Two-pion scattering amplitude from Bethe-Salpeter wave function at the interaction boundary

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References:

Namekawa and TY, PRD98:011501,1(2018); arXiv:1904.00387(accepted in PRD)

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Outline

1.1 BS wave function through LSZ reduction formula

[Lin *et al.*, NPB619:467(2001)]

BS wave function of two pions in infinite volume (Only S-wave)

$$\begin{aligned}\phi(x;k) &= \langle 0|\pi_1(\vec{x}/2)\pi_2(-\vec{x}/2)|\pi_1(\vec{k})\pi_2(-\vec{k});\text{in}\rangle \\ &= e^{i\vec{k}\cdot\vec{x}} + \int \frac{d^3p}{(2\pi)^3}\frac{H(p;k)}{p^2-k^2-i\epsilon}e^{i\vec{p}\cdot\vec{x}}\end{aligned}$$

Inelastic scattering contribution and unnecessary overall factors are neglected.

NOT exactly same as one in BS equation

Half off-shell amplitude $H(p;k)$

$$H(p;k) = \frac{E_p + E_k}{8E_pE_k}M(p;k)$$

$M(p;k)$ defined by LSZ reduction formula

$$e^{-i\mathbf{q}\cdot\mathbf{x}}\frac{-i\sqrt{Z}M(p;k)}{-\mathbf{q}^2+m^2-i\varepsilon} = \int d^4zd^4y_1d^4y_2K(\mathbf{p},\mathbf{z})K(-\mathbf{k_1},\mathbf{y_1})K(-\mathbf{k_2},\mathbf{y_2})G_4(\mathbf{z},\mathbf{x},\mathbf{y_1},\mathbf{y_2})$$

$$K(\mathbf{p},\mathbf{z}) = \tfrac{i}{\sqrt{Z}}e^{i\mathbf{p}\cdot\mathbf{z}}(-\mathbf{p}^2+m^2), \quad G_4(\mathbf{z},\mathbf{x},\mathbf{y_1},\mathbf{y_2}) = \langle 0|T[\pi_1(\mathbf{z})\pi_2(\mathbf{x})\pi_1(\mathbf{y_1})\pi_2(\mathbf{y_2})|0\rangle$$

$$\mathbf{p} = (E_p,\vec{p}), \quad \mathbf{k_1} = (E_k,\vec{k}), \quad \mathbf{k_2} = (E_k,-\vec{k}), \quad \underline{\mathbf{q} = (2E_k - E_p,-\vec{p})}$$

off-shell momentum

1.1 BS wave function through LSZ reduction formula

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Inelastic scattering contribution and unnecessary overall factors are neglected.

NOT exactly same as one in BS equation

At on-shell momentum $p = k$, on-shell amplitude $H(k;k)$

$$H(k;k) = \frac{4\pi}{k}e^{i\delta(k)}\sin\delta(k)$$

BS wave function was used to derive Lüscher's finite volume formula in quantum field theory [CP-PACS, PRD71:094504(2005)]

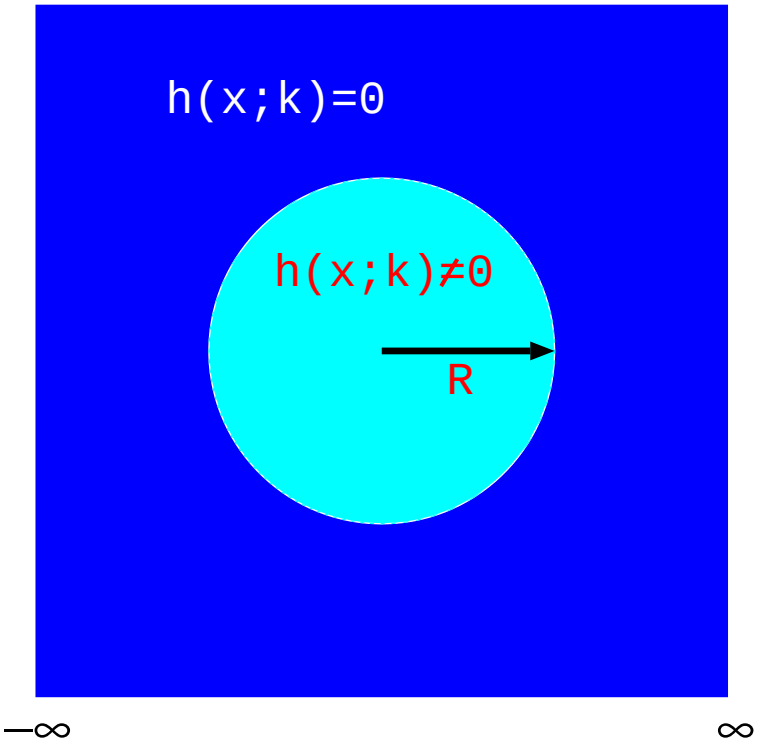
1.2 Reduced BS wave function [CP-PACS, PRD71:094504(2005)]

Reduced BS wave function

$$h(x;k) = (\Delta + k^2)\phi(x;k)$$

Important assumption

$h(x;k) = 0$ in $x > R$ (R: interaction range)



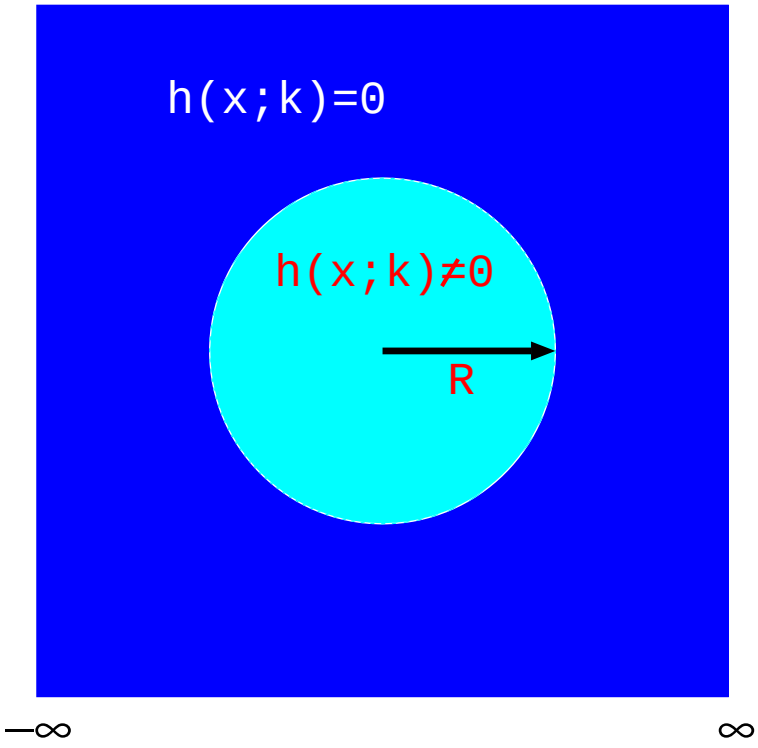
1.2 Reduced BS wave function [CP-PACS, PRD71:094504(2005)]

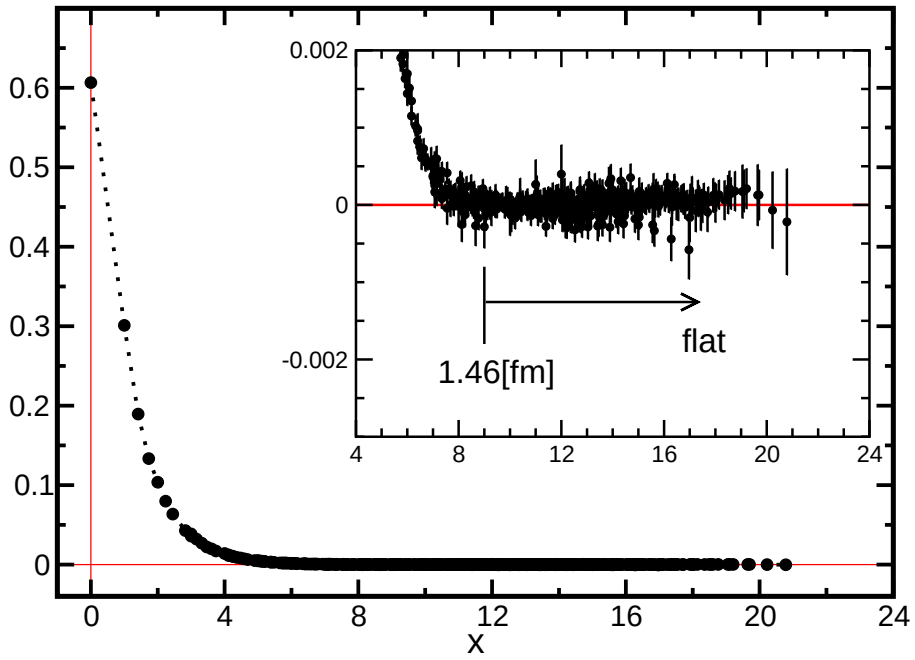
Reduced BS wave function

$$h(x;k) = (\Delta + k^2)\phi(x;k)$$

Important assumption is valid in two-pion scattering.

$h(x;k) = 0$ in $x > R$ (R: interaction range)





$I = 2$ two-pion system

$h(x;k)/\phi(x;k)$ $m_\pi = 0.52$ GeV

1.3 $H(k;k)$ from BS wave function inside R

[TY and Kuramashi, PRD96:114511,11(2017)]

Reduced BS wave function

$$h(x;k) = (\Delta + k^2)\phi(x;k) = -\int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}} H(p;k)$$

$$\because \phi(x;k) = e^{i\vec{k}\cdot\vec{x}} + \int \frac{d^3p}{(2\pi)^3} \frac{H(p;k)}{p^2 - k^2 - i\epsilon}\, e^{i\vec{p}\cdot\vec{x}}$$

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$\Downarrow$ Fourier transformation

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$\Downarrow$ Fourier transformation

Relation between $H(k;k)$ and $h(x;k)$ *i.e.* $\phi(x;k)$ inside R

$$\because h(x;k) \neq 0 \text{ in } x < R$$

$$H(k;k) = -\int d^3x\, e^{-i\vec{k}\cdot\vec{x}} h(x;k)$$

c.f. [CP-PACS, PRD71:094504(2005)]

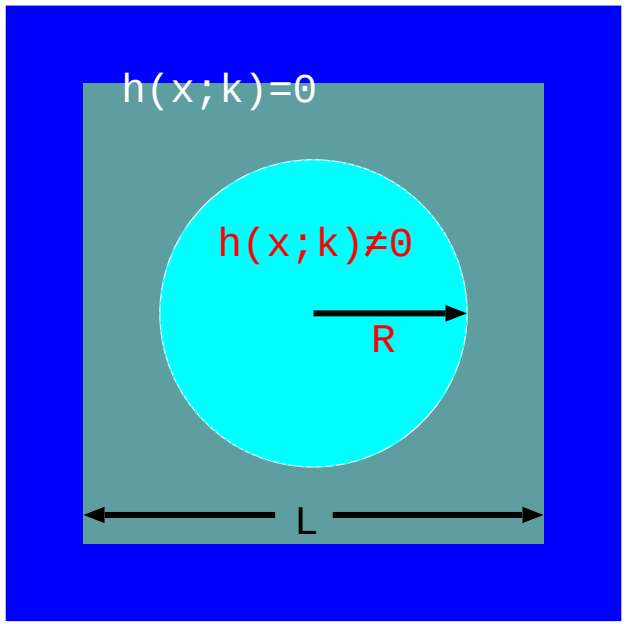
1.3 $H(k;k)$ from BS wave function inside R (cont'd)

[Namekawa and TY, PRD98:011501,1(2018)]

$H(k;k)$ in finite integration range $L/2 > R$

$$H(k;k) = -\int_{-\infty}^{\infty} d^3x\ e^{-i\vec{k}\cdot\vec{x}} h(x;k)$$

$$= -\int_{-L/2}^{L/2} d^3x\ e^{-i\vec{k}\cdot\vec{x}} h(x;k) \quad \because h(x;k) = 0\ (x > R)$$



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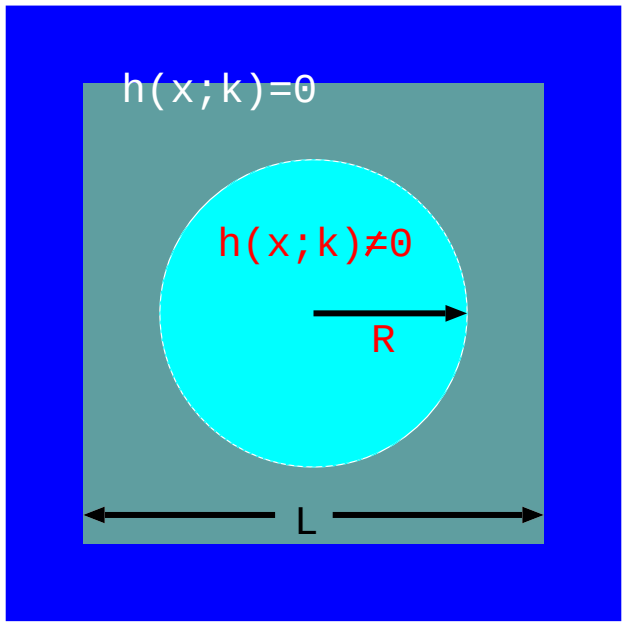


Suitable for finite volume calculation

Sufficient condition $L/2 > R$

$\rightarrow$ same as in finite volume method

First calculation using this relation

[Namekawa and TY, PRD98:011501,1(2018)]

$I = 2$ S-wave $\pi\pi$ scattering at $m_\pi = 0.86$ GeV

in quenched lattice QCD

1.4 $H(k;k)$ from lattice QCD [Namekawa and TY, PRD98:011501,1(2018)]

$I = 2$ S-wave $\pi\pi$ scattering at $m_\pi = 0.86$ GeV in quenched lattice QCD
same setup as in [CP-PACS, PRD71:094504(2005)]

Reduced BS wave function from CM frame two-pion 4-point function

$$h_L(\vec{x};k) = (\Delta + k^2)\phi_L(\vec{x};k) = (\Delta + k^2)C_{\pi\pi}(t_{\rm ref},\vec{x})$$

$$C_{\pi\pi}(t-t_0,\vec{x}) = \langle 0|\pi(t,\vec{x})\pi(t,\vec{0})\Omega_{\pi\pi}(t_0)|0\rangle$$

$$\Omega_{\pi\pi}(t) = \sum_{\vec{x}}\pi(t,\vec{x})\sum_{\vec{y}}\pi(t,\vec{y})$$

$t_{\rm ref} = 44$ for two-pion ground state: $k^2 = 0.001549(20)$ GeV^2

On-shell amplitude on lattice

$$H_L(k;k) = -\sum_{\vec{x}\in L^3} j_0(kx)h_L(\vec{x};k)$$

$j_0(kx)$: spherical Bessel function

Ratio of on-shell amplitude: overall factor cancellation

$$\frac{H_L(k;k)}{\phi_L(\vec{x}_{\rm ref};k)} = \frac{H(k;k)}{\phi(\vec{x}_{\rm ref};k)} \quad (x_{\rm ref} > R)$$

1.4 $H(k;k)$ from lattice QCD [Namekawa and TY, PRD98:011501,1(2018)]

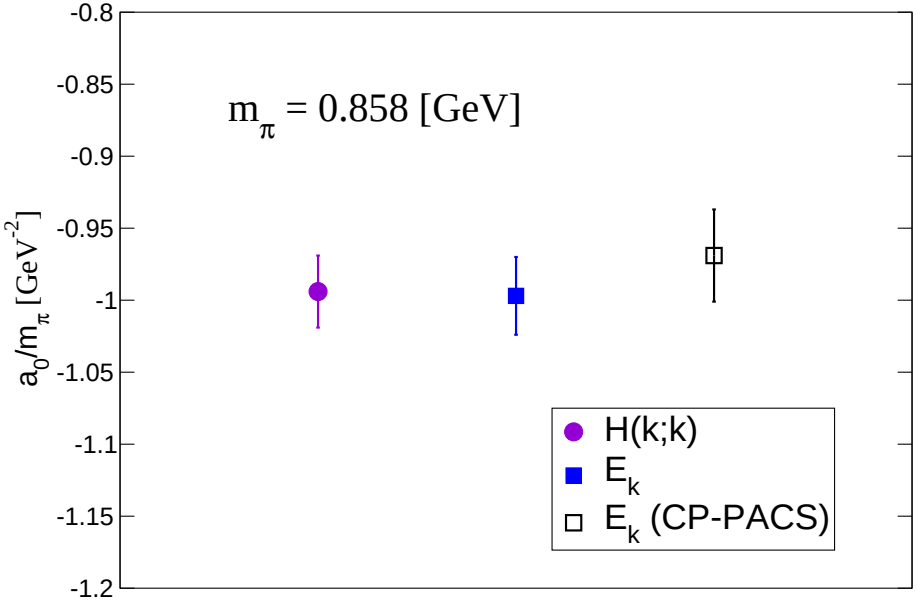
$$h_L(\vec{x};k) = (\Delta + k^2)\phi_L(\vec{x};k) = (\Delta + k^2)C_{\pi\pi}(t_{\mathrm{ref}}, \vec{x}) \qquad t_{\mathrm{ref}} = 44$$

$$H_L(k;k) \;=\; -\sum_{\vec{x}\in L^3} j_0(kx) h_L(\vec{x};k)$$

$G_{\pi\pi}(t,\vec{x})$: two-pion 4-point function, $j_0(kx)$: spherical Bessel function

$$\tan\delta(k) = \frac{\sin(kx_{\mathrm{ref}})}{4\pi x_{\mathrm{ref}}\,\phi_L(\vec{x}_{\mathrm{ref}};k)/H_L(k;k) - \cos(kx_{\mathrm{ref}})}$$

$x_{\mathrm{ref}} > R,\ \vec{x}_{\mathrm{ref}} = (12,7,2)$



$$\frac{H_L(k;k)}{\phi_L(\vec{x};k)} = \frac{H(k;k)}{\phi(\vec{x};k)}$$

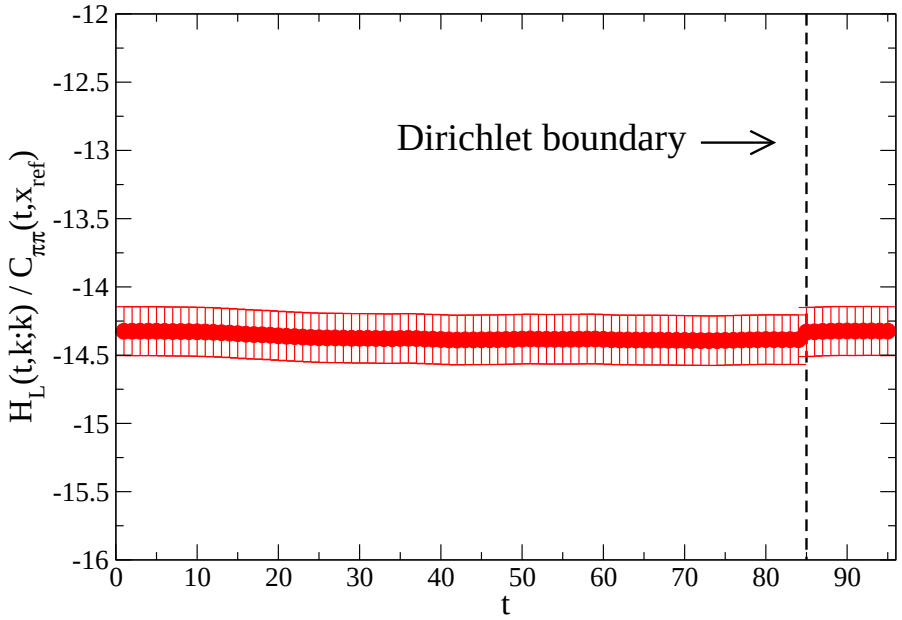
$$\frac{1}{m_\pi}\frac{\tan\delta(k)}{k} \approx \frac{a_0}{m_\pi}$$

Consistent with finite volume method (E_k in figure)

1.5 Time independence of ratio of $H(k;k)$

On-shell amplitude at each t

$$H_L(t,k;k) = -\sum_{\vec{x}} j_0(kx)(\Delta + k^2) C_{\pi\pi}(t,\vec{x})$$



$$\frac{H_L(t,k;k)}{C_{\pi\pi}(t,\vec{x}_{\mathsf{ref}})} = \left.\frac{H(k;k)}{\phi(\vec{x}_{\mathsf{ref}};k)}\right|_t \qquad (x_{\mathsf{ref}} > R)$$

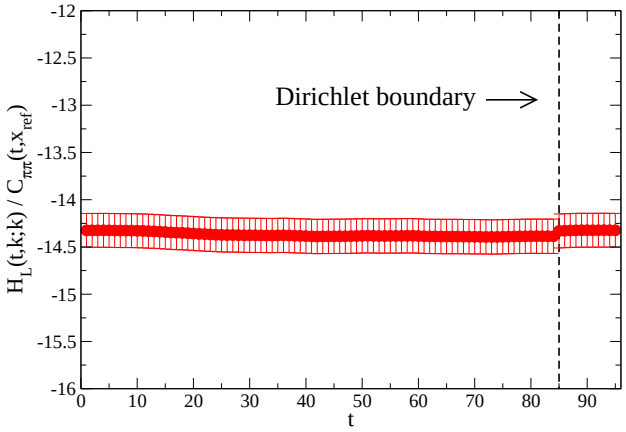
Almost independent of t

large t: only ground state in $C_{\pi\pi}(t,\vec{x}_{\mathsf{ref}})$

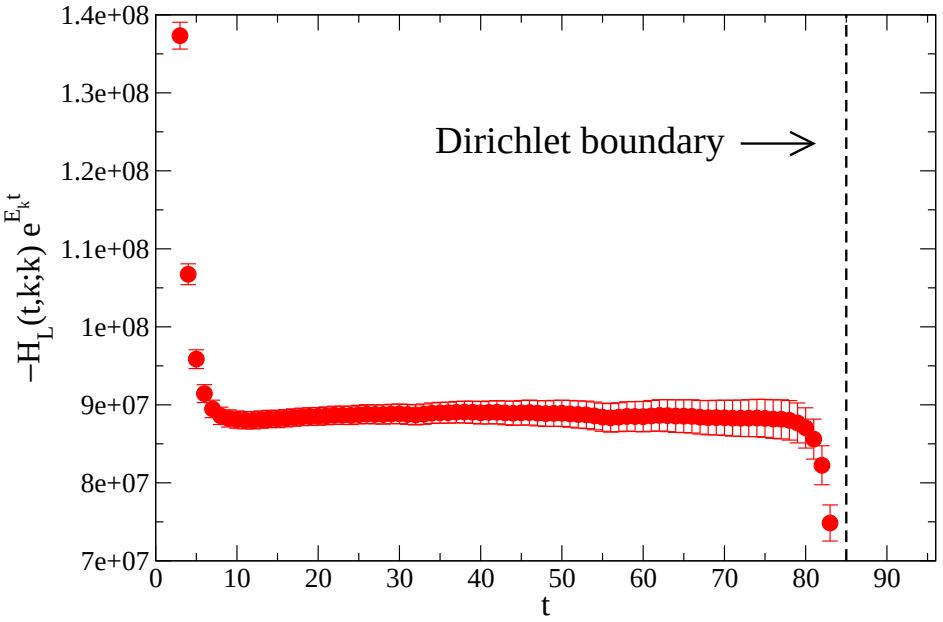
small t: non-trivial cancellation of excited states

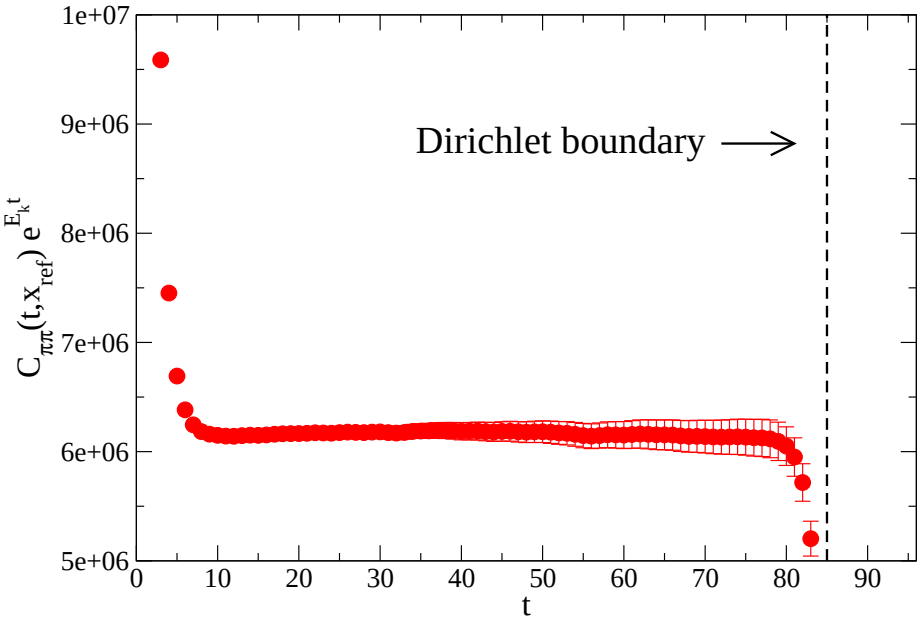
1.5 Time indep. of ratio of $H(k;k)$

$$\frac{H_L(t,k;k)}{C_{\pi\pi}(t,\vec{x}_{\rm ref})},\ H_L(t,k;k) = -\sum_{\vec{x}} j_0(kx)(\Delta + k^2)C_{\pi\pi}(t,\vec{x})$$



Significant excited state contribution in $t < 10$





Purpose

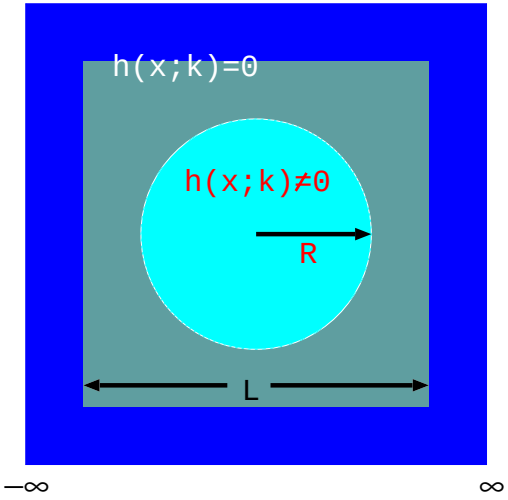
Discuss condition for excited state cancellation in ratio

using another relation between $H(k;k)$ and $\phi(x;k)$

2. $H(k;k)$ from interaction boundary

Relation between $H(p;k)$ and $\phi(x;k)$

$$H(k;k) \;=\; -\int_{-L/2}^{L/2} d^3x\; e^{-i\vec{p}\cdot\vec{x}}(\Delta + k^2)\phi(x;k)$$



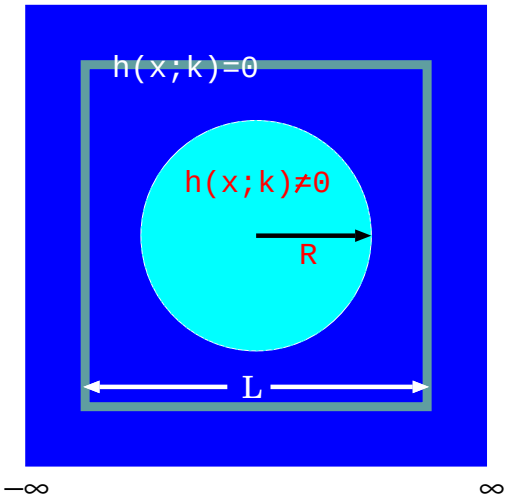
2. $H(k;k)$ from interaction boundary

Relation between $H(p;k)$ and $\phi(x;k)$

$$H(k;k) = -\int_{-L/2}^{L/2} d^3x\; e^{-i\vec{p}\cdot\vec{x}}(\Delta + k^2)\phi(x;k)$$

$$= (k^2 - k^2)\int_{-L/2}^{L/2} d^3x\; e^{-i\vec{p}\cdot\vec{x}}\phi(x;k)$$

$$-\sum_j \int_{-L/2}^{L/2} d^2x \left[e^{-i\vec{k}\cdot\vec{x}}\left(\partial_j\phi(x;k) + ik_j\phi(x;k)\right)\right]_{-L/2}^{L/2}$$



Surface term surf$(k;k)$: integration at boundary $(j = 1,2,3)$

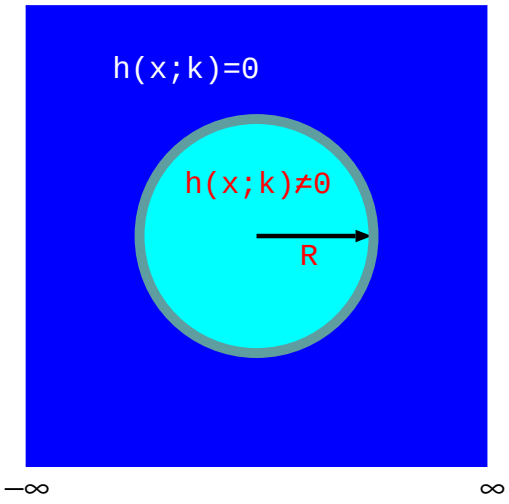
using partial integration

$H(k;k)$ from surf$(k;k)$

$$H(k;k) = \text{surf}(k;k)$$

2. $H(k;k)$ from interaction boundary

Integration range can be replaced by R



$H(k;k)$ from $\phi(x;k)$ at interaction boundary

$$H(k;k) = \mathrm{surf}(k;k)$$

$$\mathrm{surf}(k;k) = -\frac{4\pi}{k}\left[R\sin(kR)\left.\frac{\partial\phi(x;k)}{\partial x}\right|_{x=R} - (kR\cos(kR) - \sin(kR))\phi(R;k)\right]$$

surf$(k;k)$ is given by $\phi(x;k)$ at boundary $(x \geq R)$.

2. $H(k;k)$ from interaction boundary

Similar integration expressed by $\mathrm{surf}(k;p)$

$$-\int_{-L/2}^{L/2} d^3x\; e^{-i\vec{k}\cdot\vec{x}}(\Delta + k^2)\phi(x;p)$$

$$= (k^2 - k^2)\int_{-L/2}^{L/2} d^3x\; e^{-i\vec{k}\cdot\vec{x}}\phi(x;p) + \mathrm{surf}(k;p)$$

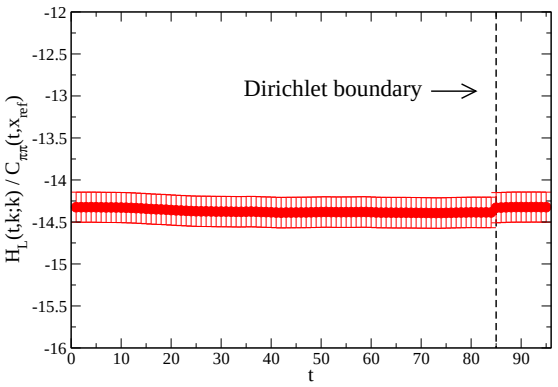
Relation on lattice assuming $(\Delta + k^2)j_0(kx) = 0$

$$-\sum_{\vec{x}\in L^3} j_0(kx)(\Delta + k^2)\phi_L(\vec{x};p) = \mathrm{surf}_L(k;p)$$

Explicit form of $\mathrm{surf}_L(k;p)$ in [Namekawa and TY, arXiv:1904.00387]

At on-shell momentum $p = k$, $H_L(k;k) = \mathrm{surf}_L(k;k)$

3. $\mathrm{surf}_L(k;q)$ and $C_{\pi\pi}(t,\vec{x})$



CM frame two-pion 4-point function in small t

Simply we consider only two contributions;

$$C_{\pi\pi}(t,\vec{x}) = \sum_q A_q(t)\phi_L(\vec{x};q) + \sum_{q'} A'_{q'}(t)\phi_L(\vec{x};q')$$

$\pi\pi$ scattering $\pi\pi'$ scattering (π': radial excitation of π)

$$A_q^{(\prime)}(t) = A_q^{(\prime)}\exp(-E_q^{(\prime)}t),\ E_q = 2\sqrt{m_\pi^2+q^2},\ E'_q = \sqrt{m_\pi^2+q^2}+\sqrt{m_{\pi'}^2+q^2}$$

$q, q' \sim 2\pi/L\cdot\sqrt{n}$ ($n \geq 0$: integer), [lowest q] $= k$

easy to extend with $\pi'\pi'$ scattering

$H_L(t,k;k)$ in terms of $\mathrm{surf}_L(k;q)$

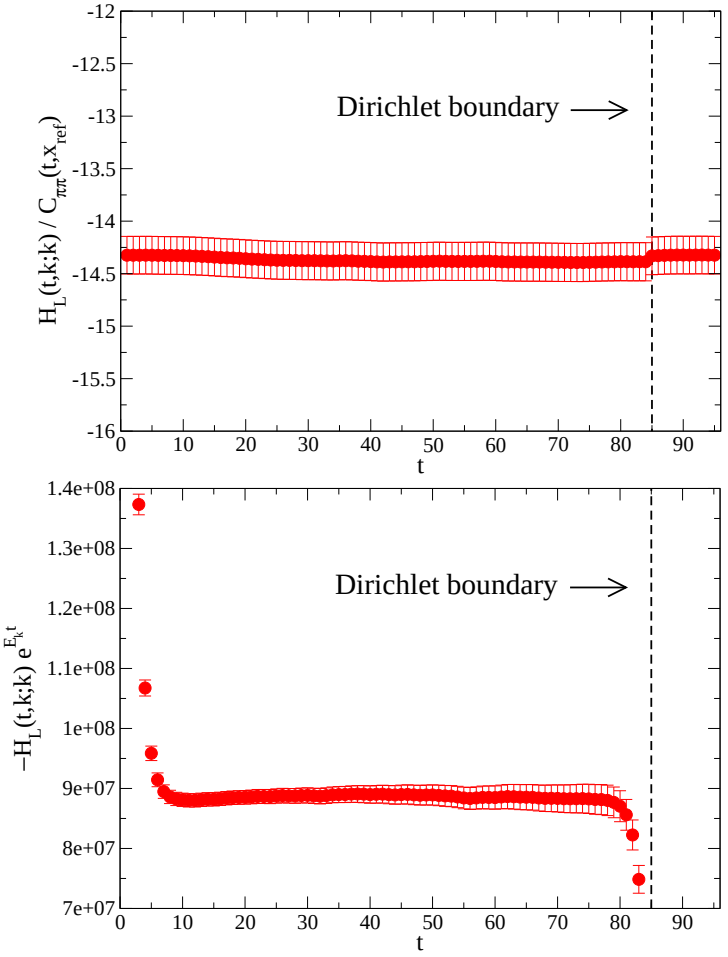
$$\begin{aligned} H_L(t,k;k) &= -\sum_{\vec{x}\in L^3} j_0(kx)(\Delta+k^2)C_{\pi\pi}(t,\vec{x}) \\ &= \sum_q A_q(t)\mathrm{surf}_L(k;q) + \sum_{q'} A'_{q'}(t)\mathrm{surf}_L(k;q') \end{aligned}$$

using $-\sum_{\vec{x}\in L^3} j_0(kx)(\Delta+k^2)\phi(\vec{x};p) = \mathrm{surf}_L(k;p)$

4. Conditions for time independence

$$\frac{H_L(t,k;k)}{C_{\pi\pi}(t,\vec{x}_{\text{ref}})} = \frac{H_L(k;k)}{\phi_L(\vec{x}_{\text{ref}};k)} \frac{1+\delta H(t)}{1+\delta C(t)}$$

$$H_L(k;k) = \text{surf}_L(k;k)$$



Excited state contributions

$$\delta H(t) = \frac{\sum_{q\neq k} A_q(t)\text{surf}_L(k;q) + \sum_{q'} A'_{q'}(t)\text{surf}_L(k;q')}{A_k(t)\text{surf}_L(k;k)}$$

$$\delta C(t) = \frac{\sum_{q\neq k} A_q(t)\phi_L(\vec{x}_{\text{ref}};q) + \sum_{q'} A'_{q'}(t)\phi_L(\vec{x}_{\text{ref}};q')}{A_k(t)\phi_L(\vec{x}_{\text{ref}};k)}$$

$$\frac{H_L(t,k;k)}{C_{\pi\pi}(t,\vec{x}_{\text{ref}})} \sim \text{const.} \Rightarrow \delta H(t) \sim \delta C(t)$$

$$\Rightarrow \frac{\text{surf}_L(k;q^{(\prime)})}{\text{surf}_L(k;k)} \sim \frac{\phi_L(\vec{x}_{\text{ref}};q^{(\prime)})}{\phi_L(\vec{x}_{\text{ref}};k)} \text{ and } A^{(\prime)}_{q^{(\prime)}}(t) \ll A_k(t) \text{ in other } q^{(\prime)}$$

Simple scenario: some states satisfy this condition

to observe excited state contribution in $t < 10$

4. Conditions for time independence

In order to examine $\dfrac{\mathrm{surf}_L(k;q^{(\prime)})}{\mathrm{surf}_L(k;k)} \sim \dfrac{\phi_L(\vec{x}_{\mathrm{ref}};q^{(\prime)})}{\phi_L(\vec{x}_{\mathrm{ref}};k)}$

Define $R_{\mathrm{surf}}(k;p) = \dfrac{\mathrm{surf}_L(k;p)}{\phi_L(\vec{x}_{\mathrm{ref}};p)} \Rightarrow R_{\mathrm{surf}}(k;p) \sim R_{\mathrm{surf}}(k;k)$

Since $\mathrm{surf}_L(k;p)$ and $\phi_L(\vec{x}_{\mathrm{ref}};p)$ given by $\phi_L(\vec{x};p)$ in $x > R$

$R_{\mathrm{surf}}(k;p)$ can be estimated by $G(\vec{x};p)$

$G(\vec{x};p) \propto \phi_L(\vec{x};p)$ in $x > R$ where $(\Delta + p^2)G(\vec{x};p) = 0$

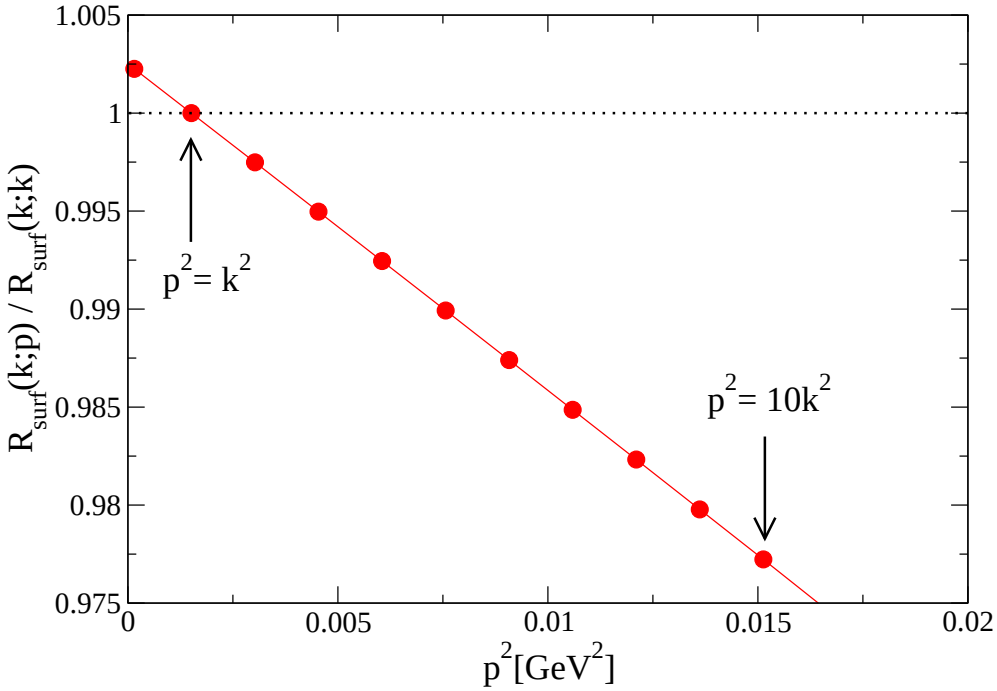
$$G(\vec{x};p) = \frac{1}{L^3}\sum_{\vec{n}} \frac{e^{i2\pi\vec{n}\cdot\vec{x}/L}}{(2\pi/L)n^2 - p^2}$$ [Lüscher, NPB354:531(1991)]

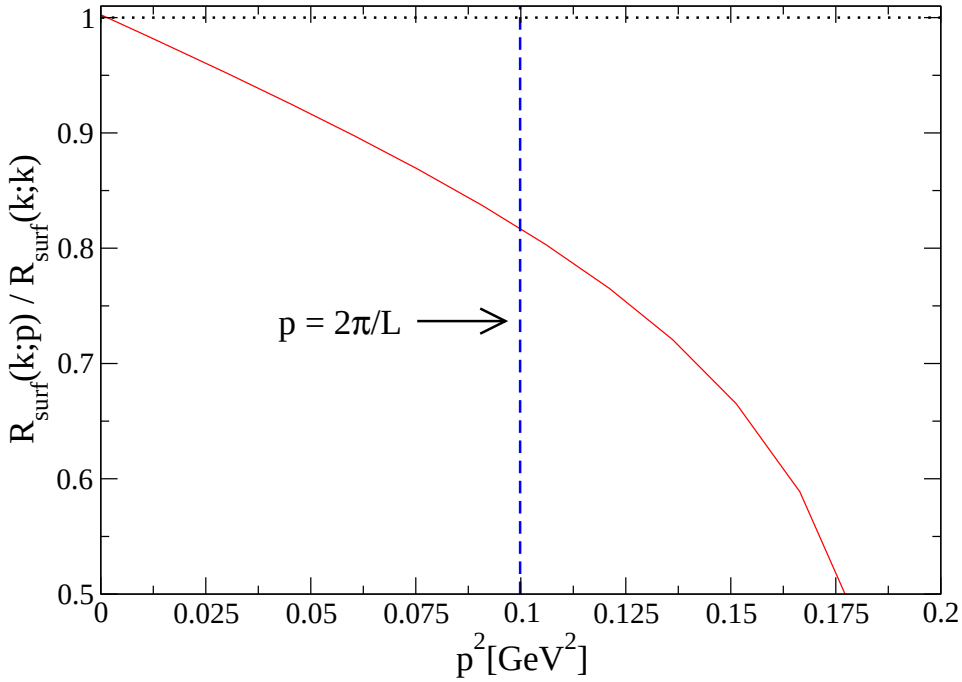
$$R_{\mathrm{surf}}(k;p) = \frac{\mathrm{surf}_L^G(k;p)}{G(\vec{x}_{\mathrm{ref}};p)} \qquad \mathrm{surf}_L^G(k;p) = \mathrm{surf}_L(k;p)\,[\phi_L(\vec{x};p) \to G(\vec{x};p)]$$

4. Conditions for time independence

Evaluation of $R_{\mathrm{surf}}(k;p) = \dfrac{\mathrm{surf}^G_L(k;p)}{G(\vec{x}_{\mathrm{ref}};p)}$

calculation method of $G(\vec{x};p)$ in [CP-PACS, PRD71:094504(2005)]





$R_{\mathrm{surf}}(k;p) = R_{\mathrm{surf}}(k;k)$ in less than 3% when $p^2 \leq 10k^2$

$R_{\mathrm{surf}}(k;p)$ largely differs from $R_{\mathrm{surf}}(k;k)$ in $p \geq 2\pi/L$.

4. Conditions for time independence

$R_{\mathrm{surf}}(k;p) \sim R_{\mathrm{surf}}(k;k)$ in $p \sim k$

$$C_{\pi\pi}(t,\vec{x}) = \sum_q A_q(t)\phi_L(\vec{x};q) + \sum_{q'} A'_{q'}(t)\phi_L(\vec{x};q')$$

$\pi\pi$ scattering $\quad$ $\pi\pi'$ scattering (π': radial excitation of π)

$q, q' \sim 2\pi/L \cdot \sqrt{n}$ ($n \geq 0$: integer), [lowest q] $= k$

Conditions for time independence of $H_L(t,k;k)/C_{\pi\pi}(t,\vec{x}_{\mathrm{ref}})$

$\pi\pi'$ state with $q' \sim k$ has $A'_{q'}(t) \sim A_k(t)$.

Other states have $A^{(\prime)}_{q^{(\prime)}}(t) \ll A_k(t)$ in $t < 10$.

Conditions seem reasonable, because

π' contribution in effective m_π in $t < 10$

Source operator with $p = 0$ projected pions

$p \neq 0$ states suppressed

Variational analysis helps to confirm these conditions

5. Summary

On-shell amplitude from BS wave function

$$\begin{aligned} H_L(k;k) &= -\sum_{\vec{x}} j_0(kx)(\Delta + k^2)\phi_L(\vec{x};k) \\ &= \text{surf}_L(k;k) \text{ at (interaction) boundary} \end{aligned}$$

Time independence of $H_L(t,k;k)/C_{\pi\pi}(t,\vec{x}_{\text{ref}})$ through $\text{surf}_L(k;q)$

- Simply consider only $\pi\pi$ and $\pi\pi'$ scatterings in $C_{\pi\pi}(t,\vec{x})$
- $H_L(t,k;k)$ given by sum of $\text{surf}_L(k;q)$
- Conditions for time independence

  $\pi\pi'$ ground state has sizable contribution in $t < 10$.
  Other states with larger momentum are smaller or negligible.

- Reasonable conditions from effective m_π and source operators

  variational analysis helps to confirm these conditions

Back up

Exploratory study in $I = 2$ two-pion scattering

[Namekawa and TY, (1)PRD98:011501,1(2018); (2)arXiv:1904.00387]

Simulation parameters

- Quenched QCD
- Iwasaki gauge action at $\beta = 2.334$ ($a^{-1} = 1.2$ GeV)
- Clover quark action at $m_\pi = 0.86, 0.67, 0.52$ GeV
- 200 configurations on (1) $24^3 \times 64$, (2) $24^3 \times 96$
- S-wave scattering (A_1^+ in cubic group) at $k \sim 0$

  $$C(\vec{x}, t) = \langle 0|\pi(\vec{x}, t)\pi(\vec{0}, t)\Omega_{\pi\pi}(0)|0\rangle \xrightarrow[t \gg 1]{} C_k \phi(\vec{x}; k) e^{-E_k t}$$

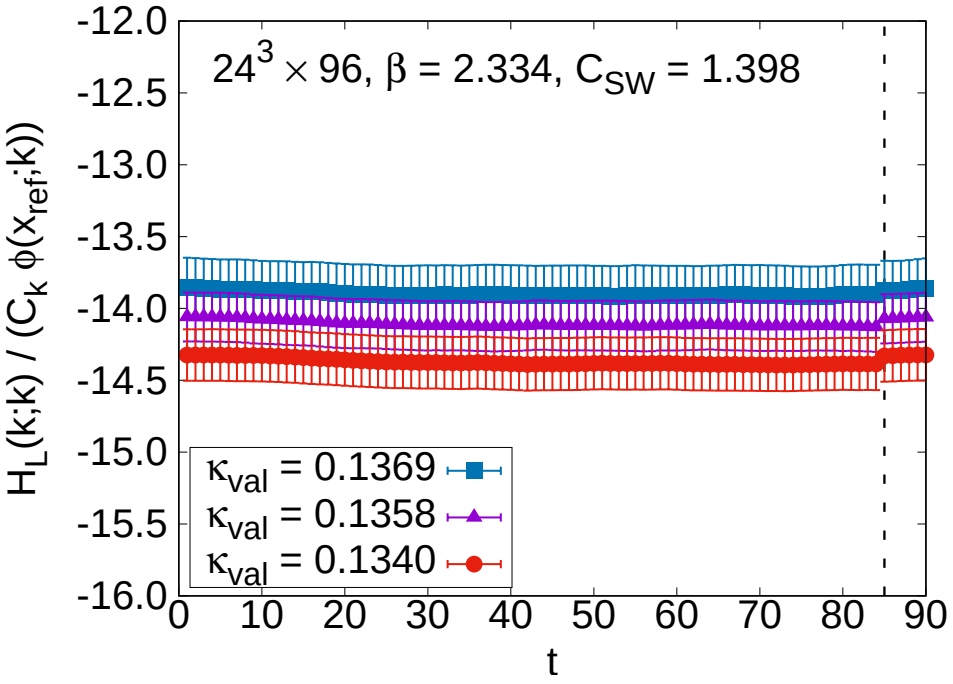
- Random $Z(2)$ source
  Four sources at every 2 time slices
- Periodic boundary condition in spatial direction
  Dirichlet boundary condition in temporal direction

Time independence of ratio of $H(k;k)$

On-shell amplitude at each t

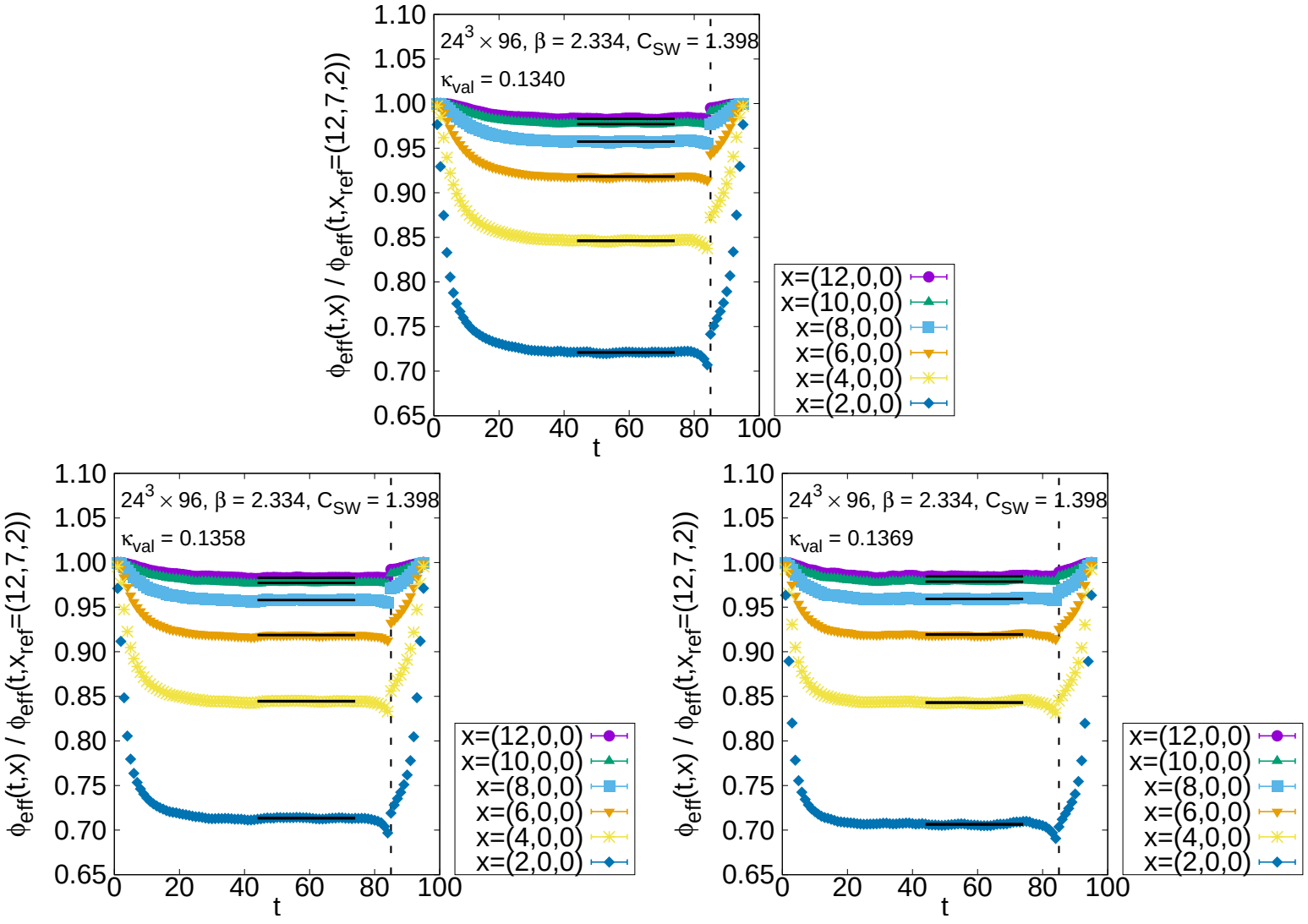
$$H_L(t,k;k) = -\sum_{\vec{x}} j_0(kx)(\Delta + k^2) C_{\pi\pi}(t,\vec{x})$$

$$\frac{H_L(t,k;k)}{C_{\pi\pi}(t,\vec{x}_{\text{ref}})} = \frac{H(k;k)}{\phi(\vec{x}_{\text{ref}};k)}\bigg|_t$$

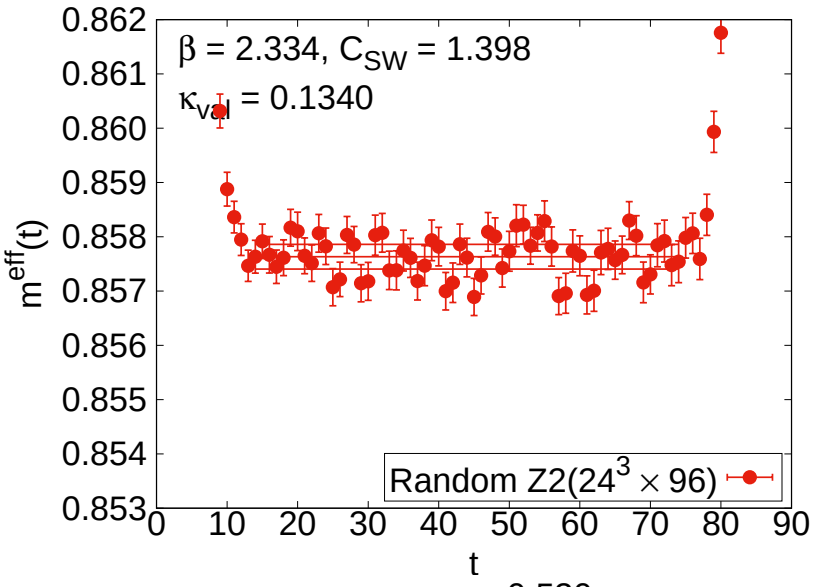


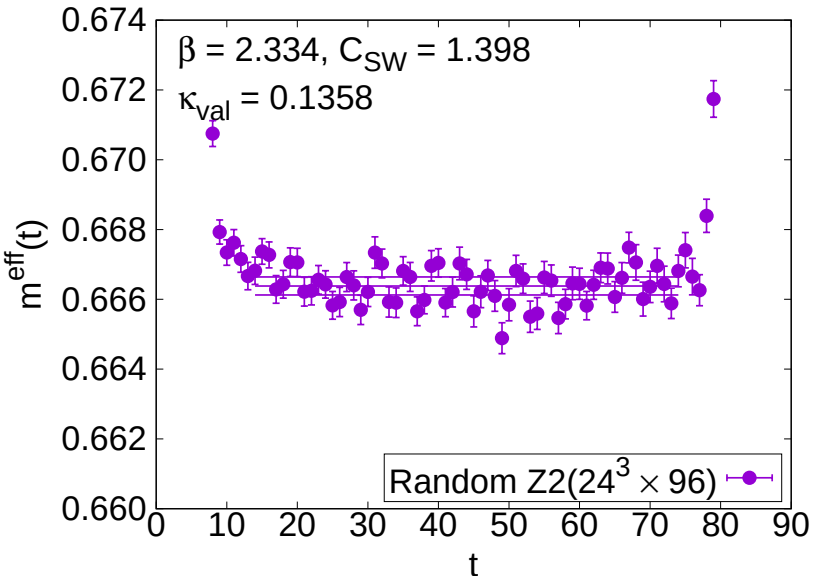
Almost independent of t

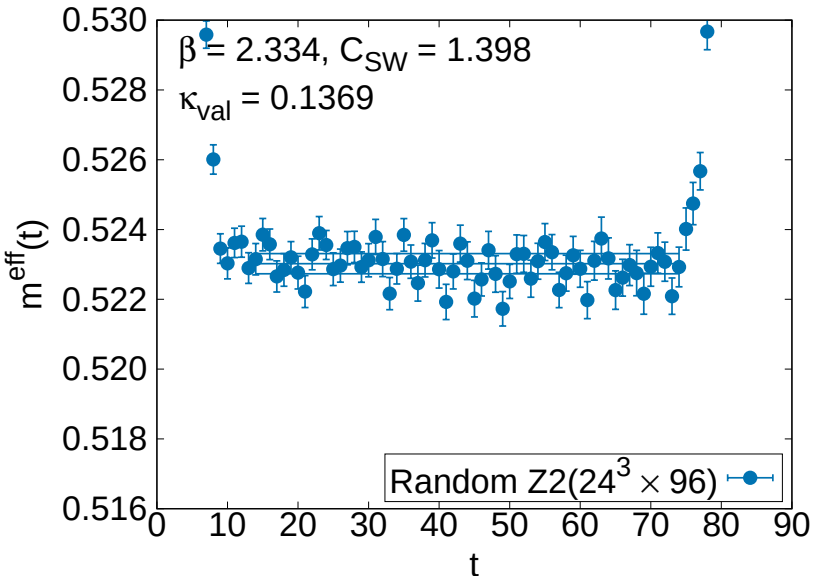
Time dependence of $\phi(t, x)$



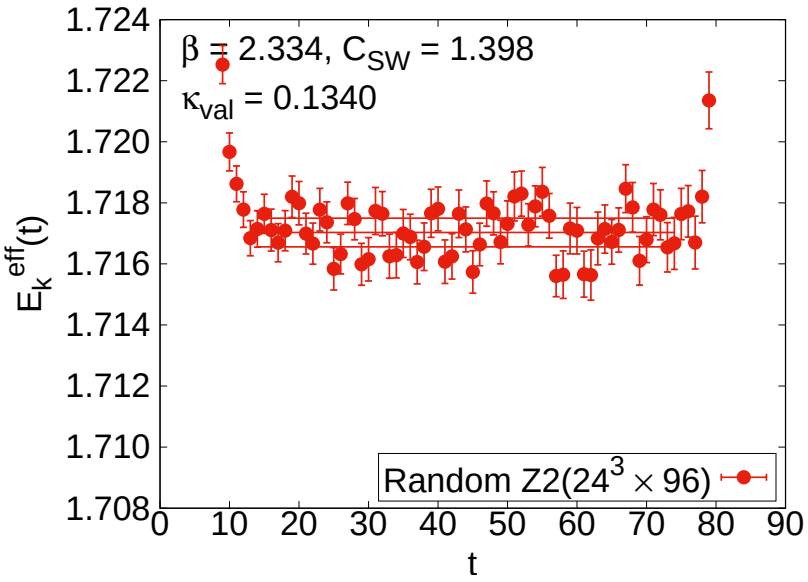
Effective pion mass

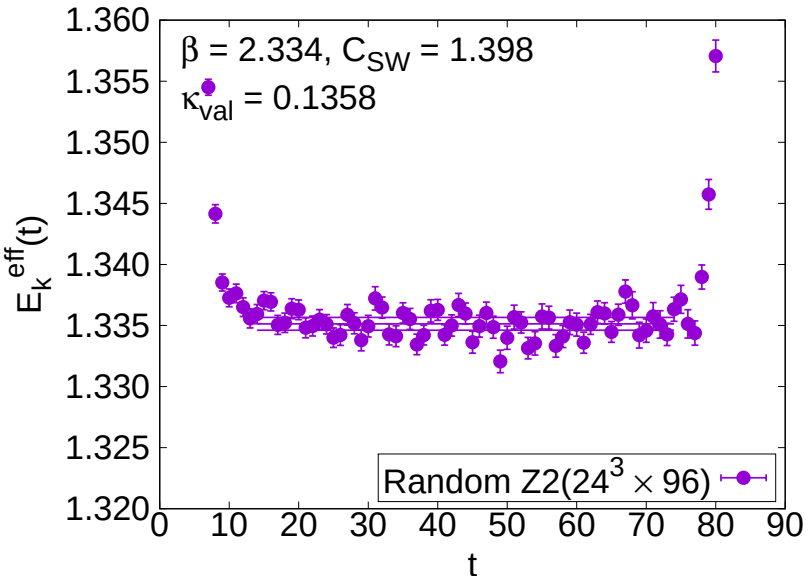


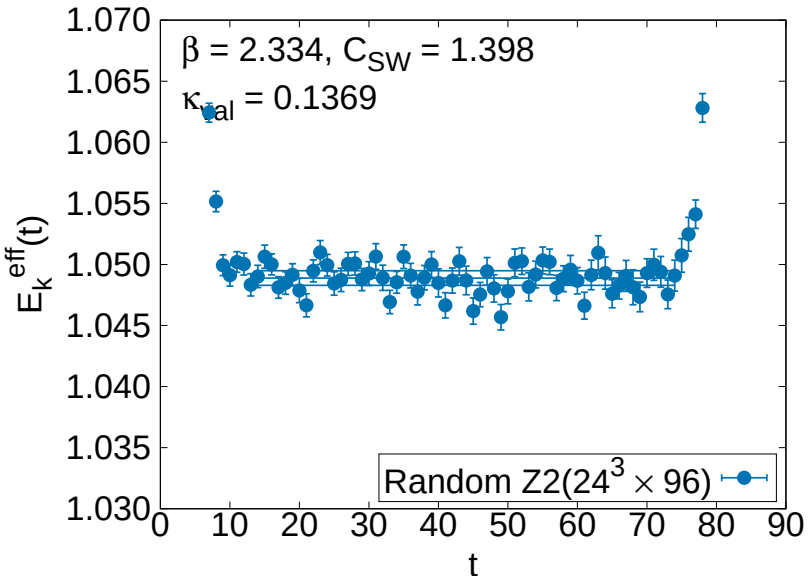




Effective two-pion energy



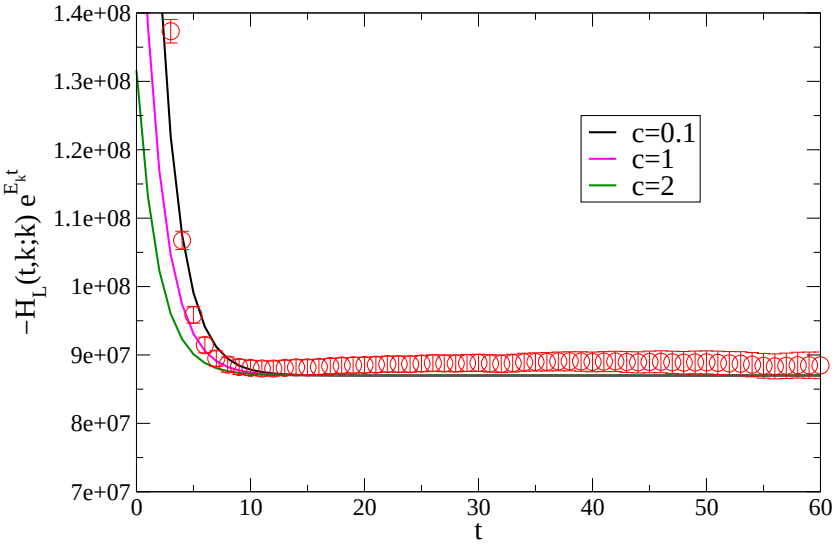


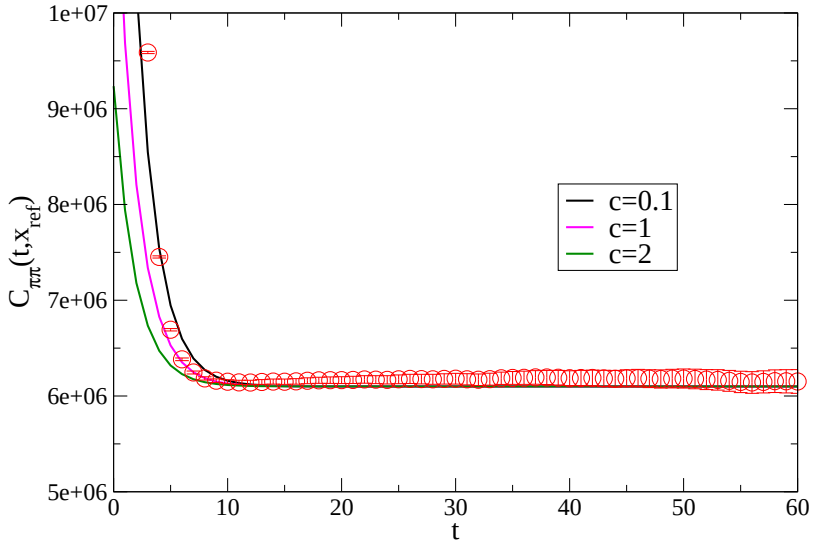


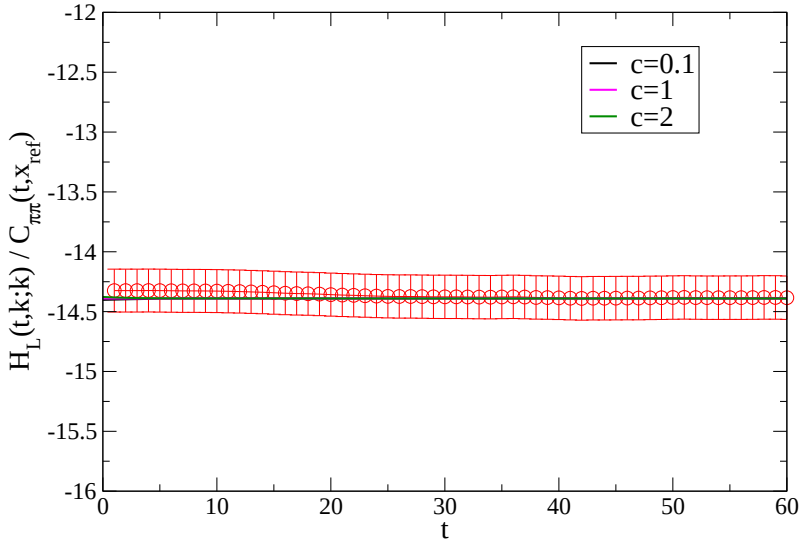
Reproduced data with radial excitation

$m_\pi = 0.86$ GeV, $m_{\pi'} = 1.5$ GeV, $p^2 = ck^2$

$$\delta H(t) = e^{-(E'_p - E_k)t} \mathrm{surf}(k;p)/\mathrm{surf}(k;k)$$



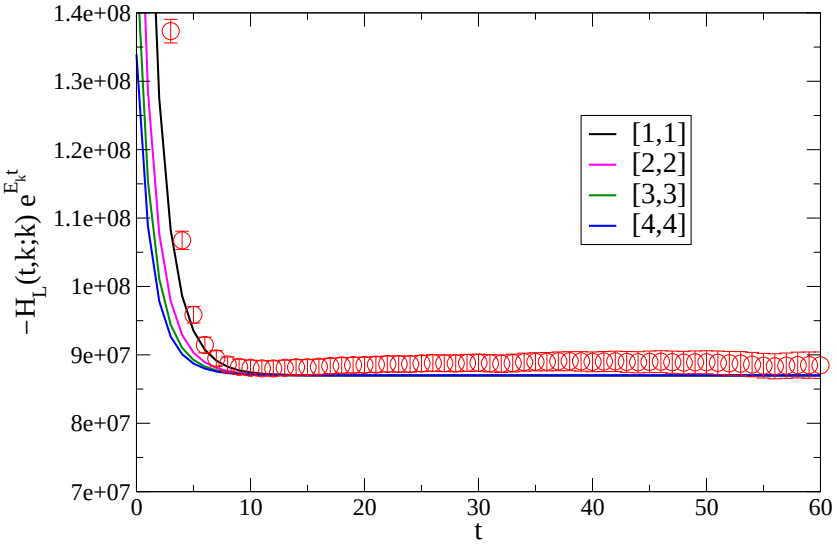


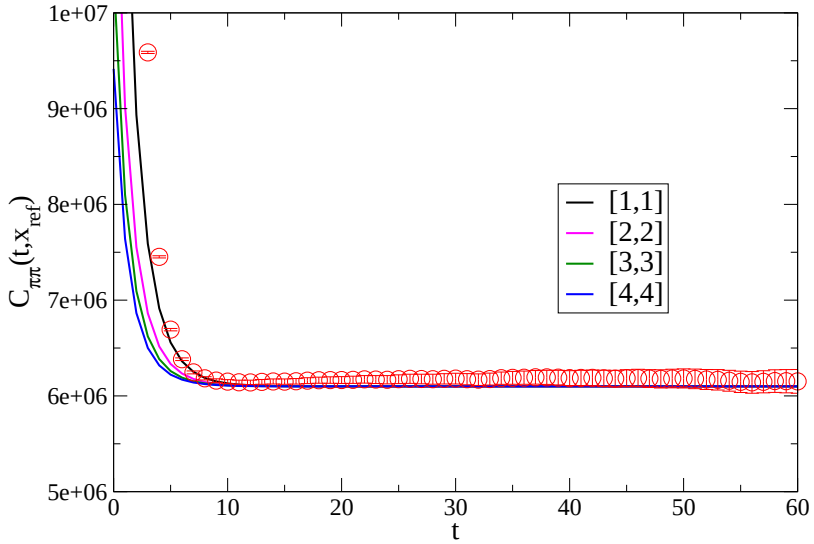


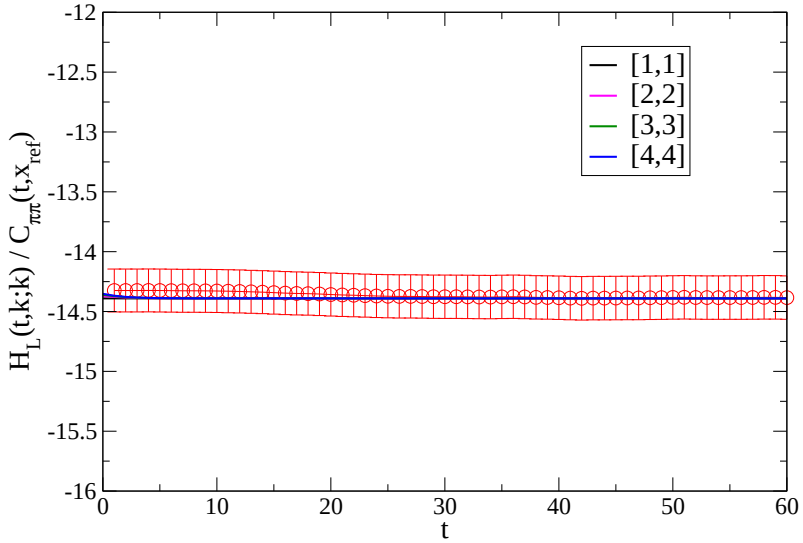
Reproduced data with radial excitations

$m_\pi = 0.86$ GeV, $m_{\pi'} = 1.5$ GeV, $p^2 = c_1 k^2$, $q^2 = c_2 k^2$

$$\delta H(t) = \left(e^{-(E'_p - E_k)t} \mathrm{surf}(k;p) + e^{-(E''_q - E_k)t} \mathrm{surf}(k;q) \right) / \mathrm{surf}(k;k)$$







$[c_1, c_2]$ in legend

Reproduced data with momentum excitations

$m_\pi = 0.86$ GeV, $p^2 = (2\pi/La)^2 + k^2$

$\delta H(t) = ce^{-(E_p - E_k)t}\text{surf}(k;p)/\text{surf}(k;k)$