The hadronic contribution to $\Delta \alpha_{QED}$

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Wuhan, 17/06/2019
Most of the uncertainty of $\alpha(\omega^2)$ comes from non-perturbative QCD. An improvement of the precision is needed, for example, for:

- Future experiments around $M_Z$ pole energy (future $e^+e^-$ collider).
- Precision tests of the Standard Model, like:
  - $(g - 2)_\mu$ (talk by Antoine Gérardin, tomorrow at 14:40)
  - $\sin^2(\theta_W)$ (explained in the previous talk by Marco Cè)
  - consistency checks of the SM, e.g. $\alpha_{QED}(M_Z^2), \sin^2(\theta_W), m_{Higgs}$.  

Introduction

We study the running of the QED coupling,

\[ \alpha(Q^2) = \frac{\alpha}{1 - \Delta\alpha(Q^2)} \]

In particular, its hadronic contributions at low energies. It is computed as

\[ \Delta\alpha_{QED}^{\text{had}}(\omega^2) = 4\pi\alpha \left( \Pi(\omega^2) - \Pi(0) \right) \]

where the substracted vacuum polarization function is defined as

\[ \Pi(\omega^2) - \Pi(0) = \int_0^\infty dt G(t)K(\omega, t) \]

\[ K(\omega, t) = \frac{1}{\omega^2} \left( \omega^2 t^2 - 4\sin^2 \left( \frac{\omega t}{2} \right) \right) \]
The hadronic contribution resides in the two-point function

\[ G(t)\delta_{kl} = -\int d^3x \langle J_k(t, x)J_l(0) \rangle \]

The electromagnetic current, \( J \) is defined as

\[ J_i = \frac{2}{3} \bar{u} \gamma_i u - \frac{1}{3} d \gamma_i d - \frac{1}{3} \bar{s} \gamma_i s + \frac{2}{3} \bar{c} \gamma_i c \]

\( J \) can be decomposed in the isospin basis, so we can divide \( G(t) \) in two components,

\[ G(t) = G^{I=1}(t) + G^{I=0}(t) \]
Lattice setup

- For the gauge fields the Lüscher-Weisz action with tree level coefficients is used.

- Almost all ensembles have open boundary conditions.

- For the fermionic part of the action, we use the Wilson-Dirac operator with $O(a)$ improvement.

- Our ensembles are $N_f = 2 + 1$, with quenched charm quark.

- The ensembles are generated at

$$trace \ M = const$$
We use two common discretizations,

- Local current,
  \[ V^I_\mu(x) = \bar{q}(x) \gamma_\mu q(x) \]

- Conserved current,
  \[
  V^c_\mu(x) = \frac{1}{2} \left( \bar{q}(x + a\hat{\mu}) (1 + \gamma_\mu) U^\dagger_\mu(x) q(x) \right. \\
  \left. - \bar{q}(x) (1 - \gamma_\mu) U_\mu(x) q(x + a\hat{\mu}) \right)
  \]

with \( q = u, d, s, c \).
Improvement and renormalization

In this work we combine the $u$, $d$, $s$ flavors in the isospin basis,

$$V^a_\mu = \bar{\psi} \gamma_\mu \frac{\lambda^a}{2} \psi,$$

with $\lambda^a$ the Gell-Mann matrices, and corresponding expression for the conserved current. The $O(a)$ improved isovector vector correlator for the local-conserved discretization is

$$V^{3, l\dagger}_{\mu, l} V^{3, c}_{\mu, l} = V^{3, l\dagger}_{\mu} V^{3, c}_{\mu} +$$

$$ac^3_{\nu} V^{3, l\dagger}_{\mu} \tilde{\nabla}_\nu \Sigma^{3, l}_{\mu \nu} + ac^3_{\nu} \tilde{\nabla}_\nu \Sigma^{3, l}_{\mu \nu} V^{3, c\dagger}_{\nu} + O(a^2)$$

The tensor current is

$$\Sigma^a_{\mu \nu} = -\bar{\psi} \left[ \gamma_\mu, \gamma_\nu \right] \frac{\lambda^a}{2} \psi$$

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\(^2\)Gérardin, Harris, and Meyer 2019.
Improvement and renormalization

The renormalization\(^3\) of the isovector local correlator is

\[
V_{\mu,R}^3 = Z_V \left( 1 + 3\bar{b}_V a m_q^{a\nu} + b_V a m_{q,l} \right) V_{\mu}^{3,l}
\]

And the renormalization of the isoscalar local contribution has a mixing with the flavor-singlet current,

\[
V_{\mu,R}^8 = Z_V \left[ \left( 1 + 3\bar{b}_V a m_q^{a\nu} + \frac{b_V}{3} a (m_{q,l} + 2m_{q,s}) \right) V_{\mu}^{8,l} + \left( \frac{b_V}{3} + f_V \right) \frac{2}{\sqrt{3}} a(m_{q,l} - m_{q,s}) V_{\mu}^{0,l} \right]
\]

with \( V_{\mu}^0 = \frac{1}{2} \bar{\psi} \gamma_{\mu} \psi \).

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\(^3\)Gérardin, Harris, and Meyer 2019.
Set of $N_f = 2 + 1$ CLS ensembles$^4$

<table>
<thead>
<tr>
<th>id</th>
<th>$\beta$</th>
<th>$L^3 \times T$</th>
<th>$a$ (fm)</th>
<th>$m_\pi$ (MeV)</th>
<th>$m_\pi L$</th>
<th>L (fm)</th>
<th>conf.</th>
</tr>
</thead>
<tbody>
<tr>
<td>H101</td>
<td>3.40</td>
<td>$32^3 \times 96$</td>
<td>0.08636</td>
<td>416(5)</td>
<td>5.8</td>
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<td>2000</td>
</tr>
<tr>
<td>H102</td>
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<td>354(5)</td>
<td>5.0</td>
<td>2.8</td>
<td>1900</td>
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<tr>
<td>H105*</td>
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<tr>
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<td>4.1</td>
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<tr>
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<tr>
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<td>5.2</td>
<td>2.4</td>
<td>1600</td>
</tr>
<tr>
<td>S400</td>
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<td>$32^3 \times 128$</td>
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<td>351(4)</td>
<td>4.3</td>
<td>2.4</td>
<td>2800</td>
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<td>N401</td>
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<td>287(4)</td>
<td>5.3</td>
<td>3.7</td>
<td>1100</td>
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<tr>
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<td>410(5)</td>
<td>6.4</td>
<td>3.1</td>
<td>900</td>
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<tr>
<td>N203</td>
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<td></td>
<td>345(4)</td>
<td>5.4</td>
<td>3.1</td>
<td>1500</td>
</tr>
<tr>
<td>N200</td>
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<td>$48^3 \times 128$</td>
<td></td>
<td>282(3)</td>
<td>4.4</td>
<td>3.1</td>
<td>1700</td>
</tr>
<tr>
<td>D200</td>
<td></td>
<td>$64^3 \times 128$</td>
<td></td>
<td>200(2)</td>
<td>4.2</td>
<td>4.1</td>
<td>1900</td>
</tr>
<tr>
<td>E250</td>
<td></td>
<td>$96^3 \times 192$</td>
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<td>130(1)</td>
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<td>6.2</td>
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</tr>
<tr>
<td>N300</td>
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<td>$48^3 \times 128$</td>
<td>0.04981</td>
<td>421(4)</td>
<td>5.1</td>
<td>2.4</td>
<td>1700</td>
</tr>
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<td>N302</td>
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<td>346(4)</td>
<td>4.2</td>
<td>2.4</td>
<td>2200</td>
</tr>
<tr>
<td>J303</td>
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<td>$64^3 \times 192$</td>
<td></td>
<td>257(3)</td>
<td>4.2</td>
<td>3.2</td>
<td>600</td>
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</tbody>
</table>

$^4$Gérardin et al. 2019.
We use the \textbf{Γ-method}\textsuperscript{5, 6} of Ulli Wolff to estimate autocorrelations. The basic object that we want to compute is the autocorrelation function of the correlator,

\[
\Gamma_{\alpha\beta}(d) = \frac{1}{N - d} \sum_{i=1}^{N-d} \left( a^i_{\alpha} - \bar{a}_{\alpha} \right) \left( a^{i+d}_{\beta} - \bar{a}_{\beta} \right)
\]

where \(\alpha, \beta\) run over the time indices and \(i\) runs over the configurations.

For \(d = 0\) we recover the usual covariance matrix.

\textsuperscript{5}Wolff 2004.
\textsuperscript{6}De Palma et al. 2019.
Autocorrelations

The autocorrelation function of a derived quantity, $F$, is

$$
\Gamma_F(d) = \sum_{\alpha,\beta} f_\alpha f_\beta \Gamma_{\alpha\beta}(d) \quad f_\alpha = \frac{\partial F}{\partial a_\alpha}
$$

The next step is to select a window, $W$, such that

$$
C_F(W) = \Gamma_F(0) + 2 \sum_{d=1}^{W} \Gamma_F(d)
$$

accounts for autocorrelations.

- $W$ too large: statistical noise enters.
- $W$ too small: autocorrelation estimated incorrectly.
Autocorrelations

After choosing the window $W$, define the integrated autocorrelation time of $F$, $\tau_{int,F}$ as

$$\tau_{int,F}(W) = \frac{C_F(W)}{2\Gamma_F(0)}$$

which is related with the statistical error,

$$\sigma_F^2 = \frac{2\tau_{int,F}}{N} \Gamma_F(0)$$

Therefore, $N/(2\tau_{int,F})$ is the number of configurations with the true error, and $2\tau_{int,F}$ acts as a bin size.
ensemble B450. Light flavor. \( \rho = \Gamma_F(W)/\Gamma_F(0) \). \( Q^2 = 4.5 \text{ GeV}^2 \)
We have performed an autocorrelation analysis for all ensembles, both discretizations and flavors light and strange.

<table>
<thead>
<tr>
<th>correlator</th>
<th>cut (fm)</th>
<th>light $\tau_{int}$</th>
<th>strange $\tau_{int}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>VV</td>
<td>1.0</td>
<td>0.8(0.1)</td>
<td>1.0(0.1)</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td>1.4(0.2)</td>
<td>1.7(0.3)</td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>1.3(0.2)</td>
<td>2.1(0.4)</td>
</tr>
<tr>
<td></td>
<td>2.5</td>
<td>1.2(0.2)</td>
<td>2.0(0.4)</td>
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<td></td>
<td>None</td>
<td>1.1(0.2)</td>
<td>1.6(0.3)</td>
</tr>
<tr>
<td>VVc</td>
<td>1.0</td>
<td>1.5(0.3)</td>
<td>1.6(0.3)</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td>1.8(0.4)</td>
<td>2.3(0.5)</td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>1.6(0.3)</td>
<td>2.5(0.5)</td>
</tr>
<tr>
<td></td>
<td>2.5</td>
<td>1.4(0.2)</td>
<td>2.4(0.5)</td>
</tr>
<tr>
<td></td>
<td>None</td>
<td>1.2(0.2)</td>
<td>1.8(0.3)</td>
</tr>
</tbody>
</table>

ensemble H102
Autocorrelations

However, most of our ensembles do not show autocorrelations.

<table>
<thead>
<tr>
<th>correlator</th>
<th>cut (fm)</th>
<th>light $\tau_{int}$</th>
<th>strange $\tau_{int}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>VV</td>
<td>1.0</td>
<td>0.5(0.1)</td>
<td>0.6(0.1)</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td>0.5(0.1)</td>
<td>0.5(0.0)</td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>0.5(0.1)</td>
<td>0.6(0.1)</td>
</tr>
<tr>
<td></td>
<td>2.5</td>
<td>0.5(0.1)</td>
<td>0.6(0.1)</td>
</tr>
<tr>
<td></td>
<td>None</td>
<td>0.7(0.3)</td>
<td>0.5(0.1)</td>
</tr>
<tr>
<td>VVc</td>
<td>1.0</td>
<td>0.5(0.1)</td>
<td>0.5(0.1)</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td>0.5(0.1)</td>
<td>0.6(0.1)</td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>0.5(0.1)</td>
<td>0.7(0.1)</td>
</tr>
<tr>
<td></td>
<td>2.5</td>
<td>0.5(0.1)</td>
<td>0.8(0.1)</td>
</tr>
<tr>
<td></td>
<td>None</td>
<td>0.5(0.2)</td>
<td>0.7(0.1)</td>
</tr>
</tbody>
</table>

ensemble N101
Signal/noise problem

ensemble J303, isovector correlator, local-conserved discretization
We have performed a fit of the tail of the correlator, $G(t)$, to the form

$$G(t) = \begin{cases} 
\text{data}, & x_0 < x_{\text{cut}} \\
Ae^{-m_\rho t}, & x_0 \geq x_{\text{cut}} 
\end{cases}$$

<table>
<thead>
<tr>
<th>ensemble</th>
<th>cut (fm)</th>
<th>$am_\rho$</th>
<th>ensemble</th>
<th>cut (fm)</th>
<th>$am_\rho$</th>
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<tr>
<td>H101</td>
<td>2.68</td>
<td>0.3770(16)</td>
<td>N202</td>
<td>1.92</td>
<td>0.2756(27)</td>
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<tr>
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<td>2.59</td>
<td>0.3623(21)</td>
<td>N200</td>
<td>2.24</td>
<td>0.2588(22)</td>
</tr>
<tr>
<td>H105</td>
<td>2.42</td>
<td>0.3475(22)</td>
<td>N203</td>
<td>2.38</td>
<td>0.2752(12)</td>
</tr>
<tr>
<td>N101</td>
<td>2.42</td>
<td>0.3385(29)</td>
<td>D200</td>
<td>2.31</td>
<td>0.2500(17)</td>
</tr>
<tr>
<td>C101</td>
<td>1.72</td>
<td>0.3283(17)</td>
<td>E250</td>
<td>1.93</td>
<td>0.2144(31)</td>
</tr>
<tr>
<td>B450</td>
<td>2.44</td>
<td>0.3399(10)</td>
<td>N300</td>
<td>2.49</td>
<td>0.2252(16)</td>
</tr>
<tr>
<td>S400</td>
<td>2.52</td>
<td>0.3147(26)</td>
<td>N302</td>
<td>1.74</td>
<td>0.2194(15)</td>
</tr>
<tr>
<td>N401</td>
<td>1.90</td>
<td>0.3101(15)</td>
<td>J303</td>
<td>1.74</td>
<td>0.1977(28)</td>
</tr>
</tbody>
</table>
Finite size effects

The energy levels in a box differ from their infinite volume counterparts.

To remedy this we compute the expected finite volume and infinite volume correlators, and take the difference as correction.

The infinite volume correlator at long distances can be computed as

\[ G(x_0) = \int_{2m_\pi}^{\infty} d\omega \omega^2 \rho(\omega) e^{-\omega|x_0|} \]

where the spectral function is dominated by the \( \pi \pi \) contribution,

\[ \rho(\omega) = \frac{1}{48\pi^2} \left( 1 - 4 \frac{m_\pi^2}{\omega^2} \right)^{3/2} |F_\pi(\omega)|^2 \]

\(^7\)Francis et al. 2013.
The correlator in finite volume,

\[ G(x_0, L) = \sum_n |A_n|^2 e^{-\omega_n x_0} \]

can be computed using the Lüscher method\textsuperscript{8, 9} at long distances. First, solve numerically the following equation to obtain \( \omega_n \).

\[ \delta_1(k) + \phi \left( \frac{kL}{2\pi} \right) = n\pi, \quad n = 1, 2, \ldots \]

\[ \omega_n = 2\sqrt{m^2_\pi + k^2} \]

where \( \phi \) is a known function.

\textsuperscript{8}Luscher 1991a.
\textsuperscript{9}Luscher 1991b.
Then, the amplitudes are computed\textsuperscript{10} as

\[ |A_n|^2 = \frac{2k^5|F_\pi(\omega_n)|^2}{3\pi \omega_n^2 \mathbb{I}(k)} \]

where the Lellouch-Lüscher factor \( \mathbb{I}(k) \) is known\textsuperscript{11},

\[ \mathbb{I}(k) = \frac{kL}{2\pi} \phi' \left( \frac{kL}{2\pi} \right) + k \frac{\partial \delta_1(k)}{\partial k} \]

Both, \( |F_\pi| \) and \( \delta_1 \) are parametrized using the Gounaris-Sakurai model.

\textsuperscript{10}Meyer 2011.

\textsuperscript{11}Lellouch and Luscher 2001.
The pion form factor $F_\pi$ with its phase shift, $\delta_1$ can be parametrized by the Gounaris-Sakurai model\(^\text{12}\), which only depends on two parameters, the $\rho$ meson mass, $m_\rho$ and its decay width, $\Gamma_\rho$.

\[
F_\pi(\omega) = \frac{f_0}{k^3 \left( \frac{\cot \delta_1(k) - i}{\omega} \right)}
\]

\[
k^3 \frac{\cot \delta_1(k)}{\omega} = k^2 h(\omega) - k^2 \rho h(m_\rho) + b(k^2 - k^2_\rho)
\]

where $f_0$, $b$ depend on $m_\rho$ and $\Gamma_\rho$. All of them, $f_0$, $b$ and $h$ have a closed form.

\(^\text{12}\) Gounaris and Sakurai 1968.
Finally, the correction at short times can be computed from a non-interacting pion model\textsuperscript{13}.

\[ G(x_0) - G(x_0, L) \overset{x_0 \geq 0}{=} -\frac{m_\pi^4 x_0}{3\pi^2} \sum_{n \neq 0} \left( \frac{K_2(m_\pi \sqrt{L^2 n^2 + 4t^2})}{m_\pi^2 \left( L^2 n^2 + 4t^2 \right)} \right) \]

\[ -\frac{1}{m_\pi L |n|} \int_1^\infty dy K_0(m_\pi y \sqrt{L^2 n^2 + 4t^2}) \sinh(m_\pi L |n| (y - 1)) \]

where \( K_n \) are modified Bessel functions of the second kind. The correction is positive for \( m_\pi L >> 1 \).

\textsuperscript{13}Della Morte et al. 2017.
For the ensemble J303, at $Q^2 = 0.5$ GeV$^2$. $K = \frac{1}{\omega^2} \left( \omega^2 t^2 - 4 \sin^2 \left( \frac{\omega t}{2} \right) \right)$
We compare two ensembles with the same parameters, except the volume. After FSE corrections both agree.

\[ G^I = K, \quad Q^2 = 0.5 \text{ GeV}^2 \]

H105, \( m_L = 2.8 \)
N101, \( m_L = 4.1 \)
H105, \( m_L = 2.8 + \text{Long FSE} \)
N101, \( m_L = 4.1 + \text{Long FSE} \)
$G^I = 1 K, \; \; Q^2 = 0.5 \text{ GeV}^2$

The hadronic contribution to $\Delta \alpha_{QED}$
At $Q^2 = 0.5$ GeV$^2$

<table>
<thead>
<tr>
<th>id</th>
<th>$\Delta \alpha \times 10^6$</th>
<th>FSE $\times 10^6$</th>
<th>id</th>
<th>$\Delta \alpha \times 10^6$</th>
<th>FSE $\times 10^6$</th>
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</thead>
<tbody>
<tr>
<td>H101</td>
<td>1688.5(5.7)</td>
<td>6.3(0.1)</td>
<td>N202</td>
<td>1642.1(7.2)</td>
<td>2.8(0.1)</td>
</tr>
<tr>
<td>H102</td>
<td>1800.3(7.3)</td>
<td>15.7(0.5)</td>
<td>N203</td>
<td>1743.3(6.9)</td>
<td>8.8(0.3)</td>
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<td>H105</td>
<td>1948.9(9.7)</td>
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<td>N200</td>
<td>1861.8(7.8)</td>
<td>24.6(0.8)</td>
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<td>N101</td>
<td>1954.0(8.8)</td>
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<td>D200</td>
<td>2039.4(10.0)</td>
<td>28.7(0.9)</td>
</tr>
<tr>
<td>C101</td>
<td>2100.0(8.5)</td>
<td>17.5(0.9)</td>
<td>E250</td>
<td>2276.5(16.1)</td>
<td>30.0(1.1)</td>
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<td>B450</td>
<td>1640.0(6.5)</td>
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<td>N300</td>
<td>1548.8(7.3)</td>
<td>14.4(2.0)</td>
</tr>
<tr>
<td>S400</td>
<td>1765.9(8.2)</td>
<td>33.0(1.4)</td>
<td>N302</td>
<td>1653.6(7.1)</td>
<td>38.9(1.0)</td>
</tr>
<tr>
<td>N401</td>
<td>1910.7(8.1)</td>
<td>9.4(0.3)</td>
<td>J303</td>
<td>1869.4(11.8)</td>
<td>31.8(0.7)</td>
</tr>
</tbody>
</table>
The next step is to extrapolate the results of $\Delta \alpha_{QED}$ to the physical point for each energy separately. For the isovector component we have used several models,

$$\Delta \alpha(a, \phi_2, d) = \Delta \alpha(0, \phi_2, i) + \delta_d a^2 + \gamma_1 (\phi_2 - \phi_2, i) + \gamma_2 (\log \phi_2 - \log \phi_2, i)$$

$$\Delta \alpha(a, \phi_2, d) = \Delta \alpha(0, \phi_2, i) + \delta_d a^2 + \gamma_3 (\phi_2 - \phi_2, i) + \gamma_4 (\phi_2^2 - \phi_2^2, i)$$

$$\Delta \alpha(a, \phi_2, d) = \Delta \alpha(0, \phi_2, i) + \delta_d a^2 + \gamma_5 (\phi_2 - \phi_2, i) + \gamma_6 (1/\phi_2 - 1/\phi_2, i)$$

$$\Delta \alpha(a, \phi_2, d) = \Delta \alpha(0, \phi_2, i) + \delta_d a^2 + \gamma_7 (\phi_2 - \phi_2, i) + \gamma_8 (\phi_2 \log \phi_2 - \phi_2, i \log \phi_2, i)$$

where $\phi_2 = 8t_0 m^2_\pi$, $\phi_2, i$ uses the flavour-symmetric, neutral pion mass, and $d$ indicates the discretization.
We have performed a simultaneous extrapolation of both discretizations to the physical point.

We have included the uncertainty of the pion mass in the computation enlarging the covariance matrix. In our case, the $\chi^2 = V^t C^{-1} V$ function is formed by

$$V = \begin{pmatrix}
\vdots \\
\phi_{2,\text{param}} - \phi_{2,e_i} \\
\Delta \alpha_{\text{param}}^{\|} - \Delta \alpha_{e_i}^{\|} \\
\Delta \alpha_{\text{param}}^{lc} - \Delta \alpha_{e_i}^{lc} \\
\vdots 
\end{pmatrix} \text{ set of ensembles}$$

where all entries are dimensionless.
Preliminary results. Isovector extrapolation

For $Q^2 = 0.5 \text{ GeV}^2$, model with $\log 8t_0m^2_\pi$ term.

\[ \Delta \alpha_{\text{QED}} = 0.002140 (13) \]

The hadronic contribution to $\Delta \alpha_{\text{QED}}$ is given by:

- Continuum
- $0.06426\text{ fm}$
- $0.07634\text{ fm}$
- $0.08636\text{ fm}$

\[ \chi^2 = 3.33 \]
Preliminary results. Isovector component

\[ \Delta \alpha^I_{QED, \text{had}} \]

With \( \phi_2 = 8 t_0 m^2_\pi \)
Summary and outlook

We have computed the isovector component of $\Delta \alpha_{QED}$ in the energy range $0.5 - 4.5 \text{ GeV}^2$, taking into account statistical and systematic uncertainties.

The next steps that we want to take are:

- Investigate the systematics of the chiral and continuum extrapolation.
- Analyze the isoscalar (including disconnected) and charm contributions.
- Perform a comparison with Phenomenology of $\Delta \alpha_{QED}$ including the four lightest flavors.
How to choose $W$? Consider the two sources of uncertainty in the autocorrelations:

- Finite statistics, $\propto \sqrt{\frac{W}{N}}$
- Truncation of $\Gamma_F$, $\propto \exp(-W/\tau)$

$\tau$ dictates the decay of autocorrelations. Suppose $\tau = S\tau_{int}$, with $S$ set by the user (reasonable ansatz $\sim 1 - 3$).

$W_{optimal}$ minimizes the total error. After $W$ is determined, we need to check that we have included just the right amount of autocorrelations.
Chiral and continuum extrapolation with other models

For $Q^2 = 0.5$ GeV$^2$, model with $\log \phi_2$ term.

![Graph showing the hadronic contribution to $\Delta \alpha_{QED}$ with various models and their fits.]
Chiral and continuum extrapolation with other models

For $Q^2 = 0.5 \text{ GeV}^2$, model with $\phi^2$ term.
Chiral and continuum extrapolation with other models

For $Q^2 = 0.5 \text{ GeV}^2$, model with $1/\phi_2$ term.
Chiral and continuum extrapolation with other models

For $Q^2 = 0.5 \text{ GeV}^2$, model with $\phi_2 \log \phi_2$ term.

The hadronic contribution to $\Delta \alpha_{\text{QED}}$