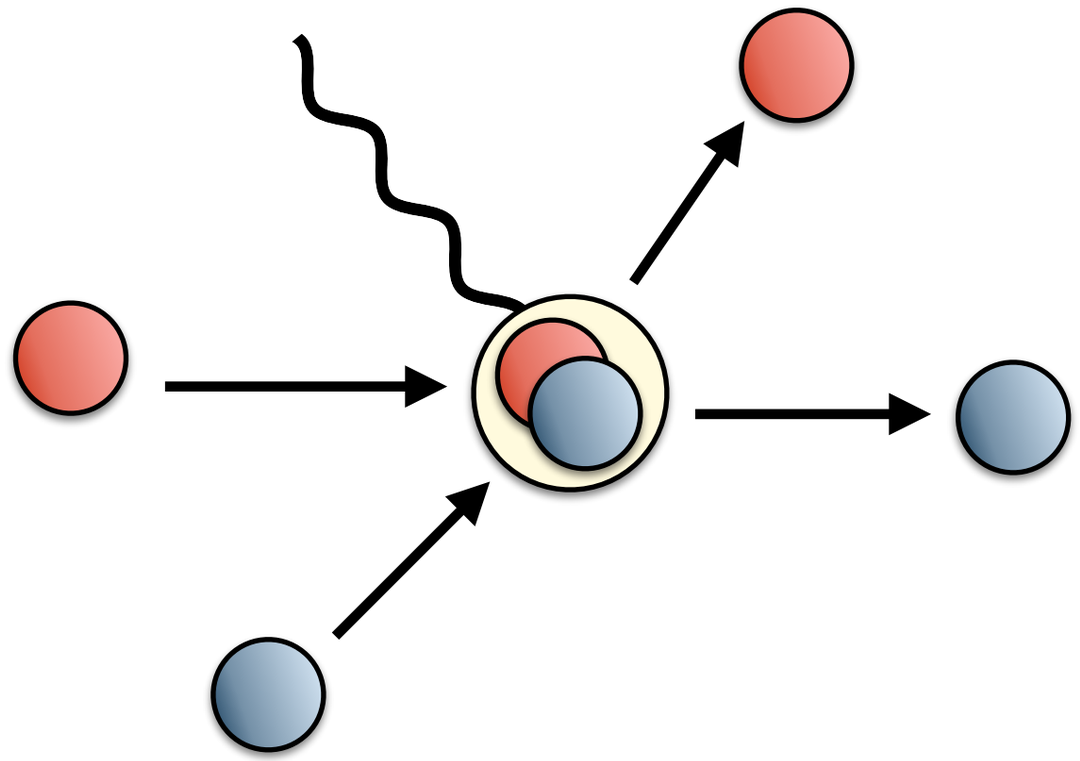
Matrix Elements of Bound States in a Finite Volume



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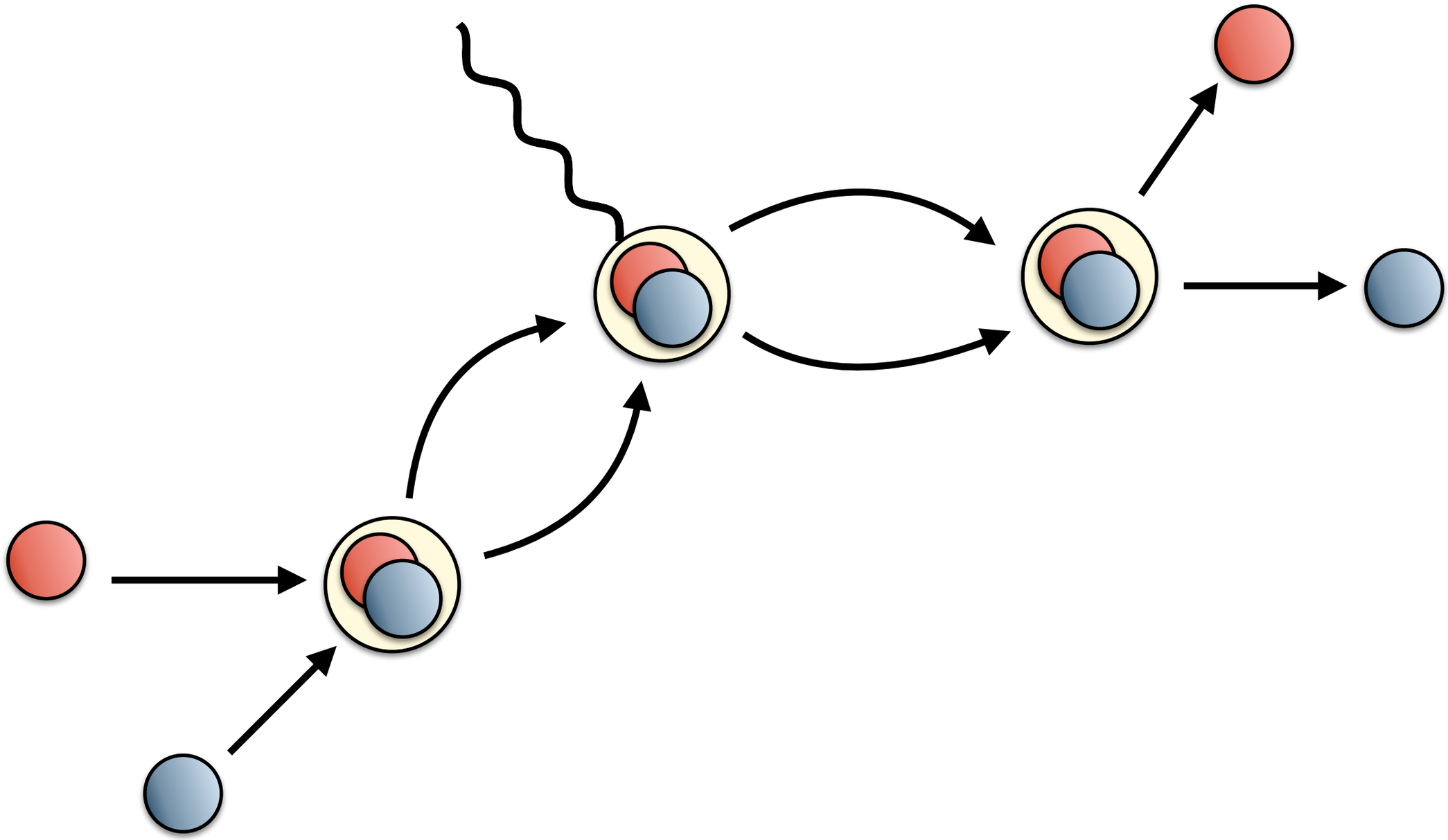
Old Dominion University

Lattice 2019
June 16-22, 2019

Structure of composite states

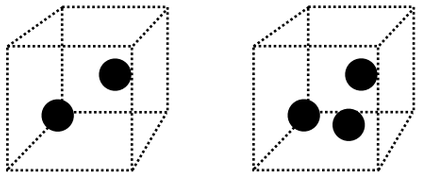
- Have seen a plethora of studies determining hadronic bound-states/ resonances from QCD
- How can we systematically determine their structure?
- Experimentally, form-factors, charge radii, pdfs, …, give us insight
- Can we determine these in Lattice QCD for resonances/ bound states?



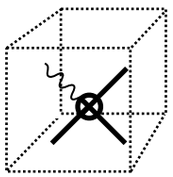
Roadmap

lattice QCD

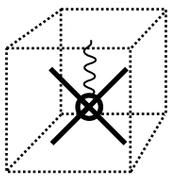
finite-volume spectrum



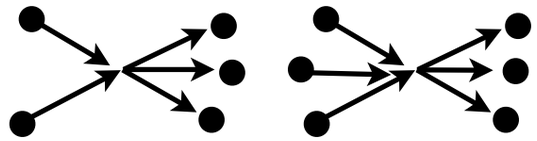
1-to-2 FV matrix elements



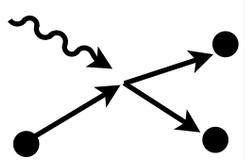
2-to-2 FV matrix elements



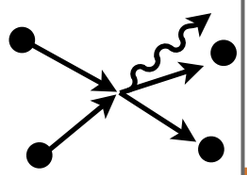
PW amplitudes



electroweak amplitudes



electroweak amplitudes

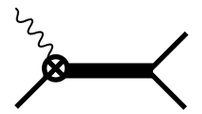


analytic continuation

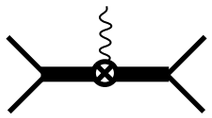
identification of
- states [masses & widths],
- production/decay mechanisms

resonance poles

transition form factors



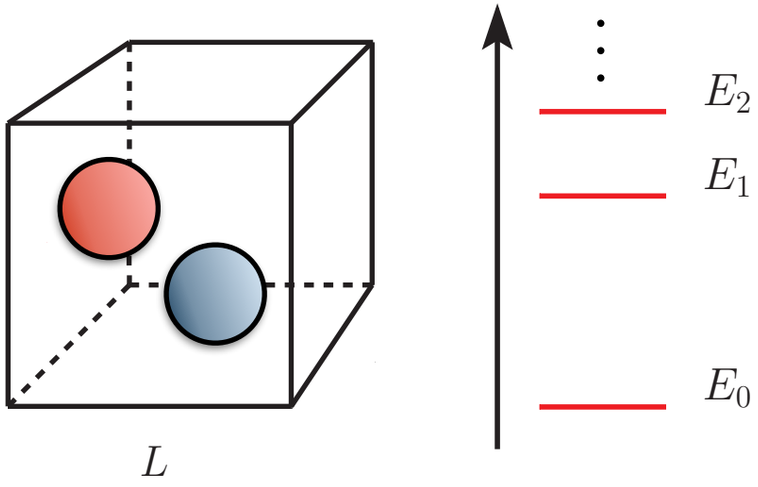
elastic form factors



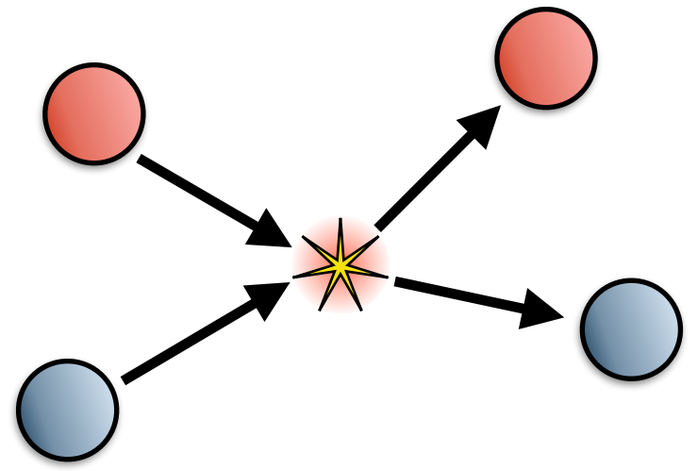
Structure & nature of states
- Form-factors, charge radii
- pdfs

Spectroscopy

Lüscher Quantization Condition $\det\left[\,1 + \mathcal{M}(E_n^\star)\, F(P_n; L)\,\right] = 0$



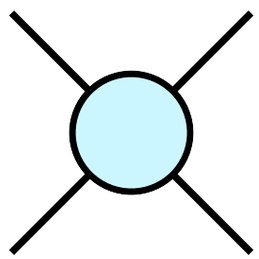
$\Longleftrightarrow$



- Lüscher (1986, 1991)
- Rummukainen & Gottlieb (1995)
- Kim, Sachrajda, & Sharpe/Christ, Kim & Yamazaki (2005)
- Feng, Li, & Liu (2004)
- Hansen & Sharpe / Briceño & Davoudi (2012)
- Briceño (2014)

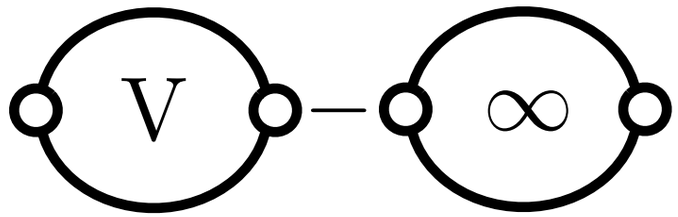
$\mathcal{M} =$ Scattering amplitude

$=$



$F =$ Finite volume function

$=$



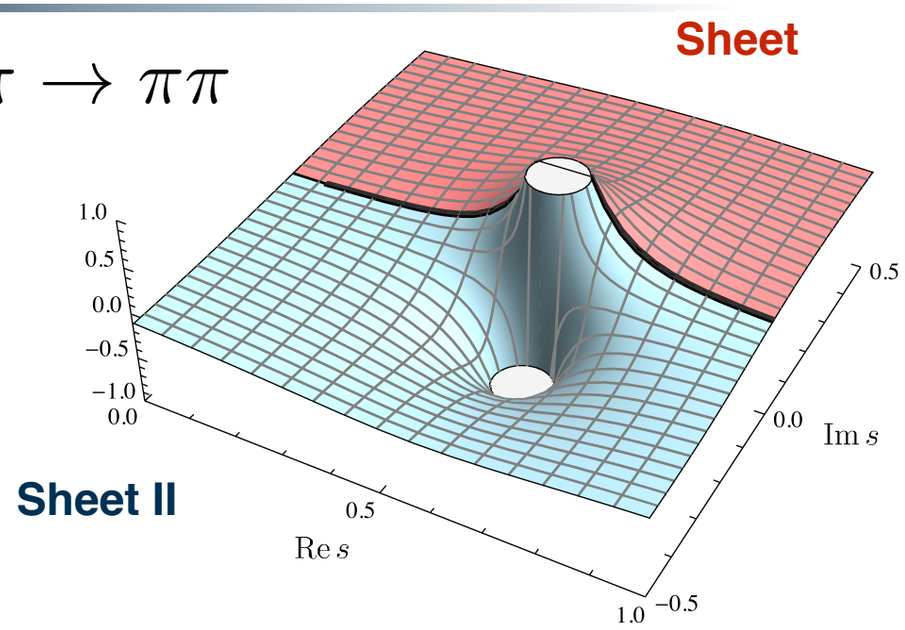
Spectroscopy

- Many examples of successful determination of resonances

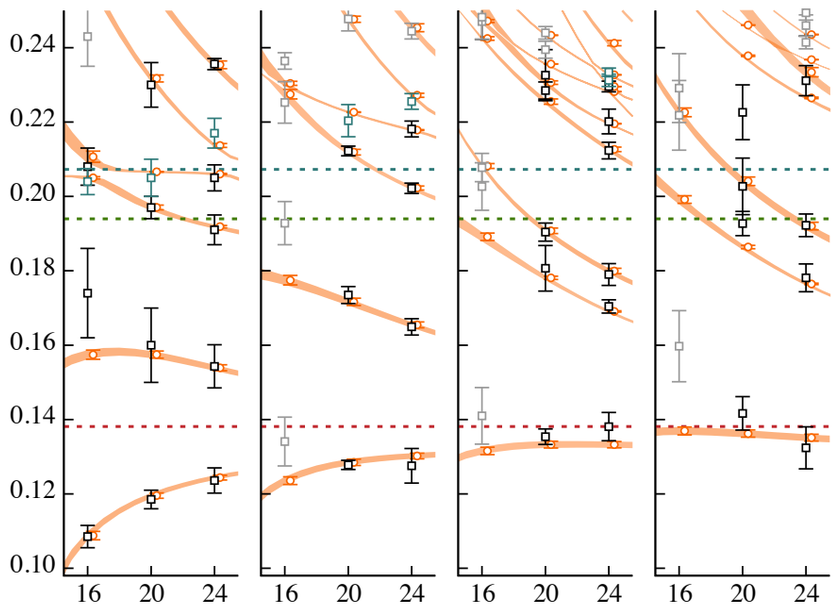
$$i\mathcal{M} = i\mathcal{K}\,\frac{1}{1 - i\rho\,\mathcal{K}}$$

$$\mathcal{M} \sim \frac{g^2}{s - s_{\rm res}}$$

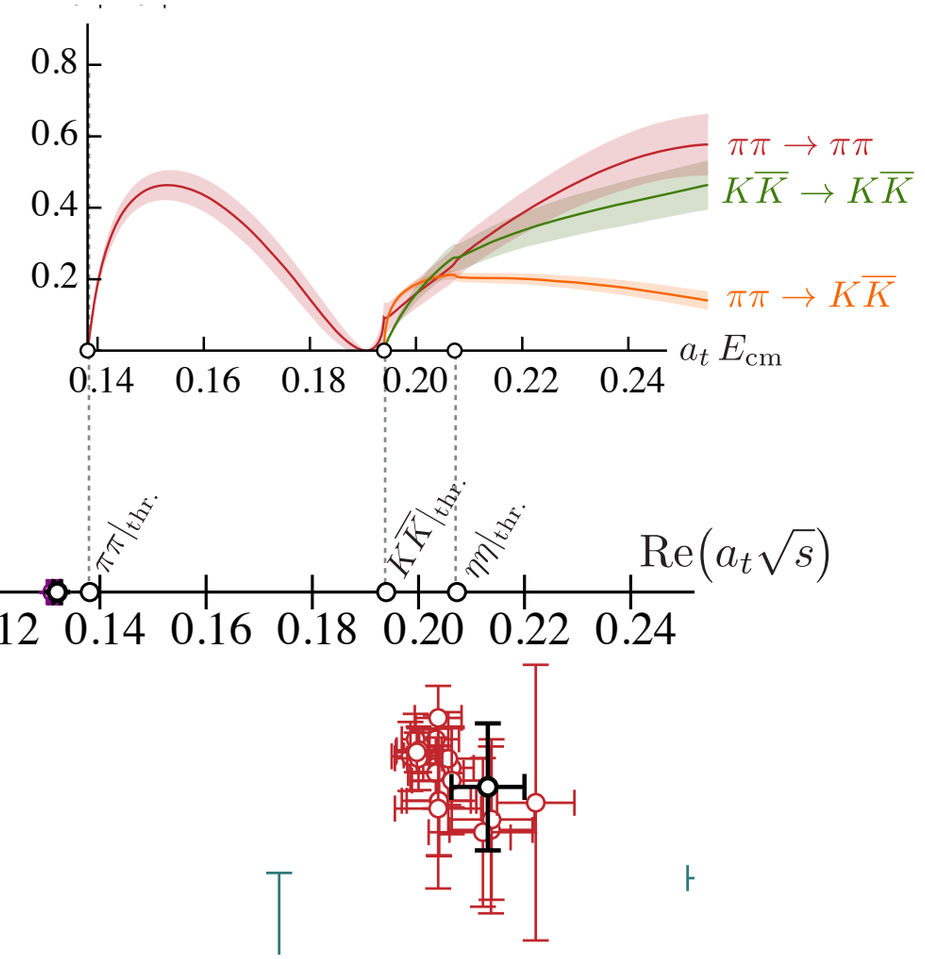
$\pi\pi \to \pi\pi$



had spec



$\Longrightarrow$



Briceño, Dudek, Edwards, & Wilson

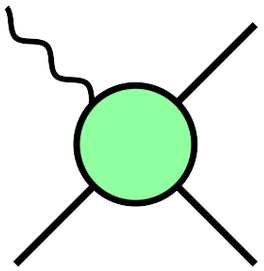
Transition Amplitudes

- Investigate production of resonances via external currents $\pi\gamma^* \to \pi\pi$

$$\langle \mathbf{2} | \, \mathcal{J} \, | \mathbf{1} \rangle_L \sim \mathcal{A}\sqrt{\mathcal{R}}$$

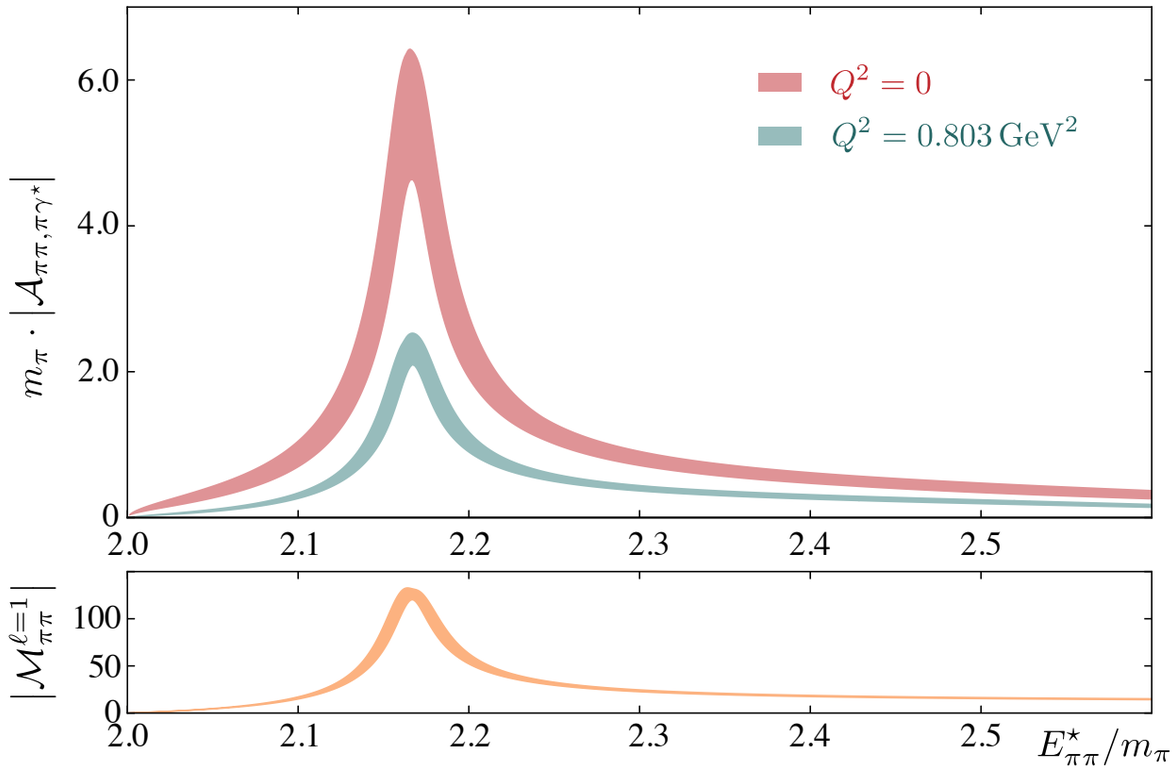
$\mathcal{A} =$ Transition amplitude

$=$



Lellouch-Lüscher factor

$$\mathcal{R}(P_n; L) = \lim_{E \to E_n} \left[\frac{(E - E_n)}{F^{-1}(P, L) + \mathcal{M}(s)} \right]$$



- Lellouch & Lüscher (2000)
- Kim, Sachrajda, & Sharpe
- Christ, Kim & Yamazaki (2005)
- Hansen & Sharpe (2012)
- Briceño, Hansen Walker-Loud (2014)
- Briceño & Hansen (2015)
- … and many others

Briceño, Dudek, Edwards, Shultz, Thomas, & Wilson

Two-Hadron Form Factors

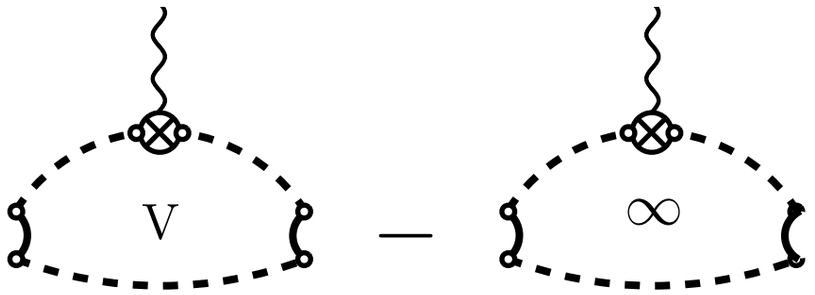
- Extend ideas to external currents coupling to two-hadron systems
- Systematic method to study structure of resonances/ bound states

$$\pi\pi\gamma^* \to \pi\pi$$

$$\langle \mathbf{2}| \mathcal{J} |\mathbf{2}\rangle_L \sim \mathcal{R} \cdot [\mathcal{W}_{\mathrm{df}} + f\,\mathcal{M}\,G\,\mathcal{M}]$$

$f =$ Single hadron form factor

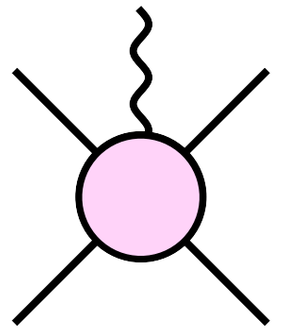
$G =$ New finite volume function

$=$ 

Generalized form factor

$$\mathcal{W}_{\mathrm{df}} = \frac{1}{1 - \mathcal{K}_f\, i\rho_f} \mathcal{F} \frac{1}{1 - i\rho_i\, \mathcal{K}_i}$$

- Bernard, Hoja, Meißner, Rusetsky (2012)
- Briceño, Hansen (2016)
- Baroni, Briceño, Hansen, and Ortega-Gama (2018)

$=$ 

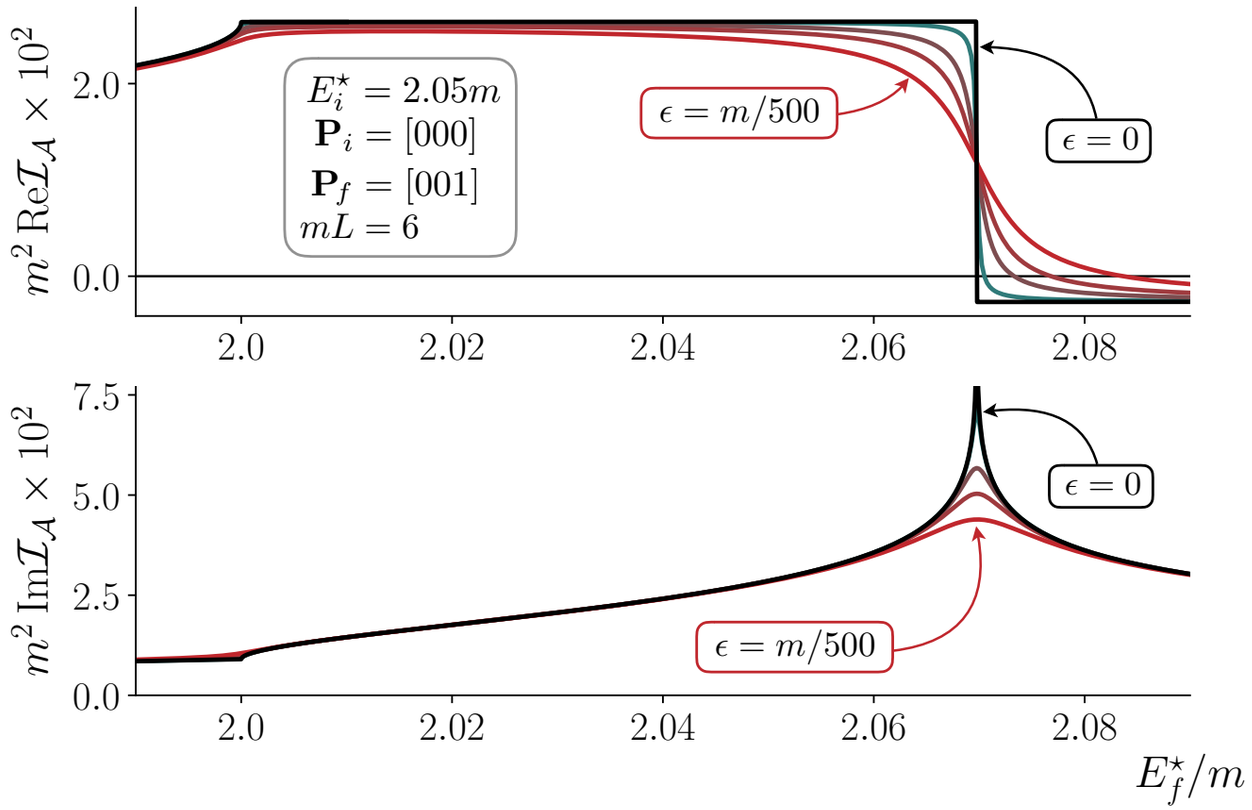
Finite Volume function — *G*

- New finite volume function — *G* (Here *S*-wave)

$$G^{\mu_1\cdots\mu_n}(P_f,P_i;L) = \left[\frac{1}{L^3}\sum\!\!\!\!\!\!\int_{\mathbf{k}}\right] \frac{k^{\mu_1}\cdots k^{\mu_n}}{2\omega_{\mathbf{k}}((P_f-k)^2-m^2+i\epsilon)((P_i-k)^2-m^2+i\epsilon)}\Big|_{k^0=\omega_{\mathbf{k}}}$$

$$G^{\mu_1\cdots\mu_n}(P_f,P_i;L) = \quad \text{V} \quad - \quad \infty$$

- Analytic behavior understood
  - Threshold cusp
  - Triangle singularities



- Baroni, Briceño, Hansen, and Ortega-Gama (2018)

Finite Volume function — *G*

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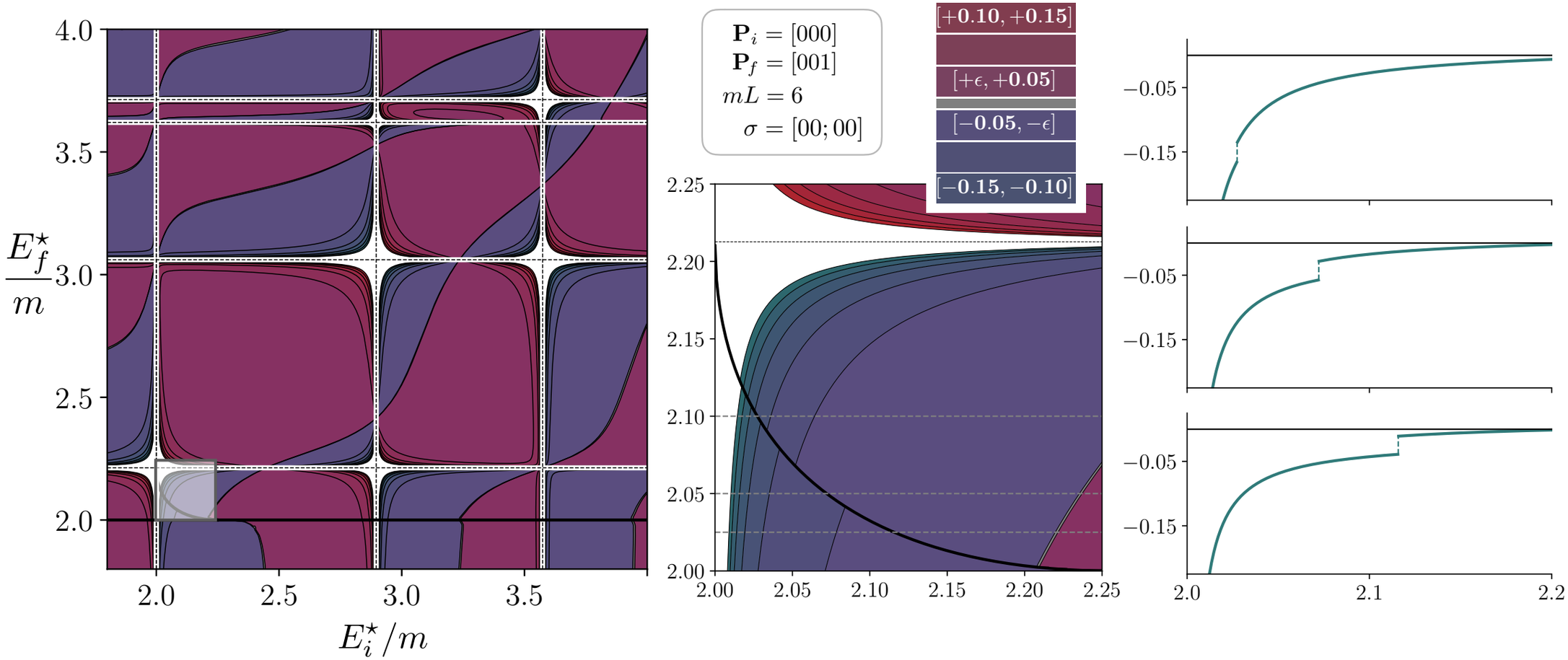
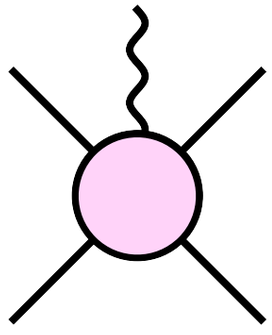
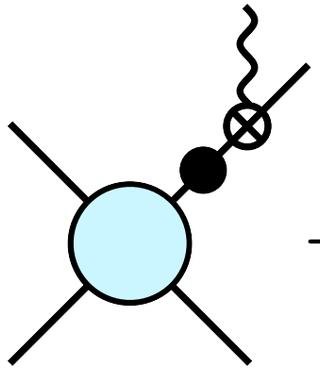
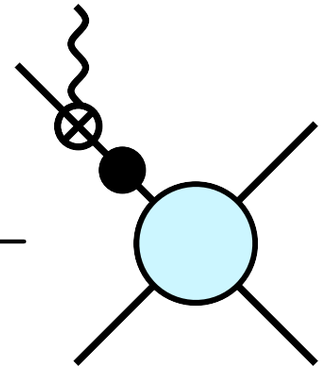


2+*J*→2 Matrix Elements

- Infinite volume matrix elements $\langle \mathbf{2}| \mathcal{J} |\mathbf{2}\rangle_\infty = \mathcal{W}$

- Remove long-range effects on external legs

$$\mathcal{W}_{\rm df} =$$  $$= \mathcal{W} -$$  $$-$$ 

- Consider simplest cases — *S*-wave systems coupling to vector and scalar currents

  - Vector current — Charge distribution
  - Scalar current — Compactness of constituents

Vector $$\mathcal{W}^{\mu}_{{\rm df},L} = \mathcal{W}^{\mu}_{\rm df} + f\mathcal{M}\left[(P_i+P_f)^{\mu}G - 2G^{\mu}\right]\mathcal{M}$$

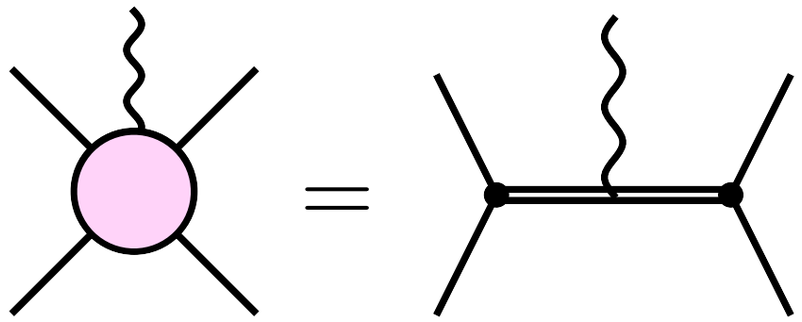
Scalar $$\mathcal{W}_{{\rm df},L} = \mathcal{W}_{\rm df} + f\,\mathcal{M}G\mathcal{M}$$

Matrix elements of bound states

- Near the bound state pole

$$\lim_{s_i, s_f \to s_{\mathrm{B}}} \mathcal{W}^{\mu}_{\mathrm{df}}(P_f, P_i) = ig \, \frac{i}{s_f - s_{\mathrm{B}}} \, (P_i + P_f)^{\mu} \, F_{\mathrm{B}}(Q^2) \, \frac{i}{s_i - s_{\mathrm{B}}} \, ig$$

$$\lim_{s_i, s_f \to s_{\mathrm{B}}}$$



- Consistency of the electric charge $F_{\mathrm{B}}(0) = Q_0$

- In the finite volume $\langle \mathbf{2} | \, \mathcal{J} \, | \mathbf{2} \rangle_L \sim \mathcal{W}_{\mathrm{df},L} \cdot \mathcal{R}$

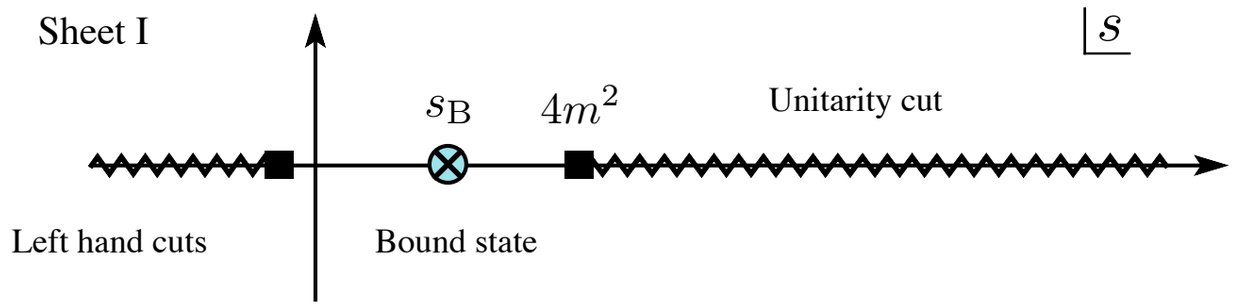
- Require — Finite volume spectra of bound state

  — Analytic representation for scattering amplitude

  — Analytic parameterization for form-factor

Bound States

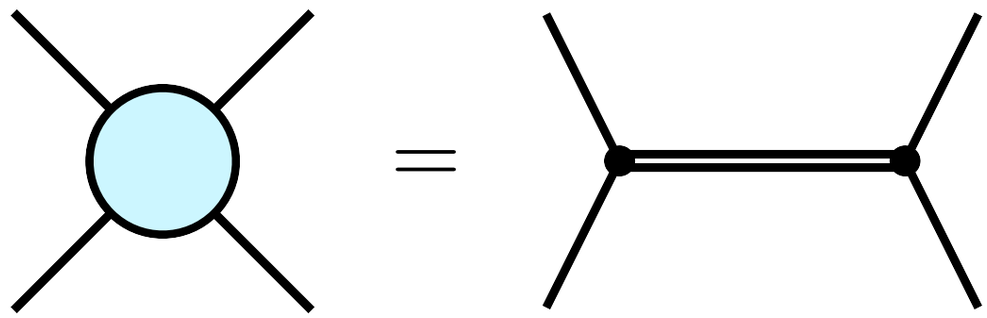
- Apply framework to an *S*-wave bound state
- Use deuteron in *pn*-scattering as toy example

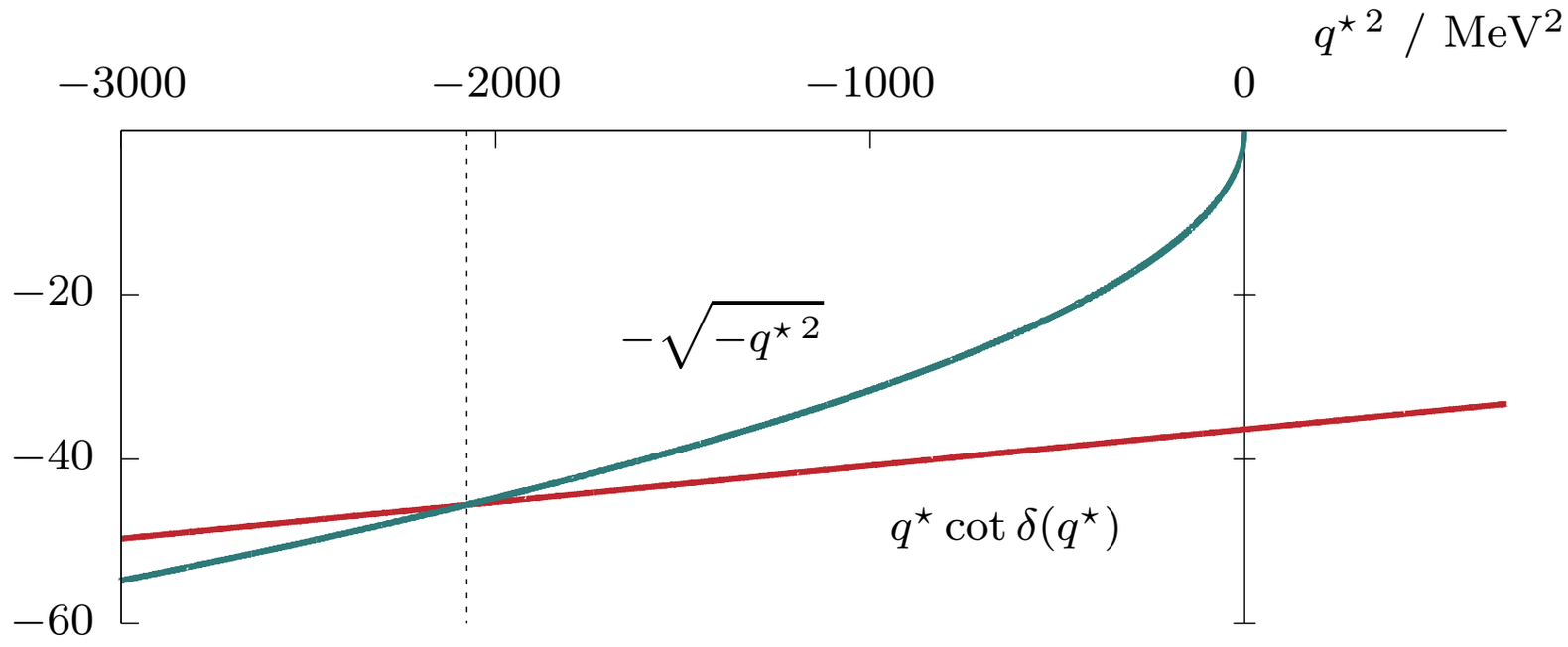


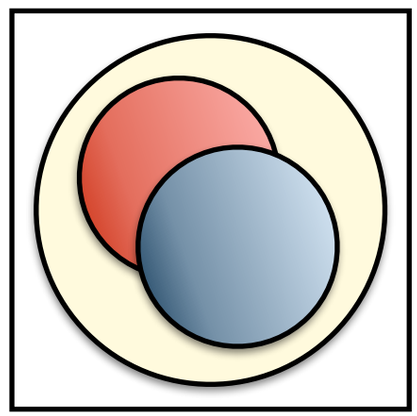
$$\lim_{s\to s_{\rm B}} \mathcal{M}(s) = \frac{(ig)^2}{s - s_{\rm B}}$$

$$q^\star \cot\delta(q^\star)\big|_{q^\star = i\kappa_{\rm B}} + \kappa_{\rm B} = 0$$

$$\lim_{s\to s_{\rm B}}$$







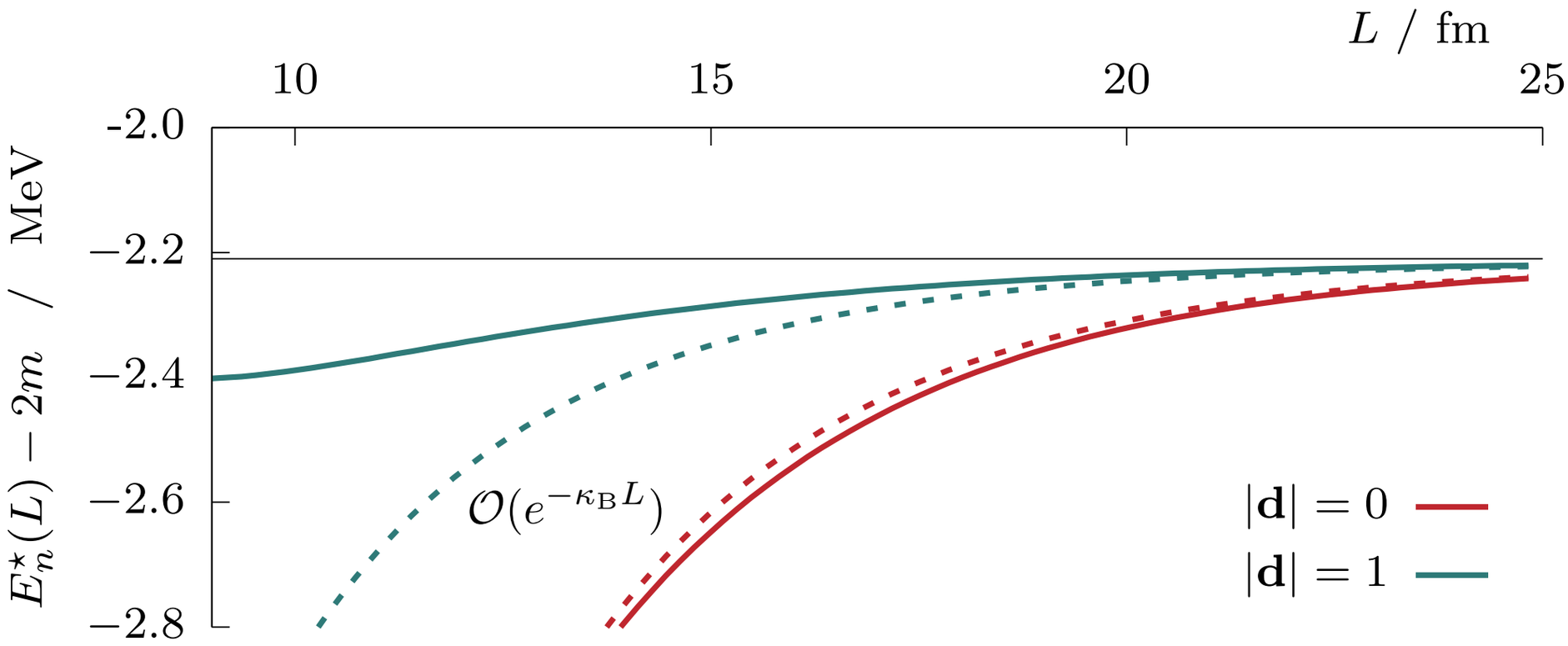
L

Finite volume spectrum

- Bound state exponentially close to Lüscher pole

$$\delta s_{\mathrm{B}} = \mathcal{O}(e^{-\kappa_{\mathrm{B}} L})$$

$$\mathcal{M}^{-1}(s_n) = -F(P_n; L) \quad \Longrightarrow \quad s_n = s_{\mathrm{B}} + \delta s_{\mathrm{B}}$$



(Here *S*-wave)

- Davoudi & Savage (2011)

$$F(P;L) = \left[\frac{1}{L^3}\sum\!\!\!\!\!\!\int_{\mathbf{k}}\right] \frac{1}{2\omega_{\mathbf{k}}((P-k)^2 - m^2 + i\epsilon)}\Big|_{k^0=\omega_{\mathbf{k}}}$$

Matrix elements of vector currents

- First investigate *S*-wave system coupling to a vector current
- At zero-momentum transfer, matrix element defines the charge — $\langle \mathbf{2} | \, \mathcal{J}^{\mu=0} \, | \mathbf{2} \rangle_L \xrightarrow[Q^2 \to 0]{} \frac{Q_0}{L^3}$
- Deeply bound state — Finite volume dependence vanishes $\kappa_{\mathrm{B}} L \to \infty$
  - Lellouch-Lüscher factor $\mathcal{R}^{-1} = (\mathcal{M})' = -\frac{2Eg^2}{(s - s_{\mathrm{B}})^2}$
  - Matrix element $\langle \mathbf{2} | \, \mathcal{J}^{\mu} \, | \mathbf{2} \rangle_L = \frac{1}{2\sqrt{E_i E_f} L^3} (P_i + P_f)^{\mu} \, F_{\mathrm{B}}(Q^2)$
- Temporal component at zero-momentum transfer

$$\langle \mathbf{2} | \, \mathcal{J}^{\mu} \, | \mathbf{2} \rangle_L \xrightarrow[Q^2 \to 0]{} \frac{F_{\mathrm{B}}(0)}{L^3} \quad \Longrightarrow \quad F_{\mathrm{B}}(0) = Q_0$$ ✔

First check that formalism reproduces expectations

Matrix elements of vector currents

- Next, consider a shallow bound state
- Require that matrix element has no non-trivial volume dependence

$$\langle \mathbf{2} | \, \mathcal{J}^{\mu=0} \, | \mathbf{2} \rangle_L \xrightarrow[Q^2 \to 0]{} \frac{Q_0}{L^3}$$

- Should see volume independence
- Formalism yields the matrix element

$$\langle \mathbf{2} | \, \mathcal{J} \, | \mathbf{2} \rangle_L = \frac{1}{L^3} \frac{1}{(i\rho\mathcal{K})' - (\mathcal{K}F)'} \left[\frac{\mathcal{F}^\mu}{\mathcal{K}} + Q_0 \, \mathcal{K} \Big(2P^\mu \, G - 2G^\mu \Big) \right]$$

Same finite volume piece!

- Lellouch-Lüscher factor simplifies

$$(i\rho\mathcal{K})' - (\mathcal{K}F)' = \frac{\mathcal{K}}{\mathcal{M}^2} \frac{\mathrm{d}}{\mathrm{d}E} \mathcal{M} + \mathcal{K} \Big(2E \, G - 2G^{\mu=0} \Big) \qquad (\,)' = \frac{\mathrm{d}}{\mathrm{d}E}$$

- Constrained form-factor at zero-momentum transfer

$$\mathcal{F}^{\mu=0} = Q_0 \frac{\mathcal{K}^2}{\mathcal{M}^2} \frac{\mathrm{d}}{\mathrm{d}E} \mathcal{M} \implies \mathcal{W}_{\mathrm{df}}^{\mu=0} = Q_0 \frac{\mathrm{d}}{\mathrm{d}E} \mathcal{M}$$

Recover WTI!

Scalar Currents - non-trivial volume dependence

- Compare vector current to scalar current

Scalar current form factor

- Deeply bound states — $\langle \mathbf{2} | \, \mathcal{J} \, | \mathbf{2} \rangle_L = \dfrac{F_{\mathrm{B}}(0)}{2EL}$

- Shallow bound states — look at leading order effects $\quad \delta s_{\mathrm{B}} = \mathcal{O}(e^{-\kappa_{\mathrm{B}} L})$

- Look at leading order effects

$$\langle \mathbf{2} | \, \mathcal{J} \, | \mathbf{2} \rangle_L = \frac{F_{\mathrm{B}}(0)}{2EL} \left[1 + \delta s_{\mathrm{B}} \left(\cdots \right) + \frac{g^2}{E} G^{\mu=0} \right] + \mathcal{O}(\delta s_{\mathrm{B}}^2)$$

Infinite volume quantities

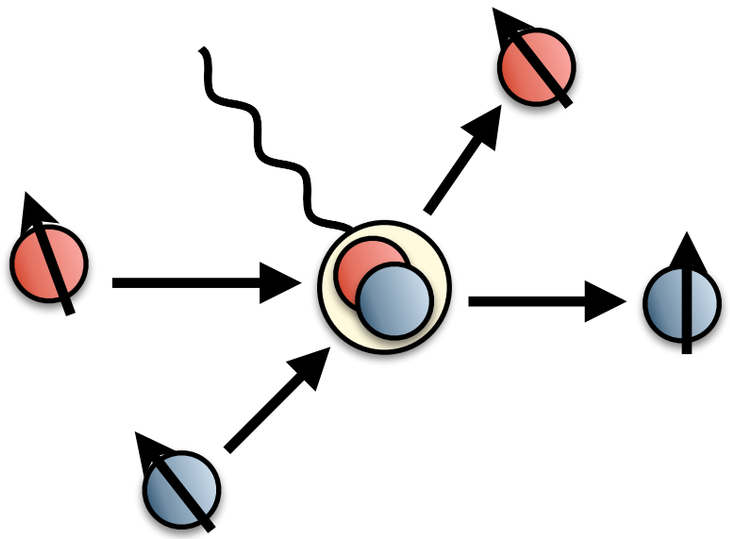
Non-trivial volume dependence!

$$G^{\mu=0}(P_{\mathrm{B}}; L) = \mathcal{O}(e^{-\kappa_{\mathrm{B}} L})$$

- Mass known to have finite volume dependence — manifestation in matrix elements

- Similar features for general tensor currents

Third non-trivial check

Summary and Outlook

- Formalism to study structure of resonant/ bound state systems
- First studies to *S*-wave bound-states — provide consistency checks
    - Volume independence of charge in electromagnetic form-factors
    - Recover infinite volume constraints via WTI
    - Volume dependence of scalar source — probe mass distribution
- Near future applications
    - Threshold expansion — continued consistency checks against results in literature
    - Numerical applications, e.g. $\pi\pi\gamma^* \to \pi\pi$
- Near future Formal developments
    - Extend formalism to particles with spin