Three particles on the lattice

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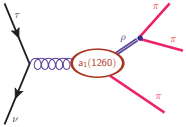
Plan

- Motivation
- Essentials:
  - What can be extracted from lattice calculations and what cannot?
  - What is the best strategy?
- Work that has already been done:
  - Formalism
  - Lattice
- Outlook

Why three particles on the lattice?

- The presence of the multi-particle inelastic channels imposes a major limitation on the applicability of the Lüscher approach for 2 particles
- Decays into the three-particle final states:
  - How does one extract the mass and width of a resonance?
  - Is there an analog of the Lellouche-Lüscher formula?
    - $K \to 3\pi\,, \quad \eta \to 3\pi\,, \quad \omega \to 3\pi$
    - $a_1(1260) \to \rho\pi \to 3\pi$
    - $a_1(1420) \to f_0(980)\pi \to 3\pi$
    - XYZ states, e.g., $X(3872)$, $Y(4260)$, ...
    - Roper resonance: πN and $\pi\pi N$ final states

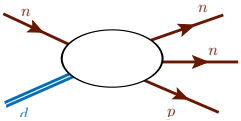


- Nuclear physics on the lattice, e.g., $nd \to nd$

Lattice vs. continuum: observables

Infinite volume:

- Three-particle bound states
- Elastic scattering
- Rearrangement reactions
- Breakup
- The mass and width of the three-particle resonances
- Resonance matrix elements (complex): e.g., $\langle K|H_W|\pi\pi\pi\rangle$



Finite volume:

- Two- and three-particle energy levels
- Matrix elements between eigenstates (real)

How does one connect these two sets?

What does one extract on the lattice (2-body sector)?

Power-law (gapless) vs. Exponentially suppressed (gap)

$E > 2m$, elastic scattering

$E < 4m$, contracted to a point

Two-body scattering: Lippmann-Schwinger equation $\rightarrow$ Lüscher equation (finite L)

K-matrix

Effective-range expansion: $K^{-1}(p) = p \cot \delta(p) = -\frac{1}{a} + \frac{1}{2} r p^2 + O(p^4)$

3-body sector

Gapless 2- and 3-body diagrams

$E > 2m$, elastic scattering

$E > 3m$

$\cdots$

Irreducible 2- and 3-body diagrams, contracted to a point

- Irreducible 2- and 3-body diagrams: couplings of the effective Lagrangian
- Exponential corrections in a finite volume are neglected

What does one extract in the 2- and 3-body sectors?

- The extraction of the infinite-volume observables proceeds differently in the 2- and 3-body sectors:

- In the 2-body sector, one extracts scattering phase $\leftrightarrow$ S-matrix directly

- In the 3-body sector, one follows two-step procedure:
- First, one extracts the 3-body couplings (not observable)
- At the next step, the equations in the infinite volume with input couplings are solved to arrive at the S-matrix elements
- Reason: 3-body K-matrix in a finite volume is singular, depends non-trivially on L

History

Perturbative calculations of the energy levels in a finite volume, non-relativistic quantum mechanics: K. Huang and C.N. Yang (1957), T.T. Wu (1959), S. Tan (2007), S.R. Beane, W. Detmold and M.J. Savage (2007), W. Detmold and M.J. Savage (2008), ...

Non-relativistic 3-body equation in a finite volume, numerical solution: S. Kreuzer and H.W. Hammer (2009,2010,2011), S. Kreuzer and H. Grießhammer (2012), P. Klos *et al.* (2018),...

History (formalism, quantization condition)

3-body S-matrix determines the finite-volume spectrum: K. Polejaeva and AR (2012)

3-body QC, 2→3 transitions [relativistic diagrammatic approach]:
M.T Hansen and S. Sharpe (2014,2015,2016,2017,2019),
R. Briceño, M.T. Hansen and S. Sharpe (2016,2017,2018,2019),
T. Blanton *et al.* (2018), T. Blanton, F. Romero-Lopez and S. Sharpe (2019),
R. Briceño *et al.* (2019), A. Jackura *et al.* (2019)...

3-body QC [NREFT approach]: U.-G. Meißner, G. Ríos and AR (2016),
H.W. Hammer, J.-Y. Pang and AR (2017), M. Döring...AR *et al.* (2018),
J.-Y. Pang...AR *et al.* (2018), Y. Meng...AR *et al.* (2018),...

3-body QC [unitarity + dispersion relations]: M. Mai and M. Döring (2017, 2018),...

3-body QC [particle-dimer picture]: R. Briceño and Z. Davoudi (2013)

Alternative approaches: S. Aoki *et al.* (2014), P. Guo (2017), ...

History (lattice calculations)

Relativistic, non-relativistic effective field theory (NREFT), particle-dimer and dispersion approaches are all generally equivalent, can be used to analyze lattice data!

H.W. Hammer, J.-Y. Pang and AR (2017), M.T. Hansen and S. Sharpe (2019), A. Jackura *et al.* (2019)

Multi-pion systems: S.R. Beane *et al.* (2007), W. Detmold *et al.* (2008), B. Hörz and A. Hanlon (2019),...

φ^4-theory: F. Romero-Lopez, AR and C. Urbach (2018), P. Guo and T. Morris (2019),...

Masses, matrix elements of light nuclei: NPLQCD coll. (2013,2014,2016,2017,2018),...

Nd scattering in nuclear lattice EFT: S. Elhatisari *et al.* (2016)

NREFT approach (H.W. Hammer...AR *et al.*, JHEP 1709 (2017) 109; JHEP 1710 (2017) 115)

No explicit particle creation & annihilation!

$$\mathcal{L}_2 = \psi^\dagger \left(i\partial_0 - \frac{\nabla^2}{2m} \right) \psi - \frac{C_0}{2} \psi^\dagger \psi^\dagger \psi \psi - \frac{C_2}{4} \left(\psi^\dagger \nabla^2 \psi^\dagger \psi \psi + \text{h.c.} \right) + \cdots$$

$C_0, C_2, \ldots$ matched to $p \cot \delta(p) = -\frac{1}{a} + \frac{r}{2} p^2 + \cdots$

dimer : 

$$\mathcal{L}_2 \to \mathcal{L}_2^{\text{dimer}} = \psi^\dagger \left(i\partial_0 - \frac{\nabla^2}{2m} \right) \psi + \sigma T^\dagger T + \left(T^\dagger \left[f_0 \psi \psi + f_1 \psi \nabla^2 \psi + \cdots \right] + \text{h.c.} \right)$$

- Higher partial waves can be included: dimers with arbitrary spin
- Can be generalized to the non-rest frames (in progress)

3-body Lagrangian

$$\mathcal{L}_3 = -\frac{D_0}{6}\,\psi^\dagger\psi^\dagger\psi^\dagger\psi\psi\psi - \frac{D_2}{12}\left(\psi^\dagger\psi^\dagger\nabla^2\psi^\dagger\psi\psi\psi + \text{h.c.}\right) + \cdots$$

Dimer picture in the 3-body sector:

$$\begin{aligned}
\mathcal{L}_3 \to \mathcal{L}_3^{\text{dimer}} &= h_0\, T^\dagger T\psi^\dagger\psi + h_2\, T^\dagger T\left(\psi^\dagger\nabla^2\psi + \text{h.c.}\right) + \cdots \\
h_0 &= -\frac{2D_0}{3f_0^2}\,, \qquad H_0 = \frac{h_0}{mf_0^2}\,, \qquad \cdots
\end{aligned}$$

The scattering equation in the infinite volume

Bethe-Salpeter eq. $\rightarrow$ Skornyakov-Ter-Martirosian eq.

$$\mathcal{M}(\mathbf{p},\mathbf{q};E) = Z(\mathbf{p},\mathbf{q};E) + \int^{\Lambda} \frac{d^3\mathbf{k}}{(2\pi)^3} Z(\mathbf{p},\mathbf{k};E)\tau(\mathbf{k};E)\mathcal{M}(\mathbf{k},\mathbf{q};E)$$

$$Z(\mathbf{p},\mathbf{q};E) = \frac{1}{\mathbf{p}^2+\mathbf{q}^2+\mathbf{p}\mathbf{q}-mE} + H_0 + H_2(\mathbf{p}^2+\mathbf{q}^2) + \cdots$$

2-body amplitude: $\tau^{-1}(\mathbf{k};E) = k^* \cot\delta(k^*) + \underbrace{\sqrt{\frac{3}{4}\mathbf{k}^2 - mE}}_{=k^*}$

Finite volume

$$\mathcal{M}_L(\mathbf{p},\mathbf{q};E) = Z(\mathbf{p},\mathbf{q};E) + \frac{8\pi}{L^3}\sum_{\mathbf{k}}^{\Lambda} Z(\mathbf{p},\mathbf{q};E)\tau_L(\mathbf{k};E)\mathcal{M}_L(\mathbf{k},\mathbf{q};E)$$

$$\tau_L^{-1}(\mathbf{k};E) = k^* \cot\delta(k^*) - \frac{4\pi}{L^3}\sum_{\mathbf{l}} \frac{1}{\mathbf{k}^2+\mathbf{l}^2+\mathbf{kl}-mE}$$

- Poles in the amplitude $\rightarrow$ finite-volume energy spectrum
- Quantization condition: $\det(\tau_L^{-1} - Z) = 0$
- $k^* \cot\delta(k^*)$ fitted in the two-particle sector
- $H_0, H_2, \ldots$ should be fitted to the three-particle energies
- Finally, solve the equations in the infinite volume to arrive at the S-matrix!

Cubic symmetries (M. Döring... AR *et al.*, PRD 97 (2018) 114508)

- Octahedral group O_h, including inversions, little groups thereof
- Reduction: an analog of the partial-wave expansion in a finite volume
- Analog for a sphere $|\mathbf{k}| = \text{const}$ for a cube: *shells*

$$s = \left\{ \mathbf{k} \ : \ \mathbf{k} = g\mathbf{k}_0 \, , \ g \in O_h \right\}$$

- For an arbitrary function of the momentum $\mathbf{p}$, belonging to a shell s,

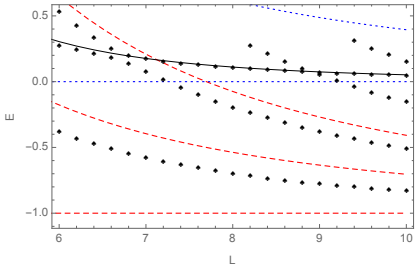
$$f(\mathbf{p}) = f(g\mathbf{p}_0) = \sum_{\Gamma} \sum_{ij} T_{ij}^{(\Gamma)}(g) f_{ji}^{(\Gamma)}(\mathbf{p}_0) \, , \quad \Gamma = A_1^{\pm}, A_2^{\pm}, E^{\pm}, T_1^{\pm}, T_2^{\pm}$$

- The quantization condition partially diagonalizes:

$$\det\left(\tau(s)^{-1} \vartheta(s)^{-1} \delta_{rs} \delta_{ij} - \frac{8\pi}{L^3} \frac{1}{G} Z_{ij}^{(\Gamma)}(r,s) \right) = 0$$

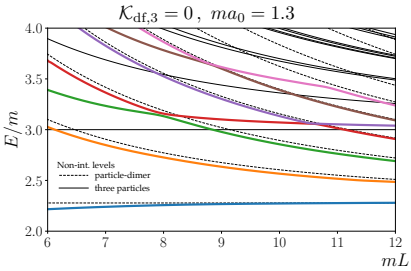
How does the three-particle spectrum looks like?

NREFT: Energy levels $\rightarrow H_0, H_2, \ldots$



M. Döring. . . AR *et al.* (2018)

Relativistic: energy levels $\rightarrow K_{3,df}$



F. Romero-Lopez *et al.* (poster session)

Avoided level crossing:

- 3-particle states and particle-dimer states
- 3-particle resonances . . .

Perturbative shift of the energy levels

Away from the avoided level crossings, the energy level shifts can be treated in perturbation theory in $1/L$:

$$\Delta E_2 = \frac{4\pi a}{mL^3}\left(1 + c_1\left(\frac{a}{\pi L}\right) + c_2\left(\frac{a}{\pi L}\right)^2 + c_3\left(\frac{a}{\pi L}\right)^3 + \frac{2\pi r a^2}{L^3} - \underbrace{\frac{\pi a}{m^2 L^3}}_{\text{relativistic}}\right)$$

$$\Delta E_3 = \frac{12\pi a}{mL^3}\left(1 + d_1\left(\frac{a}{\pi L}\right) + d_2\left(\frac{a}{\pi L}\right)^2 + \underbrace{\frac{3\pi a}{m^2 L^3}}_{\text{relativistic}} + \frac{6\pi r a^2}{L^3}\right.$$

$$\left. + \; d_3\left(\frac{a}{\pi L}\right)^3 \ln\frac{mL}{2\pi}\right) - \underbrace{\frac{D}{48 m^3 L^6}}_{\text{3-body force}}$$

Results with 4 and more particles, excited and particle-dimer levels are available

Relativistic corrections in the NREFT (F. Romero-Lopez. . . AR *et al.*, in progress)

- Use non-relativistic perturbation theory from S.R. Beane, W. Detmold and M.J Savage, PRD 76 (2007) 074507

$$\Delta E_n = \langle n|V|n\rangle + \sum_{m\neq n} \frac{|\langle n|V|m\rangle|^2}{E_n - E_m} + \cdots$$

- Leading relativistic corrections are caused by the perturbation

$$H = H_{0,\mathrm{NR}} + V + \Delta H\,, \qquad \langle \mathbf{p}|\Delta H|\mathbf{k}\rangle = -\frac{1}{2}\sum_i \frac{\mathbf{p}_i^4 + \mathbf{k}_i^4}{8m^3}(2\pi)^3\delta^3(\mathbf{p}_i - \mathbf{k}_i)$$

- a, r are obtained from the matching to the relativistic 2-body amplitude
- All results from M.T. Hansen and S.R. Sharpe, PRD 93 (2016) 014506; PRD 93 (2016) 096006 are easily reproduced
- In addition, the results for higher excited levels, multiparticle systems are available

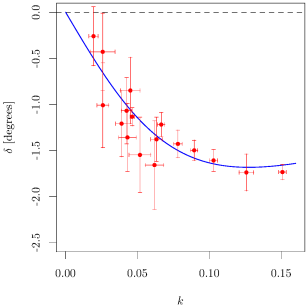
The three-body force

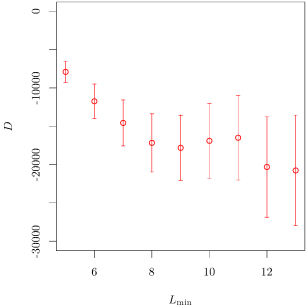
- The power counting:
  - The 2-body force contributes at $O(L^{-3})$
  - The 3-body force contributes at $O(L^{-6})$
  - Multiparticle forces are even more suppressed
- The three-body coupling constant D contains the threshold particle-dimer amplitude, singularities subtracted:

$$\mathcal{M}(\mathbf{0},\mathbf{0};E) = \frac{A_{-2}}{-E} + \frac{A_{-1}}{\sqrt{-E}} + A_0 \ln\frac{-E}{m} + \hat{\mathcal{M}}\,, \qquad \hat{\mathcal{M}} = H_0 + \cdots$$

- Use different irreps, excited states, multiparticle levels, to extract $\hat{\mathcal{M}}$ reliably

Three-body force in the φ^4-theory (F. Romero-Lopez. . . AR *et al.*, EPJC 78 (2018) 846)

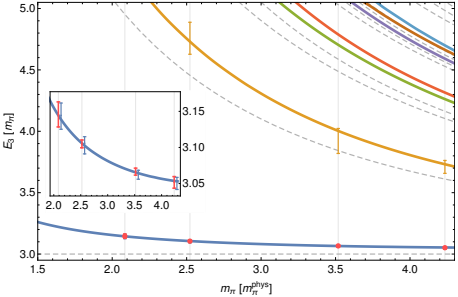




- Simultaneous fit to the 1-, 2- and 3-particle energy levels
- D is non-vanishing at 4σ for $L_{min} = 9$

Extracting 3-pion force (M. Mai and M. Döring, PRL 122 (2019) 062503)

Data from: S. Beane et al., PRL 100 (2008) 082004; W. Detmold et al., PRD78 (2008) 014507



Dispersion approach:

Three-body coupling: $c = (0.2 \pm 1.5) \cdot 10^{-10}$ [analog of H_0]

Strategies to extract the 3-body force

- In the three-body levels, the three-body force comes at N^3LO in $1/L$ ...
  - Use moderately large values of L, excited levels, different irreps
  - In order to suppress the exponential contributions, use the one-particle masses determined at the same L
- Determining the three-body force from the particle-dimer scattering

$$\Delta E_{1+D} = \frac{\mathcal{M}(\mathbf{0},\mathbf{0};-E_D)}{L^3} + O(L^{-4})$$

$\mathcal{M}(\mathbf{0},\mathbf{0};-E_D) =$  $+$  $+$ logarithmic terms $+ \cdots$

H_0 $\sim 1/E$

- Two-body scattering: $\pi\sigma,\ \pi\rho,\ N\sigma,\ N\rho,\ \pi\Delta, \ldots$
- Higher quark masses + extrapolation (A. Woss et al. (2018,2019))

Outlook

Technical improvements:

- Systematic inclusion of the relativistic effects in the NREFT approach
- Higher partial waves, spins, excited states, moving frames, twisted boundary conditions, ...
- Including fermions, cubic symmetry in the presence of fermions
- Perturbative relativistic corrections to the 4-, 5-, ... particle systems

Development of the formalism:

- Analog of the Lellouche-Lüscher formula for the three-particle decays
- Framework to analyze data on the Roper resonance

Outlook

Calculations on the lattice:

- Fitting excited state shifts, different irreps, multiparticle states, in order to reliably extract the three-particle force – both in the lattice QCD and in the models
- Study of the $\pi\sigma$, $\pi\rho$ scattering at different quark masses + extrapolation in the pion mass
- Light nuclei

Conclusion:

An important progress has been achieved in the three-particle problem during the last few years, concerning both formalism and lattice calculations. We expect much more to follow in the nearest future