The general formalism of momentum trans-formation in the moving finite volume

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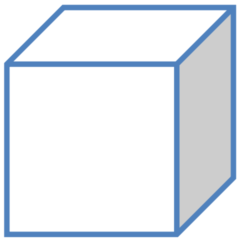
Outline

- Motivation
- General formalism for momentum transformation
- Three typical transformation formalisms
- The test in S-wave of $\pi\pi$ scattering
- Summary

Motivation

Finite Volume

Lattice Simulation



Spectrum

Infinite Volume

Experimental Events

Differential Cross Section

Differential Partial Width

S matrix or T matrix

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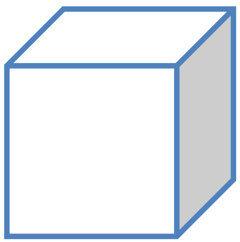

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Two Body: Lüscher Equation

Spectrum ⟷ S matrix or T matrix

Commun. Math. Phys. 105, 153 (1986);
Nucl.Phys. B364, 237 (1991).

Motivation

Two Body: Lüscher Equation (one channel case)

$$\tan\delta(k) = -\frac{q\pi^{3/2}}{Z_{00}\left(1; \frac{k^2L^2}{4\pi^2}\right)}$$

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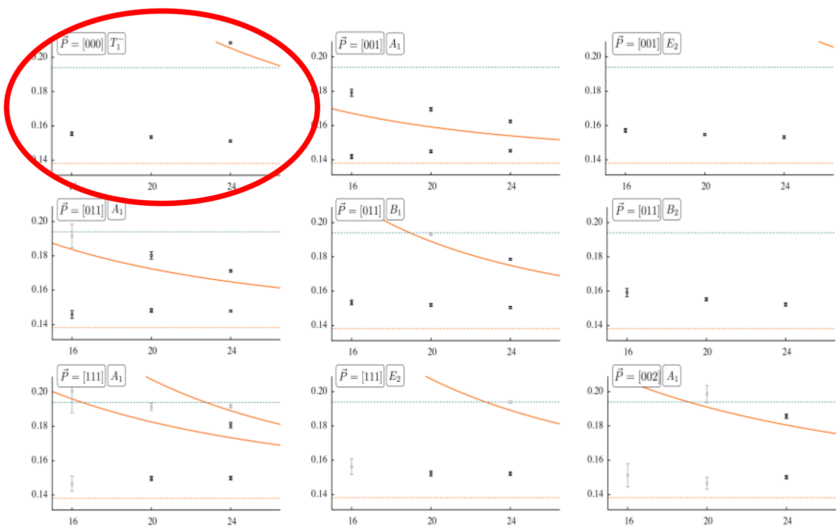
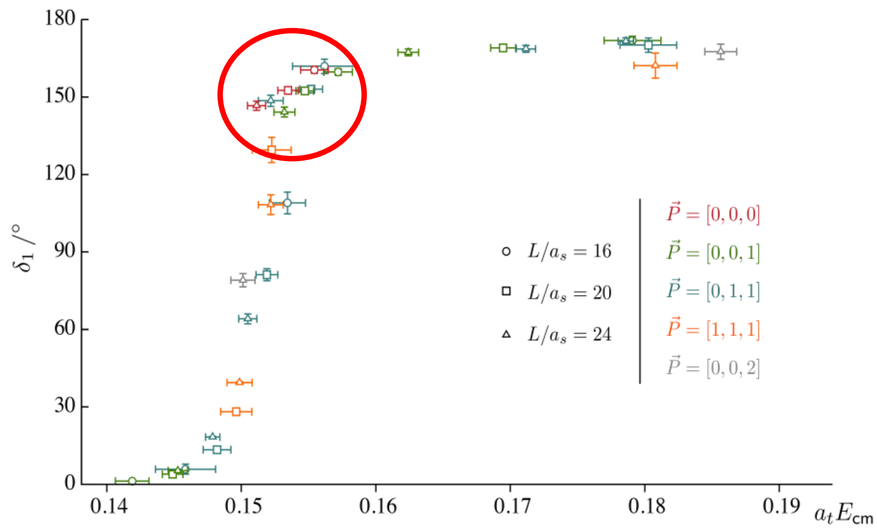
ρ -> ππ case **Dudek， Edwards and Thomas, PRD 87 034505(2013)**





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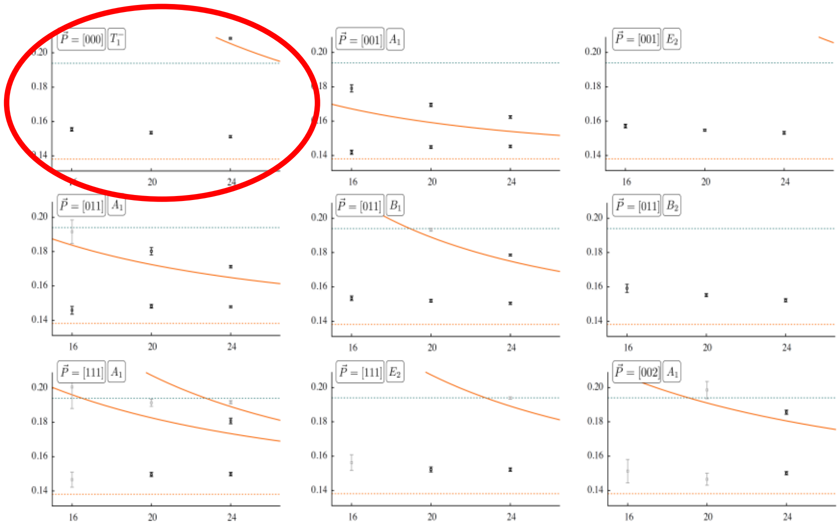

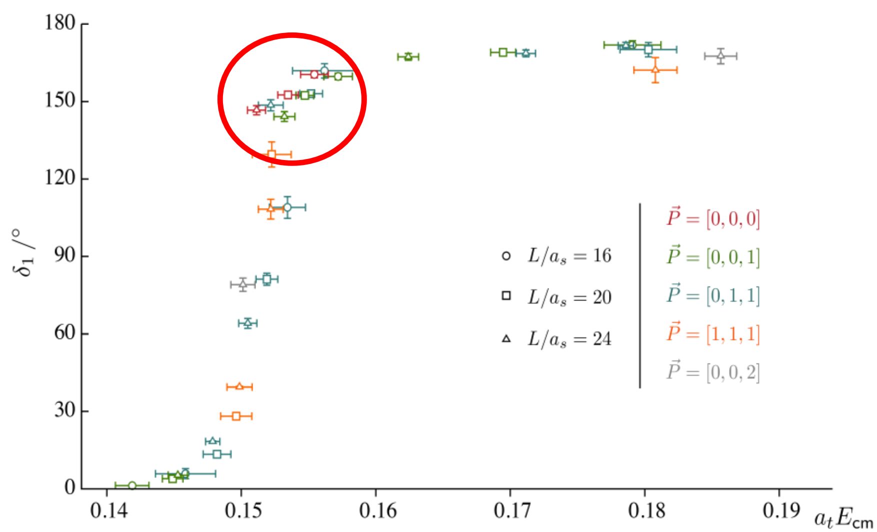

The Spectra in the Moving Frame can extend our knowledge of S matrix in the energy region.

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Rummukainen and Gottlieb NPB 450 397 (1997)

in the energy range of interest, the phase shift δ_0 is related to the momentum p^* by

$$\delta_0(p^*) = -\phi^d(q) \bmod \pi, \qquad q = \frac{p^*L}{2\pi}, \tag{17}$$

where ϕ^d is a continuous function defined by the equation

$$\tan(-\phi^d(q)) = \frac{\gamma q\pi^{3/2}}{Z_{00}^d(1;q^2)} \qquad \phi^d(0) = 0. \tag{18}$$

Function Z_{00}^d is generalized zeta function, and is formally given by

$$Z_{00}^d(s;q^2) = \frac{1}{\sqrt{4\pi}}\sum_{r\in P_d}(r^2-q^2)^{-s}, \tag{19}$$

where the set P_d is

$$P_d = \{\boldsymbol{r}\in\mathbb{R}^3 | \boldsymbol{r} = \boldsymbol{\gamma}^{-1}(\boldsymbol{n}+\boldsymbol{d}/2),\ \boldsymbol{n}\in\mathbb{Z}^3\}. \tag{20}$$

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$$\det\left[e^{2i\delta}(M-i) - (M+i)\right] = 0. \tag{100}$$

$$\Rightarrow \quad \det[\cot\delta + M] = 0$$

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where we have defined

$$q = \frac{pL}{2\pi}. \tag{90}$$

The tensor $C_{lm,js,l'm'}$ can be written in terms of Wigner $3j$-symbols

$$C_{lm,js,l'm'} = (-1)^{m'} i^{l-j+l'}\sqrt{(2l+1)(2j+1)(2l'+1)}\begin{pmatrix} l & j & l' \\ m & s & -m' \end{pmatrix}\begin{pmatrix} l & j & l' \\ 0 & 0 & 0 \end{pmatrix}. \tag{91}$$

The generalized zeta function in Eq. (89) is defined through the equation

$$Z^d_{lm}(s;q^2) = \sum_{r\in P_d}\mathcal{Y}_{lm}(\boldsymbol{r})(r^2-q^2)^s. \tag{92}$$

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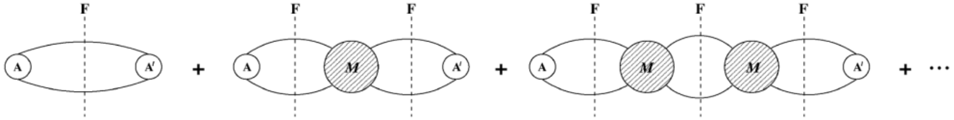
Kim, Sachrajda and Sharpe NPB 727 218 (2005)



$$\det(1 + i\,M\,F/2) = 0.$$

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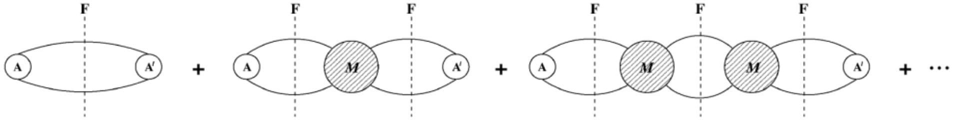

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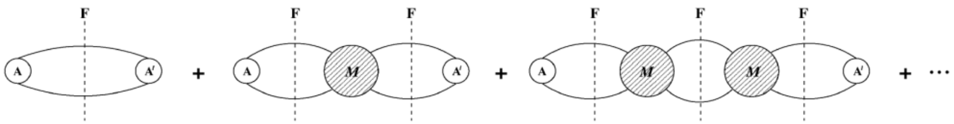

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Göckeler,Horsley,Lage,Meißner,Rakow,Rusetsky,Schierholz,and Zanotti PRD 86 094513 (2012)

$$\Gamma_{\boldsymbol{\Delta}} = \left\{\boldsymbol{k}|\boldsymbol{k} = \frac{2\pi}{L}\boldsymbol{\gamma}^{-1}\left(\boldsymbol{n}-\frac{1}{2}\boldsymbol{\Delta}\right), \boldsymbol{n}\in\mathbb{Z}^3\right\} \qquad M^{\boldsymbol{\Delta}}_{Jl\mu,J'l'\mu'} = \sum_{\substack{m,\sigma\\ m',\sigma'}}\left\langle lm,\frac{1}{2}\sigma|J\mu\right\rangle\left\langle l'm',\frac{1}{2}\sigma'|J'\mu'\right\rangle M^{\boldsymbol{\Delta}}_{lm,l'm'}$$

$$\boldsymbol{\Delta} = \boldsymbol{d}\left(1+\frac{m_1^2-m_2^2}{E^2}\right).$$

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Two Body: Lüscher Equation $\tan\delta(k) = -\frac{q\pi^{3/2}}{Z_{00}\left(1; \frac{k^2L^2}{4\pi^2}\right)}$ $Z_{00}(1;q^2) = \frac{1}{\sqrt{4\pi}} \sum_{\vec{n}\in Z^3} \frac{1}{\vec{n}^2 - q^2}$

How to extract S matrix from the Spectra in the Moving Frame ?

Rummukainen and Gottlieb NPB 450 397 (1997)

Kim, Sachrajda and Sharpe NPB 727 218 (2005)

Göckeler,Horsley,Lage,Meißner,Rakow,Rusetsky,Schierholz,and Zanotti PRD 86 094513 (2012)

$$\det[\cot\delta + M] = 0$$

$$Z^d_{lm}(s;q^2) = \sum_{r\in P_d} \mathcal{Y}_{lm}(\boldsymbol{r})\left(\boldsymbol{r}^2 - q^2\right)^s$$

$$\Gamma_{\boldsymbol{\Delta}} = \left\{\boldsymbol{k}|\boldsymbol{k} = \frac{2\pi}{L}\boldsymbol{\gamma}^{-1}\left(\boldsymbol{n} - \frac{1}{2}\boldsymbol{\Delta}\right), \boldsymbol{n}\in\mathbb{Z}^3\right\}$$

$$M^d_{lm,l'm'}(k) = \gamma^{-1}\frac{(-1)^l}{\pi^{3/2}} \sum_{j=|l-l'|}^{l+l'} \sum_{s=-j}^{j} \frac{(i2\pi)^{j+1}}{i\,(kL)^{j+1}} Z^d_{js}\left(1; \left(\frac{kL}{2\pi}\right)^2\right) C_{lm,js,l'm'}$$

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Lattice Momentum in Moving Frame

$$\vec{k} \to \vec{k}^*???$$

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Lattice Momentum in Moving Frame

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General formalism for momentum transformation

$$M^{d}_{lm,l'm'}(q) = \frac{8\pi E^*(q)}{q}\left[\frac{1}{L^3}\sum_{\frac{\vec{k}L}{2\pi}\in Z^3} - \mathbf{P}\int\frac{d^3\vec{k}}{(2\pi)^3}\right]\frac{4\pi Y^*_{lm}(\hat{\vec{k}}^*)Y_{l'm'}(\hat{\vec{k}}^*)\left(|\vec{k}^*|/q\right)^{l+l'}}{2\omega_{\vec{k}}2\omega_{\vec{P}-\vec{k}}\left(E(q)-\omega_{\vec{k}}-\omega_{\vec{P}-\vec{k}}\right)}$$

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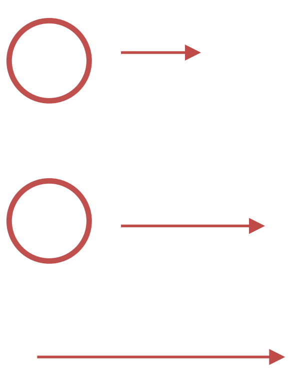
$$\vec{k} \to \vec{k}^* ???$$

$$\frac{1}{k^{*\,2} - q^2}$$



$\vec{k}_1 = \vec{k}$

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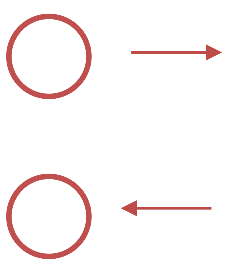
$E, \vec{P}$

$m = m_1 = m_2$ is the on-shell mass, but in the loop, we do not know two particles on-shell or not.



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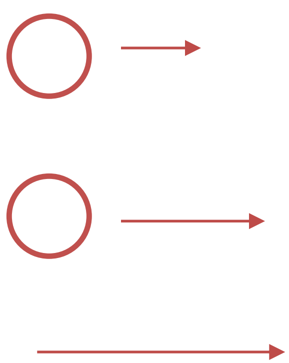
$$\vec{k} \to \vec{k}^*???$$

$$\frac{1}{k^{*\,2} - q^2}$$

We **can not fix** the exact relationship between $\vec{k}$ and $\vec{k}^*$



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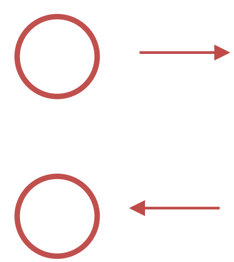

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$$\vec{k} \to \vec{k}^*???$$

$$\frac{1}{k^{*\,2} - q^2}$$

But, the main contribution of [summation – integration] is just from the singularity of function, i.e., $E(q) - \omega_{\vec{k}} - \omega_{\vec{P}-\vec{k}} = 0$ or $\vec{k}^{*2} = q^2$.

Non-singularity part contribution suppress by Exp[-mL] order.

Singularity part contribution suppress by 1/L order.

The meaning of $E(q) - \omega_{\vec{k}} - \omega_{\vec{P}-\vec{k}} = 0$ or $\vec{k}^{*2} = q^2$: two particles are both on-shell.

$$\vec{k} \to \vec{k}^* \quad \Rightarrow \quad \sqrt{\vec{k}^2+m^2} + \sqrt{\left(\vec{P}-\vec{k}\right)^2+m^2} = E$$

$$E^* = \sqrt{E^2 - \vec{P}^2} = 2\sqrt{\vec{k}^{*\,2}+m^2}$$

General formalism for momentum transformation

$$\det[\cot\delta + M] = 0$$

$$M^{d}_{lm,l'm'}(q) = \frac{8\pi E^{*}(q)}{q}\left[\frac{1}{L^3}\sum_{\frac{\vec{k}L}{2\pi}\in Z^3} - \mathbf{P}\int\frac{d^3\vec{k}}{(2\pi)^3}\right]\frac{4\pi Y^{*}_{lm}(\hat{\vec{k}}^{*})Y_{l'm'}(\hat{\vec{k}}^{*})\left(|\vec{k}^{*}|/q\right)^{l+l'}}{2\omega_{\vec{k}}2\omega_{\vec{P}-\vec{k}}\left(E(q)-\omega_{\vec{k}}-\omega_{\vec{P}-\vec{k}}\right)}$$

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$$\vec{k} \to \vec{k}^* \quad \Rightarrow$$

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Three typical transformation formalisms

$$\vec{k}^* = \vec{q}_i$$

Method A:

$$\vec{q}_A = \hat{\gamma}_A(\vec{k} - \frac{\omega_k}{P_0}\vec{P}) = \hat{\gamma}_A^{-1}(\vec{k} - \frac{\omega_{q_A}}{P_0^*}\vec{P}) \quad (43)$$

$$\gamma_A = \frac{1}{\sqrt{1-(|\vec{P}|/P_0)^2}} = \frac{P_0}{P_0^*}, \quad (44)$$

$$d^3\vec{q}_A = \frac{\omega_{q_A}}{\omega_k} d^3\vec{k}, \quad (45)$$

The first particle always on-shell

Method B:

$$\vec{q}_B = \hat{\gamma}_B^{-1}(\vec{k} - \frac{\vec{P}}{2}), \quad (46)$$

$$\gamma_B = \frac{1}{\sqrt{1-(|\vec{P}|/P_0)^2}} = \frac{P_0}{P_0^*}. \quad (47)$$

$$d^3\vec{q}_B = \gamma_B^{-1} d^3\vec{k}, \quad (48)$$

The energies of two particles are always the same.

Method C:

$$\vec{q}_C = \hat{\gamma}_C(\vec{k} - \vec{P}\frac{\omega_k}{\omega_k + \omega_{Pk}}) = \hat{\gamma}_C^{-1}(\vec{k} - \vec{P}\frac{1}{2}), \quad (49)$$

$$\gamma_C = \frac{1}{\sqrt{1-(|\vec{P}|/(\omega_k+\omega_{Pk}))^2}} = \frac{\omega_k + \omega_{Pk}}{\sqrt{(\omega_k+\omega_{Pk})^2 - \vec{P}^2}}, \quad (50)$$

$$d^3\vec{q}_C = \frac{\omega_{q_C}}{2}\frac{\omega_k + \omega_{Pk}}{\omega_k\omega_{Pk}} d^3\vec{k}, \quad (51)$$

Two particles are both always on-shell, but energy is off-shell.

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New one, but it will be very useful in Hamiltonian approach or other cases.

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S-wave case

$$cot\delta_0 = \frac{4\pi}{\tilde{k}}\tilde{\Delta}_{00}$$

$$\tilde{\Delta}_{00} = \mathscr{P}\int\frac{d^3\vec{q}}{(2\pi)^3}\frac{e^{\alpha(\tilde{k}^2-q^2)}}{\tilde{k}^2-q^2} - \frac{1}{L^3}\sum_{\vec{k}=\frac{2\pi}{L}\vec{n},\,\vec{n}\in\mathbb{Z}}\frac{\omega_{q_A}}{\omega_k}\frac{e^{\alpha(\tilde{k}^2-q_A^2)}}{\tilde{k}^2-q_A^2}$$
$$+\frac{1}{4\pi^2 L}\sum_{\vec{n}\in\mathbb{Z},\vec{n}\neq 0}\int_0^{\alpha\left(\frac{2\pi}{L}\right)^2} dt e^{t\left(\frac{L\tilde{k}}{2\pi}\right)^2}\int dq 2q e^{-tq^2}\cos\left[\pi\frac{2\sqrt{m^2+\left(\frac{2\pi q}{L}\right)^2}}{P_0^*}\vec{n}\cdot\vec{d}\right]\frac{\sin\left[2\pi q\sqrt{\vec{n}^2+\left(\frac{2\pi\vec{n}\cdot\vec{d}}{LP_0^*}\right)^2}\right]}{\sqrt{\vec{n}^2+\left(\frac{2\pi\vec{n}\cdot\vec{d}}{LP_0^*}\right)^2}}$$

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Method A:

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The first particle always on-shell

$$\det[\cot\delta + M] = 0$$

$$M^d_{lm,l'm'}(q) = \frac{8\pi}{q}\left[\frac{1}{L^3}\sum_{\frac{\vec{k}L}{2\pi}\in Z^3} - \mathbf{P}\int\frac{d^3\vec{k}}{(2\pi)^3}\right]\mathbf{J}\frac{4\pi Y^*_{lm}(\hat{\vec{k}}^*)Y_{l'm'}(\hat{\vec{k}}^*)\left(|\vec{k}^*|/q\right)^{l+l'}}{\vec{k}^{*\,2} - q^2}$$

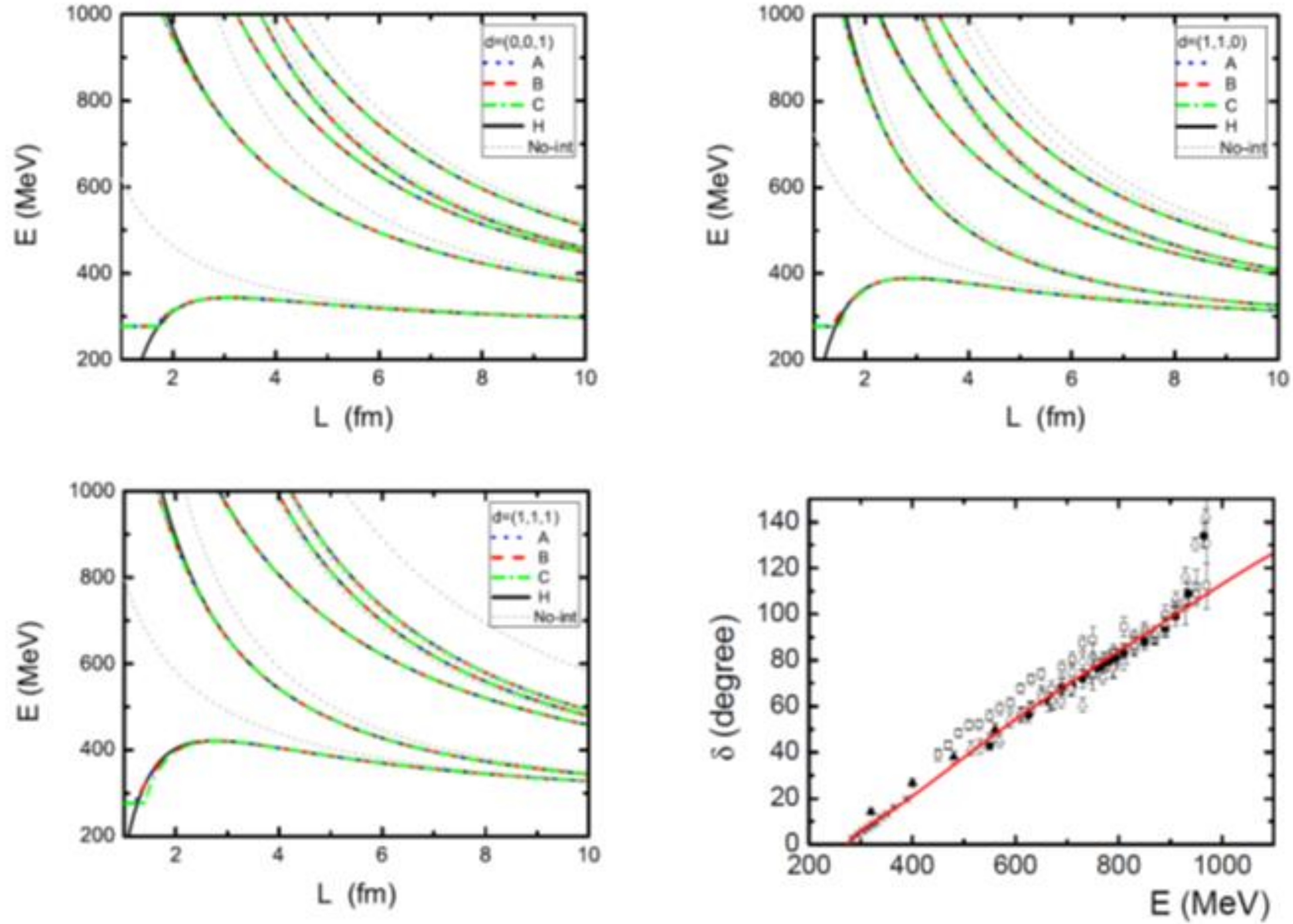
S-wave case

$$cot\delta_0 = \frac{4\pi}{\tilde{k}}\tilde{\Delta}_{00}$$

$$\begin{aligned}\tilde{\Delta}_{00} = {} & \mathscr{P}\int\frac{d^3\vec{q}}{(2\pi)^3}\frac{e^{\alpha(\tilde{k}^2-q^2)}}{\tilde{k}^2-q^2} - \frac{1}{L^3}\sum_{\vec{k}=\frac{2\pi}{L}\vec{n},\,\vec{n}\in\mathbb{Z}}\frac{\omega_{q_A}}{\omega_k}\frac{e^{\alpha(\tilde{k}^2-q_A^2)}}{\tilde{k}^2-q_A^2} \\ & + \frac{1}{4\pi^2 L}\sum_{\vec{n}\in\mathbb{Z},\vec{n}\neq 0}\int_0^{\alpha\left(\frac{2\pi}{L}\right)^2} dt e^{t\left(\frac{L\tilde{k}}{2\pi}\right)^2}\int dq 2q e^{-tq^2}\cos\left[\pi\frac{2\sqrt{m^2+\left(\frac{2\pi q}{L}\right)^2}}{P_0^*}\vec{n}\cdot\vec{d}\right]\frac{\sin\left[2\pi q\sqrt{\vec{n}^2+\left(\frac{2\pi\vec{n}\cdot\vec{d}}{LP_0^*}\right)^2}\right]}{\sqrt{\vec{n}^2+\left(\frac{2\pi\vec{n}\cdot\vec{d}}{LP_0^*}\right)^2}} \\ = {} & \frac{-1}{2\sqrt{\pi}^3 L\gamma_B}\mathcal{Z}^{\vec{d}}_{00}(1;\left(\frac{L\tilde{k}}{2\pi}\right)^2)\end{aligned}$$

Model A and Model B will give the exact same results although they use different transformations.

The test in S-wave of ππ scattering



Summary

- We discuss the general formalism of momentum transformation in the finite volume.

$$\vec{k}^* = \hat{\mathbf{A}}\left(\vec{k} + \mathbf{B}\vec{P}\right) = \mathbf{A}\left(\frac{\vec{k}\cdot\vec{P}}{\vec{P}^2} - \mathbf{B}\right)\vec{P} + \vec{k} - \frac{\vec{k}\cdot\vec{P}}{\vec{P}^2}\vec{P} \qquad d^3\vec{k}^* = \mathbf{J}d^3\vec{k}$$

$$\sqrt{\vec{k}^2 + m^2} + \sqrt{\left(\vec{P} - \vec{k}\right)^2 + m^2} = E \qquad E^* = \sqrt{E^2 - \vec{P}^2} = 2\sqrt{\vec{k}^{*\,2} + m^2}$$

$$\det[\cot\delta + M] = 0 \qquad M^d_{lm,l'm'}(q) = \frac{8\pi}{q}\left[\frac{1}{L^3}\sum_{\frac{\vec{k}L}{2\pi}\in Z^3} - \mathbf{P}\int\frac{d^3\vec{k}}{(2\pi)^3}\right]\mathbf{J}\,\frac{4\pi Y^*_{lm}(\hat{\vec{k}}^*)Y_{l'm'}(\hat{\vec{k}}^*)\left(|\vec{k}^*|/q\right)^{l+l'}}{\vec{k}^{*\,2} - q^2}$$

- Three different transformation ways are discussed. The first two ways provide the exact same results.
- The S-wave pp scattering case is checked in detailed.

Thank very much！

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