Theoretical Development of the LaMET Approach to Parton Physics

Yong Zhao
Massachusetts Institute of Technology
June 18, 2019

The 37th International Symposium on Lattice Field Theory
Wuhan, Hubei, China 06/17-22, 2019
Outline

• Large-momentum effective theory
  • Physical picture and theoretical framework
  • From the quark PDFs to the gluon PDFs

• Transverse hadron structures from lattice QCD
  • Generalized parton distributions
  • Transverse momentum dependent distributions

• Summary and outlook
Three-dimensional partonic hadron structures

- Longitudinal Parton Distribution Functions (PDFs): \( q_{i=q,q,ar{q},g}(x) \)

- Generalized Parton Distributions (GPDs):
  \( F_{i}(x, \xi = 0, \vec{b}_T) \)
  - \( \vec{b}_T \): transverse position of the parton.

- Transverse momentum dependent (TMD) PDFs
  \( q_i(x, \vec{k}_T) \)
  - \( \vec{k}_T \): transverse momentum of the parton.

- Wigner distributions or generalized transverse momentum dependent distributions
  \( W_{i}(x, \xi = 0, \vec{k}_T, \vec{b}_T) \)

The longitudinal and transverse PDFs provide complete 3D structural information of the proton.
The measurement of partonic hadron structures will enter next level of precision and comprehensiveness with the future generation of colliders.


Lattice QCD calculation of partonic hadron structures?

PDF:
\[ q(x, \mu) = \int \frac{db^-}{2\pi} e^{-ib^- (xP^+)} \langle P | \bar{\psi}(b^-) \frac{\gamma^+}{2} W[b^-,0] \psi(0) | P \rangle \]

- Minkowski space, real time;
- Defined on the light-cone which depends on the real time.

Lattice QCD:
\[ t = i \tau, \quad e^{iS} \rightarrow e^{-S}, \quad \langle O \rangle = \int D\psi D\bar{\psi} D\Lambda O(x) e^{-S} \]

- Euclidean space, imaginary time;
- Sign problem in simulating real-time dynamics.

Light-cone PDFs not directly accessible from the lattice!
Large-momentum effective theory

PDF $q(x, \mu)$: Cannot be calculated on the lattice

Quasi-PDF $\tilde{q}(x, P^z, \mu)$: Directly calculable on the lattice

- Ji, PRL110 (2013);
- Ji, SCPMA57 (2014).
Large-momentum effective theory

\[ \tilde{q}(x, P^z, \mu) \]

PDF \[ q(x, \mu) \]: Cannot be calculated on the lattice

Quasi-PDF \[ \tilde{q}(x, P^z, \mu) \]: Directly calculable on the lattice

- Ji, PRL110 (2013);
- Ji, SCPMA57 (2014).
Large-momentum effective theory

PDF \( q(x, \mu) \): Cannot be calculated on the lattice

Quasi-PDF \( \tilde{q}(x, P^z, \mu) \): Directly calculable on the lattice

Calculating the quasi-PDF at hadron momentum \( P^z \) is equivalent to boosting it.

• Ji, PRL110 (2013);
• Ji, SCPMA57 (2014).
Large-momentum effective theory

\[ \lim_{P^z \to \infty} \tilde{q}(x, P^z, \mu) \neq q(x, \mu) \]
Large-momentum effective theory

\[
\lim_{P^z \to \infty} \tilde{q}(x, P^z, \mu) = q(x, \mu)?)
\]

Instead of taking \(P^z \to \infty\) limit, one can perform an expansion for large but finite \(P^z\):

\[
\tilde{q}(x, P^z, \mu) = \int \frac{dy}{|y|} C \left( \frac{x}{y}, \frac{\mu}{yP^z} \right) q(y, \mu) + O \left( \frac{M^2}{P^2_z}, \frac{\Lambda^2_{QCD}}{x^2P^2_z} \right)
\]

- Ji, PRL110 (2013);
- X. Xiong, X. Ji, J.-H. Zhang and Y.Z., PRD90 (2014);
- Y.-Q. Ma and J. Qiu, PRD98 (2018), PRL 120 (2018);

- \(\tilde{q}(x, P^z)\) and \(q(x)\) have the same infrared physics (nonperturbative), but different ultraviolet (UV) physics (perturbative);
- Therefore, the matching coefficient \(C\) is perturbative, which controls the logarithmic dependences on \(P^z\).
Systematic procedure of calculating the PDFs

1. Simulation of the quasi PDF in lattice QCD

\[ \tilde{q}(x, P^z, \mu) = \int \frac{dy}{|y|} C \left( \frac{x}{y}, \frac{\mu}{yP^z} \right) q(y, \mu) + O \left( \frac{M^2}{P^2_\perp}, \frac{\Lambda_{QCD}^2}{x^2 P^2_\perp} \right) \]
Systematic procedure of calculating the PDFs

\[
\bar{\psi}_0(z) \frac{\Gamma}{2} W_0[z,0] \psi_0(0) = e^{\delta m|z|} Z_{j_1} Z_{j_2} Z_\psi Z_Q \left[ \bar{\psi}(z) \frac{\Gamma}{2} W[z,0] \psi(0) \right]_R
\]

\[
\bar{q}(x, P^z, \mu) = \int \frac{dy}{|y|} C \left( \frac{x}{y}, \frac{\mu}{y P^z} \right) q(y, \mu) + O \left( \frac{M^2}{P^z}, \frac{\Lambda_{QCD}^2}{x^2 P^z} \right)
\]

2. Renormalization of the lattice quasi PDF, and then taking the continuum limit

Proof of renormalizability:
- X. Ji, J.-H. Zhang, and Y.Z., PRL120 (2018);
- J.-W. Chen, Y.Z. et al., LP3 Collaboration, PRD97 (2018);

Nonperturbative renormalization in the regularization invariant momentum subtraction (RI/MOM) scheme:
- I. Stewart and Y.Z., PRD97 (2018);
- J.-W. Chen, Y.Z. et al., LP3 Collaboration, PRD97 (2018);
- Constantinou and Panagopoulos, PRD96 (2017);
- C. Alexandrou et al., ETM Collaboration, NPB923 (2017).
Systematic procedure of calculating the PDFs

\[ \tilde{q}(x, P^z, \mu) = \int \frac{dy}{|y|} C \left( \frac{x}{y}, \frac{\mu}{yp^z} \right) q(y, \mu) + O \left( \frac{M^2}{P^2_z}, \frac{\Lambda_{QCD}^2}{x^2P^2_z} \right) \]

3. Subtraction of power corrections

- O Nachtmann, NPB63 (1973);
- J.W. Chen et al. (LP3), NPB911 (2016).

Renormalon contribution to the power correction:
Systematic procedure of calculating the PDFs

\[ \tilde{q}(x, P^z, \mu) = \int \frac{dy}{|y|} C\left(\frac{x}{y}, \frac{\mu}{yP^z}\right) q(y, \mu) + O\left(\frac{M^2}{P^2_z}, \frac{\Lambda_{QCD}^2}{x^2 P^2_z}\right) \]

4. Matching to the PDF.

- Matching for the quasi-PDF:
  - X. Xiong, X. Ji, J.-H. Zhang and Y.Z., PRD90 (2014);
  - I. Stewart and Y.Z., PRD97 (2018);
  - Y.-S. Liu, Y.Z. et al. (LP3), arXiv:1807.06566;
  - T. Izubuchi, X. Ji, L. Jin, I. Stewart, and Y.Z., PRD98 (2018);
  - Y.-S. Liu, Y.Z. et al., arXiv:1810.10879;
  - Y.Z., Int.J.Mod.Phys. A33 (2019);
  - C. Alexandrou et al. (ETMC), arXiv: 1902.00587.
Systematic procedure of calculating the PDFs

\[ \bar{q}(x, P^z, \mu) = \int \frac{dy}{|y|} C \left( \frac{x}{y}, \frac{\mu}{y P^z} \right) q(y, \mu) + O \left( \frac{M^2}{P_z^2}, \frac{\Lambda_{QCD}^2}{x^2 P_z^2} \right) \]

5. Extract \( q(y) \)

4. Matching to the PDF.

- Matching for the quasi-PDF:
  - X. Xiong, X. Ji, J.-H. Zhang and Y.Z., PRD90 (2014);
  - I. Stewart and Y.Z., PRD97 (2018);
  - Y.-S. Liu, Y.Z. et al. (LP3), arXiv:1807.06566;
  - T. Izubuchi, X. Ji, L. Jin, I. Stewart, and Y.Z., PRD98 (2018);
  - Y.-S. Liu, Y.Z. et al., arXiv:1810.10879;
  - Y.Z., Int.J.Mod.Phys. A33 (2019);
  - C. Alexandrou et al. (ETMC), arXiv: 1902.00587.
Systematic procedure of calculating the PDFs

\[ \tilde{q}(x, P^\perp, \mu) = \int \frac{dy}{|y|} C \left( \frac{x}{y}, \frac{\mu}{yP^\perp} \right) q(y, \mu) + O \left( \frac{M^2}{P_\perp^2}, \frac{\Lambda_{QCD}^2}{x^2P_\perp^2} \right) \]

4. Matching to the PDF.

- Matching for the quasi-PDF:
  - X. Xiong, X. Ji, J.-H. Zhang and Y.Z., PRD90 (2014);
  - I. Stewart and Y.Z., PRD97 (2018);
  - Y.-S. Liu, Y.Z. et al. (LP3), arXiv:1807.06566;
  - T. Izubuchi, X. Ji, L. Jin, I. Stewart, and Y.Z., PRD98 (2018);
  - Y.-S. Liu, Y.Z. et al., arXiv:1810.10879;
  - Y.Z., Int.J.Mod.Phys. A33 (2019);
  - C. Alexandrou et al. (ETMC), arXiv: 1902.00587.

5. Extract \( q(y) \)

For recent progress on the lattice calculation of quark PDFs, see N. Karthik’s plenary talk.
Other proposals in recent years

- Restoration of rotational symmetry to calculate higher moments
  Z. Davoudi and M. Savage, PRD86 (2012)

- Current-current correlation with a fictitious heavy quark

- Operator product expansion of the Compton amplitude
  A. J. Chambers et al. (QCDSF), PRL 118 (2017)

- Hadronic tensor from lattice QCD

- Smeared Quasi-PDF with Gradient flow
  C. Monahan and K. Orginos, JHEP 1703 (2017)

- Pseudo-PDF (alternative to quasi-PDF)
  A. Radyushkin, PRD96 (2017); K. Orginos et al., PRD96 (2017).

- Lattice cross section

- Factorization of Euclidean correlations in coordinate space
Gluon quasi-PDFs

- Gluon light-cone PDF:
  \[ g(x, \mu) = \int \frac{db^-}{2\pi x P^+} e^{-i(xP^+)b^-} \langle P | F^{+\alpha}(b^-) W[b^-,0] F^+_{\alpha}(0) | P \rangle \]

- Gluon quasi-PDF:
  \[ \tilde{g}(x, P^z, \mu) = \int \frac{dz}{2\pi x P^z} e^{i x P^z z} \frac{1}{N_{n_1,n_2}} \langle P | \tilde{O}^g(z) | P \rangle, \]
  \[ \tilde{O}^g(z) = g_{\perp,\mu\nu} F^{n_1 \mu}(z) W[z,0] F^{n_2 \nu}(0) \]

- Proof of renormalizability:
  \[ \tilde{O}^g_0(z) = e^{\delta m|z|} Z_{j_1} Z_{j_2} Z_A Z_Q \left[ \tilde{O}^g(z) \right] R \]
  \[ + e^{\delta m|z|} Z_{\text{mix}} \left[ g_{\perp}^{\mu\nu} A_{\mu}(z) W[z,0] A_{\nu}(0) \right] R \delta(z) \delta_{n_1 \hat{z}} \delta_{n_2 \hat{z}} \]

No mixing with quarks under renormalization!

- Zhang, Ji, Schaefer et al., PRL122 (2019);
- Li, Ma, and Qiu, PRL122 (2019).
- Aslo see Jianhui Zhang’s talk in parallel session.
Gluon and singlet quark PDFs from LaMET

• Factorization formula:

\[
\tilde{q}_i(x, P_z, \mu) = \int_{-1}^{1} dy \left[ \sum_j C_{q_i q_j} \left( \frac{x}{y}, \frac{\mu}{yP_z} \right) q_j(y, \mu) + C_{qg} \left( \frac{x}{y}, \frac{\mu}{yP_z} \right) g(y, \mu) \right] + \mathcal{O} \left( \frac{M^2}{P_z^2}, \frac{\Lambda_{QCD}^2}{x^2 P_z^2} \right)
\]

\[
\tilde{g}(x, P_z, \mu) = \int_{-1}^{1} dy \left[ \sum_j C_{gq} \left( \frac{x}{y}, \frac{\mu}{yP_z} \right) q_j(y, \mu) + C_{gg} \left( \frac{x}{y}, \frac{\mu}{yP_z} \right) g(y, \mu) \right] + \mathcal{O} \left( \frac{M^2}{P_z^2}, \frac{\Lambda_{QCD}^2}{x^2 P_z^2} \right)
\]

• Nonperturbative renormalization in the RI/MOM scheme:

\[
Z^{-1}(z, p_R^2, \mu_R^2) \frac{\langle F_0^{n_1\mu}(z) W_0[z,0] F_0^{n_2\nu}(0) A^\rho_a(p) A^\sigma_a(-p) \rangle}{4 \langle A^\rho(p)_a A^\sigma(-p)_a \rangle g_{\perp,\rho\sigma}} g_{\perp,\rho\sigma}
\]

\[
= \langle F_0^{n_1\mu}(z) F_0^{n_2\nu}(0) A^\rho_a(p) A^\sigma_a(-p) \rangle_{\text{tree}} g_{\perp,\rho\sigma}
\]

• \( A^\mu(p) \) can be defined from gauge links;

• There is also finite mixings with quark quasi-PDF.

• Matching in the RI/MOM scheme:

- Wang et al., EPJC78 (2018), JHEP1805 (2018);
- Shanahan and Detmold, PRD99 (2019).

Outline

• Large-momentum effective theory
  • Physical picture and theoretical framework
  • From the quark PDFs to the gluon PDFs

• Transverse hadron structures from lattice QCD
  • Generalized parton distributions
  • Transverse momentum dependent distributions

• Summary and outlook
GPD

- Light-cone GPD:

\[
\bar{P} = \frac{P + P'}{2}, \quad \xi = \frac{P^+ - P'^+}{P^+ + P'^+}, \quad t = (P' - P)^2 \equiv \Delta^2
\]

\[
F_\Gamma(x, \xi, t, \mu) = \int \frac{db^-}{4\pi} e^{-i\bar{P}^+b^-} \langle P', S' | \bar{\psi} \left( \frac{b^-}{2} \right) \Gamma U \left( \frac{b^-}{2} \right., \left. - \frac{b^-}{2} \right) \psi \left( - \frac{b^-}{2} \right) | P, S \rangle
\]

Color charge density in impact parameter space:

\[
F(x, \xi = 0, \vec{b}_T) = \int \frac{d^2\Delta_T}{(2\pi)^2} e^{i\Delta_T \cdot \vec{b}_T} F(x, \xi = 0, t)
\]

- Measurable in hard exclusive processes such as deeply virtual Compton scattering: \( l + p \rightarrow l' + \gamma + p' \)

\[
\sim \int dx \ C(x, \xi) F(x, \xi, t)
\]
Quasi-GPD

• Definition:
\[ \tilde{F}_{\bar{F}}(x, \xi, t, \mu) = \int \frac{dz}{4\pi} e^{-ix\bar{P}z} \langle P', S' | \bar{\psi}(\frac{z}{2}) \tilde{\Gamma} U(\frac{z}{2}, -\frac{z}{2}) \psi(-\frac{z}{2}) | P, S \rangle \]

• Renormalization:
  • Same operator as the quasi-PDF, can be renormalized by the same factors!

• Factorization formula:
\[ \tilde{F}_{\bar{F}}(x, \xi, t, \mu) = \int_{-1}^{1} \frac{dy}{|\xi|} \tilde{C} \left( \frac{x}{\xi}, \frac{y}{\xi}, \frac{\mu}{\xi Pz} \right) F_{\bar{F}}(y, \xi, t, \mu) + O\left( \frac{M^2}{P_z^2}, \frac{t}{P_z^2}, \frac{\Lambda_{QCD}^2}{x^2 P_z^2} \right) \]

  • Mass correction and O(t/P_z^2) corrections: Y.-S. Liu, Y.Z. et al., in progress.
  
  • Preliminary results for quasi-GPDs (ETMC), M. Constantinou’s talk at QCD Evolution 2019.
TMD processes:

### Semi-Inclusive DIS

\[ \sigma \sim f_{q/P}(x, k_T) D_{h/q}(x, k_T) \]

- Fragmentation
  \[ D_{h/q}(x, k_T) \]

### Drell-Yan

\[ \sigma \sim f_{q/P}(x, k_T) f_{q/P}(x, k_T) \]

- \( q_T \ll Q \)

### Dihadron in e^+e^-

\[ \sigma \sim D_{h_1/q}(x, k_T) D_{h_2/q}(x, k_T) \]

Many different schemes for TMD factorization in literature:

- Collins, Soper and Sterman, NPB250 (1985); Collins, 2011;
- Becher and Neubert, EPJC71 (2011);
- Echevarria, Idilbi and Scimemi, JHEP07 (2012), PLB726 (2013);
- Chiu, Jain, Neil and Rothstein, JHEP05 (2012), PRL108 (2012);

Picture credit, Iain Stewart
• Collinear factorization (e.g., for Drell-Yan):

\[
\frac{d\sigma}{dQdY} = \sum_{a,b} \sigma_{ab}(Q, \mu, Y) f_a(x_1, \mu) f_b(x_2, \mu)
\]

• TMDPDF factorization:

\[
\frac{d\sigma}{dQdYd^2q_T} = \sum_{i,j} H_{ij}(Q, \mu) \int d^2b_T e^{i\vec{b}_T \cdot \vec{q}_T} f_i^{\text{TMD}}(x_a, \vec{b}_T, \mu, \zeta_a) f_j^{\text{TMD}}(x_b, \vec{b}_T, \mu, \zeta_b)
\]

$q_T$: Net transverse momentum of the color-singlet final state, and $q_T \ll Q$;

$\zeta$: Collins-Soper Scale. $\zeta_a \zeta_b = Q^4$

• The definition of TMDPDF involves a collinear beam function (or un-subtracted TMD) and soft function:

\[
f_i^{\text{TMD}}(x, \vec{b}_T, \mu, \zeta) = \lim_{\epsilon \to 0, \tau \to 0} Z_{\text{UV}}(\epsilon, \mu, xP^+) B_i(x, \vec{b}_T, \epsilon, \tau, xP^+) \Delta S_i(b_T, \epsilon, \tau)
\]
Evolution of TMDPDF

• Evolution of TMDPDF:

\[ f_{i}^{\text{TMD}}(x, \vec{b}_T, \mu, \zeta) = f_{i}^{\text{TMD}}(x, \vec{b}_T, \mu_0, \zeta_0) \exp\left[\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma_{\mu}(\mu', \zeta_0)\right] \exp\left[\frac{1}{2} \gamma_{\zeta}(\mu, b_T) \ln \frac{\zeta}{\zeta_0}\right] \]

• \( \mu \sim Q, \zeta \sim Q^2 >> \Lambda_{\text{QCD}}^2 \);

• \( \mu_0, \zeta_0 \): initial or reference scales, measured in experiments or determined from lattice (~2 GeV).

\( \gamma_{\mu}(\mu', \zeta_0) \) Anomalous dimension for \( \mu \) evolution, perturbatively calculable;

\( \gamma_{\zeta}(\mu, b_T) \) Collins-Soper kernel, becomes nonperturbative when \( b_T \sim 1/\Lambda_{\text{QCD}} \).

Both Initial-scale TMDPDF and the Collins-Soper kernel must be modeled in global fits of TMDPDF from experimental data.

• Bachetta et al., JHEP 1706 (2017);
• Scimemi and Vladimirov, EPJC78 (2018);
• Bertone, Scimemi and Vladimirov, JHEP 1906 (2019).
Towards a lattice calculation of the TMDPDF

- Beam function:

\[
B_q(x, \vec{b}_T, \epsilon, \tau) = \int \frac{db^+}{2\pi} e^{-i(xP^+)b^-} \langle P | \bar{q}(b^\mu) W(b^\mu) \gamma^+ \rangle \frac{1}{2} W_T(-\infty \vec{n}; \vec{b}_T, \vec{0}_T) W^\dagger(0) q(0) \mid P \rangle
\]

- Lorentz boost and \( L \to \infty \)

- Quasi-beam function on lattice:

\[
\tilde{B}_q(x, \vec{b}_T, a, L, P^z) = \int \frac{db^z}{2\pi} e^{ib^z(xP^z)} \tilde{B}_q(b^z, \vec{b}_T, a, L, P^z)
\]

\[
= \int \frac{db^z}{2\pi} e^{ib^z(xP^z)} \langle P | \bar{q}(b^\mu) W_{\tilde{z}}(b^\mu, L-b^\tilde{z}) \frac{1}{2} W_T(L\tilde{z}; \vec{b}_T, \vec{0}_T) W_{\tilde{z}}^\dagger(0) q(0) \mid P \rangle
\]

- References:

- Ji, Sun, Xiong and Yuan, PRD91 (2015);
- Ji, Jin, Yuan, Zhang and Y.Z., PRD99 (2019);
- M. Ebert, I. Stewart, Y.Z., PRD99 (2019);
Towards a lattice calculation of the TMDPDF

- Soft function:

\[ S_q(b_T, \epsilon, \tau) = \frac{1}{N_c} \langle 0 | \text{Tr} \left[ S_n^\dagger(\vec{b}_T)S_n(\vec{b}_T)S_T(-\infty \vec{n}; \vec{b}_T, \vec{0}_T)S_n^\dagger(\vec{0}_T)S_n(\vec{0}_T)S_T^\dagger(-\infty \vec{n}; \vec{b}_T, \vec{0}_T) \right] \rangle_\tau | 0 \rangle \]

- Quasi-soft function on lattice (naive definition):

\[ \tilde{S}_q(b_T, a, L) = \frac{1}{N_c} \langle 0 | \text{Tr} \left[ S_{\vec{z}}^\dagger(\vec{b}_T; L)S_{-\vec{z}}(\vec{b}_T; L)S_T(L\vec{z}; \vec{b}_T, \vec{0}_T)S_{\vec{z}}^\dagger(\vec{0}_T; L)S_{\vec{n}}(\vec{0}_T; L)S_T^\dagger(-L\vec{z}; \vec{b}_T, \vec{0}_T) \right] | 0 \rangle \]
Quasi-TMDPDF and its relation to TMDPDF

• Quasi-TMDPDF (in the MSbar scheme):

\[
\tilde{f}_q^\text{TMD}(x, \vec{b}_T, \mu, P^z) = \lim_{L \to \infty} \int \frac{db^z}{2\pi} e^{ib^z(xP^z)} \tilde{Z}'(b^z, \mu, \tilde{\mu}) \tilde{Z}_{\text{UV}}(b^z, \mu, a) \frac{\tilde{B}_q(b^z, \vec{b}_T, a, L, P^z)}{\sqrt{\tilde{S}_q(b_T, a, L)}}
\]

• Relation to TMDPDF:

\[
\tilde{f}_{\text{ns}}^\text{TMD}(x, \vec{b}_T, \mu, P^z) = C_{\text{ns}}^\text{TMD}(\mu, xP^z) g_q^S(b_T, \mu) \exp \left[ \frac{1}{2} \gamma_q^q(\mu, b_T) \ln \frac{(2xP^z)^2}{\xi} \right] \times \tilde{f}_{\text{ns}}^\text{TMD}(x, \vec{b}_T, \mu, \zeta) + O \left( \frac{b_T}{L}, \frac{1}{b_T P^z}, \frac{1}{P^z L} \right)
\]

Hierarchy of scales: \(b^z \sim \frac{1}{P^z} \ll b_T \ll L\)

For \(b_T \ll \Lambda_{\text{QCD}}^{-1}\)

\[
g_q^{\text{naive}}(b_T, \mu) = 1 + \frac{\alpha_s C_F}{2\pi} \ln \frac{b_T^2 \mu^2}{4e^{-2\gamma_E}} + O(\alpha_s^2)
\]

• Ji, Jin, Yuan, Zhang and Y.Z., PRD99 (2019);
Perturbatively matchable quasi-TMDPDF

• A bent quasi-soft function?

\[ g_q^{S_{\text{bent}}}(b_T, \mu) = 1 + O(\alpha_s^2) \]

However, this should be checked to all orders of perturbation theory.

• An ultimate solution relies on having

\[ g_q^S(b_T, \mu) = 1 \]
Collins-Soper kernel from lattice

\[ \gamma^q_\zeta(\mu, b_T) = \frac{1}{\ln(P_1^z/P_2^z)} \ln \frac{C_{ns}^{\text{TMD}}(\mu, xP_2^z) \tilde{f}_{ns}^{\text{TMD}}(x, \vec{b}_T, \mu, P_1^z)}{C_{ns}^{\text{TMD}}(\mu, xP_1^z) \tilde{f}_{ns}^{\text{TMD}}(x, \vec{b}_T, \mu, P_2^z)} \]

\[ = \frac{1}{\ln(P_1^z/P_2^z)} \ln \frac{\int dB^z e^{ib^z x P_1^z} \tilde{Z}'(b^z, \mu, \tilde{\mu}) \tilde{Z}_{\text{UV}}(b^z, \mu, a) \tilde{B}_{ns}(b^z, \vec{b}_T, a, L, P_1^z)}{\int dB^z e^{ib^z x P_2^z} \tilde{Z}'(b^z, \mu, \tilde{\mu}) \tilde{Z}_{\text{UV}}(b^z, \mu, a) \tilde{B}_{ns}(b^z, \vec{b}_T, a, L, P_2^z)} \]

- \( g^S \) as well as the quasi-soft function are canceled in the ratio;
- Does not depend on the external state hadron, could be calculated with pion for simplicity;
- Independent of the choice of \( x \) and \( P^z \), which provides a window to control systematic uncertainties;
- One can also calculate ratios of TMDPDFs with different spin structures.

The idea of forming ratios has been used in the calculation of \( x \)-moments of TMDPDFs:

A first look at the Collin-Soper kernel

Work in progress with Phiala Shanahan and Michael Wagman

Caveat: quenched approximation, $N_{cfg}=7$, valence $m_\pi\sim1.2$ GeV.

\[ \gamma_\xi(b_T, \mu = 2 \text{ GeV}) \]

\[ \gamma_\mu^q [\alpha_s(\mu)] = -\frac{\alpha_s(\mu) C_F}{\pi} \ln \frac{b_T^2 \mu^2}{4e^{-2\gamma_E}} + O(\alpha_s^2) \]

Different colored points correspond to the Collins Soper kernel calculated at $x = 0.4, 0.45, 0.5, 0.55, 0.6$. 
Summary and outlook

• The complete renormalization and matching for the quark and gluon PDFs have been derived;

• The nonperturbative renormalization and matching for the GPDs have been derived;

• The Collins-Soper kernel can be calculated with ratios of quasi-beam functions. A first look shows encouraging results with present-day resources.

• Future work will include (much) larger statistics, different lattice spacings, and more systematic treatment than the naive Fourier transform;

• The ultimate solution to calculating TMDPDF on the lattice relies on the soft sector, which will be pursued vigorously in the future.