_{\text{cut}}(\Gamma=\mathbf{A}_1^+, l_{\text{cut}}=4)=2$$
$$\mathbf{1}^- = \mathbf{T}_1^- \,,$$
$$\mathbf{2}^+ = \mathbf{E}^+ \oplus \mathbf{T}_2^+ \,,$$
$$\mathbf{3}^- = \mathbf{A}_2^- \oplus \mathbf{T}_1^- \oplus \mathbf{T}_2^- \,,$$
$$\mathbf{4}^+ = \mathbf{A}_1^+ \oplus \mathbf{E}^+ \oplus \mathbf{T}_1^+ \oplus \mathbf{T}_2^+ \,,$$

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$|N;l,m\rangle$

$\downarrow \quad \sum_{m} [C_l]_{\Gamma,f,\alpha;m}\,|N;l,m\rangle$

$|N,l;\Gamma,f,\alpha\rangle$

$\downarrow$ Orthonormalization

$|N;\Gamma,F,\alpha\rangle$

$N=\lvert\mathbf{n}\rvert^2$	0	1	2	3	$\cdots$
$\lvert\mathbf{n}\rangle$	1	6	12	8	$\cdots$
$\lvert N;l,m\rangle$	1	∞	∞	∞	$\cdots$
with l_{cut}	1	$(l_{\text{cut}}+1)^2$	$(l_{\text{cut}}+1)^2$	$(l_{\text{cut}}+1)^2$	$\cdots$
$\lvert N,l;\Gamma,f,\alpha\rangle$	1	F_{cut}	F_{cut}	F_{cut}	$\cdots$
$\lvert N;\Gamma,F,\alpha\rangle$	1	$\leq F_{\text{cut}}$	$\leq F_{\text{cut}}$	$\leq F_{\text{cut}}$	$\cdots$

$$\mathbf{0}^+ = \mathbf{A}_1^+ \,, \qquad F_{\text{cut}}(\Gamma=\mathbf{A}_1^+, l_{\text{cut}}=4)=2$$
$$\mathbf{1}^- = \mathbf{T}_1^- \,,$$
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$$\mathbf{4}^+ = \mathbf{A}_1^+ \oplus \mathbf{E}^+ \oplus \mathbf{T}_1^+ \oplus \mathbf{T}_2^+ \,,$$

Dimension: big number $\rightarrow$ at most $F_{\text{cut}}\times N_{\text{cut}}+1$

$\sim 2\times 600$ for $l_{\text{cut}}=4,\ N_{\text{cut}}=600$

$|\mathbf{n}\rangle$ $\qquad \hat{V}_L = \sum_{\mathbf{n}',\mathbf{n}\in\mathbb{Z}^3} V_L(\frac{2\pi\mathbf{n}'}{L}, \frac{2\pi\mathbf{n}}{L}) |\mathbf{n}'\rangle \langle\mathbf{n}| = \sum_{\mathbf{n}',\mathbf{n}\in\mathbb{Z}^3} L^{-3} V(\frac{2\pi\mathbf{n}'}{L}, \frac{2\pi\mathbf{n}}{L}) |\mathbf{n}'\rangle \langle\mathbf{n}|$

$\downarrow \sum_{\hat{\mathbf{n}}} Y_{lm}(\hat{\mathbf{n}}) |\mathbf{n}\rangle$ $\qquad \overbrace{\sum_{l,m} v_l(k_{N'}, k_N) Y_{lm}(\hat{\mathbf{n}}') Y^*_{lm}(\hat{\mathbf{n}})}$

$|N; l, m\rangle$ $\qquad \hat{V}_L = \sum_{N',N} \sum_{l} L^{-3} v_l(k_{N'}, k_N) \sum_{m} |N'; l, m\rangle \langle N; l, m|$

$\downarrow \sum_{m} [C_l]_{\Gamma,f,\alpha;m} |N; l, m\rangle$

$|N, l; \Gamma, f, \alpha\rangle$ $\qquad \hat{V}_L = \sum_{N',N} \sum_{l} L^{-3} v_l(k_{N'}, k_N) \sum_{\Gamma,f,\alpha} |N', l; \Gamma, f, \alpha\rangle \langle N, l; \Gamma, f, \alpha|$

$\downarrow$ Orthonormalization

$|N; \Gamma, F, \alpha\rangle$ $\qquad \hat{V}_L = \sum_{N',N} \sum_{\Gamma,F',F} v_{\Gamma,F',F}(k_{N'}, k_N) \sum_{\alpha} |N'; \Gamma, F', \alpha\rangle \langle N; \Gamma, F, \alpha|$

$|\mathbf{n}\rangle$ $\hat{V}_L = \sum_{\mathbf{n}',\mathbf{n}\in\mathbb{Z}^3} V_L(\frac{2\pi\mathbf{n}'}{L}, \frac{2\pi\mathbf{n}}{L}) |\mathbf{n}'\rangle \langle\mathbf{n}| = \sum_{\mathbf{n}',\mathbf{n}\in\mathbb{Z}^3} L^{-3} V(\frac{2\pi\mathbf{n}'}{L}, \frac{2\pi\mathbf{n}}{L}) |\mathbf{n}'\rangle \langle\mathbf{n}|$

$\sum_{\hat{\mathbf{n}}} Y_{lm}(\hat{\mathbf{n}}) |\mathbf{n}\rangle$

$\sum_{l,m} v_l(k_{N'}, k_N) Y_{lm}(\hat{\mathbf{n}}') Y^*_{lm}(\hat{\mathbf{n}})$

$|N; l, m\rangle$

$v_{\Gamma,F',F}(k_{N'}, k_N) = \sum_l L^{-3} v_l(k_{N'}, k_N) [G_{l;\Gamma}]_{N',F';N,F}$

$\sum_m [C_l]_{\Gamma,f,\alpha;m} |N; l, m\rangle$

$|N, l; \Gamma, f, \alpha\rangle$

Need to know the inner products between $|N, l; \Gamma, f, \alpha\rangle$

Orthonormalization

$|N; \Gamma, F, \alpha\rangle$ $\hat{V}_L = \sum_{N',N} \sum_{\Gamma,F',F} v_{\Gamma,F',F}(k_{N'}, k_N) \sum_\alpha |N'; \Gamma, F', \alpha\rangle \langle N; \Gamma, F, \alpha|$

$|\mathbf{n}\rangle$

$\downarrow \sum_{\hat{\mathbf{n}}} Y_{lm}(\hat{\mathbf{n}})\,|\mathbf{n}\rangle$

$|N;l,m\rangle$

$\downarrow \sum_{m} [C_l]_{\Gamma,f,\alpha;m}\,|N;l,m\rangle$

$|N,l;\Gamma,f,\alpha\rangle$

$\downarrow$ Orthonormalization

$|N;\Gamma,F,\alpha\rangle$

$$[P_N]_{l',m';l,m} := \langle N;l',m'|N;l,m\rangle = \sum_{\hat{\mathbf{n}}} Y^*_{l'm'}(\hat{\mathbf{n}})Y_{lm}(\hat{\mathbf{n}})$$

$\downarrow$

$$\begin{pmatrix} 25\times 25 \\ \text{for } l_{\text{cut}}=4 \end{pmatrix} \xrightarrow{[C]^\dagger[P][C]} \begin{pmatrix} \mathbf{A}_1^+ & & \\ & \square & \\ & & \ddots \end{pmatrix}$$

2×2 $\downarrow$

$\downarrow$

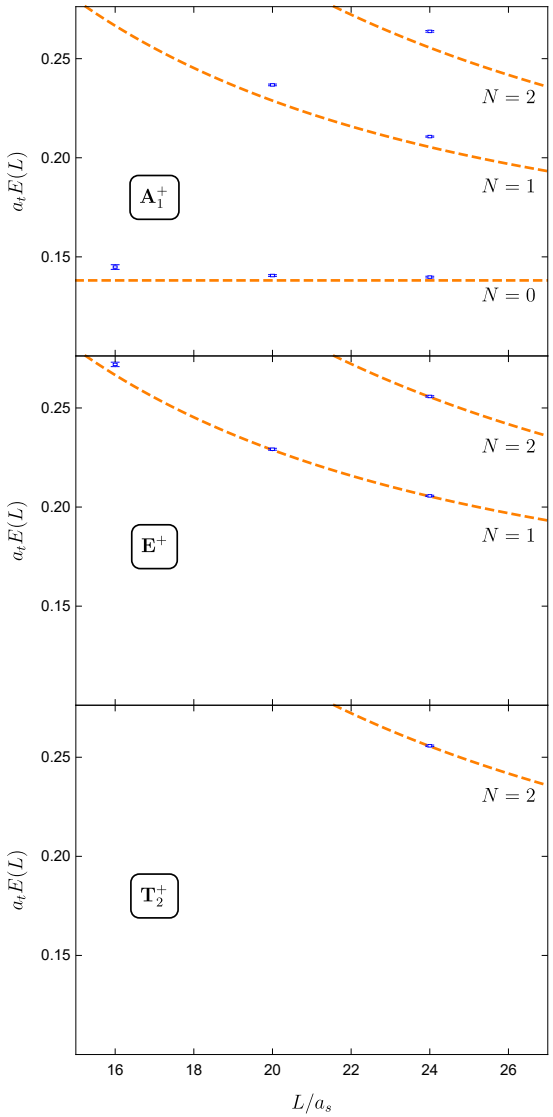
$$[P_{N;\Gamma,\alpha}]_{l',f';l,f} := \langle N,l';\Gamma,f',\alpha|N,l;\Gamma,f,\alpha\rangle = \sum_{m',m} [C_{l'}]^*_{\Gamma,f',\alpha;m'}[P_N]_{l',m';l,m}[C_l]_{\Gamma,f,\alpha;m}$$

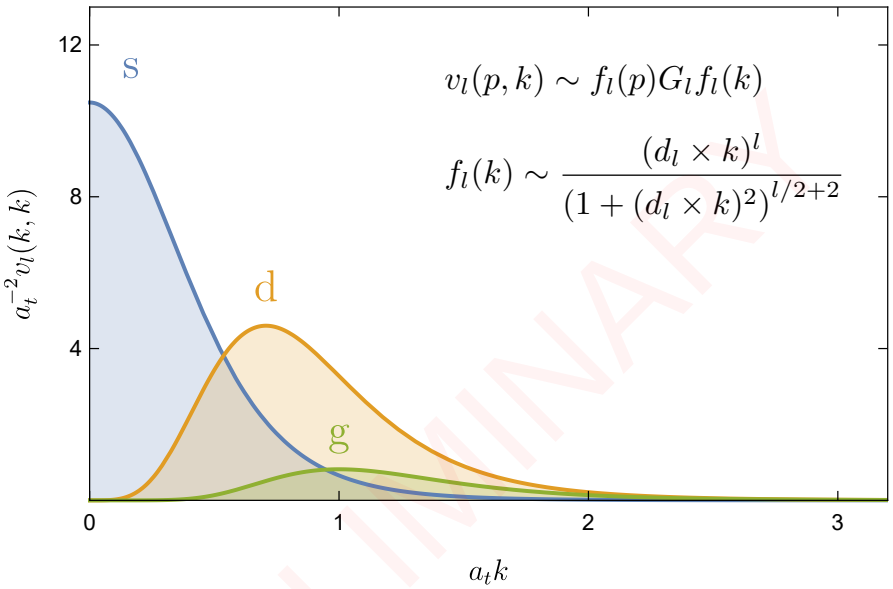
Combination Coefficients: $[G_{l;\Gamma}]_{N',F';N,F} = \sum_f \langle N';\Gamma,F',\alpha|N';l,\Gamma,f,\alpha\rangle\,\langle N;l,\Gamma,f,\alpha|N;\Gamma,F,\alpha\rangle$

Example of Isospin-2 $\pi\pi$ Scattering

Dashed orange curves show the free energy spectrum , 11 blue data points show the LQCD spectrum.

PRD 86, 034031 (2012)
Jozef J. Dudek et al.





2 parameters G_l and d_l for each v_l.

$G_2 \uparrow \Rightarrow \text{peak}_2 \uparrow$ $\quad$ $d_2 \downarrow \Rightarrow \text{peak}_2 \rightarrow$

Case	χ^2	data	G_0	d_0	G_2	d_2	G_4	d_4
A	13.4	11	68.5	4.63	56.4	d_A	74.3	d_A
B	10.5	11	67.8	4.57	90.6	d_B	$3.40 * 10^2$	d_B
C	9.77	11	67.2	4.54	$1.87 * 10^2$	d_C	$2.34 * 10^3$	d_C

$$d_A = 7.17 \sim 0.8 \text{ GeV},\ d_B = 4.78 \sim 1.2 \text{ GeV},\ d_C = 3.58 \sim 1.6 \text{ GeV}$$

Dimension of Hamiltonian:

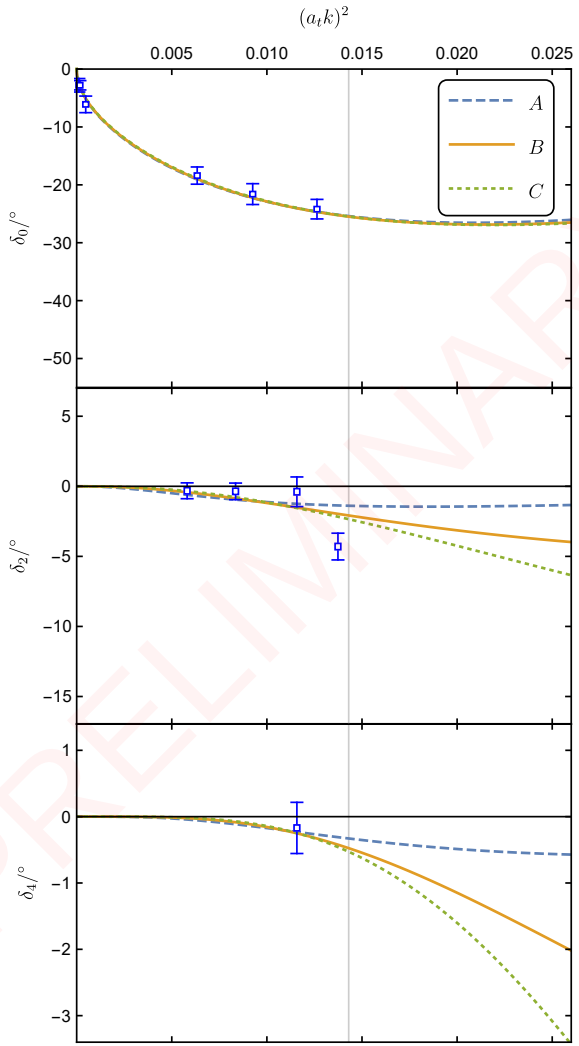
$$\mathbf{A}_1^+ :\ 923, \quad \mathbf{E}^+ :\ 965, \quad \mathbf{T}_2^+ :\ 963, \quad with\ N_{\text{cut}} = 600,\ l_{\text{cut}} = 4.$$

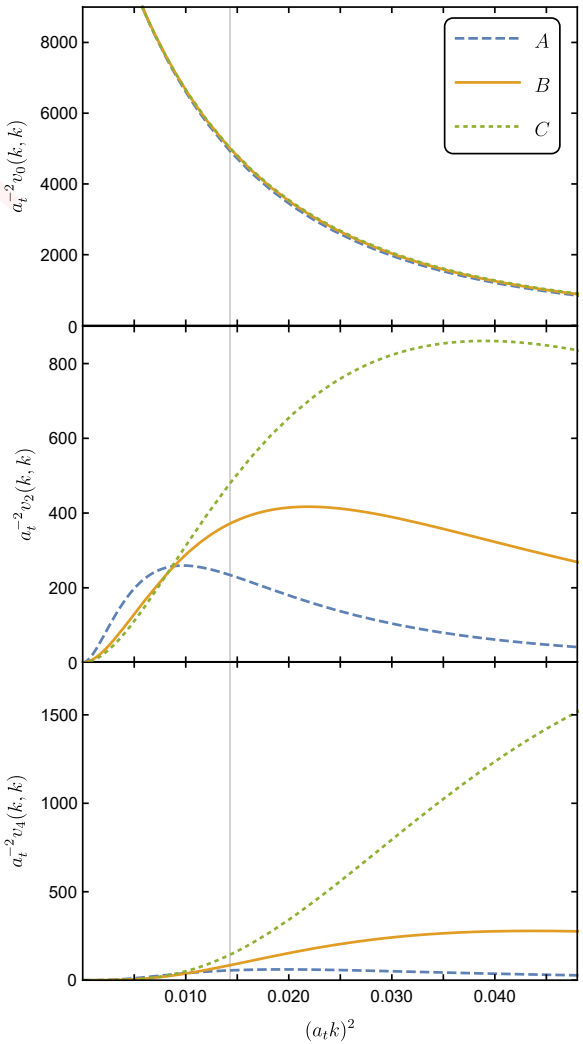
Fit with d_2 and d_4 free show an unreasonable behavior.
So we choose to fit with them fixed.

Left:
Phase shifts.

Right:
Potentials.

From up to down:
s-, d-, g-waves.

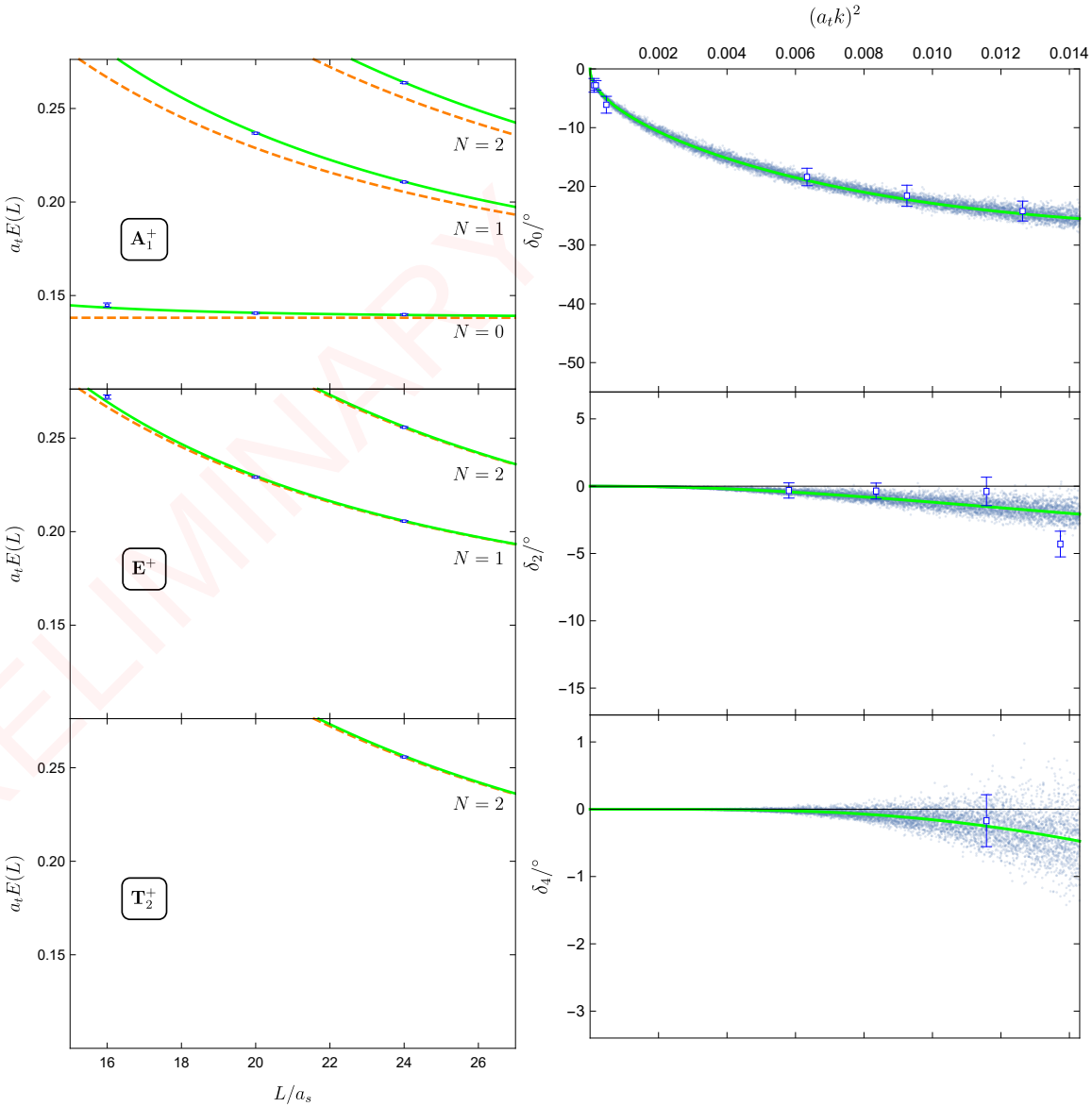




Use case B to show further results.

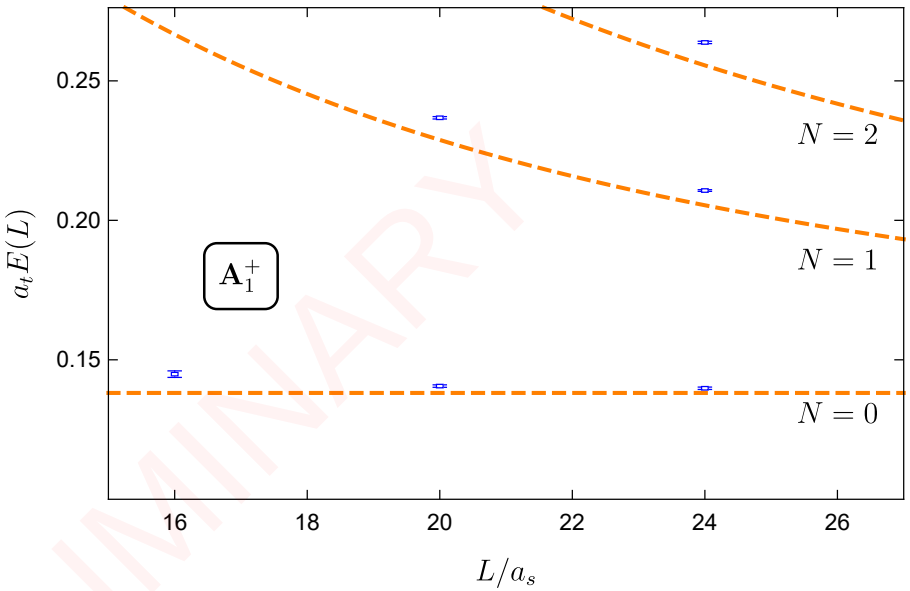
Left:
Volume dependent spectrum.

Right:
Phase shifts with errors given by resampling over parameters.



Up:
Spectrum.

Down:
The free state constituents of the eigenstates.
($\Gamma = \mathbf{A}_1^+$, $L = 24$)
(shown in percent)



	$N=0$	$N=1$	$N=2$	$N=3$	$N=4$	$\cdots$
1st	99.7	0.2	0.0	0.0	0.0	$\cdots$
2nd	0.1	97.4	1.9	0.2	0.0	$\cdots$
3rd	0.0	1.5	94.5	2.8	0.3	$\cdots$

Summary and Prospect

- ✓ Partial waves mixing in two-particle rest system.
- Moving system.
- Three-particle system.
- Model independent information involved in the eigenstates.

ACKNOWLEDGMENTS

Finite-volume energy levels taken from [PRD 86, 034031 (2012), Jozef J. Dudek et al.] were provided by the Hadron Spectrum Collaboration – no endorsement on their part of the analysis presented in the current paper should be assumed.

Thank you!