Recent Developments of Muon g-2 from Lattice QCD

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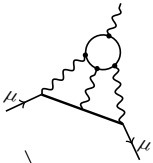
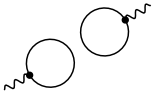
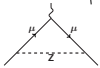
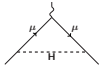
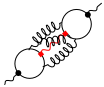
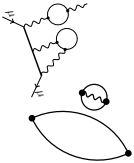


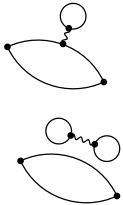


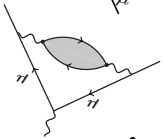


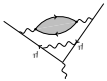


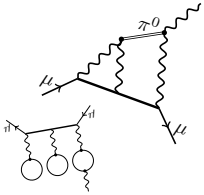


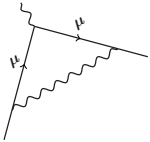












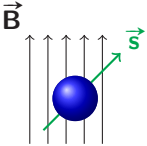
[https://upload.wikimedia.org/wikipedia/commons/f/f7/Rollercoaster_dragon_khan_universal_port_aventura_spain.jpg]

Magnetic moment of leptons (e, μ, τ)

- magnetic moment $\vec{\mu}$ of the lepton ℓ due to its spin $\vec{s}$ and electric charge e

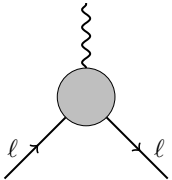
$$\vec{\mu} = g\,\frac{e}{2m_\ell}\,\vec{s}$$

torque $\quad \vec{\tau} = \vec{\mu} \times \vec{B}$



- g-factor: without quantum fluctuations for a lepton one finds $g = 2$
- deviation from the value "2" due to quantum loops $\longrightarrow$ anomalous magnetic moment of lepton ℓ

$$a_\ell = \frac{g_\ell - 2}{2}$$



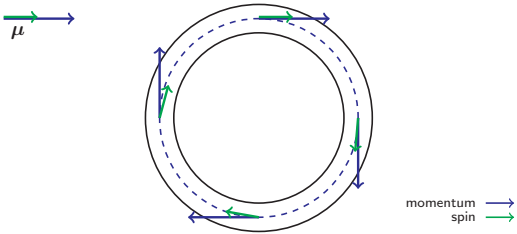
$$\langle \ell(p')|j^\gamma_\mu|\ell(p)\rangle = (-ie)\bar{u}(p')\left[\gamma_\mu F_1(q^2) + i\frac{\sigma^{\mu\nu}q_\nu}{2m_\ell}F_2(q^2)\right]u(p)$$

- $F_1(0) = 1$ (electric charge) $\qquad F_2(0) = a_\ell$ (anomalous magnetic moment)

a_μ: Experiment vs. Theory

- measured and calculated very precisely $\longrightarrow$ test of the Standard Model
- experiment: polarized muons in a magnetic field [Bennet et al., Phys.Rev. **D73**, 072003 (2006)]

$$a_\mu = 11659209.1(5.4)(3.3) \times 10^{-10}$$



$$\omega_a = a_\mu \frac{eB}{m_\mu}$$

- new experiments at Fermilab and JPARC $\rightarrow$ reduce error by ≈ 4
  $\rightarrow$ experiment at Fermilab is running
  $\rightarrow$ see [Dikai Li, Fri 09:45]

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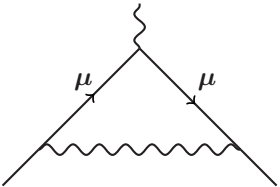
  em $\quad (11658471.895 \pm 0.008) \times 10^{-10}$ [Kinoshita et al., Phys.Rev.Lett. **109**, 111808 (2012)]



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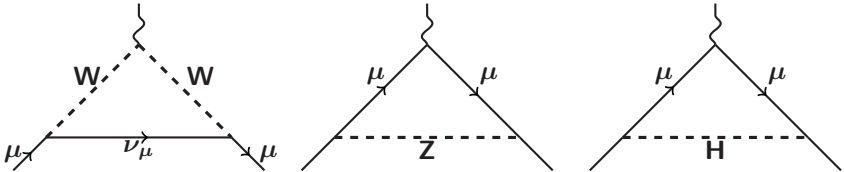

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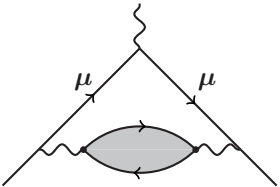

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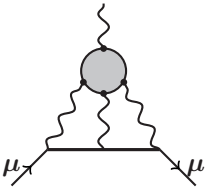

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$$\Delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = 27.9(6.3)^{\text{Exp}}(3.6)^{\text{SM}} \times 10^{-10}$$

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	Δa_μ^{hvp}
target	$\lesssim 0.2\%$
current **R**-ratio	$\approx 0.5\%$
current lattice	$\approx 2-3\%$

- Comparison of theory and experiment: 3.8σ deviation

$$\Delta a_\mu = a_\mu^{exp} - a_\mu^{SM} = 27.9(6.3)^{Exp}(3.6)^{SM} \times 10^{-10}$$

required precision to match upcoming experiments	$\Delta a_\mu^{hvp} \lesssim 0.2\%$	$\Delta a_\mu^{lbl} \lesssim 10\%$

Outline

Hadronic Vacuum Polarisation

- Introduction
- light quark contribution
- strange and charm quark contribution
- disconnected contribution
- Isospin Breaking corrections to the HVP
- Summary and Prospects

Hadronic light-by-light scattering

- Introduction
- Lattice Calculations
- Summary and Prospects

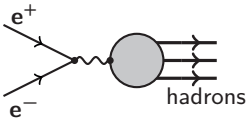
Final remarks

Outline

Hadronic Vacuum Polarisation (HVP) from the **R**-ratio

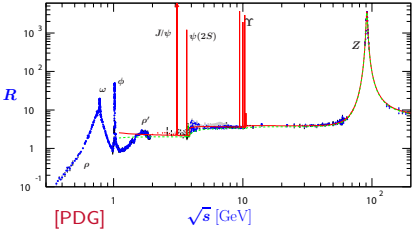
- current best theoretical estimate uses experimental data
- optical theorem

- **R**-ratio $$\mathbf{R(s)} = \frac{\sigma(e^+ e^- \to \text{hadrons}, s)}{\sigma(e^+ e^- \to \mu^+\mu^-, s)}$$



$$a_\mu^{hvp} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{m_\pi^2}^{\infty} ds \, \frac{R(s)K(s)}{s^2}$$

- first principles calculation of HVP → lattice QCD



recent results:

$$a_\mu^{hvp} = 689.46(3.25)$$ [Jegerlehner 18]

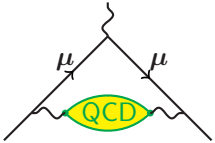
$$a_\mu^{hvp} = 693.1(3.4)$$ [DHMZ 17]

$$a_\mu^{hvp} = 693.37(2.46)$$ [KNT 18]

≈ **0.5%** precision

Hadronic Vacuum Polarisation (HVP) from the Lattice

- $\Pi_{\mu\nu}(Q) \equiv \int d^4x\, e^{iQ\cdot x} \left\langle j^\gamma_\mu(x)\, j^\gamma_\nu(0) \right\rangle = (Q_\mu Q_\nu - \delta_{\mu\nu} Q^2)\, \Pi(Q^2)$
- electromagnetic current $\quad j^\gamma_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c$
- hadronic contribution to the anomalous magnetic moment of the muon
  [T. Blum, Phys.Rev.Lett.91, 052001 (2003)]



$$a_\mu^{hvp} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2\, K(Q^2)\, \hat{\Pi}(Q^2) \qquad \text{with} \qquad \hat{\Pi}(Q^2) = 4\pi^2 \left[\Pi(Q^2) - \Pi(0)\right]$$

- subtracted HVP from vector correlator [Bernecker and Meyer, Eur.Phys.J. **A47**, 148 (2011)]

$$C(t) = \frac{1}{3}\sum_{k=0}^{2}\sum_{\vec{x}} \langle j^\gamma_k(\vec{x}, t) j^\gamma_k(0)\rangle \qquad \hat{\Pi}(Q^2) = 4\pi^2 \int_0^\infty dt\, C(t) \left[\frac{\cos(Qt) - 1}{Q^2} + \frac{1}{2}t^2\right] \qquad a_\mu^{hvp} = \int_0^\infty dt\, f(t) C(t)$$

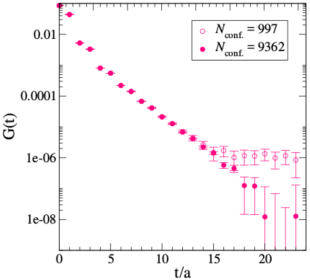
- flavour decomposition (isospin symmetric QCD)

$$C(t) = \frac{5}{9}C^\ell(t) + \frac{1}{9}C^s(t) + \frac{4}{9}C^c(t) + C^{disc}(t)$$

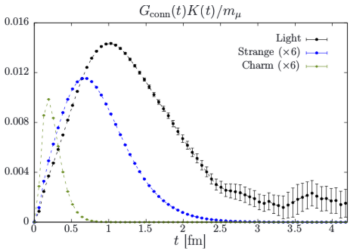
Outline

Vector correlator and long distance Signal-to-Noise problem

- examples for light-quark vector correlator at physical point



[C. Davies *et al*, arXiv:1902.04223]



[A. Gérardin *et al*, arXiv:1904.03120]

- signal deteriorates for large **t**
- need noise reduction techniques to control statistical error on raw data
  - all-mode-averaging (AMA) [T. Blum *et al.*, Phys. Rev. **D88**, 094503 (2013)], [G. Bali *et al.*, Comput.Phys.Commun. 181 (2010) 1570-1583]
  - huge reduction in error when using low-mode-averaging (LMA) [T. Blum, VG, *et al.*, Phys. Rev. Lett. 121, 022003 (2018)], [C. Aubin *et al*, arXiv:1905.09307]
- possible strategy: replace correlator by (multi-) exponential fit for $\mathbf{t} > \mathbf{t_c}$

Bounding method

- spectral representation of the vector correlator

$$C(t) = \sum_n \frac{A_n^2}{2E_n} e^{-E_n t} \qquad A_n^2 > 0$$

- bound for the correlator for $t \geq t_c$ [S. Borsanyi *et al.*, Phys. Rev. D 96, 074507 (2017)], [T. Blum, VG, *et al.*, Phys. Rev. Lett. 121, 022003 (2018)]

$$0 \leq C(t_c)\, e^{-E_{t_c}(t-t_c)} \leq C(t) \leq C(t_c)\, e^{-E_0(t-t_c)}$$

- E_{t_c}: effective mass of the correlator at t_c
- E_0: finite volume ground state energy, two pions with one unit of momentum
- use correlator data for $t < t_c$
- use upper and lower bound for $t \geq t_c$ vary t_c

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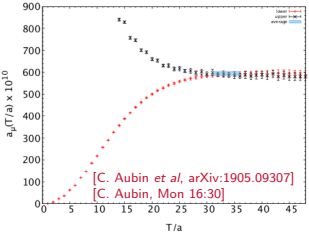
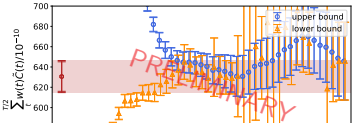
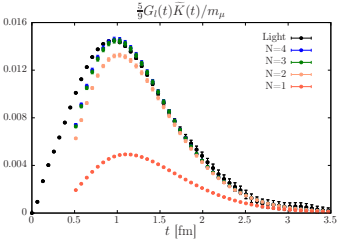




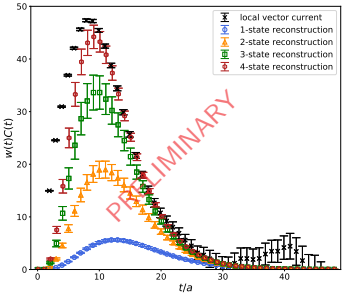
[Plot by A. Meyer], [A. Meyer, Mon 17:30]

Reconstruction of the long distance tail

- dedicated spectroscopy study, GEVP with different operators with overlap to two pions
- determine energies $\mathbf{E_n}$ and overlap factors $\mathbf{A_n}$ for lowest $\mathbf{N}$ states
- reconstruct the long distance tail of vector correlator



[A. Gérardin, Tue 14:40]
[A. Gérardin *et al*, arXiv:1904.03120]



[A. Meyer, Mon 17:30]

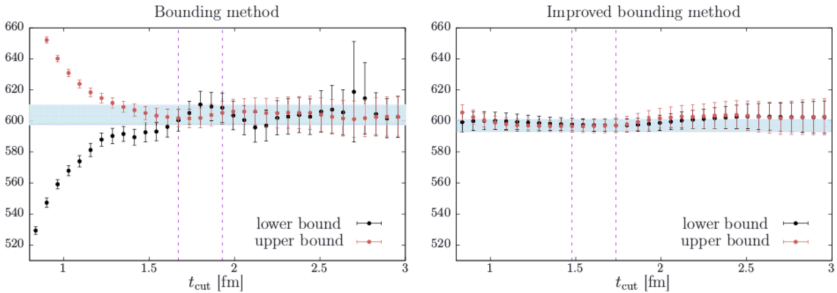
- can be used for improving the bounding method

Improved bounding method

- Improved bounding method using **N** lowest states [A. Meyer, Lattice 2018], [A. Gérardin *et al*, arXiv:1904.03120]

$$\tilde{C}(t) = C(t) - \sum_{n=0}^{N-1} \frac{A_n^2}{2E_n} e^{-E_n t} \qquad 0 \leq \tilde{C}(t_c)\, e^{-E_{t_c}(t-t_c)} \leq \tilde{C}(t) \leq \tilde{C}(t_c)\, e^{-E_N(t-t_c)}$$

- upper and lower bound overlap for smaller t_c
- a_μ^{hvp} can be extracted with smaller error



[A. Gérardin *et al*, arXiv:1904.03120], [A. Gérardin, Tue 14:40]

Finite volume (FV) effects

- dominated by two pion state - important at large t
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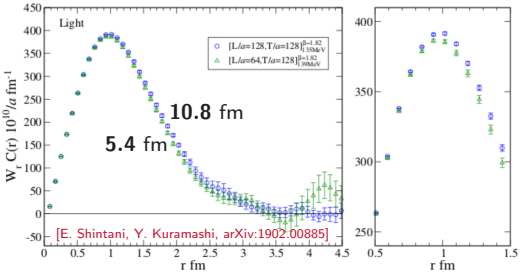

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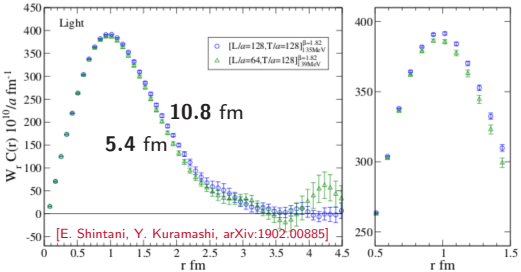

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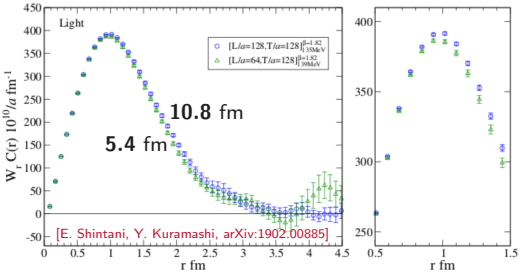

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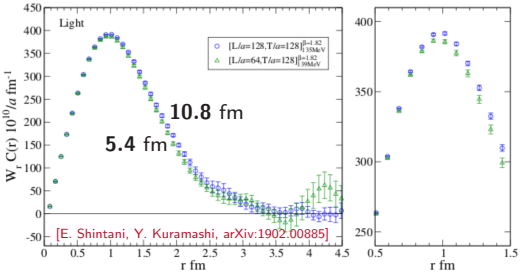

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- $\mathcal{O}(\mathbf{e^{-m_\pi L}})$ FV corrections using Hamiltonian approach (neglecting $\mathcal{O}(\mathbf{e^{-\sqrt{2}m_\pi L}})$)
  [M. Hansen, A. Patella, arXiv:1904.10010]

Finite volume effects from the timelike pion form factor

- long-distance contribution of vector correlator given in terms of the timelike pion form factor

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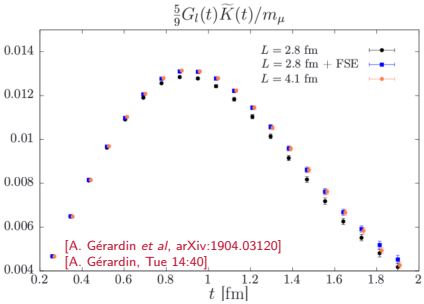

- $\mathbf{m_\pi = 280}$ MeV, two different volumes
- finite size effects (FSE) corrected using timelike pion form factor

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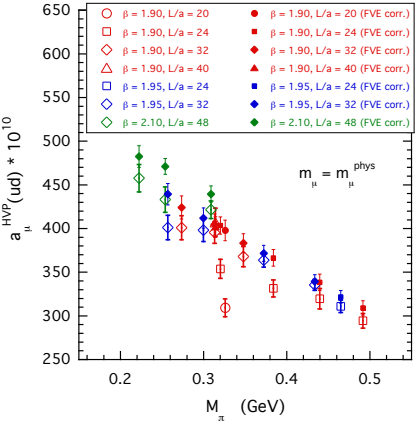

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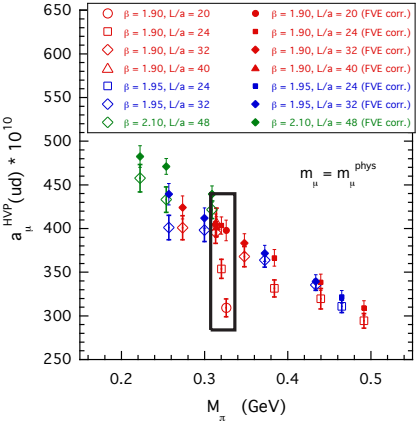

scale setting

- $\mathbf{a_\mu^{hvp}}$ depends on the scale through $\mathbf{am_\mu}$ in the kernel

- scale set by quantity $\mathbf{\Lambda}$ with error $\mathbf{\Delta\Lambda}$

$$\mathbf{\Delta a_\mu^{hvp}} = \left|\mathbf{\Lambda}\,\frac{\mathbf{da_\mu^{hvp}}}{\mathbf{d\Lambda}}\right| \cdot \frac{\mathbf{\Delta\Lambda}}{\mathbf{\Lambda}} = \left|\mathbf{M_\mu}\,\frac{\mathbf{da_\mu^{hvp}}}{\mathbf{dM_\mu}}\right| \cdot \frac{\mathbf{\Delta\Lambda}}{\mathbf{\Lambda}} \qquad \mathbf{M_\mu} = \frac{\mathbf{m_\mu}}{\mathbf{\Lambda}}$$

$\rightarrow$ relative error on $\mathbf{\Lambda}$ amplified by $\approx \mathbf{1.8}$ in relative error for $\mathbf{a_\mu}$ [M. Della Morte, VG, *et al*, JHEP 1710 (2017) 020]
$\rightarrow$ for $\mathbf{0.2\%}$ error on $\mathbf{a_\mu^{hvp}}$ need $\lesssim \mathbf{0.1\%}$ on lattice spacing

- precise scale setting, e.g. RBC/UKQCD using $\mathbf{\Omega}$-Baryon [T. Blum, VG, *et al*, Phys.Rev.Lett. 121 (2018) no.2, 022003]
$\approx \mathbf{0.2 - 0.3\%}$ on lattice spacing

extrapolation to the physical point

- chiral extrapolation
  - most calculations now done using (or including) ensembles at the physical point
  - chiral extrapolation if necessary
- continuum extrapolation
  - discretization effects depend on action used
  - ideally work in fully $\mathcal{O}(a)$ improved setup
    $\rightarrow$ actions usually $\mathcal{O}(a)$-improved
    $\rightarrow$ $\mathcal{O}(a)$-improvement of vector current, if necessary [A. Gérardin *et al*, arXiv:1904.03120]
  - ideally at least three lattice spacings

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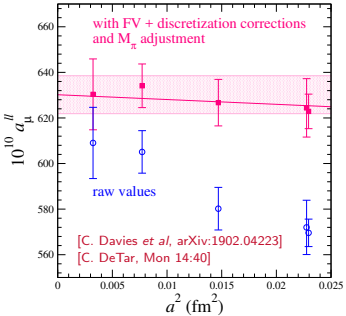
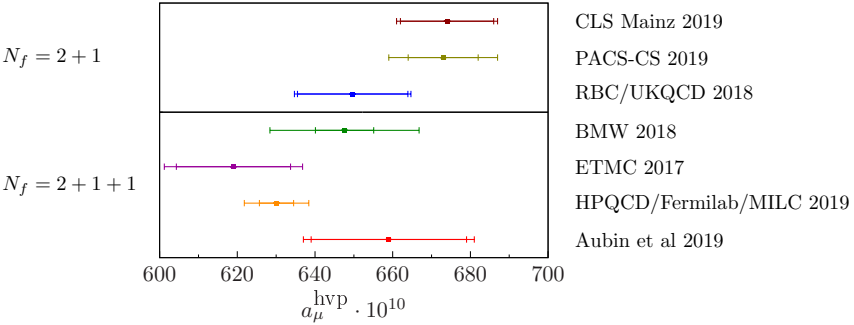


- HISQ action
- data points corrected for discretization effects from taste splitting

comparison - light quark results



- errors from **1.3% – 3.3%**
- $\approx$ **2σ** discrepancy between smallest and largest results

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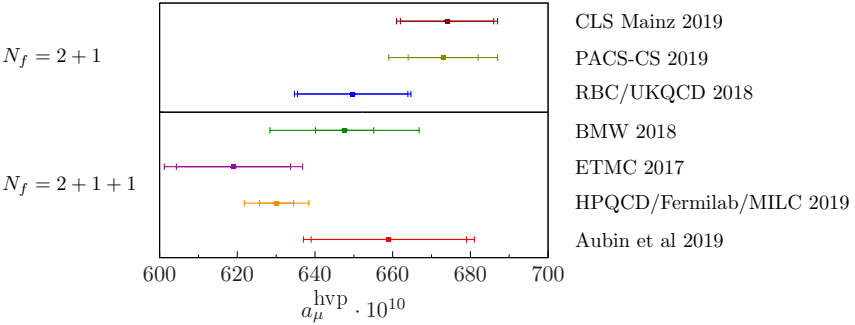
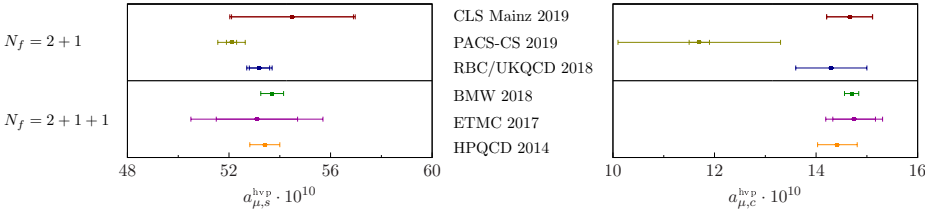


- errors from **1.3% – 3.3%**
- $\approx$ **2σ** discrepancy between smallest and largest results
- compare intermediate quantities, e.g. time-moments $\mathbf{G_{2n} = \int_{-\infty}^{\infty} dt\ t^{2n} C(t)}$ or $\mathbf{a_\mu^{hvp}}$ from time window

Outline

Strange and Charm HVP

- suffers less from long-distance noise-to-signal problem and finite volume effects than light contribution
- charm usually large discretization effects



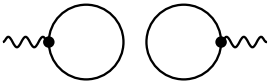
- errors on total HVP $\lesssim$ **0.4%** $\lesssim$ **0.3%**

Outline

disconnected HVP

- quark-disconnected Wick contraction
- **SU(3)** suppressed
- quark loop

$$\mathbf{\Delta^f_\mu(t) = \sum_{\vec{x}} \mathrm{Tr}\left[\gamma_\mu S^f(x,x)\right]}$$



- all-to-all propagators, calculate stochastically
- light-strange cancellation [V.G. *et al*, PoS LATTICE2014 (2014) 128]

$$\mathbf{C^{disc}(t) = \frac{1}{9}\left\langle (\Delta^\ell(t) - \Delta^s(t)) \cdot (\Delta^\ell(0) - \Delta^s(0)) \right\rangle}$$

- further noise reduction
  - [T. Blum *et al*, Phys. Rev. Lett. 116, 232002 (2016)] low-mode averaging and sparsened noise sources for high modes
  - [A. Gérardin *et al*, arXiv:1904.03120] hierarchical probing [A. Stathopoulos *et al*, arXiv:1302.4018]
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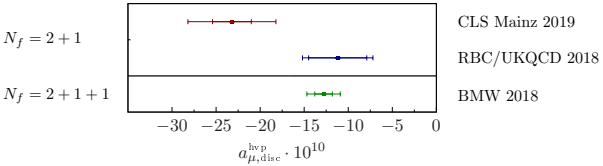

- all-to-all propagators, calculate stochastically
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- errors on total HVP **0.3 − 0.7%**
- work in progress HPQCD/FNAL/MILC [C. DeTar, Mon 14:40]

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Outline

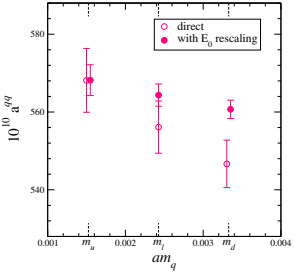
Isospin Breaking Corrections

- lattice calculations usually done in the isospin symmetric limit
- two sources of isospin breaking effects
  - different masses for up- and down quark (of $\mathcal{O}((\mathbf{m_d} - \mathbf{m_u})/\Lambda_{\mathrm{QCD}})$)
  - Quarks have electrical charge (of $\mathcal{O}(\alpha)$)
- lattice calculation aiming at $\lesssim \mathbf{1\%}$ precision requires to include isospin breaking

- separation of strong IB and QED effects requires renormalization scheme
- definition of "physical point" in a "QCD only world" also scheme dependent
  $\rightarrow$ results shown above without QED and isospin breaking for $\mathbf{m_\pi} \approx \mathbf{135}$ MeV

Strong isospin corrections from the lattice

- use different up, down quark masses
- sea quark effects:
  $\longrightarrow$ configurations with different up, down masses
- results [B. Chakraborty *et al.* Phys. Rev. Lett. **120** 152001 (2018)]
  $\delta a_\mu = 7.7(3.7) \times 10^{-10}$ $\quad$ $N_f = 2+1+1$
  $\delta a_\mu = 9.0(2.3) \times 10^{-10}$ $\quad$ $N_f = 1+1+1+1$



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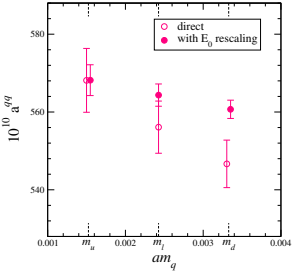
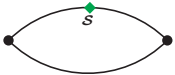
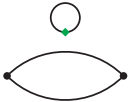


- perturbative expansion in $\Delta m = (m_u - m_d)$
  [G.M. de Divitiis *et al*, JHEP 1204 (2012) 124]

$$\langle O \rangle_{m_u \neq m_d} = \langle O \rangle_{m_u = m_d} + \Delta m \frac{\partial}{\partial m} \langle O \rangle \Big|_{m_u = m_d} + \mathcal{O}\left(\Delta m^2\right)$$



sea quark effects:



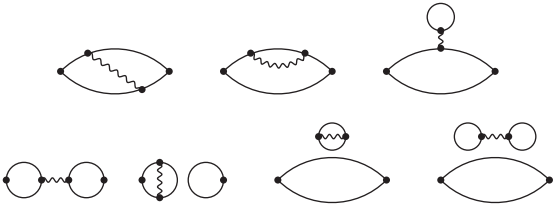
- ETMC [D. Giusti *et al*, arXiv:1901.10462]
  $\delta a_\mu = 6.0(2.3) \times 10^{-10}$
- RBC/UKQCD [T. Blum, VG *et al*, Phys.Rev.Lett. 121 (2018) no.2, 022003]
  $\delta a_\mu = 10.6(4.3)_S \times 10^{-10}$
  + work in progress
  [C. Lehner, Mon 14:20]

QED corrections from the lattice

- Euclidean path integral including QED

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}[\Psi, \overline{\Psi}] \mathcal{D}[U] \mathcal{D}[A] \; O \; e^{-S_F[\Psi, \overline{\Psi}, U, A]} \, e^{-S_G[U]} \, e^{-S_\gamma[A]}$$

- Finite Volume corrections for QED on the lattice
  $\rightarrow$ $\mathbf{1/(m_\pi L)^3}$ for QED corrections to HVP in $\mathrm{QED_L}$ [N. Hermansson Truedsson, Mon 16:50]
  [J. Bijnens *et al*, arXiv:1903.10591], [D.Giusti *et al*, JHEP 1710 (2017) 157]
  $\rightarrow$ negligible for required precision
- perturbative expansion of the path integral in α [RM123 Collaboration, Phys.Rev. **D87**, 114505 (2013)]



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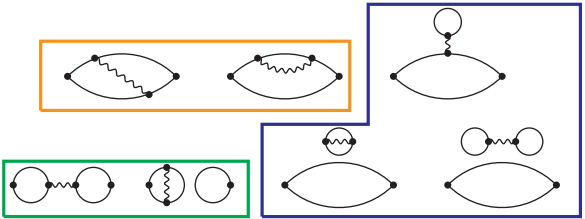


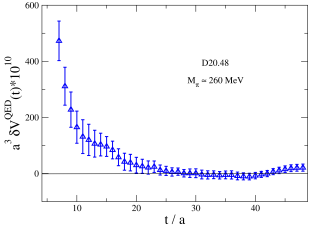
quark-connected

quark-disconnected

sea-quark effects

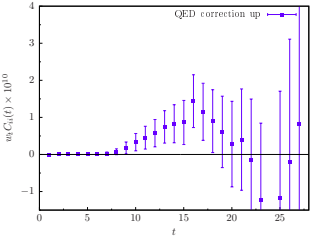
Results QED corrections

- connected contributions in electro-quenched approximation



[D. Giusti *et al*, Phys.Rev. **D99** (2019) no.11, 114502]

- several pion masses, extrapolation to physical point
  $\delta a_\mu^{\text{hvp}} = 1.1(1.0) \times 10^{-10}$



[T. Blum, VG, *et al.*, Phys. Rev. Lett. 121, 022003 (2018)]
[VG *et al.*, PoS LATTICE2018 (2018) 134]

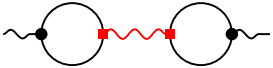
- directly at physical point, single lattice spacing
  $\delta a_\mu^{\text{hvp}} = 5.9(5.7) \times 10^{-10}$
  + work in progress
  [C. Lehner, Mon 14:20]

work in progress by BMW
[B. Toth, Tue 17:10]

work in progress by Mainz
[A. Risch, Tue 14:20]

Results QED corrections

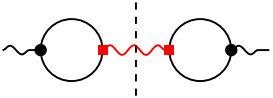
- leading QED correction to the disconnected HVP



gluons between the quarks

$\rightarrow$ QED correction to LO HVP

no gluons between the quarks



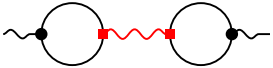
$\rightarrow$ included in NLO HVP

- QED correction to disconnected HVP [T. Blum, VG, *et al.*, Phys. Rev. Lett. 121, 022003 (2018)]

$$\mathbf{a}_{\mu}^{\text{QED, disc}} = -\mathbf{6.9(2.1)(1.4)} \times \mathbf{10^{-10}}$$

Results QED corrections

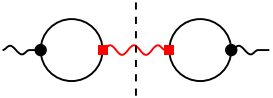
- leading QED correction to the disconnected HVP



gluons between the quarks

$\rightarrow$ QED correction to LO HVP

no gluons between the quarks



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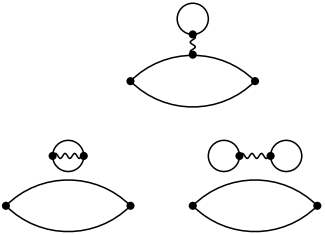
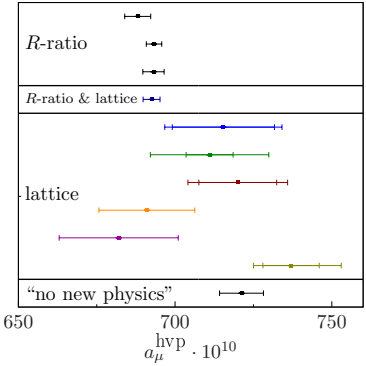
- QED corrections from sea-quark effects
- diagrams at least $\mathbf{1/N_c}$ suppressed
  $\rightarrow$ could be **33%** of connected
  $\rightarrow$ need to be studied for sub-percent precision on total HVP
- work in progress by BMW [B. Toth, Tue 17:10]
- work in progress RBC/UKQCD [C. Lehner, Mon 14:20]



Outline

Full HVP comparison



Jegerlehner 2017
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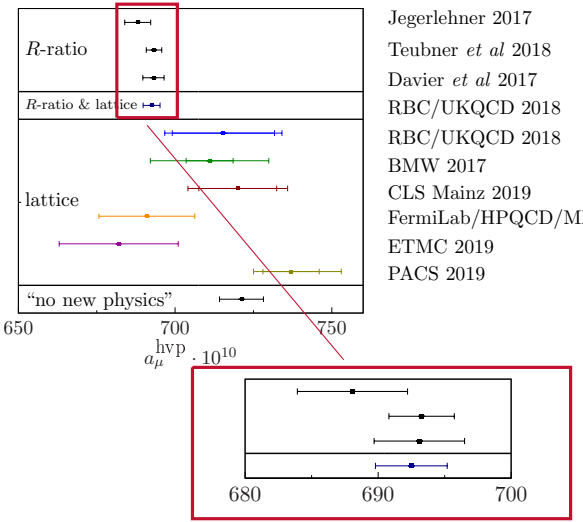

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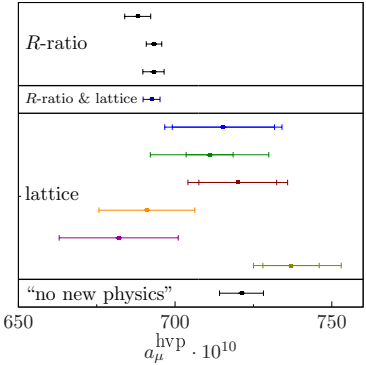

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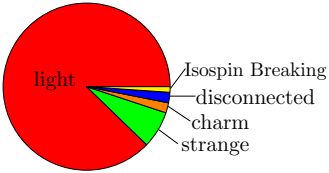
contribution to $\mathbf{a}_\mu^{\mathrm{hvp}}$



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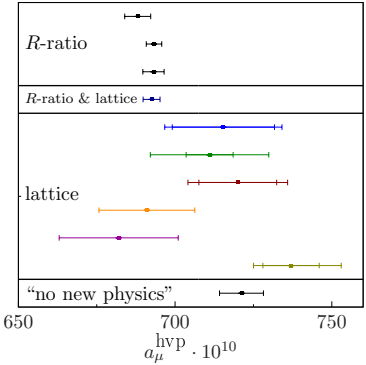

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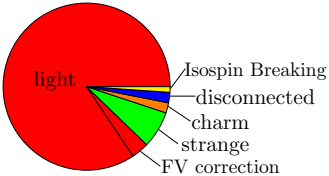

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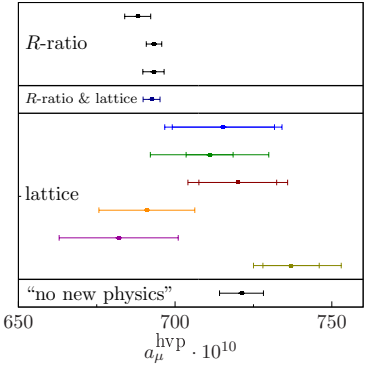

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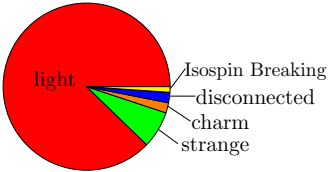
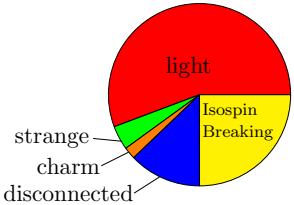


contribution to $\mathbf{\Delta a}_\mu^{\mathrm{hvp}} \approx \mathbf{2.5\%}$



Conclusions and Prospects

- most important issues:
  - noise reduction and control of long-distance tail of the light quark correlator
  - careful estimate of finite volume effects
  - first lattice calculations of isospin breaking and QED corrections
    $\rightarrow$ study also sea quark effects
  - achieve consensus between lattice results

Conclusions and Prospects

- most important issues:
  - noise reduction and control of long-distance tail of the light quark correlator
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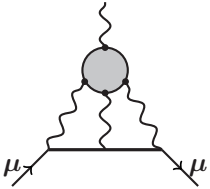
    $\rightarrow$ study also sea quark effects
  - achieve consensus between lattice results

- hadronic vacuum polarisation also enters other quantities
  - running of the electromagnetic coupling [Miguel Teseo San José Pérez, Mon 15:20]
  - running of the Weinberg angle [Marco Cè, Mon 15:00]

Outline

Introduction

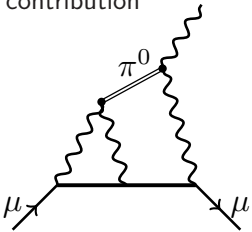
- hadronic light-by-light scattering enters at α^3



- Glasgow consensus [J. Prades, E. de Rafael, A. Vainshtein, Adv.Ser.Direct.High Energy Phys. **20** (2009) 303-317]

π^0, η, η'	$(11.4 \pm 1.3) \times 10^{-10}$
charged π loop	$(-1.9 \pm 1.9) \times 10^{-10}$
axialvector	$(1.5 \pm 1.0) \times 10^{-10}$
scalar	$(-0.7 \pm 0.7) \times 10^{-10}$
charm loops	0.2×10^{-10}
total	$(10.5 \pm 2.6) \times 10^{-10}$

e.g. π^0 contribution

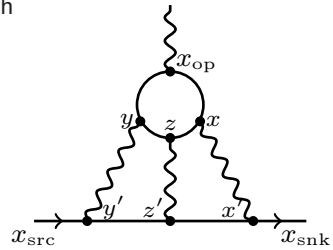


- work in progress using dispersion relations, see e.g. [G. Colangelo *et al*, JHEP 1902 (2019) 006], [G. Colangelo *et al*, Phys.Rev.Lett. 118 (2017) no.23, 232001], [G. Colangelo *et al*, JHEP 1704 (2017) 161], [M. Hoferichter *et al*, JHEP 1810 (2018) 141],[M. Hoferichter *et al*, Phys.Rev.Lett. 121 (2018) no.11, 112002], [V. Pauk and M. Vanderhaeghen, Phys.Rev. D90 (2014) no.11, 113012], . . .

Outline

light-by-light from the lattice

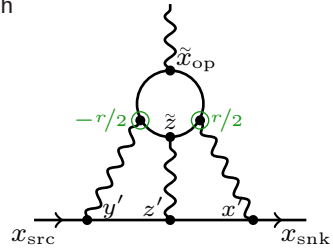
- two collaborations working on this: RBC/UKQCD and Mainz, both using position space approaches



+ 5 other permutations of $\mathbf{x}'$, $\mathbf{y}'$, $\mathbf{z}'$

light-by-light from the lattice

- two collaborations working on this: RBC/UKQCD and Mainz, both using position space approaches
- approach proposed in [T. Blum *et al*, Phys.Rev. **D93** (2016) no.1, 014503]
  - position space sampling, i.e. stochastic evaluation of sum over **r**
  - exact photon propagators
    $\longrightarrow$ photons in $\text{QED}_{\mathbf{L}}$: power-law finite volume corrections
    $\longrightarrow$ infinite volume photons [T. Blum *et al*, Phys. Rev. **D96**, 034515 (2017)]



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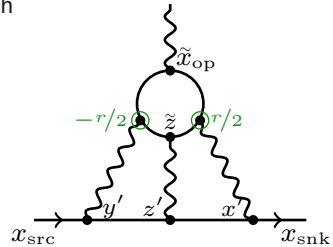
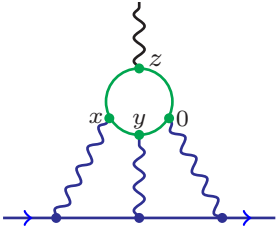


- approach proposed by Mainz [J. Green *et al*, arXiv:1510.08384],[N. Asmussen *et al*,1609.08454]

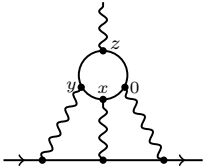
$$a_\mu^{\mathrm{lbl}} = \frac{me^6}{3} \int dx^4 dy^4 \, \overline{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) \; i\hat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)$$

  - calculate $\overline{\mathcal{L}}$ (semi-) analytical in the continuum and infinite volume

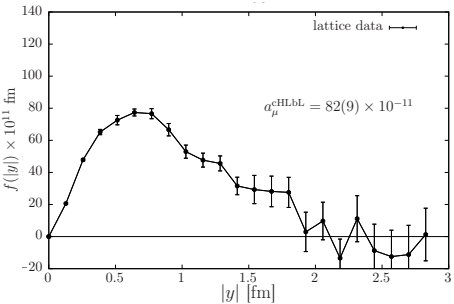


Results Mainz connected light-by-light

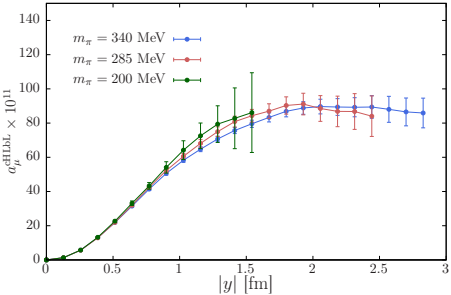
- preliminary results, see [N. Asmussen, Tue 15:00], [N. Asmussen, g-2 workshop Mainz]
- connected diagram



$\mathbf{a_\mu^{lbl}}$ integrand ($\mathbf{m_\pi = 340}$ MeV):

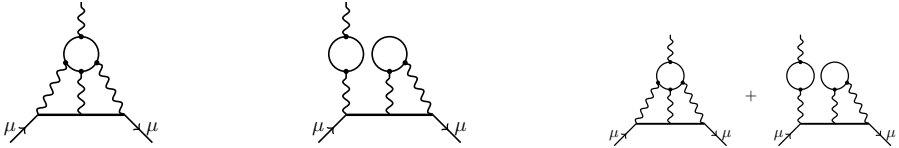


$\mathbf{a_\mu^{lbl}}$ partial integration up to $|\mathbf{y}|$:

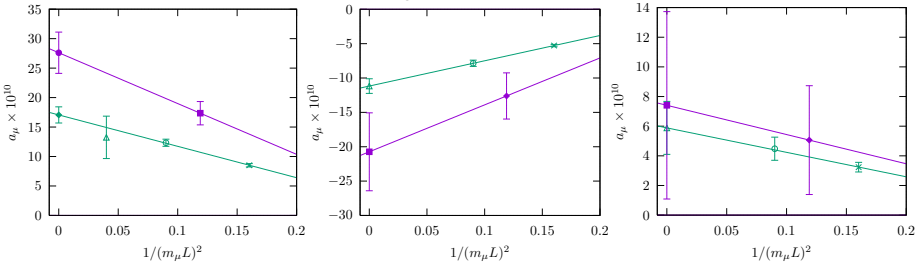


Results RBC/UKQCD connected + leading disconnected light-by-light

- preliminary results, see [T. Blum, Tue 16:30]



- continuum and infinite volume extrapolation $\mathrm{QED_L}$



$\mathbf{a_\mu^{clbl} = 27.61(3.51)(0.32)}$ $\mathbf{a_\mu^{dlbl} = -20.20(5.65)}$ $\mathbf{a_\mu^{lbl} = 7.41(6.32)(0.32)}$ $(\times 10^{-10})$

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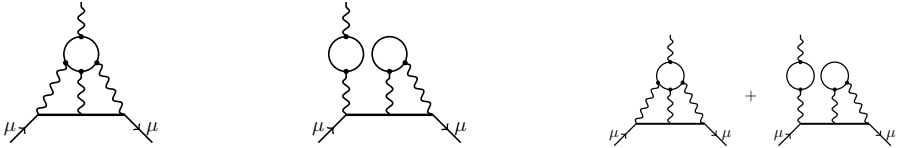
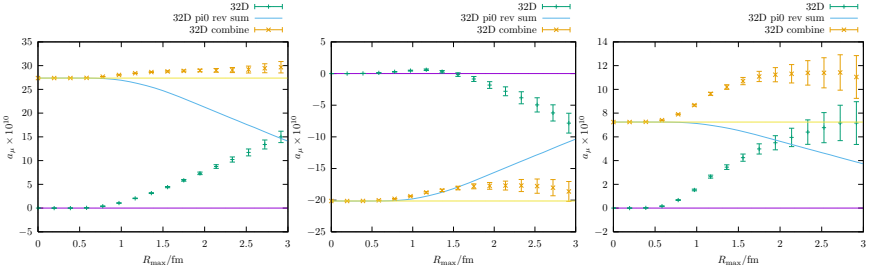


- QED$_\infty$, combined with π^0-pole contribution from model for long distances $\geq \mathbf{R}_{\max}$



- work in progress: replace model by lattice calculation of $\pi^0 \to \gamma\gamma$, see [L. Jin, Thu 10:15]

Outline

Conclusions - light-by-light

- two collaborations working on lattice calculations
  - RBC/UKQCD: first result (connected+leading disconnected) extrapolated to physical point
  - Mainz: connected contribution
- important check: consistency with Glasgow Consensus?

  $\rightarrow$ would need $\approx 3\times$ larger a_μ^{lbl} than Glasgow Consensus to explain a_μ discrepancy

  $\rightarrow$ preliminary lattice results suggest this is unlikely

- lattice calculations of the pion transition form factor $\pi^0 \rightarrow \gamma\gamma$
  [A. Gérardin *et al*, Phys.Rev. D94 (2016) no.7, 074507],[A. Gérardin *et al*, arXiv:1903.09471]

  $\rightarrow$ pion pole contribution to a_μ^{lbl}

  $\rightarrow$ constrain long-distance tail to a_μ^{lbl} lattice calculation

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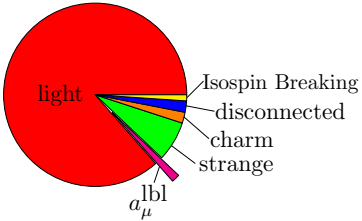

size of light-by-light vs HVP

Outline

Final remarks

- $\mathbf{a}_\mu$ measured and calculated very precisely
  $\rightarrow$ test of the Standard Model
  $\rightarrow$ new experiment running at Fermilab, see [Dikai Li, Fri 09:45]
  $\rightarrow$ largest uncertainty in Standard Model prediction from hadronic contributions
- huge effort in the lattice community to calculate hadronic contributions from first principles
  $\rightarrow$ **15** parallel talks at Lattice 2019!
- work in progress on g-2 Theory Whitepaper from the Muon g-2 Theory Initiative,
  several workshops since 2017, next workshop: September 9 - 13, 2019 at INT

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  several workshops since 2017, next workshop: September 9 - 13, 2019 at INT
- hadronic vacuum polarisation contribution to $\mathbf{a}_\mu$
  - first lattice calculations of $\mathbf{a}_\mu^{\text{hvp}}$ with $\lesssim$ **1%** precision available within $\mathcal{O}$(year)
  - $\lesssim$ **0.2%** within a few years

Thank you

Thanks for sending me material and/or discussions:

N. Asmussen, C. Aubin, T. Blum, C. DeTar, D. Giusti, C. Lehner, L. Lellouch, A. Meyer, B. Toth, G. von Hippel

VG has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme under grant agreement No 757646.

Outline

Backup

Results - total HVP

collaboration	$\mathbf{a_\mu^{hvp} \times 10^{10}}$	action	arXiv
Mainz 19	**720.0(12.4)(9.9)**	clover	1904.03120
PACS-CS 19	$\mathbf{737(9)(^{+13}_{-18})}$	StoutWilson	1902.00885
RBC/UKQCD 18	**715.4(16.3)(9.2)**	DWF	1801.07224
BMW 18	**711.1(7.5)(17.4)**	Stout4S	1711.04980
ETMC 18	**682(19)**	tm	1808.00887, 1901.10462
Flab/HPQCD/MILC 19	**691(8)(1)(13)**	HISQ	1902.04223

Results - light quark HVP

collaboration	$\mathbf{a}_{\mu,\ell}^{\mathrm{hvp}} \times \mathbf{10}^{\mathbf{10}}$	arXiv
Mainz 19	**674(12)(5)**	1904.03120
PACS-CS 19	**673(9)(11)**	1902.00885
RBC/UKQCD 18	**649.7(14.2)(4.9)**	1801.07224
BMW 18	**647.6(7.5)(17.7)**	1711.04980
ETMC 18	**619.0(14.7)(10.0)**	1808.00887
Flab/HPQCD/MILC 19	**630.1(4.4)(7.0)**	1902.04223
Aubin et al 19	**650(20)(8)**	1905.09307

Results - strange and charm quark HVP

collaboration	$a^{hvp}_{\mu,s} \times 10^{10}$	$a^{hvp}_{\mu,c} \times 10^{10}$	arXiv
Mainz 19	**54.5(2.4)(0.6)**	**14.66(0.45)(0.06)**	1904.03120
PACS-CS 19	**52.1(2)(5)**	**11.7(2)(1.6)**	1902.00885
RBC/UKQCD 18	**53.2(0.4)(0.3)**	**14.3(0.0)(0.7)**	1801.07224
BMW 18	**53.7(0.0)(0.4)**	**14.7(0.0)(0.1)**	1711.04980
ETMC 17	**53.1(16)(2.0)**	**14.75(42)(37)**	1707.03019
HPQCD14	**53.41(59)**	**1.42(39)**	1403.1778

Results - disconnected HVP

collaboration	$a^{hvp}_{\mu,disc} \times 10^{10}$	arXiv
Mainz 19	**−23.2(2.2)(4.5)**	1904.03120
RBC/UKQCD 18	**−11.2(3.3)(2.3)**	1801.07224
BMW 18	**−12.8(1.0)(1.6)**	1711.04980

Hadronic Vacuum Polarisation from time moments

- using time moments of the vector two-point function [B. Chakraborty *et al*, Phys.Rev. **D89** (2014) no.11, 114501]
- Taylor expansion for HVP function

$$\Pi(Q^2) = \Pi_0 + \sum_{j=1}^{\infty} \Pi_j Q^{2j}$$

- coefficients given by time moments of the vector correlator $C(t)$

$$\Pi_j = (-1)^{j+1} \frac{G_{2j+2}}{(2j+2)!} \qquad G_{2n} = \int_{-\infty}^{\infty} dt\ t^{2n} C(t)$$

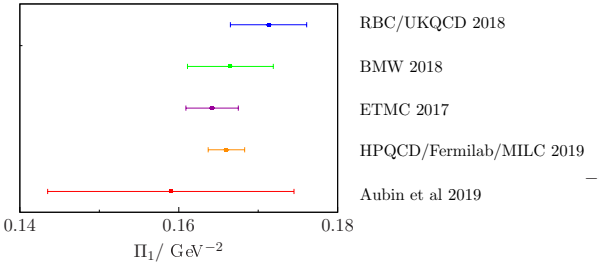
- Taylor coefficients can be used to construct, e.g., Padé approximants for $\Pi(Q^2)$

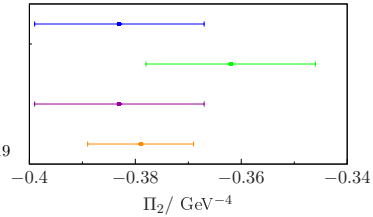
$$\Pi_{[N,M]}(Q^2) = \Pi(0) + \frac{\sum_{i=1}^{N} a_i Q^{2i}}{\sum_{k=1}^{M} b_k Q^{2k}}$$

- higher moments probe larger times in Euclidean vector correlator
  $\rightarrow$ can be used to systematically compare results from different collaborations

comparison time moments

- compare light connected time moments from different collaborations





FV effects for light quark HVP

- ETMC [D.Guisti *et al*, Phys. Rev. **D98**, 114504 (2018)]

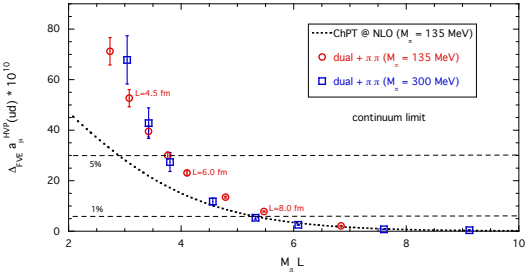


FIG. 19: *Values of* $\Delta_{FVE} a_\mu^{\rm HVP}(ud)$ *(see Eq. (34)), evaluated in the continuum limit according to our "dual + ππ" representation at the physical pion point (red circles) and at a larger pion mass equal to* $M_\pi = 300$ *MeV (blue squares). The dotted line corresponds to the predictions of ChPT at NLO [47, 60].*

- RBC/UKQCD [C. Lehner, Mon 14:20]

	$a_\mu^{\rm hvp}(L = 6.22\ \text{fm}) - a_\mu^{\rm hvp}(L = 4.66\ \text{fm})$
lattice	$21.6(6.3) \times 10^{-10}$
NLO ChPT	12.2×10^{-10}
GS	$20(3) \times 10^{-10}$

The zero-mode of the photon field

- zero-mode of the photon field
  shift symmetry of the of the photon action $\mathbf{A}_\mu(\mathbf{x}) \to \mathbf{A}_\mu(\mathbf{x}) + \mathbf{c}_\mu$
  $\to$ cannot be constrained by gauge fixing
- different prescriptions of QED:
- $\mathrm{QED}_{\mathrm{TL}}$: remove the zero-mode of the photon field, i.e. $\tilde{\mathbf{A}}_\mu(\mathbf{k}=\mathbf{0})=\mathbf{0}$
  [A. Duncan, E. Eichten, H. Thacker, Phys.Rev.Lett. **76**, 3894 (1996)]
- $\mathrm{QED}_{\mathrm{L}}$: remove all the spatial zero-modes, i.e. $\tilde{\mathbf{A}}_\mu(\mathbf{k}_0, \vec{\mathbf{k}}=\mathbf{0})=\mathbf{0}$
  [S. Uno, M. Hayakawa, Prog. Theor. Phys. **120**, 413 (2008)]

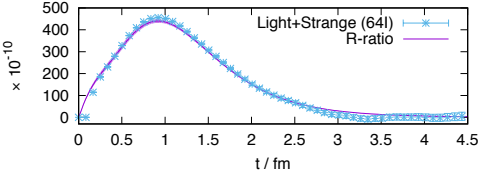
$$\mathbf{\Delta}_{\mu\nu}(\mathbf{x}-\mathbf{y}) = \frac{\mathbf{1}}{\mathbf{V}} \sum_{\mathbf{k},\, \vec{\mathbf{k}} \neq \mathbf{0}} \frac{\mathbf{e}^{\mathbf{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{\hat{\mathbf{k}}^2}$$

- $\mathrm{QED}_{\mathrm{m}}$: use a massive photon and take $\mathbf{m}_\gamma \to \mathbf{0}$ [M. Endres *et al.*,Phys. Rev. Lett. **117** (2016) 072002]
- $\mathrm{QED}_{\mathrm{C}}$: $\mathbf{C}^*$ boundary conditions in spatial direction, i.e. fields are periodic up to charge conjugation
  [B. Luchini *et al.* JHEP **02** (2016) 076]

- for detailed discussion on different prescriptions of QED see e.g. [A. Patella 1702.03857]

Window Method

- combining lattice with **R**-ratio data [RBC/UKQCD, Phys.Rev.Lett. 121 (2018) 022003]



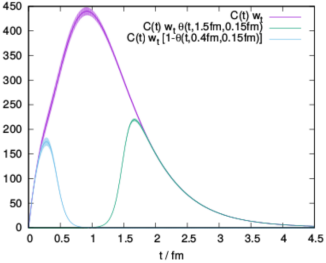
- very short and long distances from **R**-ratio, intermediate distances from lattice

$$a_\mu = a_\mu^{SD} + a_\mu^{W} + a_\mu^{LD}$$

$$a_\mu^{SD} = \sum_t w_t C(t)[1 - \theta(t, t_0, \Delta)]$$

$$a_\mu^{W} = \sum_t w_t C(t)[\theta(t, t_0, \Delta) - \theta(t, t_1, \Delta)]$$

$$a_\mu^{LD} = \sum_t w_t C(t)\, \theta(t, t_1, \Delta)$$



vector correlator and timelike pion form factor

- long-distance isovector correlator in infinite volume given by time-like pion form factor $\mathbf{F_\pi(\omega)}$

$$\mathbf{C^{I=1}(t)} = \int_0^\infty \mathrm{d}\omega\ \omega^2\, \rho(\omega^2)\, \mathrm{e}^{-\omega t} \qquad \rho(\omega^2) = \frac{1}{48\pi^2}\left(1 - \frac{4m_\pi^2}{\omega^2}\right)^{\frac{3}{2}} |F_\pi(\omega)|^2$$

- e.g. use Gounaris-Sakurai Parameterisation of $\mathbf{F_\pi(\omega)}$; two parameters $\mathbf{m_\rho}$, $\mathbf{\Gamma_\rho}$
- Lüscher: infinite volume phase shifts $\leftrightarrow$ FV energy levels
  [M. Lüscher, Nucl.Phys. **B364**, 237 (1991)], [M. Lüscher, Nucl.Phys. **B354**, 531 (1991)]

$$\delta_{11}(k_n) + \phi\left(\frac{k_n L}{2\pi}\right) = n\pi \qquad \tan(\phi(z)) = -\frac{\pi^{3/2} z}{\mathcal{Z}_{00}(1; z^2)}$$

- finite volume long-distance correlator using [H. Meyer, Phys.Rev.Lett. **107** (2011) 072002]

$$|F_\pi(\omega_n)|^2 = \left([z\phi'(z)]_{z=\frac{k_n L}{2\pi}} + k_n \frac{\partial \delta_{11}(k_n)}{\partial k_n}\right) \frac{3\pi\omega_n^2}{2k_n^5} |A_n|^2$$

and use $|F_\pi(\omega_n)|^2$ to obtain $|A_n|^2$ and

$$C(t) = \sum_n |A_n|^2 e^{-\omega_n t}$$

strong IB and QED separation prescription

- ETMC [D. Giusti *et al*, arXiv:1901.10462]

  $\rightarrow$ impose that $\mathbf{m_{ud}}$, $\mathbf{m_s}$ and $\mathbf{m_c}$ and the strong coupling constant α_s match at $\overline{\mathbf{MS}}$**(2** GeV**)** in QCD+QED and pure QCD [J. Gasser *et al*, Eur.Phys.J. C32 (2003) 97-114]

- RBC/UKQCD uses hadronic scheme [T. Blum, VG, *et al*, Phys.Rev.Lett. 121 (2018) no.2, 022003]

  $\rightarrow$ tune (**u**,**d**,**s**) masses to reproduce experimental π^+, $\mathbf{K}^+$ and $\mathbf{K_0}$ mass

$$\mathbf{m}_{\pi^+}^{\text{exp}} = \left[\mathbf{m}_{\pi}^{0} + \alpha \mathbf{m}_{\pi^+}^{\text{QED}} + \mathbf{\Delta m_d}\ \mathbf{m}_{\pi^+}^{\mathbf{\Delta m_d}} + \mathbf{\Delta m_u}\ \mathbf{m}_{\pi^+}^{\mathbf{\Delta m_u}}\right]$$

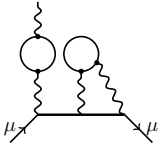
$$\mathbf{m}_{K^+}^{\text{exp}} = \left[\mathbf{m}_{K}^{0} + \alpha \mathbf{m}_{K^+}^{\text{QED}} + \mathbf{\Delta m_u}\ \mathbf{m}_{K^+}^{\mathbf{\Delta m_u}} + \mathbf{\Delta m_s}\ \mathbf{m}_{K^+}^{\mathbf{\Delta m_s}}\right]$$

$$\mathbf{m}_{K^0}^{\text{exp}} = \left[\mathbf{m}_{K}^{0} + \alpha \mathbf{m}_{K^0}^{\text{QED}} + \mathbf{\Delta m_d}\ \mathbf{m}_{K^0}^{\mathbf{\Delta m_d}} + \mathbf{\Delta m_s}\ \mathbf{m}_{K^0}^{\mathbf{\Delta m_s}}\right]$$

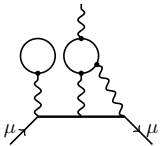
- similar scheme used by FermilabLattice/HPQCD/MILC [B. Chakraborty *et al.* Phys. Rev. Lett. **120** 152001 (2018)] using quark masses from MILC [S. Basak *et al*, Phys.Rev. D99 (2019) no.3, 034503]

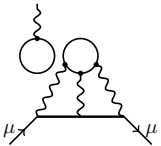
disconnected light-by-light diagrams

- leading disconnected

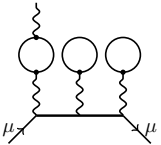


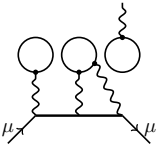
- **SU(3)**-suppressed



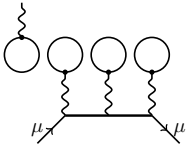


- **SU(3)**^{2}-suppressed





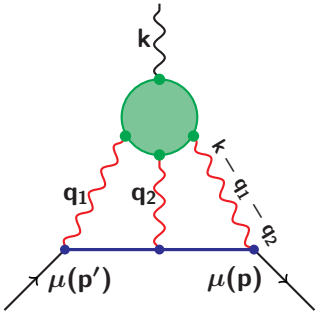
- **SU(3)**^{4}-suppressed



Hadronic light-by-light

- hadronic light-by-light matrix element

$$\langle \mu(p')|j^\gamma_\rho|\mu(p)\rangle$$
$$= -(ie)^6 \int \frac{d^4q_1}{(2\pi)^4} \int \frac{d^4q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 (q_1+q_2-k)^2}$$
$$\times\, \bar{u}(p')\gamma^\mu \frac{(i\not{p}' - i\not{q}_1 - m)}{(p'-q_1)^2+m^2}\gamma^\nu \frac{(i\not{p}' - i\not{q}_1 - i\not{q}_2 - m)}{(p'-q_1-q_2)^2+m^2}\gamma^\lambda u(p)$$
$$\times\, \Pi_{\mu\nu\lambda\rho}(q_1, q_2, k-q_1-q_2)$$

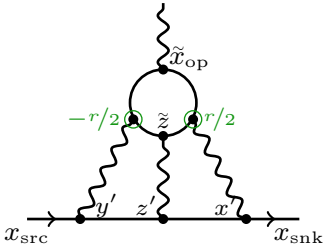


- Hadronic tensor

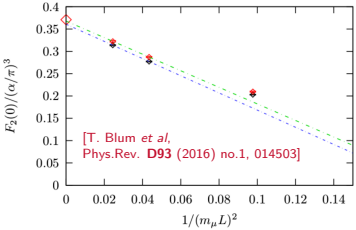
$$\Pi_{\mu\nu\lambda\rho}(q_1, q_2, q_3) = \int d^4x_1 d^4x_2 d^4x_3\, e^{-i(q_1x_1+q_2x_2+q_3x_3)} \left\langle j^\gamma_\mu(x_1) j^\gamma_\nu(x_2) j^\gamma_\lambda(x_3) j^\gamma_\rho(0)\right\rangle_{\text{QCD}}$$

light-by-light: finite volume photons

- position space calculation naively: volume sum over 6 photon vertices
- approach proposed in [T. Blum *et al*, Phys.Rev. **D93** (2016) no.1, 014503]
- position space sampling
  - stochastic evaluation of sum over **r**
    $\rightarrow$ importance sampling
  - pairs of point sources at random $-\mathbf{r}/2$ and $\mathbf{r}/2$
    $\rightarrow$ sample important region for small $|\mathbf{r}|$ more frequently
  - compute all pairs for $|\mathbf{r}| < \mathbf{r}_{\text{max}}$
  - construct hadronic part of the diagram from the two point-sources at $-\mathbf{r}/2$ and $\mathbf{r}/2$
- exact photon propagators
- finite volume, e.g. $\text{QED}_{\mathbf{L}}$
  $\rightarrow$ power law finite volume effects



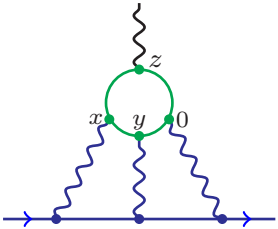
+ 5 other permutations of $\mathbf{x}'$, $\mathbf{y}'$, $\mathbf{z}'$



light-by-light: infinite volume photons

- approach proposed by Mainz [J. Green *et al*, arXiv:1510.08384],[N. Asmussen *et al*,1609.08454]

$$a_\mu^{\text{HIBI}} = \hat{F}_M(0) = \frac{me^6}{3} \int dx^4 dy^4 \, \overline{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) \; i\hat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)$$



- hadronic part

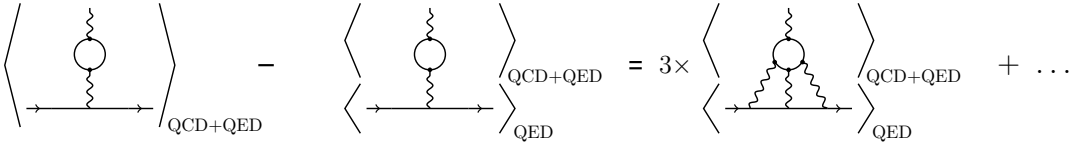
$$\hat{\Pi}_{\rho;\mu\nu\lambda\sigma} = \int dz^4 \; i z_\rho \left\langle j_\mu^\gamma(x)\, j_\nu^\gamma(y)\, j_\lambda^\gamma(0)\, j_\sigma^\gamma(z) \right\rangle$$

$\rightarrow$ calculate using sequential propagators

- QED Kernel function $\overline{\mathcal{L}}$ (averaged over direction of muon momentum)
  - calculate (semi-) analytical in the continuum and infinite volume
  - integrand $\overline{\mathcal{L}}\hat{\Pi}$ Lorentz scalar $\rightarrow$ depends only on x^2, y^2, $x \cdot y$
  - pre-calculate $\overline{\mathcal{L}}$ and store on a $(|x|, |y|, \cos\beta)$ Grid, with $\cos\beta = x \cdot y / |x||y|$
- remaining finite volume effects are $\mathcal{O}(e^{-mL})$
- similar approach developed by RBC/UKQCD [T. Blum *et al*, Phys. Rev. **D96**, 034515 (2017)]
- suitable subtraction for QED Kernel can reduce discretisation effects [T. Blum *et al*, Phys. Rev. **D96**, 034515 (2017)], [N. Asmussen, g-2 workshop Mainz], e.g. $\overline{\mathcal{L}}^{(2)}(x,y) = \overline{\mathcal{L}}(x,y) - \overline{\mathcal{L}}(0,y) - \overline{\mathcal{L}}(x,0)$

light-by-light from QCD+QED

- first strategy proposed in [M. Hayakawa *et al*, PoS LAT2005 353, hep-lat/0509016], [T. Blum *et al*, Phys. Rev. Lett. **114**, 012001 (2015)]



- using combined QCD+(quenched) QED gauge ensembles
- large subtraction of terms
- leading remaining term is (three times) hadronic light-by-light scattering

The anomalous magnetic moment of the electron

- new physics contribution to $\mathbf{a}_\ell$ factor $\mathbf{m_e^2/m_\mu^2 = 2 \times 10^{-5}}$ smaller

$$\mathbf{a_\ell^{NP} \propto \frac{m_\ell^2}{M_{NP}^2}}$$

- best experimental result [D. Hanneke, S. Fogwell, G. Gabrielse, Phys.Rev.Lett. 100 (2008) 120801]

$$\mathbf{a_e = 11596521.8073(28) \times 10^{-10}}$$

- new α measurement [R. Parker *et al*, Science 360 (2018) 191]

$$\alpha^{-1} = \mathbf{137.035999046(27)}$$

  as input for Standard Model prediction
- see, e.g. [H. Davoudiasl, W. Marciano, Phys. Rev. **D98**, 075011 (2018)

$$\mathbf{\Delta a_e = -87(28)^{exp}(23)^{\alpha}(2)^{SM} \times 10^{-14}}$$

  $\mathbf{2.4\sigma}$ discrepancy with opposite sign to $\mathbf{\Delta a_\mu}$