Taming uncertainties in staggered computations of the HVP contribution of light quarks to the muon $(g-2)$

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$a_\mu^{\text{LO-HVP}}$ from time-momentum current correlator

Bernecker et al. 11, BMWc 13, Feng et al 13, Lehner 14, ...

- Compute on $T \times L^3$ lattice

$$C_L(t) = \frac{a^3}{3} \sum_{i=1}^{3} \sum_{\vec{x}} \langle J_i(x) J_i(0) \rangle$$

w/ $J_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c + \cdots$

- Decompose ($C_L^{I=1} = \frac{9}{10} C_L^{ud}$)

$$\begin{aligned} C_L(t) &= C_L^{ud}(t) + C_L^{s}(t) + C_L^{c}(t) + C_L^{\text{disc}}(t) \\ &= C_L^{I=1}(t) + C_L^{I=0}(t) \end{aligned}$$

- Obtain (BMWc 17)

$$a_{\mu,f}^{\text{LO-HVP}}(Q^2 \le Q^2_{\max}) = \lim_{a\to 0,\, L\to\infty,\, T\to\infty} \left(\frac{\alpha}{\pi}\right)^2 \left(\frac{a}{m_\mu^2}\right) \sum_{t=0}^{T/2} W(tm_\mu, Q^2_{\max}/m_\mu^2)\, \mathrm{Re}C_L^f(t)$$

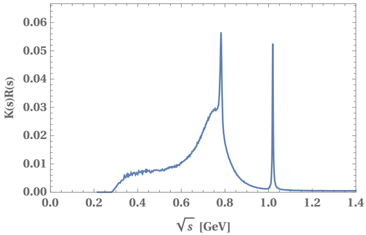
w/ $W(\tau, x_{\max})$ known kinematical function

- Focus on ud contribution: dominates $a_\mu^{\text{LO-HVP}}$ and its stat. and syst. errors

Leading systematics with staggered fermions

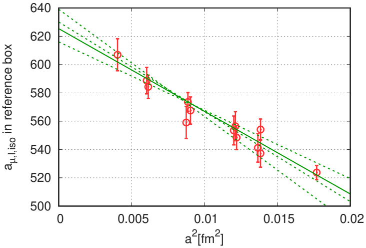
- $a_\mu^{\text{LO-HVP}}$ has strong dependence on 2-π states

$$a_\mu^{\text{LO-HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int_{s_{th}}^{\infty} ds\, K(s) R_{e^+e^- \to \text{hadrons}}(s)$$



$\Rightarrow$ FV effects may be large (Golterman et al. 16), i.e. few % for $L \sim 6\,\text{fm}$

$\Rightarrow$ Taste-breaking effects are significant: effective $M_\pi \sim M_\pi^{\text{RMS}} > M_\pi^{\text{GB}}$



Continuum extrapolation is also a chiral extrapolation

$\Rightarrow$ large a^2-dependence ($\sim$ 20% for $a \sim 0.131\,\text{fm}$ (BMWc 17))

$\Rightarrow$ possible non-linearities through $\delta_L a_{\mu,ud}^{\text{LO-HVP}} \sim \sum_{j=0}^{4} w_j \exp\left[-L\sqrt{(M_\pi^{\text{GB}})^2 + j \times a^2 \Delta^{\text{KS}}}\right]$

$\Rightarrow$ must be controlled to get $\delta_{\text{tot}} a_\mu^{\text{LO-HVP}} < 1\%$

Soln 1: LO χPT for FV effects

- FV effects are long-distance effects, determined by lightest states contributing to process
- Here $I = J = 1$, 2-π states
- Determine in χPT, to LO (Aubin et al 15), i.e.

$$C^{I=1}_{L,\text{LO-}\chi\text{PT}}(t) = \frac{1}{3L^3}\sum_{\vec{p}_\text{free}}\left(\frac{\vec{p}_\text{free}}{E_p^\text{free}}\right)^2 e^{-2E_p^\text{free}t}$$

with $E_p^\text{free} = \sqrt{M_\pi^2 + \vec{p}^{\,2}_\text{free}}$

- Then $C^{I=1}_{\infty,\text{LO-}\chi\text{PT}}(t) - C^{I=1}_{L,\text{LO-}\chi\text{PT}}(t)$ can be used to estimate FV effects
- Find, for $M_\pi \sim 135\,\text{MeV}$ and $L \sim 6\,\text{fm}$ (BMWc 17),

$$\Delta_\text{FV} a^\text{LO-HVP}_{\mu,I=1} \sim 2.3\% \times a^\text{LO-HVP}_{\mu,I=1}$$

$\rightarrow$ probably $O(50\%)$ too small (Della Morte et al 17, Shintani et al 19, Aubin et al 19 ...)

- Can do better

Soln 1: LO SχPT for taste effects

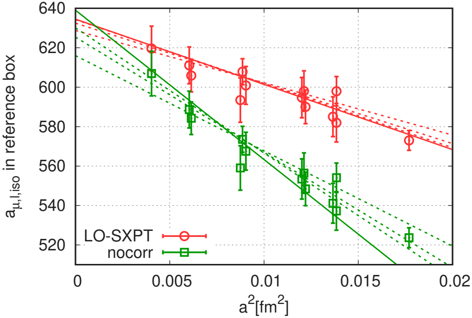
- Taste-breaking effects also mostly come from low-lying 2-π states

- Determine in SχPT, to LO (Aubin et al 15, HPQCD 17), i.e.

$$C^{I=1}_{L,\text{LO-S}\chi\text{PT}}(t, a^2\Delta^{KS}) = \frac{1}{3L^3}\sum_{\vec{p}_\text{free}}\sum_{j=0}^{4} w_j \left(\frac{\vec{p}_\text{free}}{E^\text{free}_{p,j}}\right)^2 e^{-2E^\text{free}_{p,j}t}$$

with $E^\text{free}_{p,j} = \sqrt{M^2_{\pi,j} + \vec{p}^2_\text{free}}$

- Then $C^{I=1}_{L,\text{LO-S}\chi\text{PT}}(t, 0) - C^{I=1}_{L,\text{LO-S}\chi\text{PT}}(t, a^2\Delta^{KS})$ can be used to estimate taste effects



- Helps but is it possible to do better?

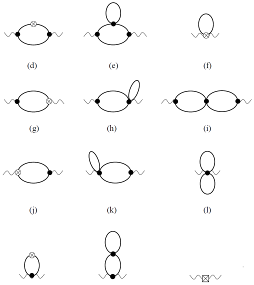
Soln 2: NLO SχPT

- At LO, the two π are free

  $\Rightarrow$ omits strong $\rho - \pi\pi$ coupling

  $\Rightarrow$ compute at NLO (Bijnens et al 99, Aubin et al 19)

- NLO includes LO 2-π rescattering and slope of $F_\pi(Q^2)$



- However NLO only obtained in continuum (Aubin et al 19)

  $\Rightarrow$ helps FV corrections: increase by $O(50\%)$ for $M_\pi \sim 135\,\mathrm{MeV}$ and $L \sim 6\,\mathrm{fm}$

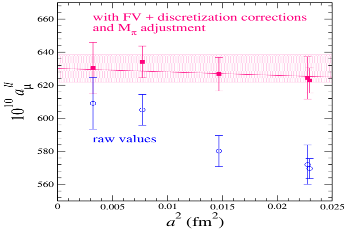
  $\Rightarrow$ does not improve taste corrections

Soln 3: add point-like ρ to SXPT and model lattice ρ

- Construct field theory that couples γ-ρ-$\pi\pi$ (Jegerlehner 11)
- Gives much better description of $\Pi(Q^2)$ in continuum
- Add taste breaking to pion loop contributions and work out coupled system to one pion loop (HPQCD 17)

$\rightarrow$ gives FV and taste corrections similar to LO SχPT

$\Rightarrow$ further correct for taste breaking by measuring m_μ in units of lattice "M_ρ" from usual correlator fits (ETMC 11, HPQCD 17) or by modelling $C_L(t)$ at large t (FNAL/HPQCD/MILC 17)



Possible issues:

- ρ not treated as a resonance in M_ρ-rescaling
- reduction of number of (staggered) states in modelling of $C_L(t)$ is done by fitting, not e.g. by systematically eliminating taste copies

Soln 4: phenomenology inspired FV corrections

Introduced for FV effects (Meyer 11). Use:

- 2-π, $\delta^{J=1}_{I=1}(p)$ from phenomenology
- Lüscher to get $E_p = \sqrt{M_\pi^2 + p^2}$ in FV, where $p = |\vec{p}|$ is momentum carried by each of the two interacting π in FV
- Lellouch-Lüscher (LL) for interacting $|\langle 0|J_i|\pi^+(p)\pi^-(p)\rangle|_L$ in FV from free amplitude $\frac{p_i^{\rm free}}{E_p^{\rm free}}$

Then

$$C^{I=1}_{L,\text{LO-}\chi\text{PT}}(t) \to C^{I=1}_{L,\text{LLGS}}(t) = \frac{1}{3L^3}\sum_i \sum_p |\langle 0|J_i|\pi^+(p)\pi^-(p)\rangle|^2_L \; e^{-2E_p t}$$

and $C^{I=1}_{\infty,\text{LLGS}}(t) - C^{I=1}_{L,\text{LLGS}}(t)$ gives estimate of FV

$\to$ good for FV corrections: increase by $O(50\%)$ over LO χPT for $M_\pi \sim 135\,\text{MeV}$ and $L \sim 6\,\text{fm}$

Soln 4: phenomenology inspired taste corrections

Here explore naive staggerization of phenomenological model

$$C^{I=1}_{L,\mathrm{LLGS}}(t) \to C^{I=1}_{L,\mathrm{SLLGS}}(t; a^2\Delta^{\mathrm{KS}}) = \frac{1}{3L^3}\sum_i \sum_p \sum_{j=0}^{4} w_j |\langle 0|J_i|\pi^+(p)\pi^-(p)\rangle|^2_{j,L}\; e^{-2E_{p,j}t}$$

- Compare model to lattice data using a sliding window

$$a^{\mathrm{LO\text{-}HVP}}_{\mu,I=1}(t_{\mathrm{win}}, \Delta t, \Delta, Q^2 \le Q^2_{\max}) = \left(\frac{\alpha}{\pi}\right)^2 \left(\frac{a}{m_\mu^2}\right) \sum_{t=0}^{T/2} [\Theta(t; t_{\mathrm{win}}, \Delta) - \Theta(t; t_{\mathrm{win}} + \Delta t, \Delta)]\, W(tm_\mu, Q^2_{\max}/m_\mu^2)\ \mathrm{Re}\, C^{I=1}_L(t)$$

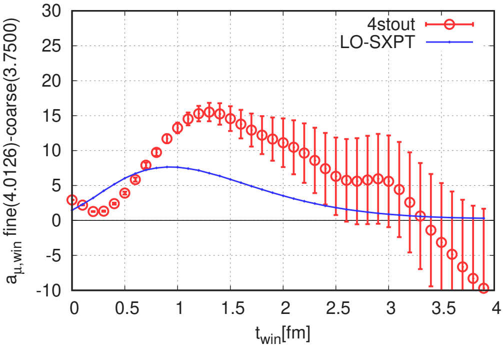
w/ $\Theta(t, t_0, \Delta) = [1 + \tanh[(t - t_0)]/\Delta]/2$ as in RBC/UKQCD 19

- Take $\Delta t = 0.5\,\mathrm{fm}$, $\Delta = 0.15\,\mathrm{fm}$ and slide window in steps of $0.1\,\mathrm{fm}$
- Implement using Gounaris-Sakurai (GS) model (Francis et al 13)
- Same comparison with LO SχPT

Sliding window: lattice vs LO-SχPT

$$\Delta_{\text{taste}}^{\text{lat}}(t_{\text{win}}) = a_{\mu,l=1,\text{lat}}^{\text{LO-HVP}}(t_{\text{win}}, L, a_{\text{fine}}) - a_{\mu,l=1,\text{lat}}^{\text{LO-HVP}}(t_{\text{win}}, L, a_{\text{coarse}})$$

$$\Delta_{\text{taste}}^{\text{LO-S}\chi\text{PT}}(t_{\text{win}}) = a_{\mu,l=1,\text{LO-S}\chi\text{PT}}^{\text{LO-HVP}}(t_{\text{win}}, L, (a^2\Delta^{\text{KS}})_{\text{fine}}) - a_{\mu,l=1,\text{LO-S}\chi\text{PT}}^{\text{LO-HVP}}(t_{\text{win}}, L, (a^2\Delta^{\text{KS}})_{\text{coarse}})$$

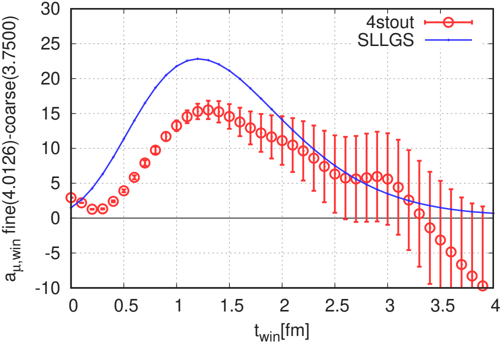


- LO SXPT describes taste-breaking corrections well for $t \gtrsim 2.0\,\text{fm}$
- Correct taste breaking in simulations with LO SXPT using either $t \geq 2.0\,\text{fm}$ or $t \geq 2.5\,\text{fm}$
- Use spread in continuum systematic error

Sliding window: lattice vs SLLGS

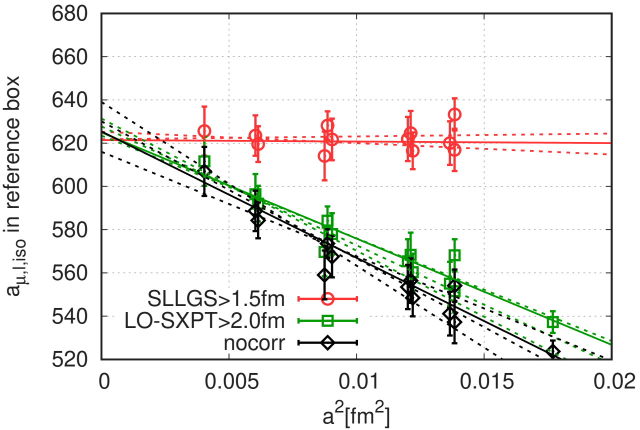
$$\Delta^{\text{lat}}_{\text{taste}}((t_{\text{win}}) = a^{\text{LO-HVP}}_{\mu,I=1,\text{lat}}(t_{\text{win}}, L, a_{\text{fine}}) - a^{\text{LO-HVP}}_{\mu,I=1,\text{lat}}(t_{\text{win}}, L, a_{\text{coarse}})$$

$$\Delta^{\text{SLLGS}}_{\text{taste}}(t_{\text{win}}) = a^{\text{LO-HVP}}_{\mu,I=1,\text{SLLGS}}(t_{\text{win}}, L, (a^2\Delta^{\text{KS}})_{\text{fine}}) - a^{\text{LO-HVP}}_{\mu,I=1,\text{SLLGS}}(t_{\text{win}}, L, (a^2\Delta^{\text{KS}})_{\text{coarse}})$$



- SLLGS describes taste-breaking corrections well for $t \gtrsim 1.5\,\mathrm{fm}$
- Correct taste breaking in simulations with SLLGS using either $t \geq 1.5\,\mathrm{fm}$ or $t \geq 2.0\,\mathrm{fm}$
- Use spread in continuum systematic error

Continuum extrapolations of $a_{\mu,ud}^{\text{LO-HVP}}$ - PRELIMINARY



errors [%]	none	LO-SχPT	SLLGS
$\delta^{\text{stat}}(a_{\mu,ud}^{\text{LO-HVP}})$	1.5	1.5	1.6
$\delta^{a\text{-extrap}}(a_{\mu,ud}^{\text{LO-HVP}})$	2.2	1.0	0.6

Conclusions

- Leading systematic errors in staggered computation of $a_\mu^{\text{LO-HVP}}$ come from FV and taste-breaking effects
- Explored 4 solutions
  - LO-SXPT: helps but can do better
  - NLO-SXPT: NLO only implemented in continuum; improves FV but not taste breaking
  - SLLGS: improves FV AND taste breaking
- SLLGS looks like a promising way to allow reducing $\delta_{\text{tot}} a_\mu^{\text{LO-HVP}}$ to below 1%
- If NLO staggered implemented, NLO-SXPT may provide a satisfactory solution also for taste breaking
- Hansen & Patella 19 presents very interesting framework for FV effects, but neglected $e^{-\sqrt{2}LM_\pi}$ are significant for parameters of current simulations with $LM_\pi \sim 4$