$N\pi$-state contamination in lattice calculations of nucleon form factors

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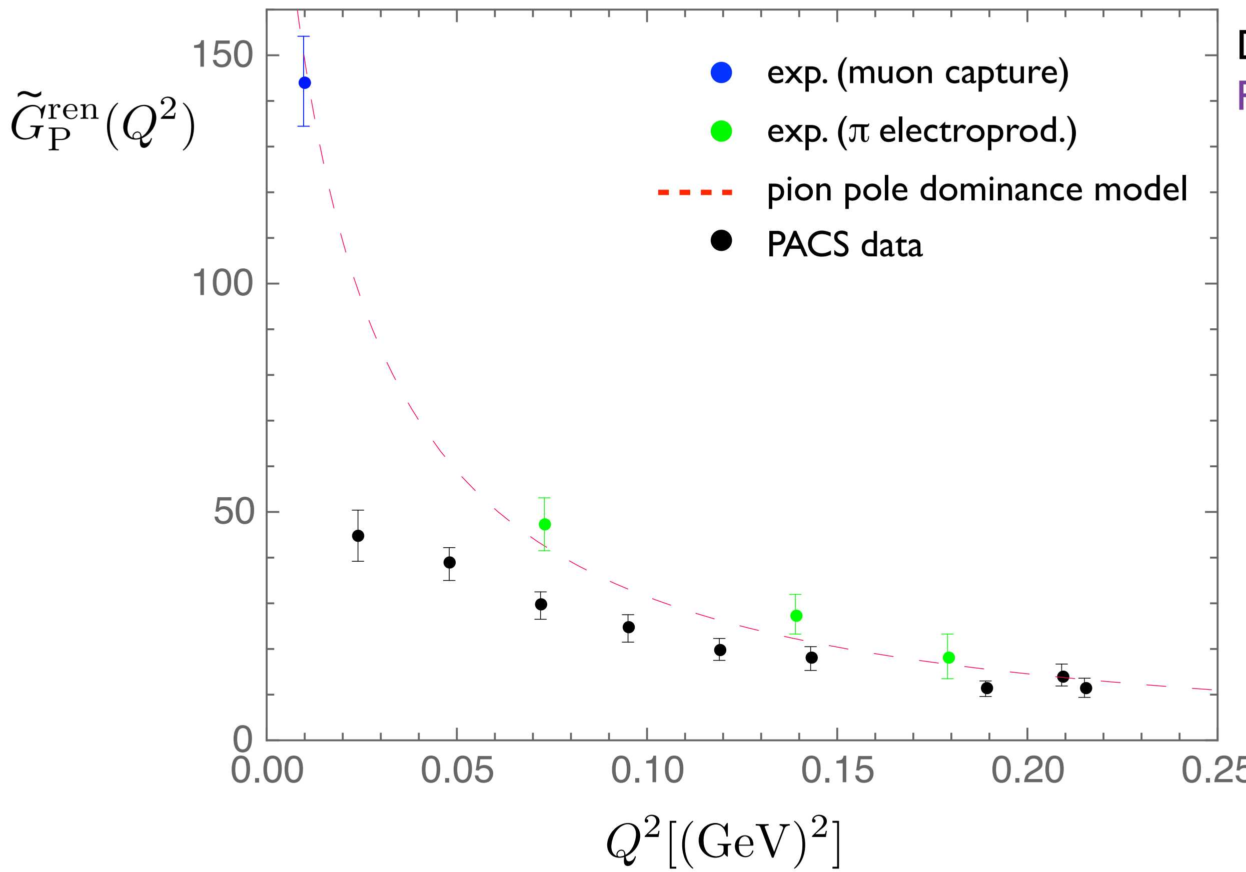
Introduction: Physical point simulations

- Advantage: No chiral extrapolation needed, i.e. one systematic error eliminated
- Problems:
  - Numerically demanding
  - Signal-to-noise problem
  - Significant impact in correlation functions of multi-particle-states involving light pions → Excited state contamination in physical observables
- ChPT can be used to estimate this multi-particle-state contamination:
  B. Tiburzi 2009; OB and M. Golterman 2013
  - Nucleon mass
  - Nucleon axial, scalar, tensor charge
  - Moments of pdfs
  - Nucleon axial form factors

OB, Lattice 2017

Example: Induced pseudoscalar form factor $\tilde{G}_P(Q^2)$



Data in
PRD 98 (2018) 074510

$a \approx 0.085\,\mathrm{fm}$

$M_\pi \approx 146\,\mathrm{MeV}$

$M_\pi L \approx 6$

$t \approx 1.3\,\mathrm{fm}$

Plot from
OB, Lattice 2018

- Plateau estimate data below exp. results and pion pole dominance model

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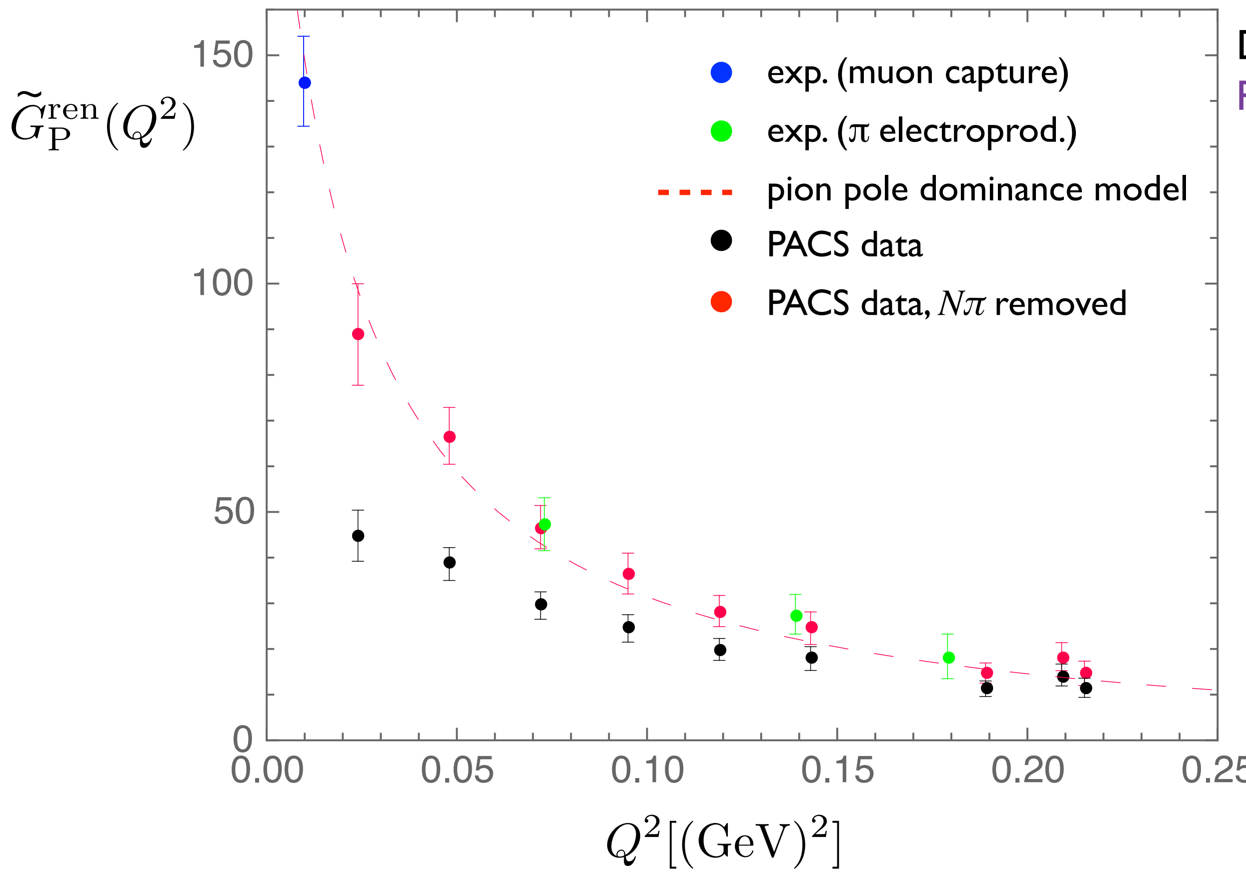

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Plot from
OB, Lattice 2018

- Plateau estimate data below exp. results and pion pole dominance model
- LO ChPT predicts <u>underestimation due to $N\pi$ contamination</u>
  Can be removed analytically ⇒ better agreement with exp. / ppd model

Outline

- In the following: Impact of the $N\pi$ contamination in the
  - 3-point function involving the temporal component A_0
  - pseudoscalar form factor $G_P(Q^2)$
  - generalized Goldberger-Treiman relation ($PCAC_{FF}$)

Details: OB, PRD 99 (2019) 054506
OB, arXiv:1906.03652 [hep-lat]

The nucleon form factors

Matrix elements of local isovector axial vector current and pseudoscalar density

isospin symmetry assumed

$$\langle N(p',s')|A^a_\mu(0)|N(p,s)\rangle = \bar{u}(p',s')\left(\gamma_\mu\gamma_5 \underbrace{G_\mathrm{A}(Q^2)}_{\text{axial ff}} - i\gamma_5 \frac{Q_\mu}{2M_N}\underbrace{\tilde{G}_\mathrm{P}(Q^2)}_{\text{induced pseudo scalar ff}}\right)\frac{\sigma^a}{2}u(p,s)$$

$$\langle N(p',s')|P^a(0)|N(p,s)\rangle = \underbrace{G_\mathrm{P}(Q^2)}_{\text{pseudo scalar ff}}\bar{u}(p',s')\gamma_5\frac{\sigma^a}{2}u(p,s)$$

Momentum transfer $\quad Q_\mu = (iE_{\vec{p}'} - iE_{\vec{p}}, \vec{q}) \qquad \vec{q} = \vec{p}' - \vec{p} \qquad \vec{p}' = 0$

euclidean space time $\qquad\qquad\qquad\qquad$ chosen here

Lattice determination

- Compute 3-pt function, e.g. $C_{3,A^3_\mu}(\vec{q},t,t') = \sum_{\vec{x},\vec{y}} e^{i\vec{q}\vec{y}}\,\Gamma_{\beta\alpha}\langle N_\alpha(\vec{x},t)A^3_\mu(\vec{y},t')\overline{N}_\beta(0,0)\rangle$

  Current / Density at t'

  Nucleon interpolating fields at $t, 0$

  Projector Γ

- Ratio with 2-pt function $R_\mu(\vec{q},t,t') = \frac{C_{3,A^3_\mu}(\vec{q},t,t')}{C_2(0,t)}\sqrt{\frac{C_2(\vec{q},t-t')}{C_2(0,t-t')}\frac{C_2(\vec{0},t)}{C_2(\vec{q},t)}\frac{C_2(\vec{0},t')}{C_2(\vec{q},t')}}$

- Consider limit t, t', t-t' → ∞: $R_\mu(\vec{q},t,t') \longrightarrow \Pi_\mu(\vec{q})$

$$\Pi_k(\vec{q}) = \frac{i}{\sqrt{2E_{N,\vec{q}}(M_N+E_{N,\vec{q}})}}\left((M_N+E_{N,\vec{q}})G_{\mathrm{A}}(Q^2)\delta_{3k} - \frac{\tilde{G}_{\mathrm{P}}(Q^2)}{2M_N}q_3q_k\right),\ k=1,2,3$$

$$\Pi_0(\vec{q}) = \frac{q_3}{\sqrt{2E_{N,\vec{q}}(M_N+E_{N,\vec{q}})}}\left(G_{\mathrm{A}}(Q^2) + \frac{M_N-E_{N,\vec{q}}}{2M_N}\tilde{G}_{\mathrm{P}}(Q^2)\right)$$

$$\Pi_{\mathrm{P}}(\vec{q}) = \frac{q_3}{\sqrt{2E_{N,\vec{q}}(M_N+E_{N,\vec{q}})}}G_{\mathrm{P}}(Q^2)$$

- Extract the form factors from the asymptotic values $\Pi_\mu(\vec{q})\,,\quad \mu=0,1,2,3,P$

Lattice determination

- In practice: finite time separations t and t'

$$R_\mu(\vec{q}, t, t') \;\to\; G_{\rm X}^{\rm eff}(Q^2, t, t')\,, \;\; X = A, P, \tilde{P}$$

- The <u>effective form factors</u> contain <u>excited-state contributions</u> and depend on t, t'

$$G_{\rm X}^{\rm eff}(Q^2, t, t') \;=\; G_{\rm X}(Q^2)\Big[1 + \epsilon_X(Q^2, t, t')\Big]$$

$$\longrightarrow\; 0 \quad \text{for} \quad t, t' \longrightarrow \infty$$

- Dominant excited state for physical pion mass and large time separations:

2-particle $N\pi$ states

$\epsilon_X^{N\pi}(Q^2, t, t')$ can be computed in ChPT

ChPT including nucleons

- SU(2) ChPT at LO

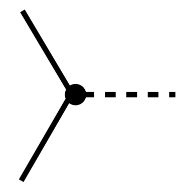
  Gasser, Sainio, Švarc 1988

  isospin symmetry, euclidean space time

  contains the three pions and the nucleon doublet $\Psi = \begin{pmatrix} p \\ n \end{pmatrix}$

$$\mathcal{L}^{(1)}_{\text{int},1\pi} = \frac{i g_A}{2f} \overline{\Psi} \gamma_\mu \gamma_5 \sigma^a \Psi \, \partial_\mu \pi^a \quad \rightarrow$$

nucleon-pion vertex



g_A axial charge $\qquad f$ pion decay constant

- Low energy constants at this order: g_A, f, M_N, M_π
  experimentally well-known

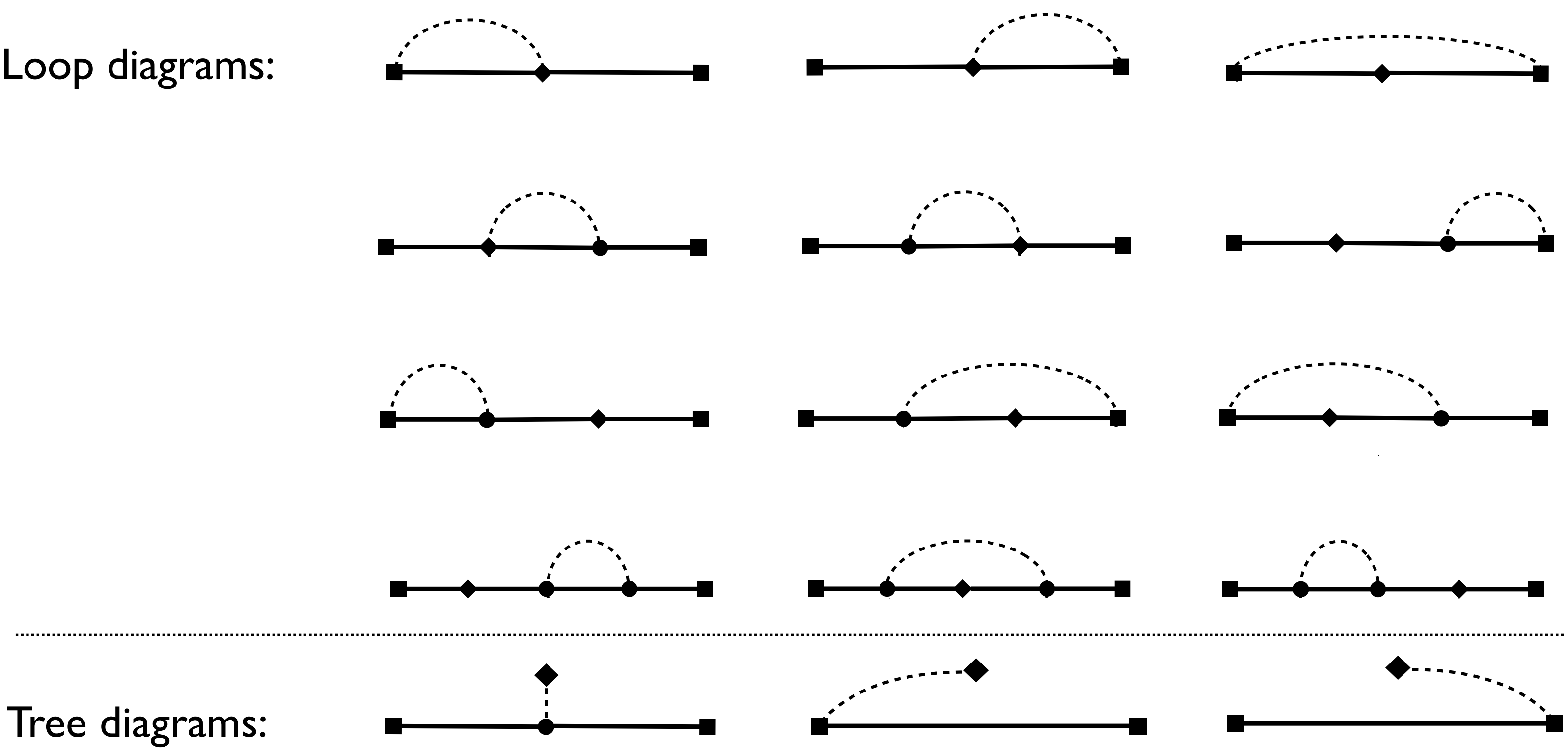
- Also known: chiral expressions for
  - axial vector current and pseudoscalar density
    Gasser, Sainio, Švarc 1988, Fettes *et al* 2000
  - nucleon interpolating fields (local and smeared)
    Nagata et al 2008; Wein, Bruns, Hemmert, Schäfer 2011; OB 2015

$N\pi$ contribution to the form factors

- To do: Compute 2-pt and 3-pt functions and the ratio R_μ in ChPT
- Example: Feynman diagrams for the 3-pt function

Loop diagrams:

Tree diagrams:



- Obtain $N\pi$ contamination to
  the ratio R_μ , the effective form factors, and the plateau / mid-point estimates

practically the same

3-pt function with A_0

Often quoted:
3pt-function and ratio involving the <u>temporal component A_0</u> are

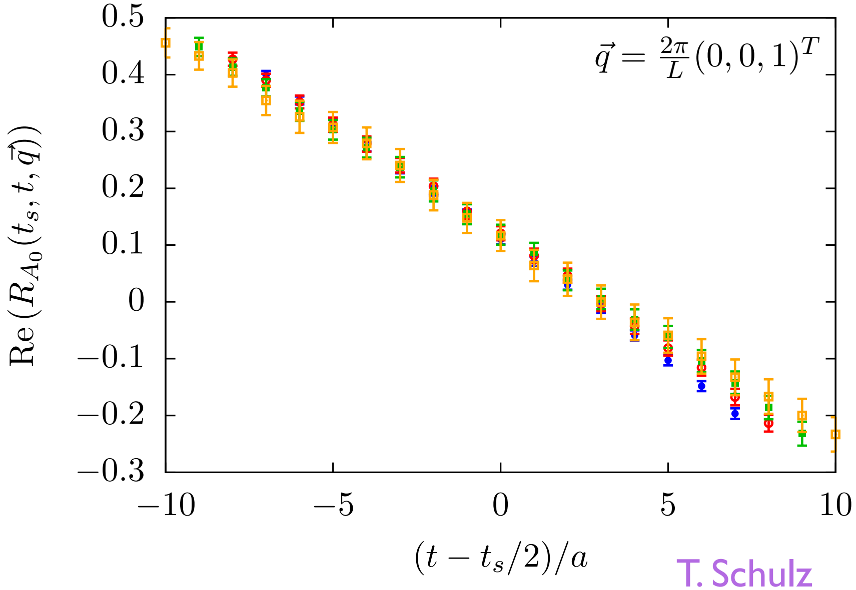
- statistically too noisy
- affected by large excited state contamination

⇒ A_0 data usually excluded from the determination of the form factors

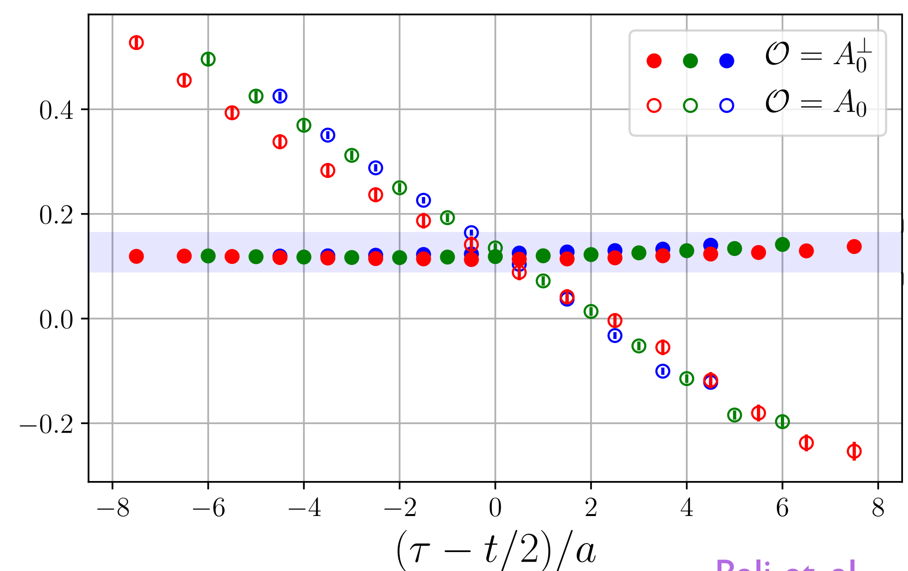
2nd statement is confirmed by the ChPT results for the $N\pi$ contamination !

Numerical data for 3-pt function with A_0

Numerical examples:



T. Schulz
talk at Lattice 2018
$M_\pi \approx 200\,\mathrm{MeV}$



Bali et. al.,
arXiv:1810.05569
$M_\pi \approx 150\,\mathrm{MeV}$

$R_{A_0}(\vec{q},t,t')$ shows a nearly linear dependence on t' (for fixed $\vec{q},t$)

note: no plateau estimate

$N\pi$ contamination

$$C_{3,\mu=0}(\vec{q},t,t') = C^{N}_{3,\mu=0}(\vec{q},t,t') + C^{N\pi}_{3,\mu=0}(\vec{q},t,t')$$

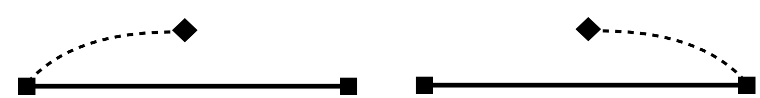
ChPT results/observations:

- Perform the non-relativistic expansion $E_{N,\vec{q}} = M_N + \dfrac{\vec{q}^{\,2}}{2M_N} + \ldots$ and find

$$C^{N}_{3,\mu=0} = \mathrm{O}(\frac{1}{M_N}) \qquad C^{N\pi}_{3,\mu=0} = \mathrm{O}(1)$$

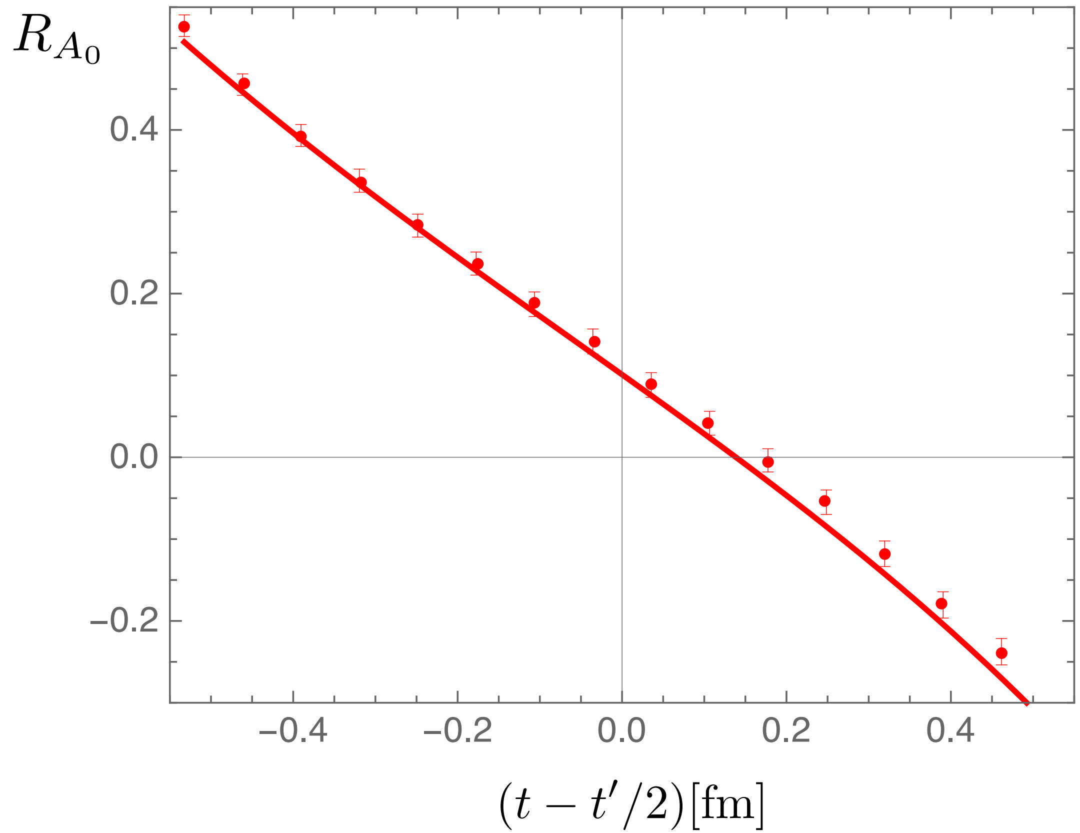
➥ $N\pi$ contamination is "O(M_{N})-enhanced" compared to single N contribution

- Dominant contribution from the tree diagrams with a low-energetic pion



- Relative sign between the two contributions
  ➥ $N\pi$ contribution reproduces the observed linear behaviour !

$N\pi$ contamination



from Bali *et. al.*, arXiv:1810.05569

$M_\pi \approx 150\,\mathrm{MeV}$

$M_\pi L \approx 3.47$

$|\vec{q}| = \frac{2\pi}{L}$

$t \approx 1.06\,\mathrm{fm}$

ChPT (no fit !)

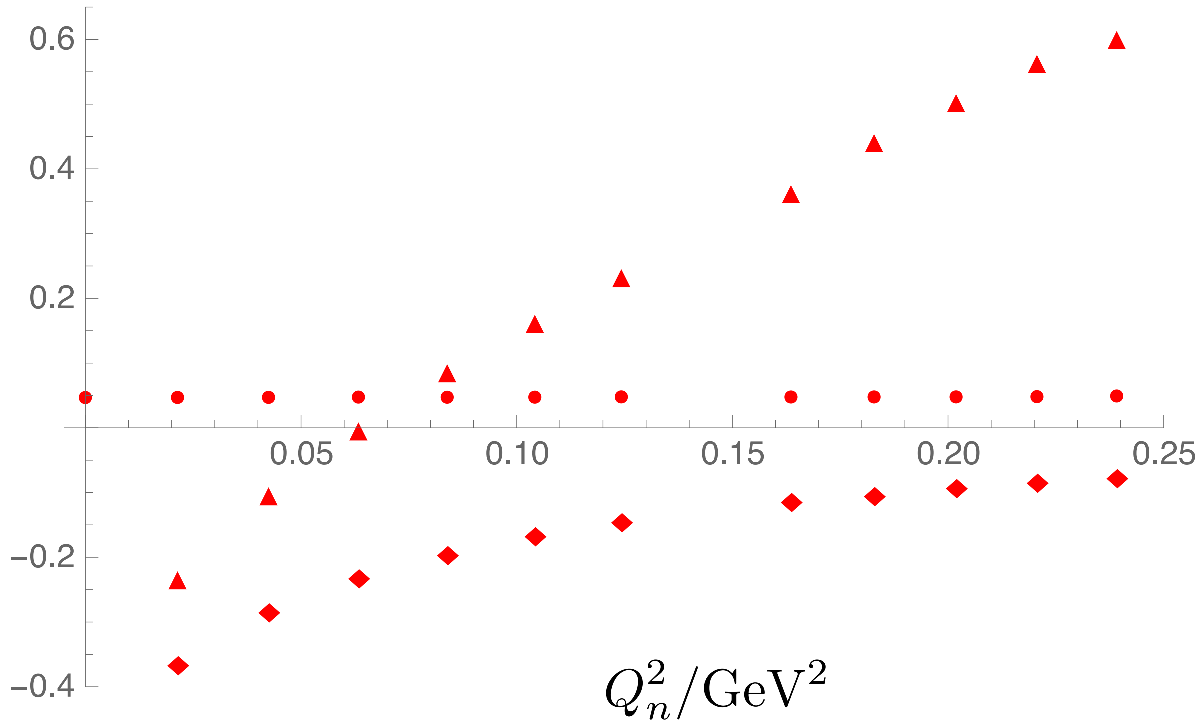
- ChPT reproduces the almost linear time dependence
- Very good agreement for all times !

  Expected: Reproduce the slope in the middle of the plot (if at all…)

ChPT works much better than expected. Why ???

$N\pi$ contamination in $G_P(Q^2)$

for t = 2 fm, t' = 1 fm (midpoint estimate)



$$\epsilon_X^{N\pi}(Q^2,t,t') = \frac{G_X^{\mathrm{eff}}(Q^2,t,t')}{G_X(Q^2)} - 1$$

▲ P

● A

◆ $\tilde{P}$

lowest discrete momenta for $M_\pi L = 6$ (e.g. PACS coll.)

- G_A^{plat} overestimates by ≈ 5% (no visible Q^2 dependence)
- $\tilde{G}_P^{\mathrm{plat}}$ underestimates by ≈10% - 40% depending on momentum transfer
- G_P^{plat} underestimates for $Q^2 \lesssim 0.06\ \mathrm{GeV}^2$ (up to ≈ -20%)

  overestimates for $Q^2 \gtrsim 0.06\ \mathrm{GeV}^2$ (up to ≈ +50%)

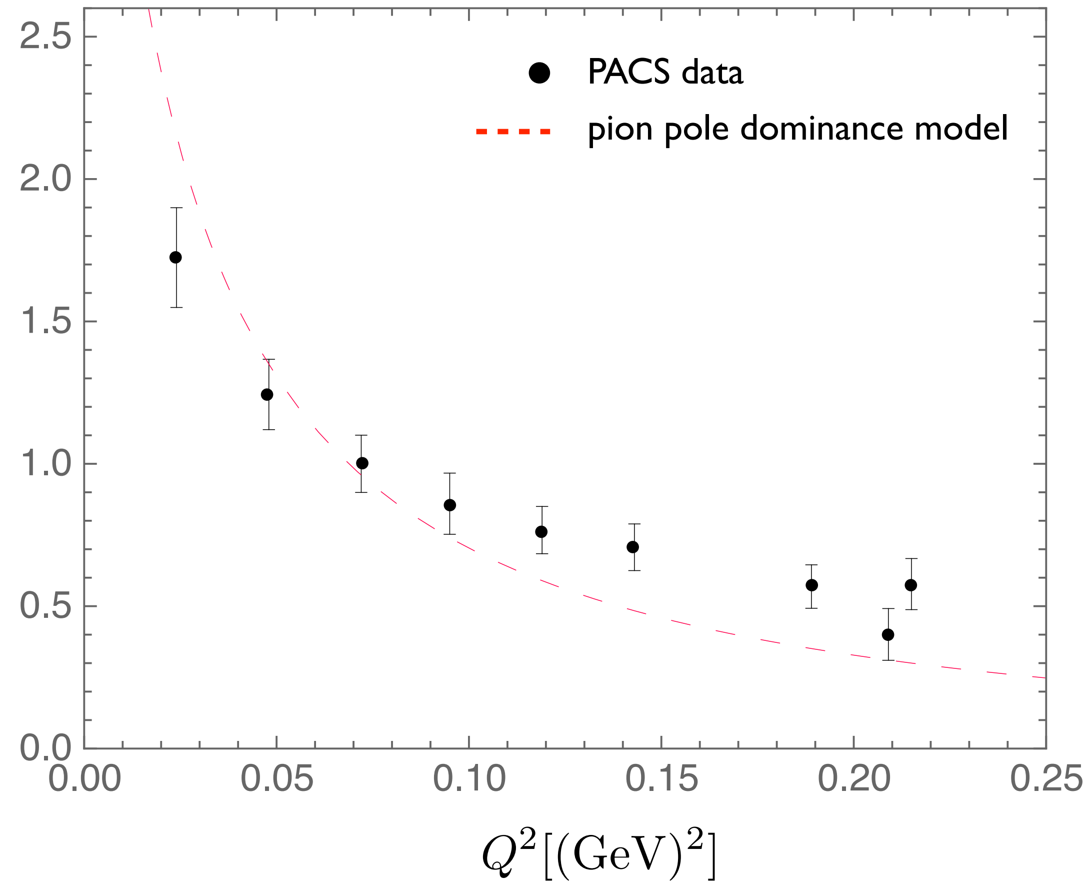
OB 2018

$N\pi$ contamination in $G_P(Q^2)$

for t = 1.3 fm

PACS plateau estimate data
for <u>normalized</u> form factor
independent of Z_P

$$G_P^{\rm norm}(Q^2, Q^2_{\rm ref}, t) \equiv \frac{G_P^{\rm plat}(Q^2, t)}{G_P^{\rm plat}(Q^2_{\rm ref}, t)}$$



- Data <u>underestimate</u> ppd model result <u>for small Q^2</u>
  <u>overestimate</u> ppd model result <u>for larger Q^2</u>

$N\pi$ contamination in $G_P(Q^2)$

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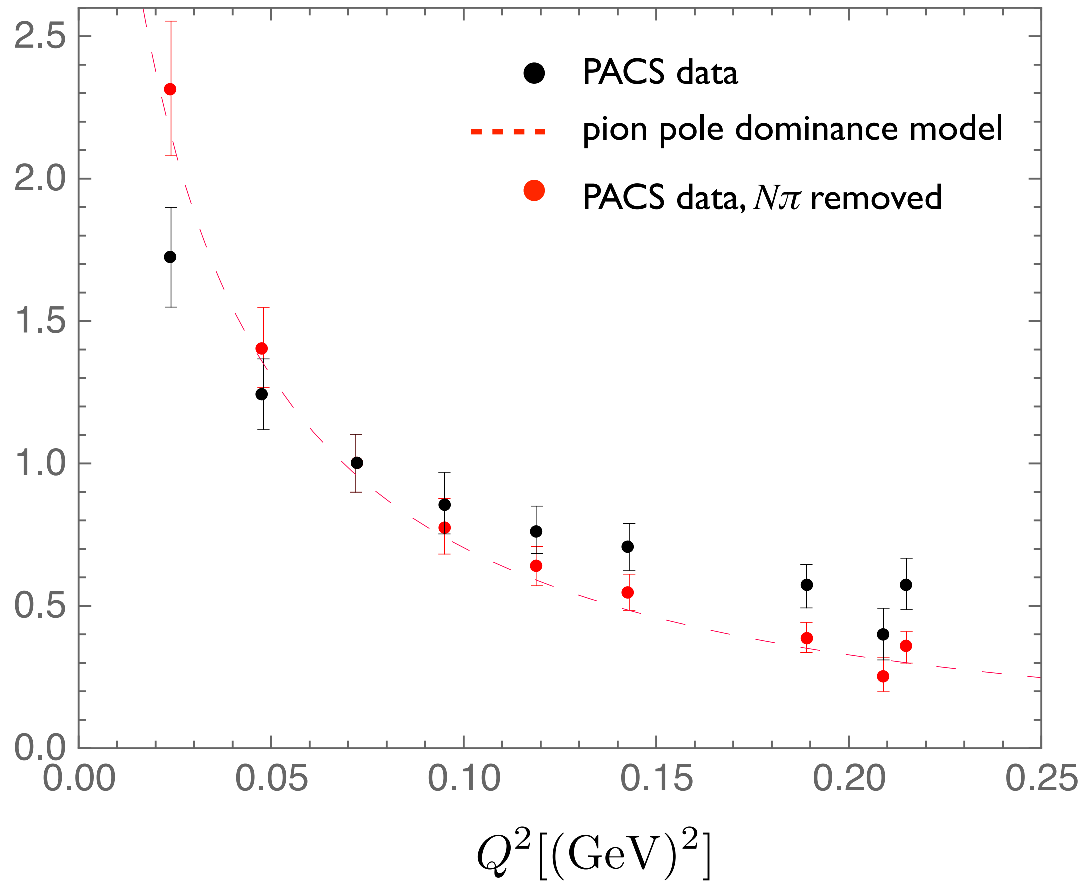


- Data <u>underestimate</u> ppd model result <u>for small Q^2</u>
  <u>overestimate</u> ppd model result <u>for larger Q^2</u>
- Remove the LO ChPT $N\pi$ contamination from data → much better agreement
  surprising since $t = 1.3$ fm

Generalized Goldberger-Treiman relation

PCAC implies a relation between the three form factors (also called PCAC$_{FF}$ relation):

$$2M_N G_A(Q^2) - \frac{Q^2}{2M_N}\tilde{G}_P(Q^2) = 2m_q G_P(Q^2)$$

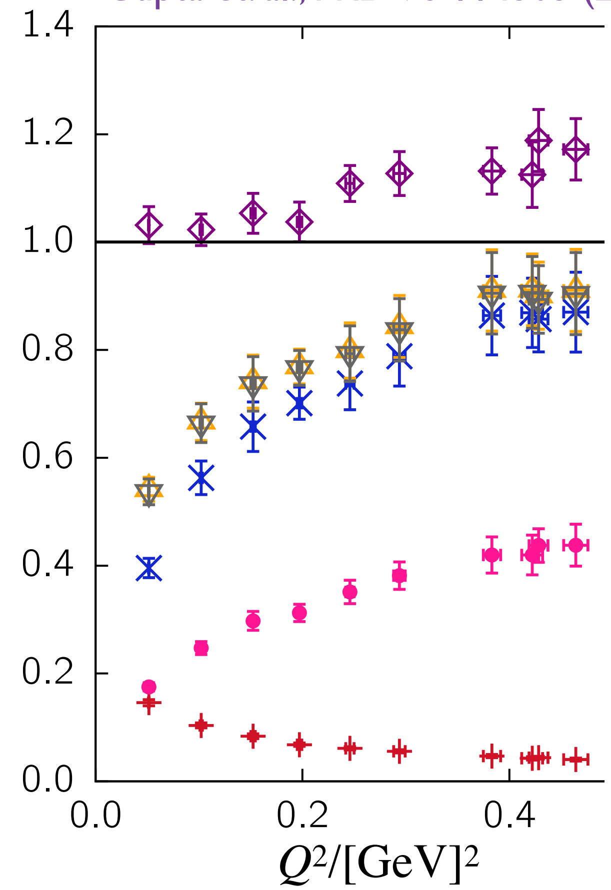
In general it is violated badly by lattice estimates, e.g. the ratio

$$r^{\rm est}_{\rm PCAC}(Q^2,t) = \frac{Q^2}{4M_N^2}\frac{\tilde{G}^{\rm est}_{\rm P}(Q^2,t)}{G^{\rm est}_{\rm A}(Q^2,t)} + \frac{2m_q}{2M_N}\frac{G^{\rm est}_{\rm P}(Q^2,t)}{G^{\rm est}_{\rm A}(Q^2,t)}$$

is typically < 1 and Q^2-dependent

Bali et. al. 2018
Gupta et. al. 2017

Plot from
Gupta et. al., PRD 96 114503 (2017)



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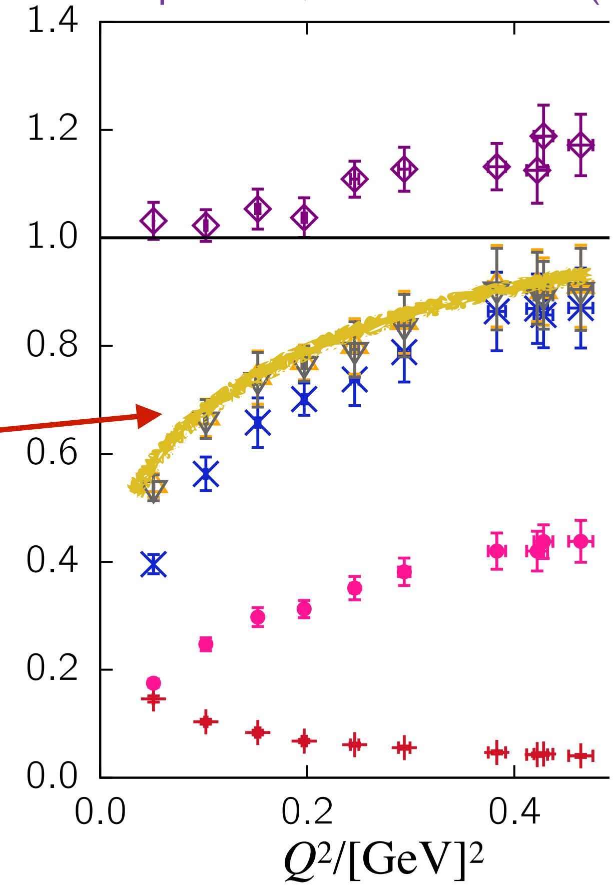
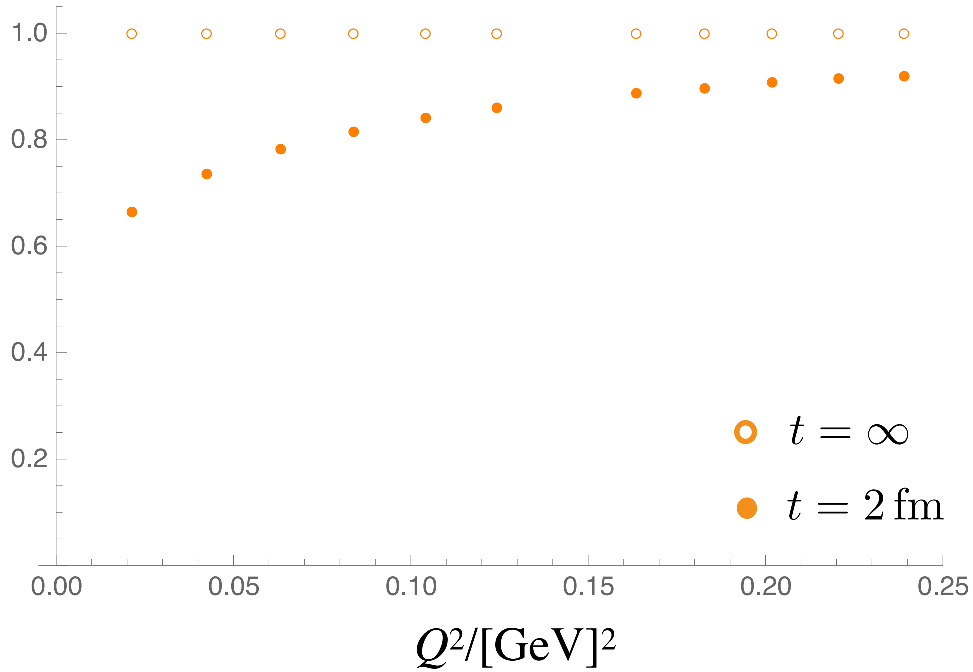
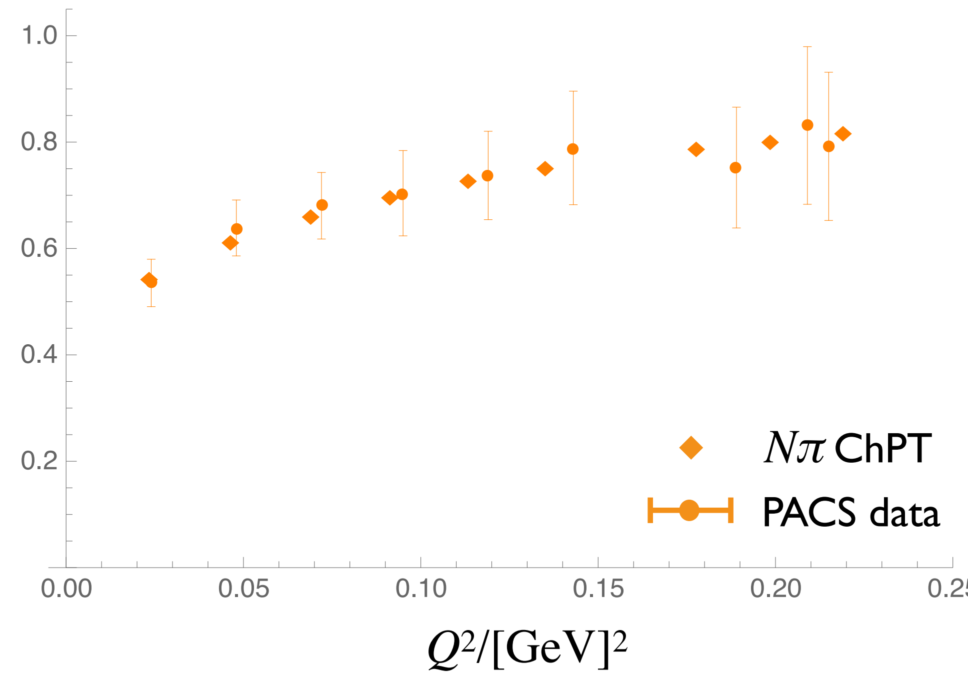


Impact $N\pi$-state contamination on r_{PCAC}

ChPT result



PACS data and ChPT, $t = 1.3$ fm



- ChPT result for $N\pi$ contamination predicts $r_{\text{PCAC}}^{\text{plat}}(Q^2, t) < 1$
- Good agreement with PACS data even for small source-sink separation $t = 1.3$fm
- Dominant source for $r_{\text{PCAC}}^{\text{plat}}(Q^2, t) < 1$: Large $N\pi$ contamination in $\tilde{G}_{\text{P}}(Q^2, t)$

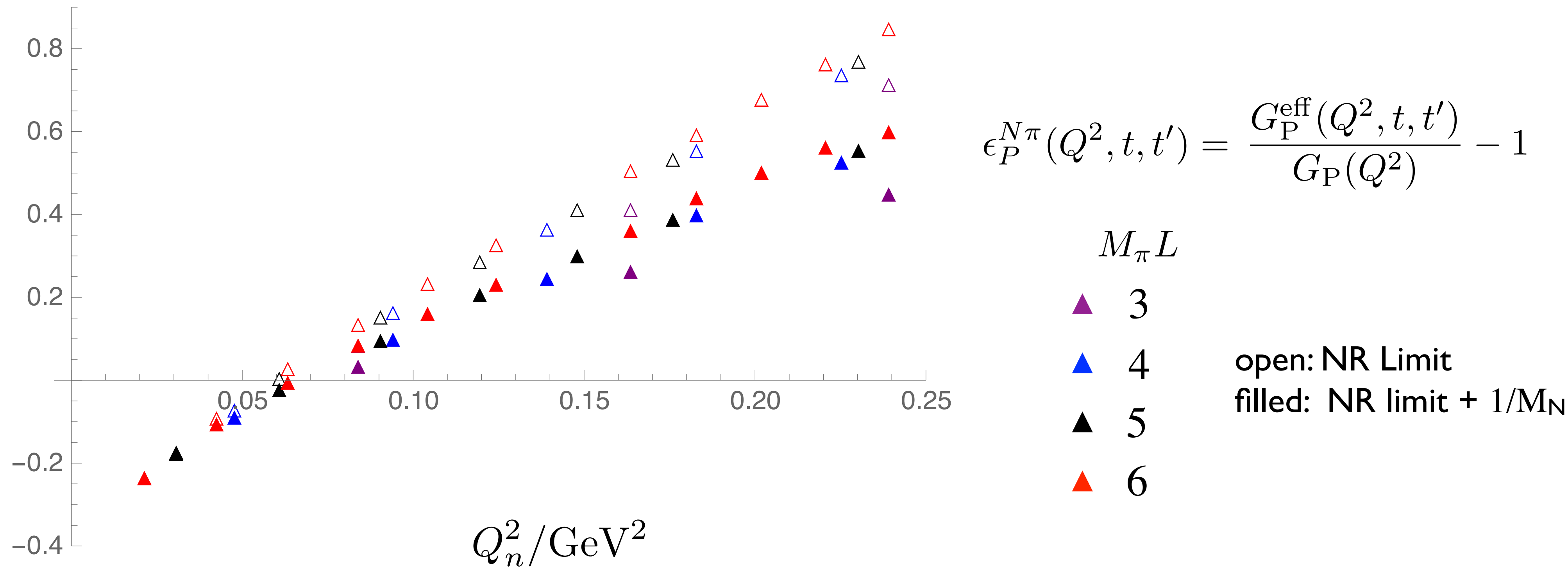
Summary

- LO ChT predicts significant $N\pi$ contamination in
  - 3-point function involving the temporal component A_0
  - pseudoscalar form factor $G_P(Q^2)$
  - generalized Goldberger-Treiman relation ($PCAC_{FF}$)
- Preliminary conclusions:
  - Deviations lattice results $\longleftrightarrow$ exp. / phen. data
    probably due to $N\pi$ excited states
    needs to be corroborated with more data
  - (Much) larger source-sink separations needed to extract form factors reliably
- Outlook: Analogous calculation for vector current form factors

Backup slides

$N\pi$ contamination in $G_P(Q^2)$

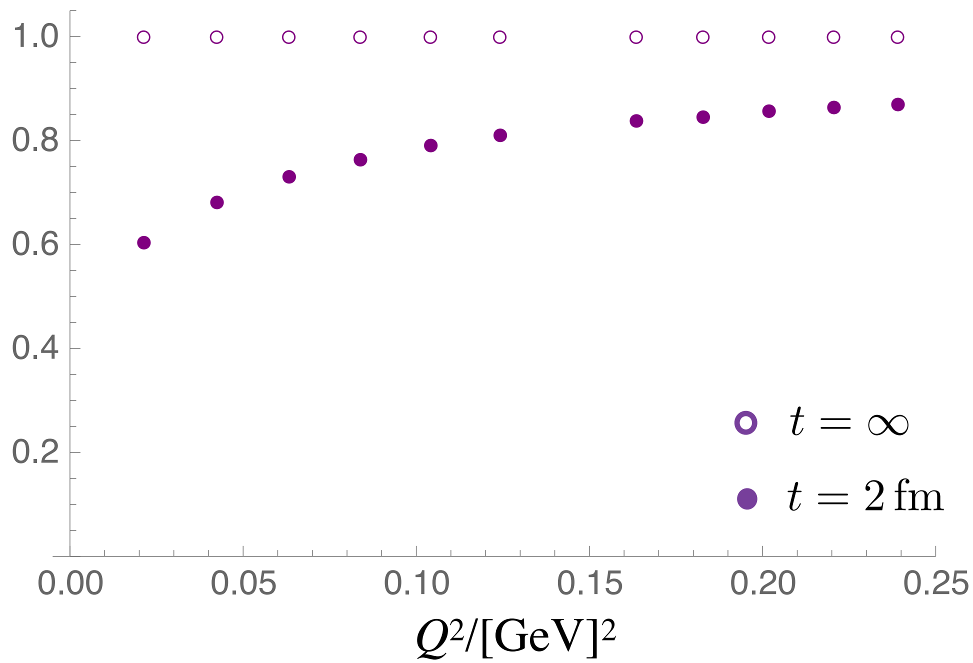
for t = 2 fm, t' = 1 fm (midpoint estimate)



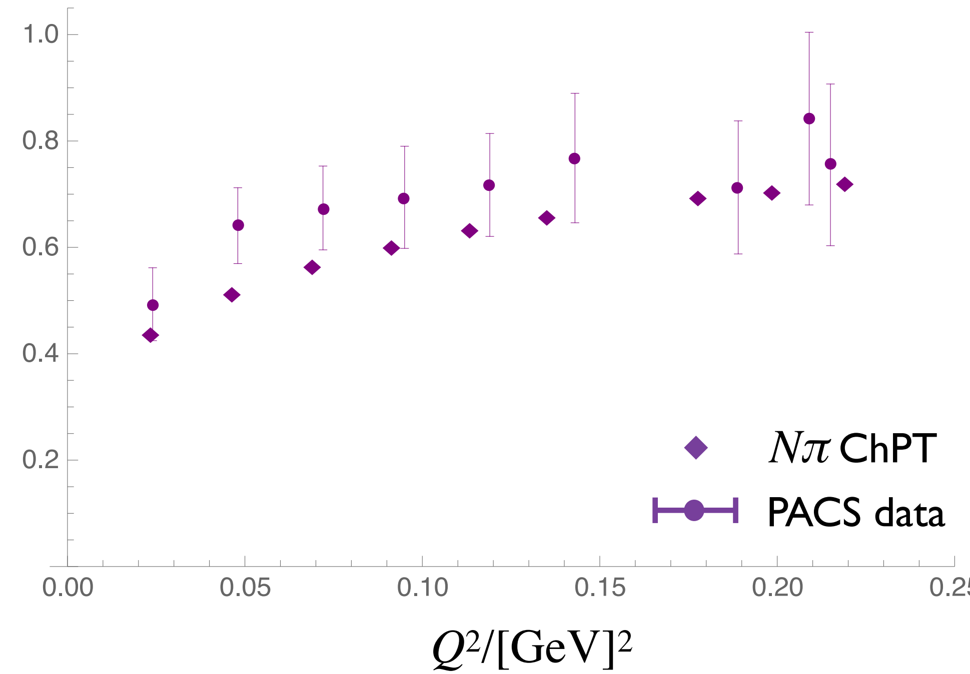
$$\epsilon_P^{N\pi}(Q^2,t,t') = \frac{G_\mathrm{P}^\mathrm{eff}(Q^2,t,t')}{G_\mathrm{P}(Q^2)} - 1$$

Impact $N\pi$-state contamination on r_3

ChPT result



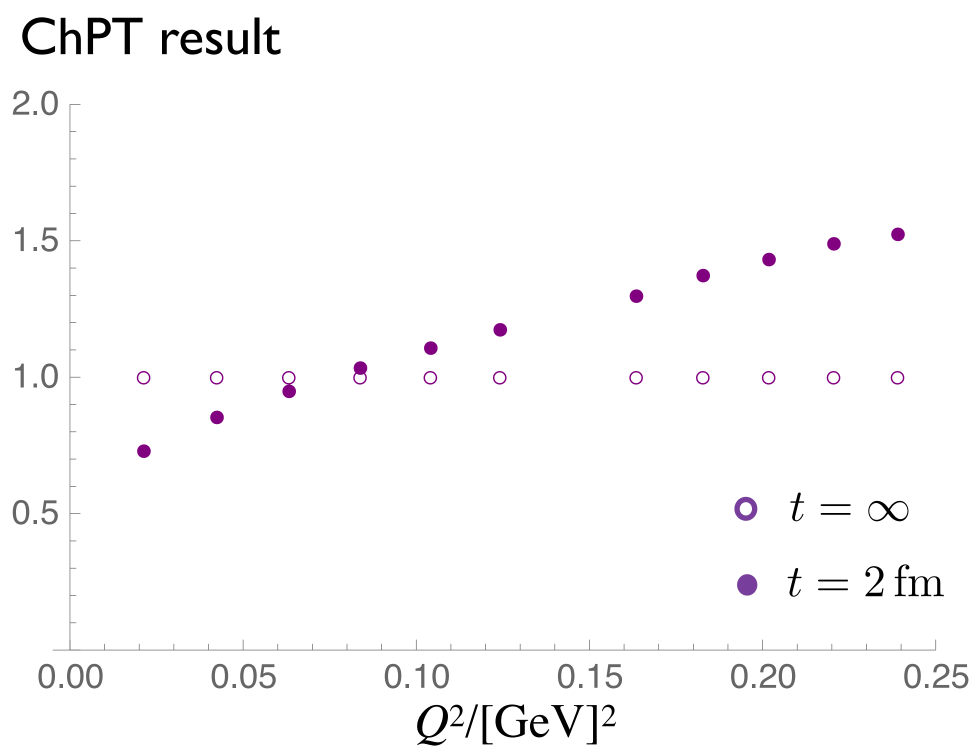
PACS data and ChPT, $t = 1.3$ fm

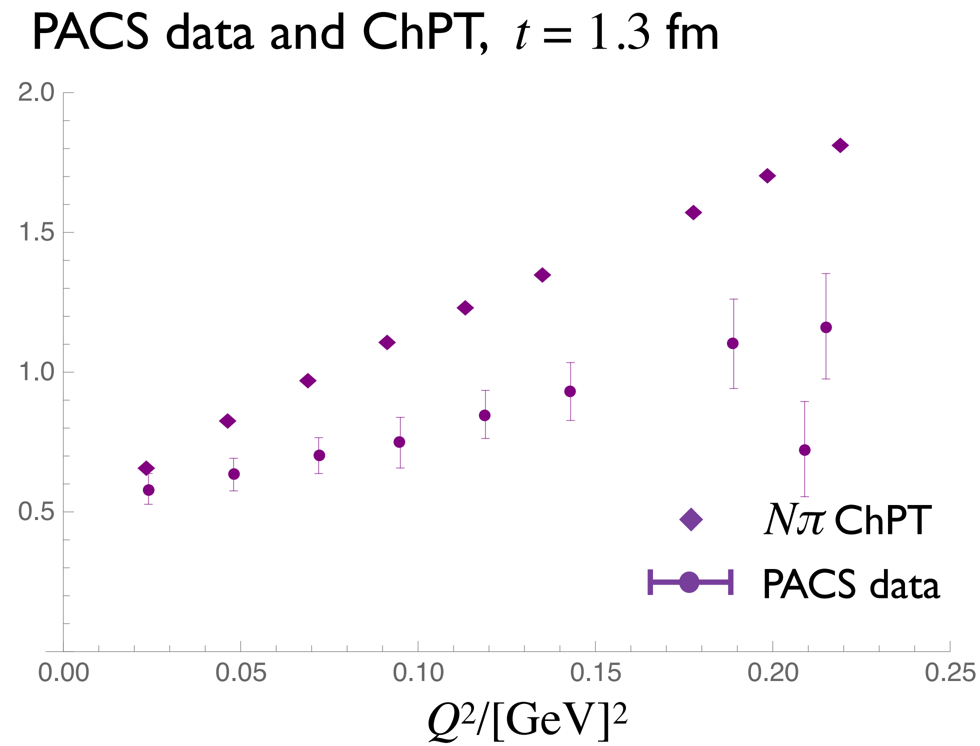


$$r_3^{\mathrm{est}}(Q^2, t) = \frac{Q^2 + M_\pi^2}{4M_N^2} \frac{\tilde{G}_{\mathrm{P}}^{\mathrm{est}}(Q^2, t)}{G_{\mathrm{A}}^{\mathrm{est}}(Q^2, t)}$$

- ChPT result for $N\pi$ contamination predicts $r_3^{\mathrm{est}}(Q^2, t) < 1$
- Good agreement with PACS data even for small source-sink separation $t = 1.3$fm
- Dominant source for $r_3^{\mathrm{est}}(Q^2, t) < 1$: Large $N\pi$ contamination in $\tilde{G}_{\mathrm{P}}(Q^2, t)$

Impact $N\pi$-state contamination on r_4

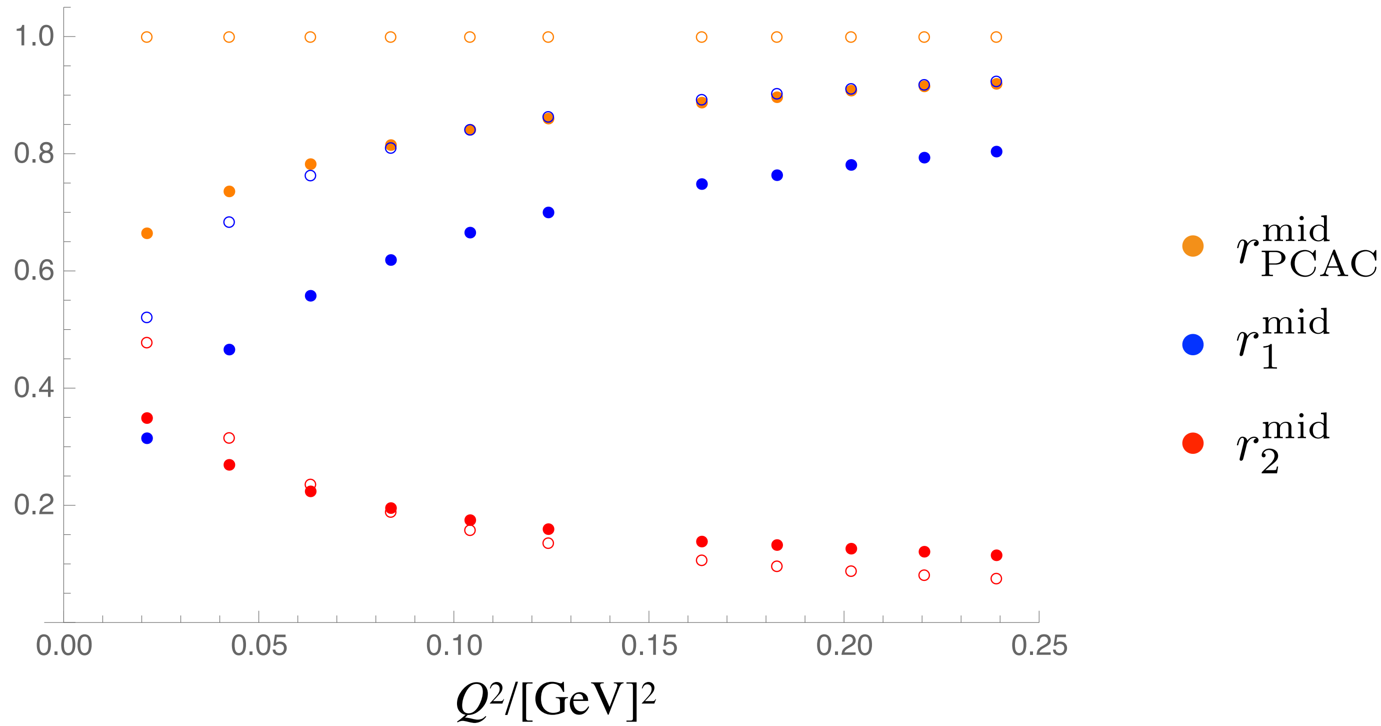




$$r_4^{\mathrm{est}}(Q^2,t) = \frac{Q^2+M_\pi^2}{2M_N M_\pi^2}\,\frac{2m_q G_{\mathrm{P}}^{\mathrm{est}}(Q^2,t)}{G_{\mathrm{A}}^{\mathrm{est}}(Q^2,t)}$$

- Poor agreement with PACS data

Impact $N\pi$-state contamination on PCAC$_{\mathrm{FF}}$



$$r_1^{\mathrm{est}}(Q^2,t) = \frac{Q^2}{4M_N^2}\frac{\tilde{G}_{\mathrm{P}}^{\mathrm{est}}(Q^2,t)}{G_{\mathrm{A}}^{\mathrm{est}}(Q^2,t)}$$

$$r_2^{\mathrm{est}}(Q^2,t = \frac{2m_q}{2M_N}\frac{G_{\mathrm{P}}^{\mathrm{est}}(Q^2,t)}{G_{\mathrm{A}}^{\mathrm{est}}(Q^2,t)}$$

$N\pi$ contamination in the correlation functions

3-pt function:

$$C_{3,\mu}(\vec{q},t,t') = C^{N}_{3,\mu}(\vec{q},t,t') + C^{N\pi}_{3,\mu}(\vec{q},t,t')$$
$$= C^{N}_{3,\mu}(\vec{q},t,t')\Big(1 + Z_\mu(\vec{q},t,t')\Big)$$

computable in ChPT

2-pt function: analogously

Ratios:

$$R_\mu(\vec{q},t,t') = \Pi_\mu(\vec{q})\left(1 + Z_\mu(\vec{q},t,t') + \frac{1}{2}Y(\vec{q},t,t')\right)$$

from 2-pt functions

$N\pi$ contamination in the correlation functions

$$Z_\mu(\vec{q},t,t') = a_\mu(\vec{q})e^{-\Delta E(0,\vec{q})(t-t')} + \tilde{a}_\mu(\vec{q})e^{-\Delta E(\vec{q},-\vec{q})t'}$$ tree diagrams

$$+\sum_{\vec{p}} b_\mu(\vec{q},\vec{p})e^{-\Delta E(0,\vec{p})(t-t')} + \tilde{b}_\mu(\vec{q},\vec{p})e^{-\Delta E(q,\vec{p})t'}$$

loop diagrams

$$+\sum_{\vec{p}} c_\mu(\vec{q},\vec{p})e^{-\Delta E(0,\vec{p})(t-t')}e^{-\Delta E(\vec{q},\vec{p})t'}$$

Energy gaps:

$$\begin{aligned}\Delta E(0,\vec{q}) &= E_{\pi,\vec{q}} + E_{N,q} - M_N \\ \Delta E(0,\vec{p}) &= E_{\pi,\vec{p}} + E_{N,p} - M_N \\ \Delta E(\vec{q},-\vec{q}) &= E_{\pi,\vec{q}} + M - E_{N,q}\end{aligned}$$

Non-trivial results of the ChPT calcultion: The coefficients in Z_μ

$N\pi$ contamination in the correlation functions

Example: Coefficients a_k from the tree-level diagrams

$$a_k(\vec{q}) = a_k^{\infty}(\vec{q}) + \frac{E_{\pi,q}}{M_N} a_k^{\mathrm{corr}}(\vec{q}) + \mathrm{O}\left(\frac{1}{M_N^2}\right)$$

NR Limit: $$a^{\infty}_{k=1,2}(\vec{q}) = -\frac{1}{2} \qquad a^{\infty}_{k=3}(\vec{q}) = \frac{1}{2}\frac{q_3^2}{E_{\pi,q}^2 - q_3^2}$$

Relevant result for approximate $\epsilon_{\tilde{P}}^{N\pi}$

Correction:

$$a^{\mathrm{corr}}_{k=1,2}(\vec{q}) = -\frac{1}{4}\left(\frac{M_\pi^2}{E_{\pi,\vec{q}}^2} - \frac{1}{g_A}\right) \qquad a^{\mathrm{corr}}_{k=3}(\vec{q}) = \frac{1}{4}\left(\frac{M_\pi^2}{E_{\pi,\vec{q}}^2} - \frac{1}{g_A}\right)\frac{q_3^2}{E_{\pi,q}^2 - q_3^2}$$

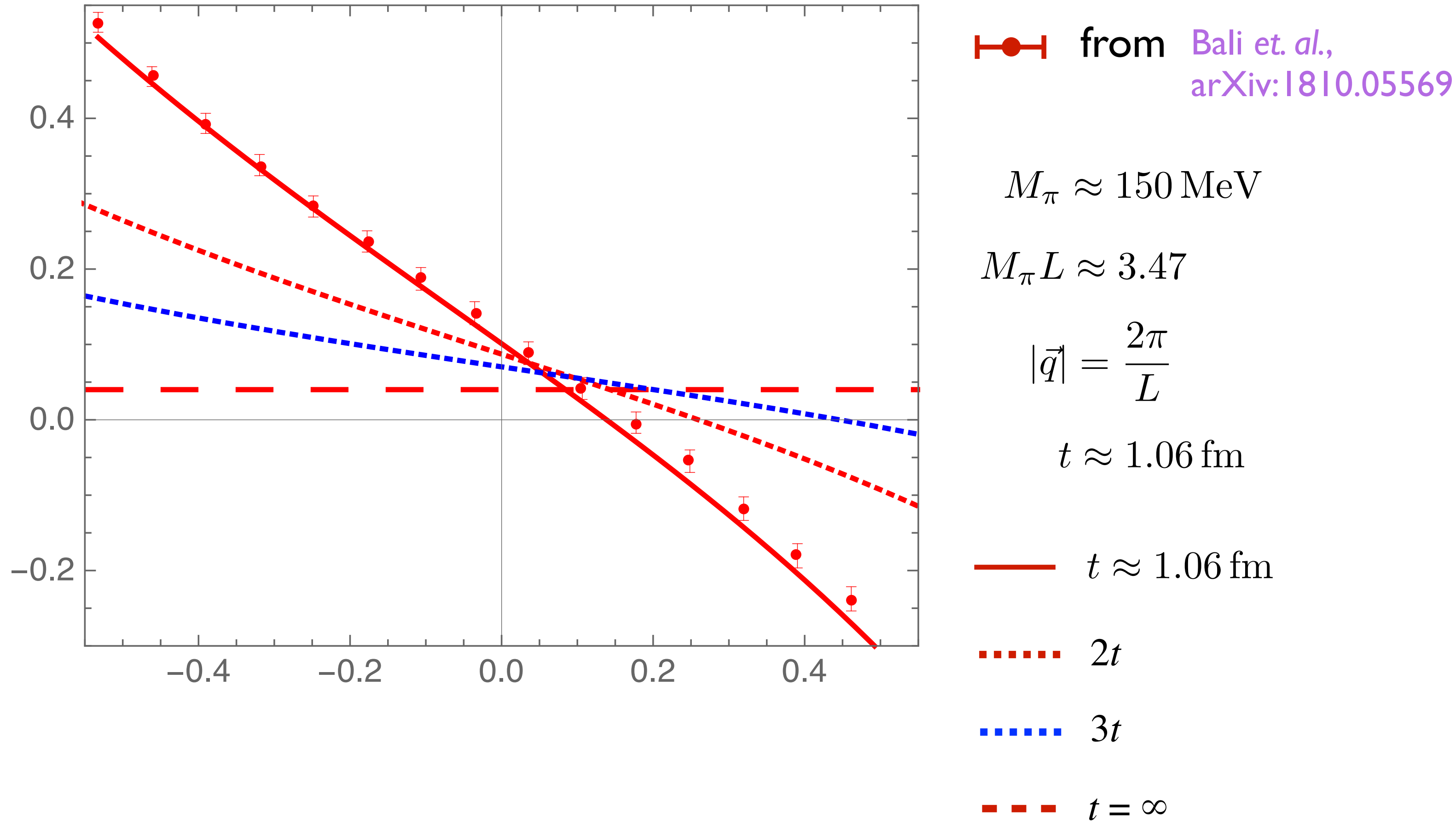
ChPT: Single nucleon contribution

2-pt function

3-pt function

➡ $$G_{\mathrm{A}}(Q^2) = g_A \qquad G_{\mathrm{P}}(Q^2) \;=\; 4M_N^2 \frac{g_A}{Q^2 + M_\pi^2}$$

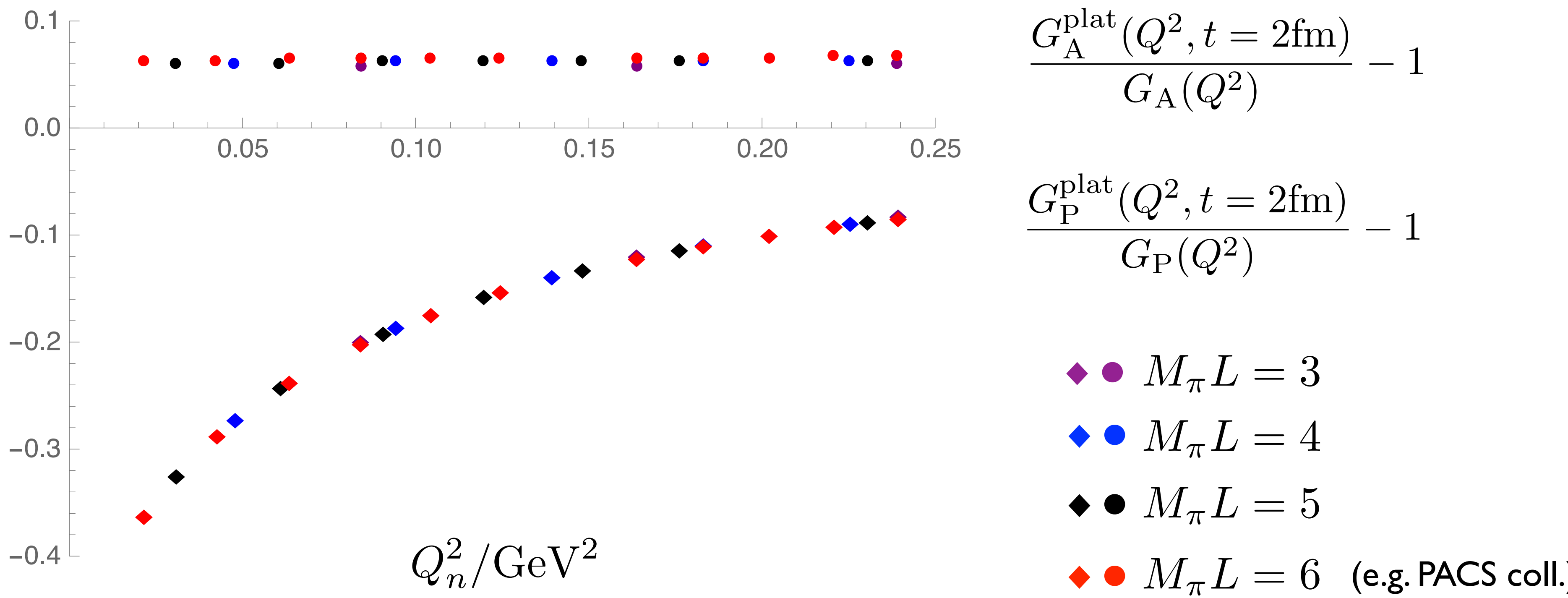
$N\pi$ contamination in A_0 correlator



Much larger t values needed to get the true plateau (N contribution) !

Results

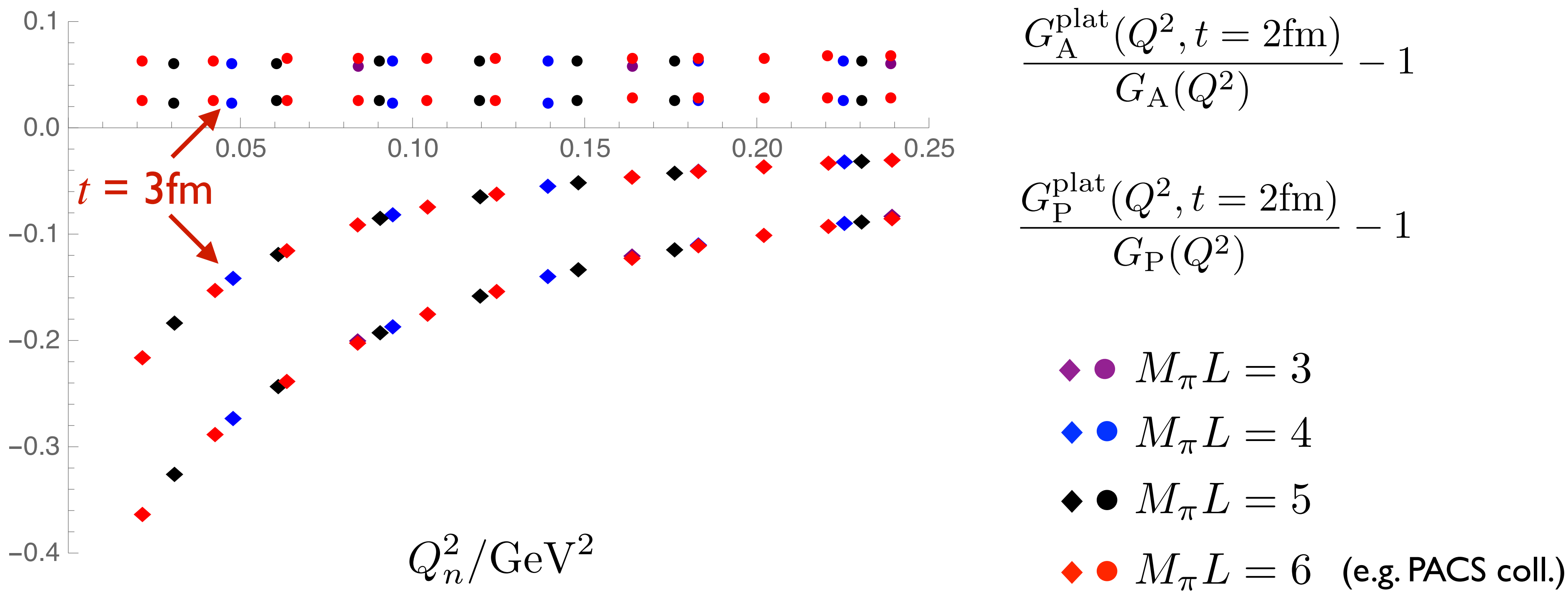
for t = 2 fm



- G_A^plat overestimates by ≈ 5% (no visible Q^2 dependence)
  - ➥ agrees with result for g_A in previous calculation
- G_P^plat underestimates by ≈10% - 40% depending on momentum transfer
- Small FV effect for $M_\pi L \geqq 3$

Results

for t = 2 fm



- G_A^plat overestimates by ≈ 5% (no visible Q^2 dependence)
  ➥ agrees with result for g_A in previous calculation
- G_P^plat underestimates by ≈10% - 40% depending on momentum transfer
- Small FV effect for $M_\pi L \geqq 3$
- Increasing t to 3 fm reduces $N\pi$ contribution roughly by a factor $1/2$