Electromagnetic finite-size effects to the hadronic vacuum polarisation

Nils Hermansson-Truedsson

Lund University

Department of Astronomy and Theoretical Physics

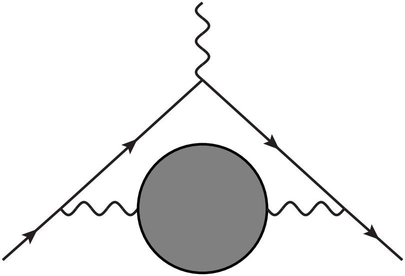
J. Bijnens, J. Harrison, T. Janowski, A. Jüttner, A. Portelli

June 17, 2019

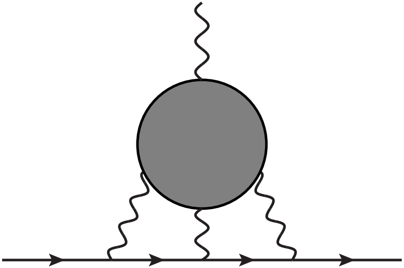
The muon anomalous magnetic moment

$$a_\mu = a_\mu^{\mathrm{QED}} + a_\mu^{\mathrm{EW}} + a_\mu^{\mathrm{had}}\,,$$

$$a_\mu^{\mathrm{had}} = a_\mu^{\mathrm{HVP}} + a_\mu^{\mathrm{HLbL}}$$



HVP



HLbL

- We are interested in the hadronic vacuum polarisation (HVP)

HVP

- Vector 2-point function

$$\Pi_{\mu\nu}(q) = \int d^4x\, e^{iq\cdot x} \langle 0 |\, T\left[j_\mu(x) j_\nu^\dagger(0)\right] |0\rangle$$

$$\Pi_{\mu\nu}(q^2) = \left(q_\mu q_\nu - q^2 \delta_{\mu\nu}\right) \Pi(q^2)$$

$$\hat{\Pi}(q^2) = \Pi(q^2) - \Pi(0)$$

- Dispersion relations: $\mathrm{Im}\,\hat{\Pi} \sim \sigma\left(e^+e^- \to \mathrm{hadrons}\right)$
- Lattice is competitive, so need electromagnetic (EM) effects to reach sub-percent level accuracy
- We want EM finite-size effects on hadronic vacuum polarisation (HVP) [hep-ph/1903.10591]

Finite volume corrections

- Finite volume effects depend on lattice size L

$$\text{Massive particles: } e^{-mL}$$

$$\text{Massless particles: } \frac{1}{L^a}$$

- Electromagnetic corrections to the HVP potentially dangerous
- How big are the effects? $a \geq 2$ but what is it really?

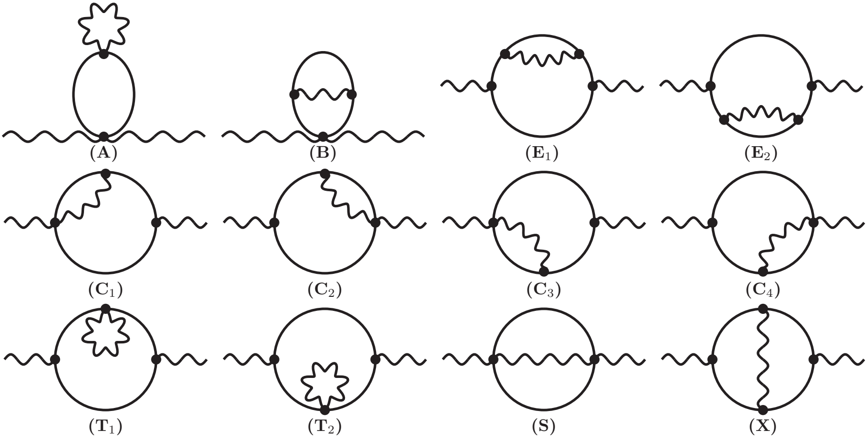
QED in finite volume

- Difficult to define charged states in finite volume with periodic boundary conditions
- Solution: QED_L [hep-lat/0804.2044; hep-lat/1810.05923]
- The global zero mode subtracted on every time slice, i.e.

$$\sum_{\mathbf{k}} \longrightarrow \sum_{\mathbf{k}}{}' = \sum_{\mathbf{k}\neq 0}$$

- We use scalar QED
- Also, $q = (q_0, \mathbf{0})$

The HVP at NLO in the electromagnetic coupling



- Scalar QED: Pure pion loops
- Compare with lattice simulations and vegas MC evaluation of loop integrals in lattice perturbation theory

Actual calculation

- 2-loop integrals for each diagram U

$$\hat{\Pi} = 2\,\hat{\Pi}_E + 2\,\hat{\Pi}_T + \hat{\Pi}_S + \hat{\Pi}_X + 4\,\hat{\Pi}_C$$

$$\hat{\Pi}_U = \int \frac{d^4k}{(2\pi)^4}\frac{d^4\ell}{(2\pi)^4}\,\hat{\pi}_U\left(q_0^2, k, \ell\right)$$

- Do energy integrals analytically

$$\hat{\rho}_U(\mathbf{k}, \ell, q_0) = \int \frac{dk_0}{2\pi}\frac{d\ell_0}{2\pi}\hat{\pi}_U\left(q_0^2, k, \ell\right)$$

- On the lattice momenta are discretised: $\mathbf{k} = (2\pi/L)\mathbf{n}$

Actual calculation

- Want the finite volume corrections, so for each diagram U we need to calculate

$$\Delta\hat{\Pi}_U(q^2) = \frac{1}{L^3}\Delta_n \int \frac{d^3\ell}{(2\pi)^3}\,\hat{\rho}_U\left(q_0^2, \frac{2\pi\,\mathbf{n}}{L}\right) + \mathcal{O}\left(e^{-mL}\right)$$

$$\Delta_{\mathbf{n}} = \left(\sum_{\mathbf{n}}{}' - \int d^3\mathbf{n}\right)$$

- Taylor expand in $1/L$
- Result will be in terms of $c_j = \Delta_{\mathbf{n}}\,|\mathbf{n}|^{-j}$ and $\Omega_{i,j}(z = q_0^2/m^2)$

$$\Omega_{i,j}(z) = \int dx\, x^2 \frac{1}{(1+x^2)^{i/2}(z+4(x^2+1))^j}$$

Finite volume corrections

$$\Delta\hat{\Pi}(q^2) = \frac{c_0}{m_\pi^3 L^3}\left(-\frac{16}{3}\Omega_{0,3} - \frac{5}{3}\Omega_{2,2} + \frac{40}{9}\Omega_{2,3} - \frac{3}{8}\Omega_{4,1} + \frac{7}{6}\Omega_{4,2} + \frac{8}{9}\Omega_{4,3}\right) + \mathcal{O}\left(\frac{1}{L^4}, e^{-m_\pi L}\right)$$

- Suppression: Leading order is $1/L^3$
- Physics: Neutral current and photon far away sees no charge – dipole effect
- Universality: adding form factors yields same cancellation

Numerical validation

- Want to validate the results with numerical methods
- Action $S[\phi, A] = S_\phi[\phi, A] + S_A[A]$

$$S_\phi\left[\phi, A\right] = \frac{a^4}{2}\sum_x \left[\sum_\mu \left|D_\mu \phi(x)\right|^2 + m_0^2\left|\phi(x)\right|^2\right] = \frac{a^4}{2}\sum_x \phi^*(x)\Delta\phi(x),$$

$$S_A[A] = \frac{a^4}{2}\sum_{x,\mu}\left[\sum_\nu \frac{1}{2}F_{\mu\nu}(x)^2 + [\delta_\mu A_\mu(x)]^2\right] = -\frac{a^4}{2}\sum_{x,\mu} A_\mu(x)\delta^2 A_\mu(x)\,,$$

$$\Delta = m^2 - \sum_\mu D_\mu^* D_\mu$$

- Integrate out the scalar field

Numerical validation

- In QED_L

$$\langle O \rangle = \frac{1}{\mathcal{Z}_\mathrm{L}} \int \mathcal{D}A \, O_\mathrm{Wick}[\Delta^{-1}] \, e^{-S_{\mathrm{L},A}[A]}$$

- Inverse propagator expanded in electromagnetic coupling

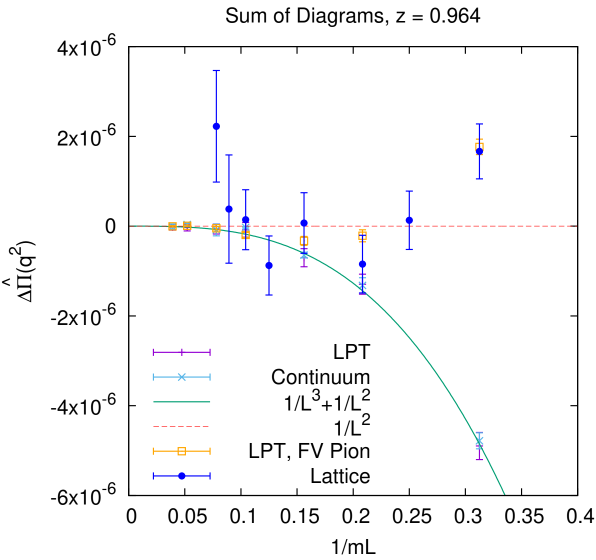
$$\Delta^{-1} = \Delta_0^{-1} - q\,\Delta_0^{-1}\Delta_1\Delta_0^{-1} + q^2\Delta_0^{-1}\Delta_1\Delta_0^{-1}\Delta_1\Delta_0^{-1} - q^2\Delta_0^{-1}\Delta_2\Delta_0^{-1} + O(q^3)$$

- $\Pi_{\mu\nu}$ is discrete Fourier transform of $C_{\mu\nu}(x) \equiv \langle V_\mu(x)\, V_\nu(0) \rangle$

Numerical validation

- Easy to obtain lattice perturbation theory (LPT) Feynman rules from the action as well
- LPT: `Cuba vegas` Monte Carlo integration for both FV and IV pions
- We do this for several lattice spacings and then make a continuum extrapolation

Numerical validation

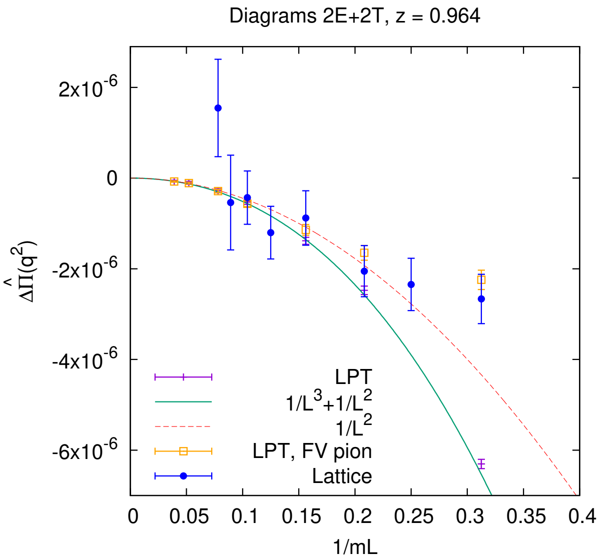


Conclusions

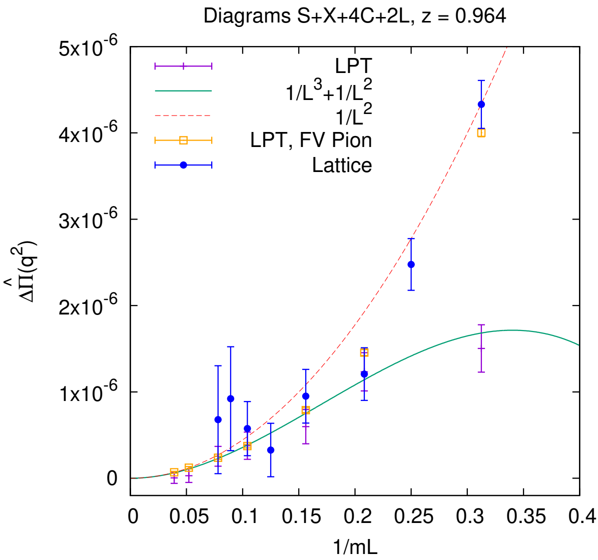
- Finite volume corrections smaller than anticipated: $c_0/(m_\pi L)^3$
- For $mL \gtrsim 4$ the effects are very small 10^{-6} (cf. LO HVP $10^{-3} \sim 10^{-5}/\alpha$)
- FV effects probably negligible in foreseeable future

Back-up slides

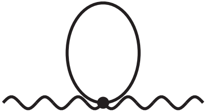
Numerical validation



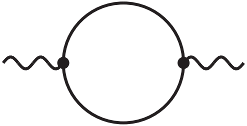
Numerical validation



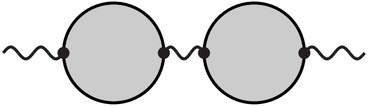
LO and NLO disconnected diagrams



(F)



(G)



(D)

Charged currents

$$
\begin{aligned}
\Delta\hat{\Pi}_{\text{charged}}\left(q^2\right) = & \frac{1}{m^3L^3}\left(-\frac{13}{24}\Omega_{2,2}+\frac{20}{9}\Omega_{2,3}-\frac{15}{64}\Omega_{4,1}+\frac{7}{24}\Omega_{4,2}+\frac{4}{9}\Omega_{4,3}\right)\\
& +\frac{c_1}{m^2L^2\pi}\left(-\frac{8}{3}\Omega_{-1,3}+\Omega_{1,2}+\frac{8}{3}\Omega_{1,3}+\frac{1}{8}\Omega_{3,1}\right)\\
& +\mathcal{O}\left(\frac{1}{L^4},e^{-mL}\right)
\end{aligned}
$$

Lattice numbers

- Time extent $T = 128$
- Eight different volumes with L between 16 and 64
- $am_0 = 0.2$
- $q^2 = 0.964\, m_0^2$
- 40 000 photon field configurations for $L \leq 48$, 10 000 for $L = 56$ and $L = 64$