The hadronic contribution to $\Delta\alpha_{QED}$

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Introduction. Motivation

Most of the uncertainty of $\alpha(\omega^2)$ comes from non-perturbative QCD. An improvement of the precision is needed, for example, for:

- Future experiments around M_Z pole energy (future e^+e^-- collider).
- Precision tests of the Standard Model, like:
  - $(g-2)_\mu$ (talk by Antoine Gérardin, tomorrow at 14:40)
  - $sin^2(\theta_W)$ (explained in the previous talk by Marco Cè)
  - consistency cecks of the SM, e.g. $\alpha_{QED}(M_Z^2)$, $sin^2(\theta_W)$, m_{Higgs}.

Introduction

We study the running of the QED coupling,

$$\alpha(Q^2) = \frac{\alpha}{1 - \Delta\alpha(Q^2)}$$

In particular, its hadronic contributions at low energies. It is computed as

$$\Delta\alpha_{QED}^{had}(\omega^2) = 4\pi\alpha\left(\Pi(\omega^2) - \Pi(0)\right)$$

where the substracted vacuum polarization function is defined as

$$\Pi(\omega^2) - \Pi(0) = \int_0^\infty dtG(t)K(\omega, t)$$

$$K(\omega, t) = \frac{1}{\omega^2}\left(\omega^2 t^2 - 4sin^2\left(\frac{\omega t}{2}\right)\right)$$

The hadronic contribution resides in the two-point function

$$G(t)\delta_{kl} = -\int d^3x \, \langle J_k(t, \boldsymbol{x}) J_l(0) \rangle$$

The electromagnetic current, J is defined as

$$J_i = \frac{2}{3}\bar{u}\gamma_i u - \frac{1}{3}\bar{d}\gamma_i d - \frac{1}{3}\bar{s}\gamma_i s + \frac{2}{3}\bar{c}\gamma_i c$$

J can be decomposed in the isospin basis, so we can divide $G(t)$ in two components,

$$G(t) = G^{I=1}(t) + G^{I=0}(t)$$

Lattice setup[1]

- For the gauge fields the Lüscher-Weisz action with tree level coefficients is used.
- Almost all ensembles have open boundary conditions.
- For the fermionic part of the action, we use the Wilson-Dirac operator with $O(a)$ improvement.
- Our ensembles are $N_f = 2 + 1$, with quenched charm quark.
- The ensembles are generated at

$$trace\ M = const$$

[1] Bruno et al. 2015.

We use two common discretizations,

- Local current,

$$V_\mu^l(x) = \bar{q}(x)\gamma_\mu q(x)$$

- Conserved current,

$$V_\mu^c(x) = \frac{1}{2}\left(\bar{q}(x+a\hat{\mu})\left(1+\gamma_\mu\right)U_\mu^\dagger(x)q(x)\right. \\ \left. -\bar{q}(x)\left(1-\gamma_\mu\right)U_\mu(x)q(x+a\hat{\mu})\right)$$

with $q = u, d, s, c$.

Improvement and renormalization

In this work we combine the u, d, s flavors in the isospin basis,

$$V_\mu^a = \bar{\psi}\gamma_\mu \frac{\lambda^a}{2}\psi,$$

with λ^a the Gell-Mann matrices, and corresponding expression for the conserved current. The $O(a)$ improved isovector vector correlator for the local-conserved discretization is[2]

$$\begin{aligned} V_{\mu,I}^{3,l\dagger} V_{\mu,I}^{3,c} =& V_\mu^{3,l\dagger} V_\mu^{3,c} + \\ & a c_v^{3,c} V_\mu^{3,l\dagger} \tilde{\nabla}_\nu \Sigma_{\mu\nu}^{3,l} + a c_v^{3,l} \tilde{\nabla}_\nu \Sigma_{\mu\nu}^{3,l} V_\nu^{3,c\dagger} + O(a^2) \end{aligned}$$

The tensor current is

$$\Sigma_{\mu\nu}^a = -\bar{\psi}\left[\gamma_\mu, \gamma_\nu\right] \frac{\lambda^a}{2}\psi$$

[2] Gérardin, Harris, and Meyer 2019.

Improvement and renormalization

The renormalization[3] of the isovector local correlator is

$$V^3_{\mu,R} = Z_V \left(1 + 3\bar{b}_V a m_q^{av} + b_V a m_{q,l}\right) V_\mu^{3,l}$$

And the renormalization of the isoscalar local contribution has a mixing with the flavor-singlet current,

$$V^8_{\mu,R} = Z_V \Bigg[\left(1 + 3\bar{b}_V a m_q^a v + \frac{b_V}{3} a \left(m_{q,l} + 2m_{q,s}\right)\right) V_\mu^{8,l} + \left(\frac{b_V}{3} + f_V\right) \frac{2}{\sqrt{3}} a(m_{q,l} - m_{q,s}) V_\mu^{0,l} \Bigg]$$

with $V^0_\mu = \frac{1}{2}\bar{\psi}\gamma_\mu\psi$.

[3] Gérardin, Harris, and Meyer 2019.

Set of $N_f = 2 + 1$ CLS ensembles[4]

id	β	$L^3 \times T$	a (fm)	$m_\pi (MeV)$	$m_\pi L$	L (fm)	conf.
H101	3.40	$32^3 \times 96$	0.08636	416(5)	5.8	2.8	2000
H102		$32^3 \times 96$		354(5)	5.0	2.8	1900
H105*		$32^3 \times 96$		284(4)	3.9	2.8	2800
N101		$48^3 \times 128$		282(4)	5.9	4.1	1500
C101		$48^3 \times 96$		221(2)	4.7	4.1	2600
B450	3.46	$32^3 \times 64$	0.07634	416(4)	5.2	2.4	1600
S400		$32^3 \times 128$		351(4)	4.3	2.4	2800
N401		$48^3 \times 128$		287(4)	5.3	3.7	1100
N202	3.55	$48^3 \times 128$	0.06426	410(5)	6.4	3.1	900
N203		$48^3 \times 128$		345(4)	5.4	3.1	1500
N200		$48^3 \times 128$		282(3)	4.4	3.1	1700
D200		$64^3 \times 128$		200(2)	4.2	4.1	1900
E250		$96^3 \times 192$		130(1)	4.1	6.2	500
N300	3.70	$48^3 \times 128$	0.04981	421(4)	5.1	2.4	1700
N302		$48^3 \times 128$		346(4)	4.2	2.4	2200
J303		$64^3 \times 192$		257(3)	4.2	3.2	600

[4] Gérardin et al. 2019.

Autocorrelations

We use the **Γ-method**[5, 6] of Ulli Wolff to estimate autocorrelations. The basic object that we want to compute is the autocorrelation function of the correlator,

$$\Gamma_{\alpha\beta}(d) = \frac{1}{N-d} \sum_{i=1}^{N-d} \left(a_\alpha^i - \bar{a}_\alpha \right) \left(a_\beta^{i+d} - \bar{a}_\beta \right)$$

where α, β run over the time indices and i runs over the configurations.

For $d = 0$ we recover the usual covariance matrix.

[5] Wolff 2004.

[6] De Palma et al. 2019.

Autocorrelations

The autocorrelation function of a derived quantity, F, is

$$\Gamma_F(d) = \sum_{\alpha,\beta} f_\alpha f_\beta \Gamma_{\alpha\beta}(d) \qquad f_\alpha = \frac{\partial F}{\partial a_\alpha}$$

The next step is to select a window, W, such that

$$C_F(W) = \Gamma_F(0) + 2\sum_{d=1}^{W} \Gamma_F(d)$$

accounts for autocorrelations.

- W too large: statistical noise enters.
- W too small: autocorrelation estimated incorrectly.

Autocorrelations

After choosing the window W, define the integrated autocorrelation time of F, $\tau_{int,F}$ as

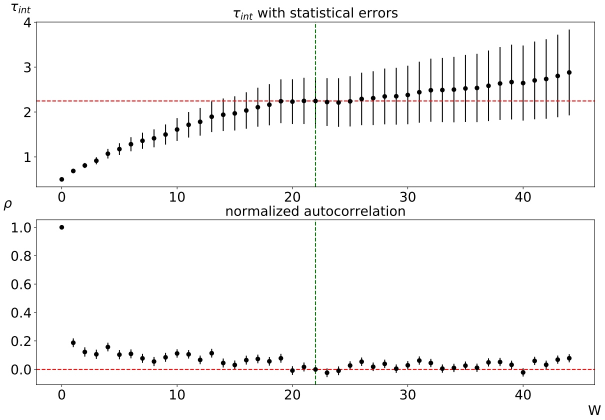
$$\tau_{int,F}(W) = \frac{C_F(W)}{2\Gamma_F(0)}$$

which is related with the statistical error,

$$\sigma_F^2 = \frac{2\tau_{int,F}}{N}\Gamma_F(0)$$

Therefore, $N/(2\tau_{int,F})$ is the number of configurations with the true error, and $2\tau_{int,F}$ acts as a bin size.

Autocorrelations



ensemble B450. Light flavor. $\rho = \Gamma_F(W)/\Gamma_F(0)$. $Q^2 = 4.5$ GeV^2

Autocorrelations

We have performed an autocorrelation analysis for all ensembles, both discretizations and flavors light and strange.

		light	strange
correlator	cut (fm)	τ_{int}	τ_{int}
VV	1.0	0.8(0.1)	1.0(0.1)
	1.5	1.4(0.2)	1.7(0.3)
	2.0	1.3(0.2)	2.1(0.4)
	2.5	1.2(0.2)	2.0(0.4)
	None	1.1(0.2)	1.6(0.3)
VVc	1.0	1.5(0.3)	1.6(0.3)
	1.5	1.8(0.4)	2.3(0.5)
	2.0	1.6(0.3)	2.5(0.5)
	2.5	1.4(0.2)	2.4(0.5)
	None	1.2(0.2)	1.8(0.3)

ensemble H102

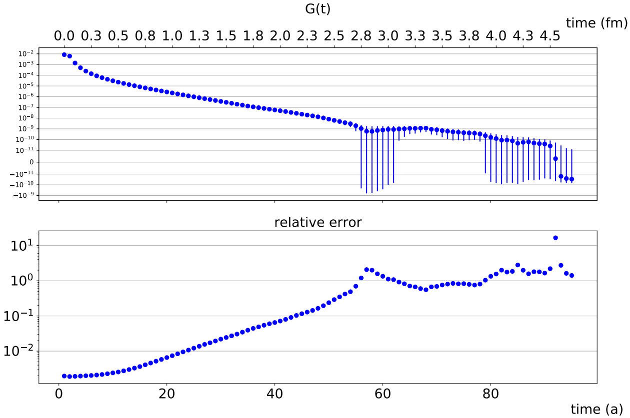
Autocorrelations

However, most of our ensembles do not show autocorrelations.

		light	strange
correlator	cut (fm)	τ_{int}	τ_{int}
VV	1.0	0.5(0.1)	0.6(0.1)
	1.5	0.5(0.1)	0.5(0.0)
	2.0	0.5(0.1)	0.6(0.1)
	2.5	0.5(0.1)	0.6(0.1)
	None	0.7(0.3)	0.5(0.1)
VVc	1.0	0.5(0.1)	0.5(0.1)
	1.5	0.5(0.1)	0.6(0.1)
	2.0	0.5(0.1)	0.7(0.1)
	2.5	0.5(0.1)	0.8(0.1)
	None	0.5(0.2)	0.7(0.1)

ensemble N101

Signal/noise problem



ensemble J303, isovector correlator, local-conserved discretization

Signal/noise problem

We have performed a fit of the tail of the correlator, $G(t)$, to the form

$$G(t) = \begin{cases} \text{data}, & x_0 < x_{\text{cut}} \\ Ae^{-m_\rho t}, & x_0 \geq x_{\text{cut}} \end{cases}$$

ensemble	cut (fm)	am_ρ	ensemble	cut (fm)	am_ρ
H101	2.68	0.3770(16)	N202	1.92	0.2756(27)
H102	2.59	0.3623(21)	N200	2.24	0.2588(22)
H105	2.42	0.3475(22)	N203	2.38	0.2752(12)
N101	2.42	0.3385(29)	D200	2.31	0.2500(17)
C101	1.72	0.3283(17)	E250	1.93	0.2144(31)
B450	2.44	0.3399(10)	N300	2.49	0.2252(16)
S400	2.52	0.3147(26)	N302	1.74	0.2194(15)
N401	1.90	0.3101(15)	J303	1.74	0.1977(28)

Finite size effects

The energy levels in a box differ from their infinite volume counterparts.

To remedy this we compute the expected finite volume and infinite volume correlators, and take the difference as correction.

The **infinite volume correlator** at long distances can be computed[7] as

$$G(x_0) = \int_{2m_\pi}^{\infty} d\omega \omega^2 \rho(\omega) e^{-\omega|x_0|}$$

where the spectral function is dominated by the $\pi\pi$ contribution,

$$\rho(\omega) = \frac{1}{48\pi^2} \left(1 - 4\frac{m_\pi^2}{\omega^2}\right)^{3/2} |F_\pi(\omega)|^2$$

[7] Francis et al. 2013.

FSE. Finite Volume

The correlator in finite volume,

$$G(x_0, L) = \sum_n |A_n|^2 e^{-\omega_n x_0}$$

can be computed using the Lüscher method[8, 9] at long distances. First, solve numerically the following equation to obtain ω_n.

$$\delta_1(k) + \phi\left(\frac{kL}{2\pi}\right) = n\pi, \quad n = 1, 2, ...$$

$$\omega_n = 2\sqrt{m_\pi^2 + k^2}$$

where ϕ is a known function.

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[8] Luscher 1991a.

[9] Luscher 1991b.

FSE. Finite Volume

Then, the amplitudes are computed[10] as

$$|A_n|^2 = \frac{2k^5 |F_\pi(\omega_n)|^2}{3\pi\omega_n^2 \mathbb{L}(k)}$$

where the Lellouch-Lüscher factor $\mathbb{L}(k)$ is known[11],

$$\mathbb{L}(k) = \frac{kL}{2\pi}\phi'\left(\frac{kL}{2\pi}\right) + k\frac{\partial\delta_1(k)}{\partial k}$$

Both, $|F_\pi|$ and δ_1 are parametrized using the Gounaris-Sakurai model.

[10] Meyer 2011.

[11] Lellouch and Luscher 2001.

FSE. Gounaris-Sakurai model

The pion form factor F_π with its phase shift, δ_1 can be parametrized by the Gounaris-Sakurai model[12], which only depends on two parameters, the ρ meson mass, m_ρ and its decay width, Γ_ρ.

$$F_\pi(\omega) = \frac{f_0}{\frac{k^3}{\omega}(cot\delta_1(k) - i)}$$

$$\frac{k^3}{\omega}cot\delta_1(k) = k^2 h(\omega) - k_\rho^2 h(m_\rho) + b(k^2 - k_\rho^2)$$

where f_0, b depend on m_ρ and Γ_ρ. All of them, f_0, b and h have a closed form.

[12] Gounaris and Sakurai 1968.

FSE. Short time correction

Finally, the correction at short times can be computed from a non-interacting pion model[13].

$$G(x_0) - G(x_0, L) \overset{x_0 \geq 0}{=}$$

$$- \frac{m_\pi^4 x_0}{3\pi^2} \sum_{\boldsymbol{n} \neq 0} \left(\frac{K_2(m_\pi \sqrt{L^2 \boldsymbol{n}^2 + 4t^2})}{m_\pi^2 \left(L^2 \boldsymbol{n}^2 + 4t^2 \right)} \right.$$

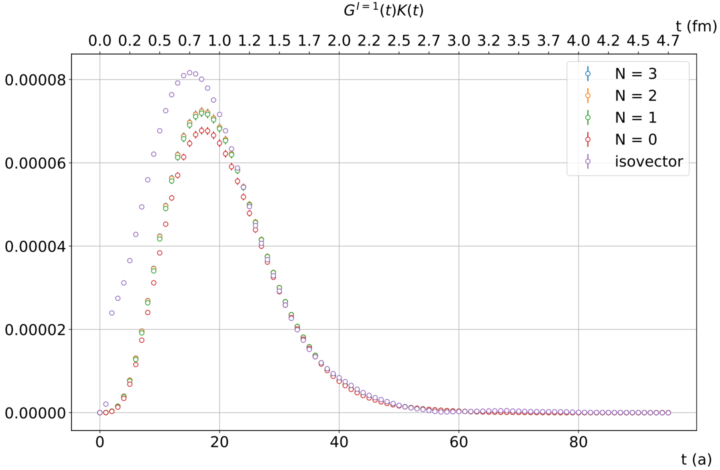
$$\left. - \frac{1}{m_\pi L |\boldsymbol{n}|} \int_1^\infty dy K_0(m_\pi y \sqrt{L^2 \boldsymbol{n}^2 + 4t^2}) sinh(m_\pi L |\boldsymbol{n}|(y-1)) \right)$$

where K_n are modified Bessel functions of the second kind. The correction is positive for $m_\pi L >> 1$.

[13] Della Morte et al. 2017.

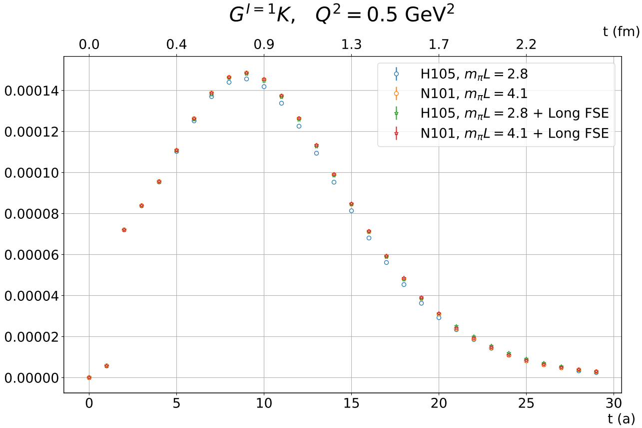
FSE. J303 long distance, finite volume

For the ensemble J303, at $Q^2 = 0.5\ \text{GeV}^2$. $K = \frac{1}{\omega^2}\left(\omega^2 t^2 - 4sin^2\left(\frac{\omega t}{2}\right)\right)$

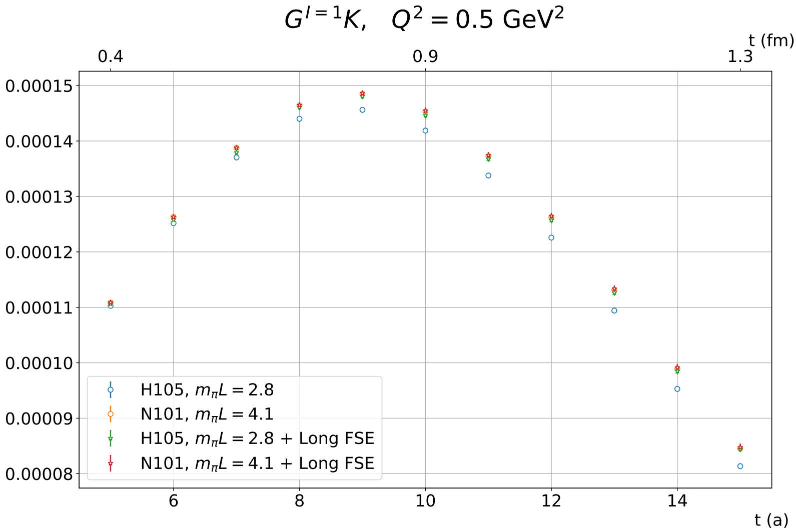


FSE. H105 vs N101

We compare two ensembles with the same parameters, except the volume. After FSE corrections both agree.



FSE. H105 vs N101



FSE

At $Q^2 = 0.5$ GeV^2

id	$\Delta\alpha \times 10^6$	FSE $\times 10^6$	id	$\Delta\alpha \times 10^6$	FSE $\times 10^6$
H101	1688.5(5.7)	6.3(0.1)	N202	1642.1(7.2)	2.8(0.1)
H102	1800.3(7.3)	15.7(0.5)	N203	1743.3(6.9)	8.8(0.3)
H105	1948.9(9.7)	46.7(3.3)	N200	1861.8(7.8)	24.6(0.8)
N101	1954.0(8.8)	4.7(0.2)	D200	2039.4(10.0)	28.7(0.9)
C101	2100.0(8.5)	17.5(0.9)	E250	2276.5(16.1)	30.0(1.1)
B450	1640.0(6.5)	13.4(0.3)	N300	1548.8(7.3)	14.4(2.0)
S400	1765.9(8.2)	33.0(1.4)	N302	1653.6(7.1)	38.9(1.0)
N401	1910.7(8.1)	9.4(0.3)	J303	1869.4(11.8)	31.8(0.7)

Chiral and continuum extrapolation

The next step is to extrapolate the results of $\Delta\alpha_{QED}$ to the physical point for each energy separately. For the isovector component we have used several models,

$$\Delta\alpha(a,\phi_2,d) = \Delta\alpha(0,\phi_{2,i}) + \delta_d a^2 + \gamma_1\left(\phi_2 - \phi_{2,i}\right) + \gamma_2\left(log\phi_2 - log\phi_{2,i}\right)$$
$$\Delta\alpha(a,\phi_2,d) = \Delta\alpha(0,\phi_{2,i}) + \delta_d a^2 + \gamma_3\left(\phi_2 - \phi_{2,i}\right) + \gamma_4\left(\phi_2^2 - \phi_{2,i}^2\right)$$
$$\Delta\alpha(a,\phi_2,d) = \Delta\alpha(0,\phi_{2,i}) + \delta_d a^2 + \gamma_5\left(\phi_2 - \phi_{2,i}\right) + \gamma_6\left(1/\phi_2 - 1/\phi_{2,i}\right)$$
$$\Delta\alpha(a,\phi_2,d) = \Delta\alpha(0,\phi_{2,i}) + \delta_d a^2 + \gamma_7\left(\phi_2 - \phi_{2,i}\right) + \gamma_8\left(\phi_2 log\phi_2 - \phi_{2,i} log\phi_{2,i}\right)$$

where $\phi_2 = 8t_0 m_\pi^2$, $\phi_{2,i}$ uses the flavour-symmetric, neutral pion mass, and d indicates the discretization.

Chiral and continuum extrapolation

We have performed a simultaneous extrapolation of both discretizations to the physical point.

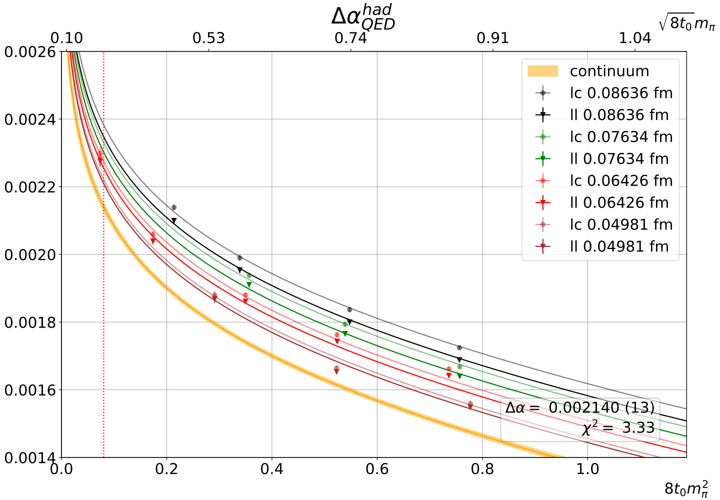
We have included the uncertainty of the pion mass in the computation enlarging the covariance matrix. In our case, the $\chi^2 = V^t C^{-1} V$ function is formed by

$$V = \left.\begin{pmatrix} \vdots \\ \phi_{2,param} - \phi_{2,e_i} \\ \Delta\alpha^{ll}_{param} - \Delta\alpha^{ll}_{e_i} \\ \Delta\alpha^{lc}_{param} - \Delta\alpha^{lc}_{e_i} \\ \vdots \end{pmatrix}\right\} \text{set of ensembles}$$

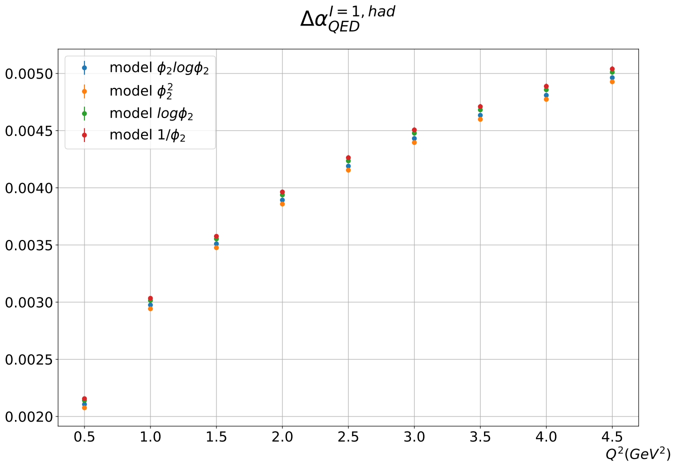
where all entries are dimensionless.

Preliminary results. Isovector extrapolation

For $Q^2 = 0.5$ GeV^2, model with *log* $8t_0m_\pi^2$ term.



Preliminary results. Isovector component



With $\phi_2 = 8t_0 m_\pi^2$

Summary and outlook

We have computed the isovector component of $\Delta\alpha_{QED}$ in the energy range $0.5 - 4.5$ GeV^2, taking into account statistical and systematic uncertainties.

The next steps that we want to take are:

- Investigate the systematics of the chiral and continuum extrapolation.
- Analyze the isoscalar (including disconnected) and charm contributions.
- Perform a comparison with Phenomenology of $\Delta\alpha_{QED}$ including the four lightest flavors.

How to choose W

How to choose W? Consider the two sources of uncertainty in the autocorrelations:

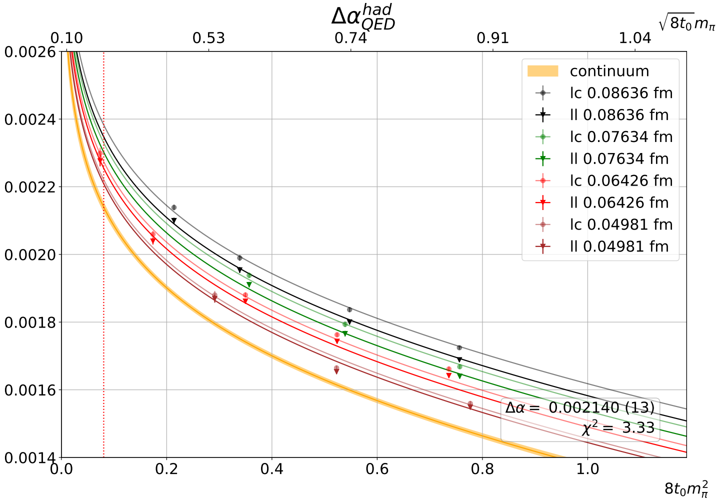
- Finite statistics, $\propto \sqrt{\frac{W}{N}}$
- Truncation of Γ_F, $\propto exp(-W/\tau)$

τ dictates the decay of autocorrelations. Suppose $\tau = S\tau_{int}$, with S set by the user (reasonable ansatz $\sim 1-3$).

$W_{optimal}$ minimizes the total error. After W is determined, we need to check that we have included just the right amount of autocorrelations

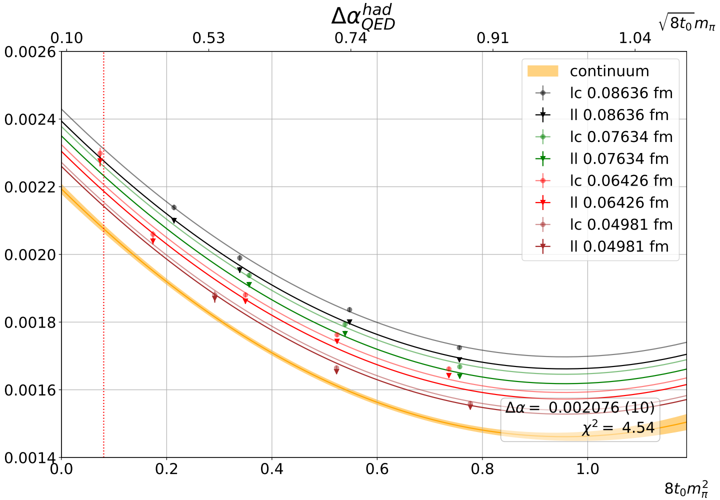
Chiral and continuum extrapolation with other models

For $Q^2 = 0.5$ GeV^2, model with $log\phi_2$ term.



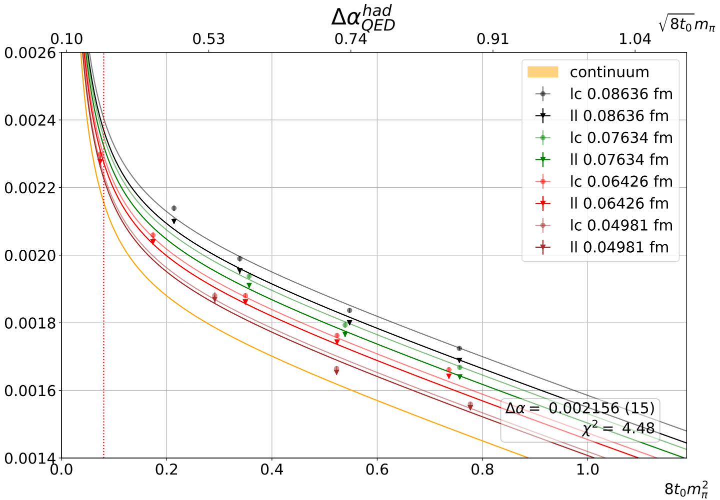
Chiral and continuum extrapolation with other models

For $Q^2 = 0.5$ GeV^2, model with ϕ_2^2 term.



Chiral and continuum extrapolation with other models

For $Q^2 = 0.5$ GeV^2, model with $1/\phi_2$ term.



Chiral and continuum extrapolation with other models

For $Q^2 = 0.5$ GeV^2, model with $\phi_2 log \phi_2$ term.