QED corrections to hadronic observables

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June 18, 2019

Motivation

- Lattice computations of hadronic observables nowadays reach below 1 % precision
- E.g. HVP contribution to muon $g-2$
- $m_u \neq m_d$ and QED effects become relevant
- These make lattice computations much more expensive

$\longrightarrow$ An efficient method to include leading order SIB and QED effects is required

Outline

Full QCD+QED

- QCD + QED

$$\langle \mathcal{C} \rangle_{\text{QCD+QED}} = \frac{\int \mathrm{d}U \int \mathrm{d}A \; \mathcal{C}(U,A) \; e^{-S_g[U]} \, e^{-S_\gamma[A]} \; \det M(U,A)}{\int \mathrm{d}U \int \mathrm{d}A \; e^{-S_g[U]} \, e^{-S_\gamma[A]} \; \det M(U,A)}$$

- Keep leading order in $\delta m = m_d - m_u$ and $e^2 = 4\pi\alpha$

$$\langle \mathcal{C} \rangle_{\text{QCD+QED}}(\delta m, e) = \langle \mathcal{C} \rangle_{\text{QCD+QED}} \Big|_{\substack{\delta m=0\\ e=0}} + \delta m \cdot \frac{\partial}{\partial \delta m} \langle \mathcal{C} \rangle_{\text{QCD+QED}} \Big|_{\substack{\delta m=0\\ e=0}} + \\ + \frac{e^2}{2} \cdot \frac{\partial^2}{\partial e^2} \langle \mathcal{C} \rangle_{\text{QCD+QED}} \Big|_{\substack{\delta m=0\\ e=0}} + \mathcal{O}(e^4, e^2 \delta m, \delta m^2)$$

1. Dynamical QED as quenched QED expectation values

$$\langle \mathcal{C} \rangle_{\mathrm{QCD+QED}} = \frac{\int \mathrm{d}U \int \mathrm{d}A \; \mathcal{C}(U,A) \; e^{-S_\mathrm{g}[U]} \, e^{-S_\gamma[A]} \, \det M(U,A)}{\int \mathrm{d}U \int \mathrm{d}A \; e^{-S_\mathrm{g}[U]} \, e^{-S_\gamma[A]} \, \det M(U,A)} =$$

$$= \frac{\int \mathrm{d}U \int \mathrm{d}A \; \mathcal{C}(U,A) \, \frac{\det M(U,A)}{\det M(U,0)} \; e^{-S_\gamma[A]} \; e^{-S_\mathrm{g}[U]} \, \det M(U,0)}{\int \mathrm{d}U \int \mathrm{d}A \; \frac{\det M(U,A)}{\det M(U,0)} \; e^{-S_\gamma[A]} \; e^{-S_\mathrm{g}[U]} \, \det M(U,0)} =$$

$$= \frac{\int \mathrm{d}U \; \left\langle \mathcal{C}(U,A) \, \frac{\det M(U,A)}{\det M(U,0)} \right\rangle_{A,\mathrm{q.}} \; e^{-S_\mathrm{g}[U]} \, \det M(U,0)}{\int \mathrm{d}U \; \left\langle \frac{\det M(U,A)}{\det M(U,0)} \right\rangle_{A,\mathrm{q.}} \; e^{-S_\mathrm{g}[U]} \, \det M(U,0)} =$$

$$= \frac{\left\langle \left\langle \mathcal{C}(U,A) \, \frac{\det M(U,A)}{\det M(U,0)} \right\rangle_{A,\mathrm{q.}} \right\rangle_U}{\left\langle \left\langle \frac{\det M(U,A)}{\det M(U,0)} \right\rangle_{A,\mathrm{q.}} \right\rangle_U}$$

2. Expansion up to $\mathcal{O}(e^2, \delta m)$

e_v: QED coupling in valence sector

e_s: QED coupling in sea sector

$$\mathcal{C}(U,A) \approx \mathcal{C}_0(U) + \frac{\delta m}{m_l} \cdot \mathcal{C}'_m(U) + e_v \cdot \mathcal{C}'_1(U,A) + \frac{e_v^2}{2} \cdot \mathcal{C}''_2(U,A)$$

$$\left(\prod_{f=u,d,s,c} \frac{\det M^{(f)}[U,A]}{\det M^{(f)}[U,0]} \right)^{1/4} \approx 1 + e_s \cdot \frac{d_1(U,A)}{d_0(U)} + \frac{e_s^2}{2} \cdot \frac{d_2(U,A)}{d_0(U)}$$

- $m_u = m_l - \frac{\delta m}{2}, \quad m_d = m_l + \frac{\delta m}{2} \quad \longrightarrow \quad \mathcal{O}(\delta m)$ sea effect vanishes

$$\begin{aligned} \langle \mathcal{C} \rangle_{\text{QCD+QED}} = {} & \langle \mathcal{C}_0(U) \rangle_U + \frac{\delta m}{m_l} \cdot \langle \mathcal{C}'_m(U) \rangle_U + \frac{e_v^2}{2} \cdot \left\langle \langle \mathcal{C}''_2(U,A) \rangle_{A,\text{q.}} \right\rangle_U + \\ & + e_v e_s \cdot \left\langle \left\langle \mathcal{C}'_1(U,A) \cdot \frac{d_1(U,A)}{d_0(U)} \right\rangle_{A,\text{q.}} \right\rangle_U + \\ & + \frac{e_s^2}{2} \cdot \left\langle \left(\mathcal{C}_0(U) - \langle \mathcal{C}_0(U) \rangle_U \right) \cdot \left\langle \frac{d_2(U,A)}{d_0(U)} \right\rangle_{A,\text{q.}} \right\rangle_U \end{aligned}$$

QCD+QED

$$\langle \mathcal{C} \rangle_{\mathrm{QCD+QED}} \approx \langle \mathcal{C} \rangle\big|_0 + \frac{\delta m}{m_l} \cdot \langle \mathcal{C} \rangle'_m + \frac{e_v^2}{2} \cdot \langle \mathcal{C} \rangle''_{20} + e_v e_s \cdot \langle \mathcal{C} \rangle''_{11} + \frac{e_s^2}{2} \cdot \langle \mathcal{C} \rangle''_{02}$$

$$\begin{aligned}
\langle \mathcal{C} \rangle\big|_0 &= \langle \mathcal{C}_0(U) \rangle_U \\
\langle \mathcal{C} \rangle'_m &= \left\langle \mathcal{C}'_m(U) \right\rangle_U \\
\langle \mathcal{C} \rangle''_{20} &= \left\langle \left\langle \mathcal{C}''_2(U,A) \right\rangle_{A,\mathrm{q.}} \right\rangle_U \\
\langle \mathcal{C} \rangle''_{11} &= \left\langle \left\langle \mathcal{C}'_1(U,A) \cdot \frac{d_1(U,A)}{d_0(U)} \right\rangle_{A,\mathrm{q.}} \right\rangle_U \\
\langle \mathcal{C} \rangle''_{02} &= \left\langle \left(\mathcal{C}_0(U) - \langle \mathcal{C}_0(U) \rangle_U \right) \cdot \left\langle \frac{d_2(U,A)}{d_0(U)} \right\rangle_{A,\mathrm{q.}} \right\rangle_U
\end{aligned}$$

Strategy:

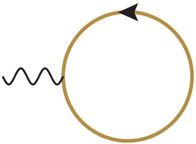
- Take isospin symmetric $SU(3)$ configurations: U
- Measure $\mathcal{C}_0(U)$ and $\mathcal{C}'_m(U)$
- For each U, generate quenched $U(1)$ fields: A
- Measure $\mathcal{C}'_1(U,A)$, $\mathcal{C}''_2(U,A)$, $\frac{d_1(U,A)}{d_0(U)}$ and $\frac{d_2(U,A)}{d_0(U)}$

Outline

1^{st} derivative of determinant

$$\frac{d_1(U,A)}{d_0(U)} = \sum_f \frac{Q_f}{4}\,\mathrm{Tr}\left[\left(M_0^{(f)}\right)^{-1}\cdot D[U\cdot iA]\right]$$

$$= \frac{1}{12}\,\mathrm{Tr}\left[\left(\underbrace{\left(M_0^{(l)}\right)^{-1} - \left(M_0^{(s)}\right)^{-1}}_{SU(3)\text{ suppression}} + 2\left(M_0^{(c)}\right)^{-1}\right)\cdot D[U\cdot iA]\right]$$



where

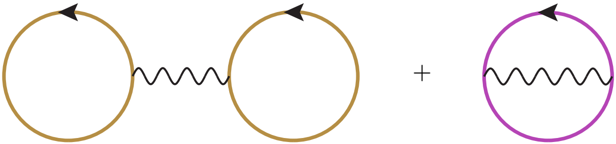
$$M^{(f)} = D[U\cdot e^{iQ_f eA}] + m_f$$

$$M_0^{(f)} = D[U] + m_f$$

2^{nd} derivative of determinant

$$\frac{d_2(U,A)}{d_0(U)} = \underbrace{\left(\frac{d_1(U,A)}{d_0(U)}\right)^2}_{\text{disconnected}} +$$

$$+\sum_f \left(\underbrace{-\frac{Q_f^2}{4}\,\mathrm{Tr}\left[\left(M_0^{(f)}\right)^{-1} D[U\cdot iA]\left(M_0^{(f)}\right)^{-1} D[U\cdot iA]\right]}_{\text{connected}} - \underbrace{-\frac{Q_f^2}{4}\,\mathrm{Tr}\left[\left(M_0^{(f)}\right)^{-1} D[U\cdot A^2]\right]}_{\text{contact}} \right)$$



Valence derivatives

$$\langle C\rangle_{\text{QCD+QED}}(t) \approx \langle C\rangle\big|_0(t) + \frac{\delta m}{m_l}\cdot\langle C\rangle'_m(t) + \frac{e_v^2}{2}\cdot\langle C\rangle''_{20}(t) + e_v e_s\cdot\langle C\rangle''_{11}(t) + \frac{e_s^2}{2}\cdot\langle C\rangle''_{02}(t)$$

- Measure correlators $C_t(0,0)$, $C_t(\delta m,0)$, $C_t(0,+e_v)$, $C_t(0,-e_v)$
- Valence derivatives as finite differences

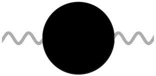
$$\begin{aligned} C'_m(t) &\approx \frac{m_l}{\delta m}\cdot[C_t(\delta m,0) - C_t(0,0)] \\ C'_1(t) &\approx \frac{C_t(0,+e_v) - C_t(0,-e_v)}{2\,e_v} \\ C''_2(t) &\approx \frac{C_t(0,+e_v)+C_t(0,-e_v)-2\,C_t(0,0)}{e_v^2} \end{aligned}$$

- Derivatives of determinant are exact derivatives: hybrid approach

$$\begin{aligned} \langle C\rangle\big|_0(t) &= \langle C_t(0,0)\rangle_U \\ \langle C\rangle'_m(t) &= \langle C'_m(t)\rangle_U \approx \left\langle \frac{m_l}{\delta m}\cdot[C_t(\delta m,0) - C_t(0,0)]\right\rangle_U \\ \langle C\rangle''_{20}(t) &= \Big\langle \langle C''_2(t)\rangle_{A,\text{q.}}\Big\rangle_U \approx \left\langle\left\langle \frac{C_t(0,+e_v)+C_t(0,-e_v)-2\,C_t(0,0)}{e_v^2}\right\rangle_{A,\text{q.}}\right\rangle_U \\ \langle C\rangle''_{11}(t) &= \left\langle\left\langle C'_1(t)\cdot\frac{d_1(U,A)}{d_0(U)}\right\rangle_{A,\text{q.}}\right\rangle_U \approx \left\langle\left\langle \frac{C_t(0,+e_v)-C_t(0,-e_v)}{2\,e_v}\cdot\frac{d_1(U,A)}{d_0(U)}\right\rangle_{A,\text{q.}}\right\rangle_U \\ \langle C\rangle''_{02}(t) &= \left\langle\Big(C_t(0,0) - \langle C_t(0,0)\rangle_U\Big)\cdot\left\langle\frac{d_2(U,A)}{d_0(U)}\right\rangle_{A,\text{q.}}\right\rangle_U \end{aligned}$$

Outline

Hadronic vacuum polarization



- EM current of quarks

$$j_\mu(x) = \sum_f Q_f \, \overline{\psi}^{(f)}(x)\gamma_\mu \psi^{(f)}(x)$$

Q_f: electric charge of flavor f.

$$Q_u = \frac{2}{3}, \quad Q_d = -\frac{1}{3}, \quad Q_s = -\frac{1}{3}, \quad Q_c = \frac{2}{3}$$

- Hadronic vacuum polarization $\Pi(Q^2)$ is defined through

$$\Pi_{\mu\nu}(Q) = (Q^2\delta_{\mu\nu} - Q_\mu Q_\nu)\,\Pi(Q^2),$$

where

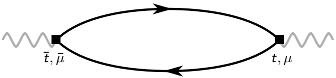
$$\Pi_{\mu\nu}(Q) = \int \mathrm{d}x \, e^{iQx} \, \langle j_\mu(x) j_\nu(0)\rangle =$$

on the lattice $\longrightarrow$ $\displaystyle \frac{1}{TV}\sum_{x,y}\left(e^{iQ(x-y)} - 1\right)\langle j_\mu(x) j_\nu(y)\rangle$

Hadronic vacuum polarization

$$\Pi_{\mu\nu}(Q) = \left(Q^2\delta_{\mu\nu} - Q_\mu Q_\nu\right)\Pi(Q^2) = \frac{1}{TV}\sum_{x,y}\left(e^{iQ(x-y)} - 1\right)\langle j_\mu(x) j_\nu(y)\rangle$$

- Connected contribution



$$C_{\mu\bar{\mu}}(t-\bar{t}) = \langle j_\mu(t) j_{\bar{\mu}}(\bar{t})\rangle_{\text{conn.}} =$$

$$= -\sum_{\underline{\bar{\mathbf{x}}}\text{ even}}\sum_{\underline{\mathbf{x}}}\sum_{f}\frac{Q_f^2}{4V}\,\mathrm{Re}\,\mathrm{tr}_c\left(M^{(f)}{}^{-1}_{\underline{\mathbf{x}}+\mu,t;\underline{\bar{\mathbf{x}}},\bar{t}}\,U^{+-}_{\bar{\mathrm{x}},\bar{t},\bar{\mu}}\,M^{(f)}{}^{-1}_{\underline{\bar{\mathbf{x}}}+\bar{\mu},\bar{t};\underline{\mathbf{x}},t}\,U^{+-}_{\underline{\mathbf{x}},t,\mu}\right)$$

with

$$U^{+-}_\mu = (\text{covariant shift in direction } \mu) + (\text{covariant shift in direction } -\mu)$$

- 1^{st} moment: slope of $\Pi(Q^2)$ at $Q^2 = 0$

$$\Pi_1 = \frac{1}{3}\sum_{\mu=1}^{3}\sum_{t}\frac{t^4}{24}\,C_{\mu\mu}(t)$$

QED corrections to connected light HVP

- Valence sector: finite differences

$$\frac{\mathrm{d}}{\mathrm{d}e}\Pi_1 \approx \frac{\Pi_1\left(\frac{e}{3}\right) - \Pi_1\left(-\frac{e}{3}\right)}{\frac{2}{3}e}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}e^2}\Pi_1 \approx \frac{\Pi_1\left(\frac{e}{3}\right) + \Pi_1\left(-\frac{e}{3}\right) - 2\,\Pi_1(0)}{\frac{1}{9}e^2}$$

- $Q_u = \frac{2}{3}, \quad Q_d = -\frac{1}{3}$

$$\Pi_1 \equiv \frac{5}{9}\,\Pi_1^{(l)} \stackrel{\text{iso}}{=} \frac{4}{9}\,\Pi_1^{(u)} + \frac{1}{9}\,\Pi_1^{(d)}$$

- Construct correction to $\Pi_1^{(u)}$ and $\Pi_1^{(d)}$ from measurements at charge $\pm\frac{1}{3}e_v$

$$(\Pi_1)_1' \approx \frac{7}{18\,e_v}\cdot\left(\Pi_1^{(l)}\left(\tfrac{e_v}{3}\right) - \Pi_1^{(l)}\left(-\tfrac{e_v}{3}\right)\right)$$

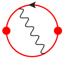
$$(\Pi_1)_2'' \approx \frac{17}{9\,e_v^2}\cdot\left(\Pi_1^{(l)}\left(\tfrac{e_v}{3}\right) + \Pi_1^{(l)}\left(-\tfrac{e_v}{3}\right) - 2\,\Pi_1^{(l)}(0)\right)$$

- Measure $\Pi_1^{(l)}\left(\frac{e_v}{3}\right)$, $\Pi_1^{(l)}\left(-\frac{e_v}{3}\right)$, $\Pi_1^{(l)}(0)$ using the same set of random vectors

QED corrections to connected light HVP

- Valence sector:

$$\langle \Pi_1 \rangle''_{20} = \Big\langle \big\langle (\Pi_1)''_2\,(U,A) \big\rangle_{A,\mathrm{q.}} \Big\rangle_U \quad =$$



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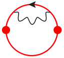
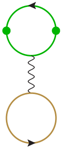


- Valence–sea sector:

$$\langle \Pi_1 \rangle''_{11} = \left\langle \left\langle (\Pi_1)'_1\,(U,A) \cdot \frac{d_1(U,A)}{d_0(U)} \right\rangle_{A,\mathrm{q.}} \right\rangle_U \quad =$$



- Sea–sea sector:

$$\langle \Pi_1 \rangle''_{02} = \left\langle \Big((\Pi_1)_0\,(U) - \big\langle (\Pi_1)_0\,(U) \big\rangle_U \Big) \cdot \left\langle \frac{d_2(U,A)}{d_0(U)} \right\rangle_{A,\mathrm{q.}} \right\rangle_U =$$

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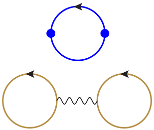

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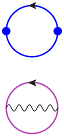
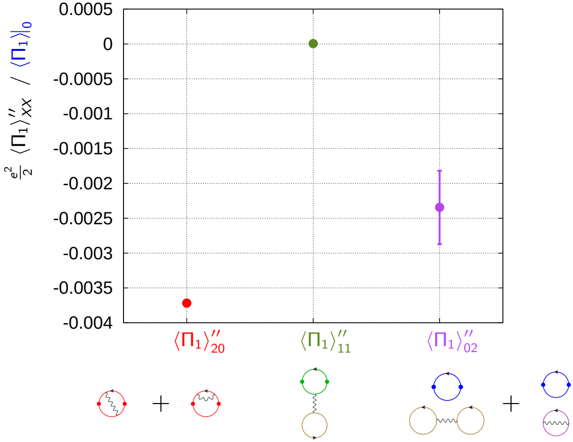


QED corrections to connected light HVP

- Example: $48^3 \times 64$, $a = 0.134\,\mathrm{fm}$



- Dynamical QED contribution is not negligible

Outline

QED correction to w_0

- Gluonic action density on configuration U at flow time t

$$E(t, U) = \frac{1}{4} G_{\mu\nu}(t, U)\, G_{\mu\nu}(t, U)$$

- Define

$$W(t, U) = t \frac{\mathrm{d}}{\mathrm{d}t} \left[t^2\, E(t, U) \right]$$

- w_0 is defined as

$$\left\langle W(t, U) \right\rangle_U \Big|_{t = w_0^2} = 0.3$$

- In case of QCD + QED, define $w_{0,e}$ through

$$\left\langle W(t, U) \right\rangle_{U,A} \Big|_{t = w_{0,e}^2} = 0.3$$

- Define $\delta w_0^2 = w_{0,e}^2 - w_0^2$, then

$$\left\langle W(w_0^2, U) + \dot{W}(w_0^2, U) \cdot \delta w_0^2 \right\rangle_{U,A} = 0.3$$

QED correction to w_0

$$\left\langle W(w_0^2,U) + \dot{W}(w_0^2,U)\cdot\delta w_0^2 \right\rangle_{U,A} = 0.3$$

- W and $\dot{W}$ do not depend on $\delta m,\ e_v$

$$\underbrace{\langle W\rangle\big|_0}_{=0.3} + \frac{e_s^2}{2}\langle W\rangle''_{02} + \left(\left\langle \dot{W}\right\rangle\Big|_0 + \frac{e_s^2}{2}\left\langle \dot{W}\right\rangle''_{02}\right)\cdot\delta w_0^2 = 0.3$$

- Solve for δw_0^2 and keep terms up to $\mathcal{O}(e_s^2)$

$$\delta w_0^2 = -\frac{e_s^2}{2}\cdot\frac{\langle W\rangle''_{02}}{\left\langle \dot{W}\right\rangle\Big|_0} \quad\longrightarrow\quad \delta w_0 = w_{0,e} - w_0 = \frac{\delta w_0^2}{2\,w_0} = -\frac{e_s^2}{4}\cdot\frac{\langle W\rangle''_{02}}{w_0\cdot\left\langle \dot{W}\right\rangle\Big|_0}$$

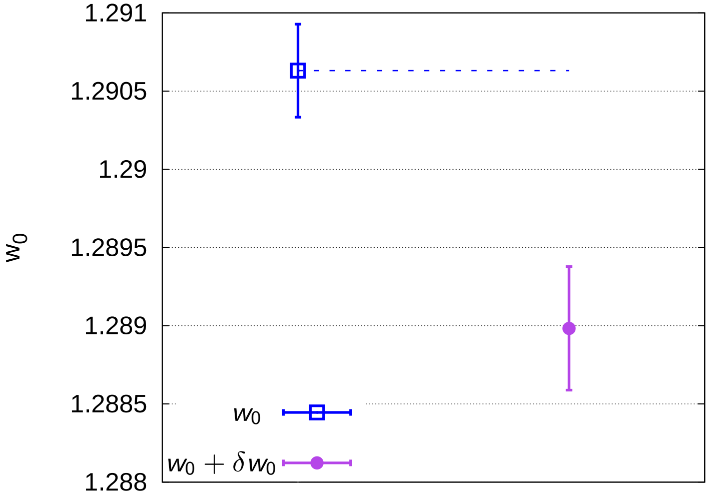
$\longrightarrow$ To obtain δw_0, one has to measure

$$\langle W\rangle''_{02} = \left\langle \left(W(w_0^2,U) - \left\langle W(w_0^2,U)\right\rangle_U \right)\cdot\left\langle \frac{d_2(U,A)}{d_0(U)}\right\rangle_{A,\mathrm{q.}} \right\rangle_U$$

$$\left\langle \dot{W}\right\rangle\Big|_0 = \left\langle \dot{W}(w_0^2,U)\right\rangle_U$$

QED correction to w_0

- Example: $48^3 \times 64, \quad a = 0.134\,\mathrm{fm}$



Outline

Summary

- Included $\mathcal{O}(\delta m)$ SIB and $\mathcal{O}(e^2)$ full QED effects
- Hybrid approach:
  - Valence sector: finite differences
  - Sea sector: exact derivatives
- Preliminary results:
  Dynamical QED effects for $(g-2)_\mu$ are relevant

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$\frac{d_1}{d_0}, \frac{d_2}{d_0}$: Noise reduction with leading order HPE

$$\mathrm{Tr}\left[\left(M_0^{(f)}\right)^{-1} \cdot D[U \cdot iA]\right] = -a_0 \cdot \underbrace{\mathrm{Tr}\left[(-D[U]) \cdot D[U \cdot iA]\right]}_{=0} +$$

$$+\left(1 + a_0 m_f^2\right) \cdot \mathrm{Tr}\left[\left(M_0^{(f)\dagger} M_0^{(f)}\right)^{-1} \cdot (-D[U]) \cdot D[U \cdot iA]\right] -$$

$$- a_0 \cdot \mathrm{Tr}\left[\left(M_0^{(f)\dagger} M_0^{(f)}\right)^{-1} \cdot (-D[U])^3 \cdot D[U \cdot iA]\right]$$

$$\mathrm{Tr}\left[\left(M_0^{(f)}\right)^{-1} \cdot D[U \cdot A^2]\right] = -a_0 \cdot \underbrace{\mathrm{Tr}\left[(-D[U]) \cdot D[U \cdot A^2]\right]}_{\text{exact part}} +$$

$$+\left(1 + a_0 m_f^2\right) \cdot \mathrm{Tr}\left[\left(M_0^{(f)\dagger} M_0^{(f)}\right)^{-1} \cdot (-D[U]) \cdot D[U \cdot A^2]\right] -$$

$$- a_0 \cdot \mathrm{Tr}\left[\left(M_0^{(f)\dagger} M_0^{(f)}\right)^{-1} \cdot (-D[U])^3 \cdot D[U \cdot A^2]\right]$$

- Choose a_0 such that noise is minimal

Effective masses

$$M_{\mathrm{QCD+QED}}(t) \approx M_0(t) + \frac{\delta m}{m_l}\cdot M'_m(t) + \frac{e_v^2}{2}\cdot M''_{20}(t) + e_v e_s \cdot M''_{11}(t) + \frac{e_s^2}{2}\cdot M''_{02}(t)$$

- Effective mass function

$$M(t) = \mathcal{M}_t\left[\langle C\rangle\right] \overset{\text{e.g.}}{=} \frac{1}{2}\left[\cosh^{-1}\left(\frac{\langle C_{t-1}\rangle}{\langle C_{N_t/2}\rangle}\right) - \cosh^{-1}\left(\frac{\langle C_{t+1}\rangle}{\langle C_{N_t/2}\rangle}\right)\right]$$

- Valence sector: finite differences

$$M'_m(t) = \frac{m_l}{\delta m}\cdot\Big[\mathcal{M}_t\left[\langle C(\delta m, 0)\rangle\right] - \mathcal{M}_t\left[\langle C(0,0)\rangle\right]\Big]$$

$$M''_{20}(t) = \frac{1}{e_v^2}\cdot\left[\mathcal{M}_t\left[\frac{\langle C(0,+e_v)+C(0,-e_v)\rangle}{2}\right] - \mathcal{M}_t\left[\langle C(0,0)\rangle\right]\right]$$

- Sea–sea sector: derivative

$$M''_{02}(t) = \sum_{t'} \left.\frac{\partial\mathcal{M}_t[C]}{\partial C_{t'}}\right|_{C=\langle C(0,0)\rangle} \cdot \langle C\rangle''_{02}(t')$$

- Valence–sea sector: mixed

$$M''_{11}(t) = \sum_{t'} \left.\frac{\partial\mathcal{M}_t[C]}{\partial C_{t'}}\right|_{C=\langle C(0,+e_v)+C(0,-e_v)\rangle/2} \cdot \langle C\rangle''_{11}(t')$$

Effective masses

- Example: M_{K^+}, $24^3 \times 48$, $a = 0.134\,\mathrm{fm}$