Magnetic moment of leptons ($e, \mu, \tau$)

- Magnetic moment $\vec{\mu}$ of the lepton $\ell$ due to its spin $\vec{s}$ and electric charge $e$

  $$\vec{\mu} = g \frac{e}{2m_\ell} \vec{s}$$

- Torque $\vec{\tau} = \vec{\mu} \times \vec{B}$

- $g$-factor: without quantum fluctuations for a lepton one finds $g = 2$

- Deviation from the value “2” due to quantum loops $\rightarrow$ anomalous magnetic moment of lepton $\ell$

  $$a_\ell = \frac{g_\ell - 2}{2}$$

- $F_1(0) = 1$ (electric charge) $\quad F_2(0) = a_\ell$ (anomalous magnetic moment)
\( a_\mu \): Experiment vs. Theory

- measured and calculated very precisely \( \rightarrow \) test of the Standard Model
- experiment: polarized muons in a magnetic field \( [\text{Bennet et al., Phys. Rev. D73, 072003 (2006)}] \)

\[
a_\mu = 11659209.1(5.4)(3.3) \times 10^{-10}
\]

\( \omega_a = a_\mu \frac{eB}{m_\mu} \)

- new experiments at Fermilab and JPARC \( \rightarrow \) reduce error by \( \approx 4 \)
- \( \rightarrow \) experiment at Fermilab is running
- \( \rightarrow \) see [Dikai Li, Fri 09:45]
The anomalous magnetic moment of the muon

$a_\mu$: Experiment vs. Theory

- measured and calculated very precisely $\rightarrow$ test of the Standard Model
- experiment: polarized muons in a magnetic field \cite{Bennet et al., Phys.Rev. D73, 072003 (2006)}

\[
a_\mu = 11659209.1(5.4)(3.3) \times 10^{-10}
\]

- Standard Model
\[ a_\mu: \text{Experiment vs. Theory} \]

- measured and calculated very precisely \(\rightarrow\) test of the Standard Model

\[ a_\mu = 11659209.1(5.4)(3.3) \times 10^{-10} \]

- Standard Model
  - \(\text{em} \quad (11658471.895 \pm 0.008) \times 10^{-10} \) [Kinoshita et al., Phys.Rev.Lett. 109, 111808 (2012)]
$a_\mu$: Experiment vs. Theory

- measured and calculated very precisely $\rightarrow$ test of the Standard Model
- experiment: polarized muons in a magnetic field $[\text{Bennet et al., Phys.Rev. D73, 072003 (2006)}]$ 
  
  $$a_\mu = 11659209.1(5.4)(3.3) \times 10^{-10}$$

- Standard Model
  
  $\text{em} \quad (11658471.895 \pm 0.008) \times 10^{-10} \quad [\text{Kinoshita et al., Phys.Rev.Lett. 109, 111808 (2012)}]$ 
  
  $\text{weak} \quad (15.36 \pm 0.10) \times 10^{-10} \quad [\text{Gnendinger et al., Phys.Rev. D88, 053005 (2013)}]$
$a_\mu$: Experiment vs. Theory

- measured and calculated very precisely $\rightarrow$ test of the Standard Model
- experiment: polarized muons in a magnetic field $[\text{Bennet et al., Phys.Rev. D73, 072003 (2006)}]$

$$a_\mu = 11659209.1(5.4)(3.3) \times 10^{-10}$$

- Standard Model
  - em $(11658471.895 \pm 0.008) \times 10^{-10}$ $[\text{Kinoshita et al., Phys.Rev.Lett. 109, 111808 (2012)}]$
  - weak $(15.36 \pm 0.10) \times 10^{-10}$ $[\text{Gnendinger et al., Phys.Rev. D88, 053005 (2013)}]$
  - HVP $(693.26 \pm 2.46) \times 10^{-10}$ $[\text{Keshavarzi et al., Phys. Rev. D97 114025 (2018)}]$
  - HVP($\alpha^3$) $(-9.84 \pm 0.06) \times 10^{-10}$ $[\text{Hagiwara et al., J.Phys. G38, 085003 (2011)}]$
The anomalous magnetic moment of the muon

\( a_\mu \): Experiment vs. Theory

- measured and calculated very precisely \( \rightarrow \) test of the Standard Model
- experiment: polarized muons in a magnetic field \([\text{Bennet et al.}, \text{Phys.Rev.} \text{D73}, 072003 (2006)]\)

\[
a_\mu = 11659209.1(5.4)(3.3) \times 10^{-10}
\]

- Standard Model

  - \( \text{em} \) \( (11658471.895 \pm 0.008) \times 10^{-10} \) \([\text{Kinoshita et al.}, \text{Phys.Rev.Lett.} \text{109}, 111808 (2012)]\)
  - \( \text{weak} \) \( (15.36 \pm 0.10) \times 10^{-10} \) \([\text{Gnendinger et al.}, \text{Phys.Rev.} \text{D88}, 053005 (2013)]\)
  - \( \text{HVP} \) \( (693.26 \pm 2.46) \times 10^{-10} \) \([\text{Keshavarzi et al.}, \text{Phys. Rev.} \text{D97} 114025 (2018)]\)
  - \( \text{HVP}(\alpha^3) \) \( (-9.84 \pm 0.06) \times 10^{-10} \) \([\text{Hagiwara et al.}, \text{J.Phys.} \text{G38}, 085003 (2011)]\)
  - \( \text{LbL} \) \( (10.5 \pm 2.6) \times 10^{-10} \) \([\text{Prades et al.}, \text{Adv.Ser.Direct.High Energy Phys.} \text{20}, 303 (2009)]\)
The anomalous magnetic moment of the muon

\( a_\mu \): Experiment vs. Theory

- measured and calculated very precisely \( \rightarrow \) test of the Standard Model

\[ a_\mu = 11659209.1(5.4)(3.3) \times 10^{-10} \]

- Standard Model
  - \( \text{em} \) \( (11658471.895 \pm 0.008) \times 10^{-10} \) \[\text{[Kinoshita et al., Phys.Rev.Lett. 109, 111808 (2012)]}\]
  - \( \text{weak} \) \( (15.36 \pm 0.10) \times 10^{-10} \) \[\text{[Gnendinger et al., Phys.Rev. D88, 053005 (2013)]}\]
  - \( \text{HVP} \) \( (693.26 \pm 2.46) \times 10^{-10} \) \[\text{[Keshavarzi et al., Phys. Rev. D97 114025 (2018)]}\]
  - \( \text{HVP}(\alpha^3) \) \( (-9.84 \pm 0.06) \times 10^{-10} \) \[\text{[Hagiwara et al., J.Phys. G38, 085003 (2011)]}\]
  - \( \text{LbL} \) \( (10.5 \pm 2.6) \times 10^{-10} \) \[\text{[Prades et al., Adv.Ser.Direct.High Energy Phys. 20, 303 (2009)]}\]

- Comparison of theory and experiment: 3.8\( \sigma \) deviation

\[ \Delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = 27.9(6.3)^{\text{Exp}}(3.6)^{\text{SM}} \times 10^{-10} \]
**$a_\mu$: Experiment vs. Theory**

- measured and calculated very precisely $\rightarrow$ test of the Standard Model
- experiment: polarized muons in a magnetic field $[\text{Bennet et al., Phys.Rev.} \ D73, \ 072003 \ (2006)]$

$$a_\mu = 11659209.1(5.4)(3.3) \times 10^{-10}$$

- Standard Model
  
<table>
<thead>
<tr>
<th>Type</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>em</td>
<td>$(11658471.895 \pm 0.008) \times 10^{-10}$</td>
</tr>
<tr>
<td>weak</td>
<td>$(15.36 \pm 0.10) \times 10^{-10}$</td>
</tr>
<tr>
<td>HVP</td>
<td>$(693.26 \pm 2.46) \times 10^{-10}$</td>
</tr>
<tr>
<td>HVP($\alpha^3$)</td>
<td>$(-9.84 \pm 0.06) \times 10^{-10}$</td>
</tr>
<tr>
<td>LbL</td>
<td>$(10.5 \pm 2.6) \times 10^{-10}$</td>
</tr>
</tbody>
</table>


[Hagiwara et al., J.Phys. G38, 085003 (2011)]


- Comparison of theory and experiment: $3.8\sigma$ deviation

$$\Delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = 27.9(6.3)^{\text{Exp}}(3.6)^{\text{SM}} \times 10^{-10}$$

required precision to match upcoming experiments $\Delta a_\mu^{\text{hvp}} \lesssim 0.2\% \quad \Delta a_\mu^{\text{lbl}} \lesssim 10\%$
The anomalous magnetic moment of the muon

\( a_\mu \): Experiment vs. Theory

- measured and calculated very precisely \( \rightarrow \) test of the Standard Model
  \[ a_\mu = 11659209.1(5.4)(3.3) \times 10^{-10} \]

- Standard Model
  \begin{align*}
  \text{em} & \quad (11658471.895 \pm 0.008) \times 10^{-10} \\
  \text{weak} & \quad (15.36 \pm 0.10) \times 10^{-10} \\
  \text{HVP} & \quad (693.26 \pm 2.46) \times 10^{-10} \\
  \text{HVP}(\alpha^3) & \quad (-9.84 \pm 0.06) \times 10^{-10} \\
  \text{LbL} & \quad (10.5 \pm 2.6) \times 10^{-10}
  \end{align*}

- Comparison of theory and experiment: \( 3.8\sigma \) deviation
  \[ \Delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = 27.9(6.3)^{\text{Exp}}(3.6)^{\text{SM}} \times 10^{-10} \]

required precision to match upcoming experiments

- \( \Delta a_\mu^{\text{hvp}} \lesssim 0.2\% \)
- \( \Delta a_\mu^{\text{lbl}} \lesssim 10\% \)
Outline

Hadronic Vacuum Polarisation
- Introduction
- light quark contribution
- strange and charm quark contribution
- disconnected contribution
- Isospin Breaking corrections to the HVP
- Summary and Prospects

Hadronic light-by-light scattering
- Introduction
- Lattice Calculations
- Summary and Prospects

Final remarks
Outline

Hadronic Vacuum Polarisation
  • Introduction
  • light quark contribution
  • strange and charm quark contribution
  • disconnected contribution
  • Isospin Breaking corrections to the HVP
  • Summary and Prospects

Hadronic light-by-light scattering
  • Introduction
  • Lattice Calculations
  • Summary and Prospects

Final remarks
Hadronic Vacuum Polarisation (HVP) from the R-ratio

- current best theoretical estimate uses experimental data
- optical theorem

\[
R(s) = \frac{\sigma(e^+ e^- \rightarrow \text{hadrons}, s)}{\sigma(e^+ e^- \rightarrow \mu^+ \mu^-, s)}
\]

- R-ratio

\[
a_{hvp}^\mu = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{m_\pi^2}^{\infty} ds \frac{R(s)K(s)}{s^2}
\]

- first principles calculation of HVP \rightarrow lattice QCD

recent results:

\[
\begin{align*}
    a_{hvp}^\mu &= 689.46(3.25) \quad \text{[Jegerlehner 18]} \\
    a_{hvp}^\mu &= 693.1(3.4) \quad \text{[DHMZ 17]} \\
    a_{hvp}^\mu &= 693.37(2.46) \quad \text{[KNT 18]}
\end{align*}
\]

\(\approx 0.5\%\) precision
Hadronic Vacuum Polarisation (HVP) from the Lattice

- $\Pi_{\mu\nu}(Q) \equiv \int d^4x \ e^{iQ\cdot x} \left< j_\mu^\gamma(x) j_\nu^\gamma(0) \right> = (Q_\mu Q_\nu - \delta_{\mu\nu} Q^2) \Pi(Q^2)$

- electromagnetic current $j_\mu^\gamma = \frac{2}{3} \bar{u}\gamma\mu u - \frac{1}{3} \bar{d}\gamma\mu d - \frac{1}{3} \bar{s}\gamma\mu s + \frac{2}{3} \bar{c}\gamma\mu c$

- hadronic contribution to the anomalous magnetic moment of the muon

[T. Blum, Phys.Rev.Lett.91, 052001 (2003)]

$$a_{\mu}^{\text{hvp}} = \left( \frac{\alpha}{\pi} \right)^2 \int_0^\infty dQ^2 K(Q^2) \hat{\Pi}(Q^2) \quad \text{with} \quad \hat{\Pi}(Q^2) = 4\pi^2 \left[ \Pi(Q^2) - \Pi(0) \right]$$


$$C(t) = \frac{1}{3} \sum_{k=0}^2 \sum_{\vec{x}} \left< j_k^\gamma(\vec{x}, t) j_k^\gamma(0) \right> \quad \hat{\Pi}(Q^2) = 4\pi^2 \int_0^\infty dt \ C(t) \left[ \frac{\cos(Qt) - 1}{Q^2} + \frac{1}{2} t^2 \right] \quad a_{\mu}^{\text{hvp}} = \int_0^\infty dt \ f(t) C(t)$$

- flavour decomposition (isospin symmetric QCD)

$$C(t) = \frac{5}{9} C_\ell(t) + \frac{1}{9} C_s(t) + \frac{4}{9} C_c(t) + C^{\text{disc}}(t)$$
Outline

Hadronic Vacuum Polarisation
- Introduction
- light quark contribution
- strange and charm quark contribution
- disconnected contribution
- Isospin Breaking corrections to the HVP
- Summary and Prospects

Hadronic light-by-light scattering
- Introduction
- Lattice Calculations
- Summary and Prospects

Final remarks
Hadronic Vacuum Polarisation

Vector correlator and long distance Signal-to-Noise problem

▶ examples for light-quark vector correlator at physical point


▶ signal deteriorates for large $t$

▶ need noise reduction techniques to control statistical error on raw data


▶ huge reduction in error when using low-mode-averaging (LMA)

▶ possible strategy: replace correlator by (multi-) exponential fit for $t > t_c$
Bounding method

- spectral representation of the vector correlator

\[
C(t) = \sum_n \frac{A_n^2}{2E_n} e^{-E_n t} \quad A_n^2 > 0
\]


\[
0 \leq C(t_c) e^{-E_{t_c}(t-t_c)} \leq C(t) \leq C(t_c) e^{-E_0(t-t_c)}
\]

- \( E_{t_c} \): effective mass of the correlator at \( t_c \)
- \( E_0 \): finite volume ground state energy, two pions with one unit of momentum

- use correlator data for \( t < t_c \)
- use upper and lower bound for \( t \geq t_c \) vary \( t_c \)
Bounding method

- spectral representation of the vector correlator

\[ C(t) = \sum_n \frac{A_n^2}{2E_n} e^{-E_n t} \quad A_n^2 > 0 \]


\[ 0 \leq C(t_c) e^{-E_{t_c}(t-t_c)} \leq C(t) \leq C(t_c) e^{-E_0(t-t_c)} \]

- \( E_{t_c} \): effective mass of the correlator at \( t_c \)
- \( E_0 \): finite volume ground state energy, two pions with one unit of momentum
- use correlator data for \( t < t_c \)
- use upper and lower bound for \( t \geq t_c \) vary \( t_c \)

[Plot by A. Meyer], [A. Meyer, Mon 17:30]
Reconstruction of the long distance tail

- dedicated spectroscopy study, GEVP with different operators with overlap to two pions
- determine energies $E_n$ and overlap factors $A_n$ for lowest $N$ states
- reconstruct the long distance tail of vector correlator

\[ \frac{g G(t) \tilde{K}(t)}{m_\mu} \]

- can be used for improving the bounding method
Improved bounding method


$$\tilde{C}(t) = C(t) - \sum_{n=0}^{N-1} \frac{A_n^2}{2E_n} e^{-E_n t}$$

$$0 \leq \tilde{C}(t_c) e^{-E_{tc}(t-t_c)} \leq \tilde{C}(t) \leq \tilde{C}(t_c) e^{-E_N(t-t_c)}$$

- upper and lower bound overlap for smaller $t_c$

- $a_{\mu}^{\text{hvp}}$ can be extracted with smaller error

![Graph showing bounding method and improved bounding method](image-url)
Finite volume (FV) effects

- dominated by two pion state - important at large $t$
- finite volume effects of $\sim \mathcal{O}(20 - 30 \times 10^{-10})$ for typical lattice sizes $\sim \mathcal{O}(5 - 6\ \text{fm})$ at physical point, see e.g. [E. Shintani, Y. Kuramashi, arXiv:1902.00885], [A. Gérardin, Tue 14:40], [C. Lehner, Mon 14:20], [C. Aubin et al, arXiv:1905.09307]

- study using ensembles with different volumes

\[ \frac{W(r)}{C(r) \times 10^{19} \text{a.fm}^{-1}} \]

\[ 10.8 \ \text{fm} \]

\[ 5.4 \ \text{fm} \]

$\rightarrow$ FV effects about $1.7 \times$ larger than NLO ChiPT
Finite volume (FV) effects

- dominated by two pion state - important at large t
- finite volume effects of $\sim O(20 - 30 \times 10^{-10})$ for typical lattice sizes $\sim O(5 - 6 \text{ fm})$ at physical point, see e.g. [E. Shintani, Y. Kuramashi, arXiv:1902.00885], [A. Gérardin, Tue 14:40], [C. Lehner, Mon 14:20], [C. Aubin et al, arXiv:1905.09307]

- study using ensembles with different volumes
- similar observation
  - RBC/UKQCD [C. Lehner, Mon 14:20] using different volumes or timelike pion form factor

$\rightarrow$ FV effects about $1.7 \times$ larger than NLO ChiPT
Finite volume (FV) effects

- dominated by two pion state - important at large $t$
- finite volume effects of $\sim \mathcal{O}(20 - 30 \times 10^{-10})$ for typical lattice sizes $\sim \mathcal{O}(5 - 6 \text{ fm})$ at physical point, see e.g. [E. Shintani, Y. Kuramashi, arXiv:1902.00885], [A. Gérardin, Tue 14:40], [C. Lehner, Mon 14:20], [C. Aubin et al, arXiv:1905.09307]

- study using ensembles with different volumes

$10.8 \text{ fm}$ $5.4 \text{ fm}$

$\rightarrow$ FV effects about $1.7 \times$ larger than NLO ChiPT

- similar observation
  - RBC/UKQCD [C. Lehner, Mon 14:20] using different volumes or timelike pion form factor

- finite volume effects in NNLO ChiPT

$\rightarrow$ additional FV effects from NNLO ChiPT $\approx 0.4 - 0.45$ of NLO FV effects [C. Aubin et al, arXiv:1905.09307]
Finite volume (FV) effects

- dominated by two pion state - important at large t
- finite volume effects of $\sim \mathcal{O}(20 - 30 \times 10^{-10})$ for typical lattice sizes $\sim \mathcal{O}(5 - 6 \text{ fm})$ at physical point, see e.g. [E. Shintani, Y. Kuramashi, arXiv:1902.00885], [A. Gérardin, Tue 14:40], [C. Lehner, Mon 14:20], [C. Aubin et al, arXiv:1905.09307]

- study using ensembles with different volumes
  - $10.8 \text{ fm} \rightarrow \text{FV effects about } 1.7 \times \text{ larger than NLO ChiPT}$

- $\mathcal{O}(e^{-m_{\pi}L})$ FV corrections using Hamiltonian approach (neglecting $\mathcal{O}(e^{-\sqrt{2}m_{\pi}L})$)

- similar observation
    - using timelike pion form factor
  - RBC/UKQCD [C. Lehner, Mon 14:20]
    - using different volumes or timelike pion form factor

- finite volume effects in NNLO ChiPT
  - additional FV effects from NNLO ChiPT $\approx 0.4 - 0.45$ of NLO FV effects [C. Aubin et al, arXiv:1905.09307]
Finite volume effects from the timelike pion form factor

- long-distance contribution of vector correlator given in terms of the timelike pion form factor

- Gounaris-Sakurai (GS) parameterisation of the timelike pion form factor

- infinite volume long distance correlator from GS

- finite volume long distance correlator from GS & Lellouch-Lüscher formalism
Finite volume effects from the timelike pion form factor

- long-distance contribution of vector correlator given in terms of the timelike pion form factor
- Gounaris-Sakurai (GS) parameterisation of the timelike pion form factor
- infinite volume long distance correlator from GS
- finite volume long distance correlator from GS & Lellouch-Lüscher formalism

- \( m_\pi = 280 \) MeV, two different volumes
- finite size effects (FSE) corrected using timelike pion form factor
Finite volume effects from the timelike pion form factor

- long-distance contribution of vector correlator given in terms of the timelike pion form factor

- Gounaris-Sakurai (GS) parameterisation of the timelike pion form factor
  

- infinite volume long distance correlator from GS

- finite volume long distance correlator from GS & Lellouch-Lüscher formalism


GS and perturbative QCD for small \( t \)

---

Gounaris-Sakurai (GS) parameterisation of the timelike pion form factor

\[ V_{\mu}^{HVP}(ud) = 10^{10} \]

---

ETMC

GS and perturbative QCD for small \( t \)
Finite volume effects from the timelike pion form factor

- long-distance contribution of vector correlator given in terms of the timelike pion form factor

- Gounaris-Sakurai (GS) parameterisation of the timelike pion form factor

- infinite volume long distance correlator from GS

- finite volume long distance correlator from GS & Lellouch-Lüscher formalism


GS and perturbative QCD for small $t$
scale setting

- $a_{\mu}^{\text{hvp}}$ depends on the scale through $a_m^\mu$ in the kernel

- scale set by quantity $\Lambda$ with error $\Delta \Lambda$

$$\Delta a_{\mu}^{\text{hvp}} = \left| \Lambda \frac{da_{\mu}^{\text{hvp}}}{d\Lambda} \right| \cdot \frac{\Delta \Lambda}{\Lambda} = \left| M_{\mu} \frac{da_{\mu}^{\text{hvp}}}{dM_{\mu}} \right| \cdot \frac{\Delta \Lambda}{\Lambda}$$

$\Rightarrow$ relative error on $\Lambda$ amplified by $\approx 1.8$ in relative error for $a_{\mu}$ [M. Della Morte, VG, et al, JHEP 1710 (2017) 020]

$\Rightarrow$ for 0.2% error on $a_{\mu}^{\text{hvp}}$ need $\lesssim 0.1\%$ on lattice spacing


$\approx 0.2 - 0.3\%$ on lattice spacing
extrapolation to the physical point

- **chiral extrapolation**
  - most calculations now done using (or including) ensembles at the physical point
  - chiral extrapolation if necessary

- **continuum extrapolation**
  - discretization effects depend on action used
  - ideally work in fully $\mathcal{O}(a)$ improved setup
    - actions usually $\mathcal{O}(a)$-improved
    - $\mathcal{O}(a)$-improvement of vector current, if necessary [A. Gérardin et al, arXiv:1904.03120]
  - ideally at least three lattice spacings
extrapolation to the physical point

- chiral extrapolation
  - most calculations now done using (or including) ensembles at the physical point
  - chiral extrapolation if necessary

- continuum extrapolation
  - discretization effects depend on action used
  - ideally work in fully $\mathcal{O}(a)$ improved setup
    -> actions usually $\mathcal{O}(a)$-improved
    -> $\mathcal{O}(a)$-improvement of vector current, if necessary
  - ideally at least three lattice spacings

- HISQ action
  - data points corrected for discretization effects from taste splitting
comparison - light quark results

\[
N_f = 2 + 1
\]

\[
N_f = 2 + 1 + 1
\]

CLS Mainz 2019
PACS-CS 2019
RBC/UKQCD 2018
BMW 2018
ETMC 2017
HPQCD/Fermilab/MILC 2019
Aubin et al 2019

- errors from $1.3\% - 3.3\%$
- $\approx 2\sigma$ discrepancy between smallest and largest results
Comparison - light quark results

- $N_f = 2 + 1$
- $N_f = 2 + 1 + 1$

- Errors from $1.3\% - 3.3\%$
- $\approx 2\sigma$ discrepancy between smallest and largest results
- Compare intermediate quantities, e.g. time-moments $G_{2n} = \int_{-\infty}^{\infty} dt \, t^{2n}C(t)$ or $a_{\mu}^{\text{hvp}}$ from time window

- CLS Mainz 2019
- PACS-CS 2019
- RBC/UKQCD 2018
- BMW 2018
- ETMC 2017
- HPQCD/Fermilab/MILC 2019
- Aubin et al 2019
Outline

Hadronic Vacuum Polarisation
- Introduction
- light quark contribution
- **strange and charm quark contribution**
- disconnected contribution
- Isospin Breaking corrections to the HVP
- Summary and Prospects

Hadronic light-by-light scattering
- Introduction
- Lattice Calculations
- Summary and Prospects

Final remarks
Strange and Charm HVP

- suffers less from long-distance noise-to-signal problem and finite volume effects than light contribution
- charm usually large discretization effects

\[
\begin{align*}
N_f &= 2 + 1 \\
N_f &= 2 + 1 + 1
\end{align*}
\]

\[
\begin{array}{c}
48 & \quad 52 & \quad 56 & \quad 60 \\
\end{array}
\]

\[
\begin{array}{c}
N_f = 2 + 1 \\
N_f = 2 + 1 + 1
\end{array}
\]

\[
\begin{array}{c}
\pm 0.4\% \\
\pm 0.3\%
\end{array}
\]

- errors on total HVP

CLS Mainz 2019
PACS-CS 2019
RBC/UKQCD 2018
BMW 2018
ETMC 2017
HPQCD 2014

\[
\begin{array}{c}
9 \quad 11 \quad 13 \quad 15 \\
\end{array}
\]

\[
\begin{array}{c}
N_f = 2 + 1 \\
N_f = 2 + 1 + 1
\end{array}
\]

\[
\begin{array}{c}
\pm 0.4\% \\
\pm 0.3\%
\end{array}
\]

- errors on total HVP
Outline

Hadronic Vacuum Polarisation
- Introduction
- light quark contribution
- strange and charm quark contribution
- disconnected contribution
- Isospin Breaking corrections to the HVP
- Summary and Prospects

Hadronic light-by-light scattering
- Introduction
- Lattice Calculations
- Summary and Prospects

Final remarks
disconnected HVP

- quark-disconnected Wick contraction
- SU(3) suppressed
- quark loop
  \[
  \Delta^f_{\mu}(t) = \sum_x \text{Tr} \left[ \gamma_\mu S^f(x, x) \right]
  \]

- all-to-all propagators, calculate stochastically

  \[
  C^{\text{disc}}(t) = \frac{1}{9} \langle (\Delta^\ell(t) - \Delta^s(t)) \cdot (\Delta^\ell(0) - \Delta^s(0)) \rangle
  \]

- further noise reduction

- Mainz: use bounding method on total isoscalar correlator \(C^{I=0}(t)\) and subtract connected contributions \([\text{A. Gérardin et al, arXiv:1904.03120}]\)
**Hadronic Vacuum Polarisation**

- **quark-disconnected Wick contraction**
- **SU(3) suppressed**
- **quark loop**
  \[ \Delta^f_\mu(t) = \sum_x Tr [\gamma_\mu S^f(x, x)] \]
- **all-to-all propagators, calculate stochastically**
- **light-strange cancellation** \cite{V.G. et al, PoS LATTICE2014 (2014) 128}
  \[ C^{\text{disc}}(t) = \frac{1}{9} \langle (\Delta^\ell(t) - \Delta^s(t)) \cdot (\Delta^\ell(0) - \Delta^s(0)) \rangle \]
- **further noise reduction**
  - \cite{T. Blum et al, Phys. Rev. Lett. 116, 232002 (2016)} low-mode averaging and sparsened noise sources for high modes
  - \cite{A. Gérardin et al, arXiv:1904.03120} hierarchical probing \cite{A. Stathopoulos et al, arXiv:1302.4018}
  - **frequency-splitting estimators** \cite{L. Giusti et al, arXiv:1903.10447}, \cite{T. Harris, Fri 14:20}
- **Mainz: use bounding method on total isoscalar correlator** \( C^{I=0}(t) \) and subtract connected contributions \cite{A. Gérardin et al, arXiv:1904.03120}

- **errors on total HVP** \( 0.3 - 0.7\% \)
- **work in progress HPQCD/FNAL/MILC**
  \cite{C. DeTar, Mon 14:40}

**Mainz:**
- **use bounding method on total isoscalar correlator** \( C^{I=0}(t) \) and subtract connected contributions

**Graphs**
- CLS Mainz 2019
- RBC/UKQCD 2018
- BMW 2018

**Tables**
- \( a_{\mu, \text{disc}}^{\text{hvp}} \cdot 10^{10} \)
- \( N_f = 2 + 1 \)
- \( N_f = 2 + 1 + 1 \)
Outline

Hadronic Vacuum Polarisation
- Introduction
- light quark contribution
- strange and charm quark contribution
- disconnected contribution
- Isospin Breaking corrections to the HVP
- Summary and Prospects

Hadronic light-by-light scattering
- Introduction
- Lattice Calculations
- Summary and Prospects

Final remarks
Isospin Breaking Corrections

- lattice calculations usually done in the isospin symmetric limit
- two sources of isospin breaking effects
  - different masses for up- and down quark (of $\mathcal{O}((m_d - m_u)/\Lambda_{QCD})$)
  - Quarks have electrical charge (of $\mathcal{O}(\alpha)$)
- lattice calculation aiming at $\lesssim 1\%$ precision requires to include isospin breaking

- separation of strong IB and QED effects requires renormalization scheme
- definition of “physical point” in a “QCD only world” also scheme dependent
  → results shown above without QED and isospin breaking for $m_\pi \approx 135$ MeV
Strong isospin corrections from the lattice

- use different up, down quark masses
- sea quark effects:
  → configurations with different up, down masses
- results [B. Chakraborty et al. Phys. Rev. Lett. 120 152001 (2018)]
  \[
  \delta a_\mu = 7.7(3.7) \times 10^{-10} \quad \text{N}_f = 2 + 1 + 1
  \]
  \[
  \delta a_\mu = 9.0(2.3) \times 10^{-10} \quad \text{N}_f = 1 + 1 + 1 + 1
  \]
Strong isospin corrections from the lattice

- use different up, down quark masses
- sea quark effects:
  - configurations with different up, down masses
- results [B. Chakraborty et al. Phys. Rev. Lett. 120 152001 (2018)]
  \[
  \delta a_\mu = 7.7(3.7) \times 10^{-10} \quad N_f = 2 + 1 + 1 \\
  \delta a_\mu = 9.0(2.3) \times 10^{-10} \quad N_f = 1 + 1 + 1 + 1 
  \]

- perturbative expansion in \( \Delta m = (m_u - m_d) \)
  [G.M. de Divitiis et al., JHEP 1204 (2012) 124]
  \[
  \langle O \rangle_{m_u \neq m_d} = \langle O \rangle_{m_u = m_d} + \Delta m \frac{\partial}{\partial m} \langle O \rangle_{m_u = m_d} + \mathcal{O} (\Delta m^2)
  \]
  sea quark effects:

- ETMC [D. Giusti et al., arXiv:1901.10462]
  \[
  \delta a_\mu = 6.0(2.3) \times 10^{-10}
  \]

  \[
  \delta a_\mu = 10.6(4.3) \times 10^{-10}
  \]
  + work in progress
  [C. Lehner, Mon 14:20]
Hadronic Vacuum Polarisation

QED corrections from the lattice

- Euclidean path integral including QED

\[ \langle O \rangle = \frac{1}{Z} \int D[\Psi, \bar{\Psi}] D[U] D[A] \, O \, e^{-S_F[\Psi, \bar{\Psi}, U, A]} \, e^{-S_G[U]} \, e^{-S_\gamma[A]} \]

- Finite Volume corrections for QED on the lattice
  \[ \rightarrow 1/(m_\pi L)^3 \]
  for QED corrections to HVP in QED\(_L\)  
  [N. Hermansson Truedsson, Mon 16:50]
  \[ \rightarrow \] negligible for required precision

- perturbative expansion of the path integral in \( \alpha \)
  [RM123 Collaboration, Phys.Rev. D87, 114505 (2013)]
QED corrections from the lattice

- Euclidean path integral including QED

\[ \langle O \rangle = \frac{1}{Z} \int \mathcal{D}[\psi, \bar{\psi}] \mathcal{D}[U] \mathcal{D}[A] O \ e^{-S_F[\psi, \bar{\psi}, U, A]} e^{-S_G[U]} e^{-S_\gamma[A]} \]

- Finite Volume corrections for QED on the lattice

→ \( \frac{1}{(m_\pi L)^3} \) for QED corrections to HVP in QED\(_L\) [[N. Hermansson Truedsson, Mon 16:50]]


→ negligible for required precision

- perturbative expansion of the path integral in \( \alpha \) [[RM123 Collaboration, Phys.Rev. \textbf{D87}, 114505 (2013)]]
Results QED corrections

- connected contributions in electro-quenched approximation

\[ \delta a_{\mu}^{\text{HVP}} = 1.1(1.0) \times 10^{-10} \]

- several pion masses, extrapolation to physical point

\[ \delta a_{\mu}^{\text{HVP}} = 5.9(5.7) \times 10^{-10} \]

  + work in progress


Results QED corrections

- **leading QED correction to the disconnected HVP**

  \[ a_{\mu}^{\text{QED, disc}} = -6.9(2.1)(1.4) \times 10^{-10} \]

- QED correction to disconnected HVP

Results QED corrections

- leading QED correction to the disconnected HVP
  - gluons between the quarks
  - no gluons between the quarks

  → QED correction to LO HVP
  → included in NLO HVP

- QED correction to disconnected HVP
  \[ a_{\mu}^{\text{QED, disc}} = -6.9(2.1)(1.4) \times 10^{-10} \]

- QED corrections from sea-quark effects
  - diagrams at least \(1/N_c\) suppressed
    → could be 33% of connected
    → need to be studied for sub-percent precision on total HVP

- work in progress by BMW  [B. Toth, Tue 17:10]
- work in progress RBC/UKQCD  [C. Lehner, Mon 14:20]
Outline

Hadronic Vacuum Polarisation
- Introduction
- light quark contribution
- strange and charm quark contribution
- disconnected contribution
- Isospin Breaking corrections to the HVP
- Summary and Prospects

Hadronic light-by-light scattering
- Introduction
- Lattice Calculations
- Summary and Prospects

Final remarks
Full HVP comparison

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Comparison of different calculations for the Hadronic Vacuum Polarisation (HVP) at various energy scales.}
\end{figure}

- Jegerlehner 2017
- Teubner \textit{et al} 2018
- Davier \textit{et al} 2017
- RBC/UKQCD 2018
- RBC/UKQCD 2018
- BMW 2017
- CLS Mainz 2019
- FermiLab/HPQCD/MILC 2019
- ETMC 2019
- PACS 2019

\textit{“no new physics”}
Full HVP comparison

- Jegerlehner 2017
- Teubner et al 2018
- Davier et al 2017
- RBC/UKQCD 2018
- RBC/UKQCD 2018
- BMW 2017
- CLS Mainz 2019
- FermiLab/HPQCD/MILC 2019
- ETMC 2019
- PACS 2019

"no new physics"
Full HVP comparison

- Jegerlehner 2017
- Teubner et al 2018
- Davier et al 2017
- RBC/UKQCD 2018
- BMW 2017
- CLS Mainz 2019
- FermiLab/HPQCD/MILC 2019
- ETMC 2019
- PACS 2019

"no new physics"

contribution to $a_{\mu}^{hvp}

Isospin Breaking
disconnected

charm

strange
Full HVP comparison

Jegerlehner 2017
Teubner et al 2018
Davier et al 2017
RBC/UKQCD 2018
RBC/UKQCD 2018
BMW 2017
CLS Mainz 2019
FermiLab/HPQCD/MILC 2019
ETMC 2019
PACS 2019

“no new physics”

650 700 750

$\alpha_{\mu}^{\text{hvp}} \cdot 10^{10}$
Full HVP comparison

<table>
<thead>
<tr>
<th>R-ratio</th>
<th>R-ratio &amp; lattice</th>
<th>“no new physics”</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

650 700 750

$\alpha_\mu \cdot 10^{10}$

Jegerlehner 2017
Teubner et al 2018
Davier et al 2017
RBC/UKQCD 2018
RBC/UKQCD 2018
BMW 2017
CLS Mainz 2019
FermiLab/HPQCD/MILC 2019
ETMC 2019
PACS 2019

“no new physics”

contribution to $a_{\mu}^{\text{hvp}}$ ≈ 2.5%

contribution to $\Delta a_{\mu}^{\text{hvp}}$
Conclusions and Prospects

- most important issues:
  - noise reduction and control of long-distance tail of the light quark correlator
  - careful estimate of finite volume effects
  - first lattice calculations of isospin breaking and QED corrections
    → study also sea quark effects
  - achieve consensus between lattice results

hadronic vacuum polarisation also enters other quantities

- running of the electromagnetic coupling
  - [Miguel Teseo San José Pérez, Mon 15:20]
- running of the Weinberg angle
  - [Marco Cé, Mon 15:00]
Conclusions and Prospects

- most important issues:
  - noise reduction and control of long-distance tail of the light quark correlator
  - careful estimate of finite volume effects
  - first lattice calculations of isospin breaking and QED corrections → study also sea quark effects
  - achieve consensus between lattice results

- hadronic vacuum polarisation also enters other quantities
  - running of the electromagnetic coupling [Miguel Teseo San José Pérez, Mon 15:20]
  - running of the Weinberg angle [Marco Cè, Mon 15:00]
Hadronic Vacuum Polarisation
- Introduction
- light quark contribution
- strange and charm quark contribution
- disconnected contribution
- Isospin Breaking corrections to the HVP
- Summary and Prospects

Hadronic light-by-light scattering
- Introduction
- Lattice Calculations
- Summary and Prospects

Final remarks
Introduction

- hadronic light-by-light scattering enters at $\alpha^3$


$\pi^0, \eta, \eta'$

- $11.4 \pm 1.3 \times 10^{-10}$
- $-1.9 \pm 1.9 \times 10^{-10}$
- $1.5 \pm 1.0 \times 10^{-10}$
- $-0.7 \pm 0.7 \times 10^{-10}$
- $0.2 \times 10^{-10}$

e.g. $\pi^0$ contribution

charged $\pi$ loop

axialvector

scalar

charm loops

- $10.5 \pm 2.6 \times 10^{-10}$

Outline

Hadronic Vacuum Polarisation
- Introduction
- light quark contribution
- strange and charm quark contribution
- disconnected contribution
- Isospin Breaking corrections to the HVP
- Summary and Prospects

Hadronic light-by-light scattering
- Introduction
- Lattice Calculations
- Summary and Prospects

Final remarks
light-by-light from the lattice

- two collaborations working on this: RBC/UKQCD and Mainz, both using position space approaches

\[ \begin{align*}
  x' & \quad y' & \quad z' \\
  x & \quad y & \quad z \\
  x_{\text{src}} & \quad x_{\text{SNK}}
\end{align*} \]

+ 5 other permutations of $x'$, $y'$, $z'$
two collaborations working on this: RBC/UKQCD and Mainz, both using position space approaches

- approach proposed in [T. Blum et al, Phys. Rev. D93 (2016) no.1, 014503]
  - position space sampling, i.e. stochastic evaluation of sum over $r$
  - exact photon propagators
    - photons in $\text{QED}_L$: power-law finite volume corrections
light-by-light from the lattice

- two collaborations working on this: RBC/UKQCD and Mainz, both using position space approaches

- approach proposed in [T. Blum et al, Phys. Rev. D93 (2016) no.1, 014503]
  - position space sampling, i.e. stochastic evaluation of sum over $r$
  - exact photon propagators
    - photons in QED$_L$: power-law finite volume corrections

  - $a_{\mu}^{\text{lbl}} = \frac{m e^6}{3} \int dx^4 dy^4 \mathcal{L}_{[\rho,\sigma];\mu \nu \lambda}(x, y) \ i\tilde{\Pi}_{\rho;\mu \nu \lambda \sigma}(x, y)$
  - calculate $\mathcal{L}$ (semi-) analytical in the continuum and infinite volume
Results Mainz connected light-by-light

- preliminary results, see [N. Asmussen, Tue 15:00],
  [N. Asmussen, g-2 workshop Mainz]
- connected diagram

\[ a_{\mu}^{\text{l.l.}} \text{ integrand (} m_\pi = 340 \text{ MeV):} \]

\[
\begin{align*}
  f(|y|) \times 10^{11} \text{ fm} \\
  \begin{array}{c}
  \text{lattice data} \\
  a_{\mu}^{\text{HLLL}} = 82(9) \times 10^{-11}
  \end{array}
\end{align*}
\]

\[ a_{\mu}^{\text{l.l.}} \text{ partial integration up to } |y|:\]

\[
\begin{array}{c}
  m_\pi = 340 \text{ MeV} \\
  m_\pi = 285 \text{ MeV} \\
  m_\pi = 200 \text{ MeV}
\end{array}
\]
Results RBC/UKQCD connected + leading disconnected light-by-light

- preliminary results, see [T. Blum, Tue 16:30]

- continuum and infinite volume extrapolation QED_L

\[ a_{\mu}^{clbl} = 27.61(3.51)(0.32) \]
\[ a_{\mu}^{dbl} = -20.20(5.65) \]
\[ a_{\mu}^{lbl} = 7.41(6.32)(0.32) \times 10^{-10} \]
Results RBC/UKQCD connected + leading disconnected light-by-light

- preliminary results, see [T. Blum, Tue 16:30]

- QED$_\infty$, combined with $\pi^0$-pole contribution from model for long distances $\geq R_{\text{max}}$

- work in progress: replace model by lattice calculation of $\pi^0 \to \gamma\gamma$, see [L. Jin, Thu 10:15]
Outline

Hadronic Vacuum Polarisation
- Introduction
- light quark contribution
- strange and charm quark contribution
- disconnected contribution
- Isospin Breaking corrections to the HVP
- Summary and Prospects

Hadronic light-by-light scattering
- Introduction
- Lattice Calculations
- Summary and Prospects

Final remarks
Conclusions - light-by-light

- two collaborations working on lattice calculations
  - RBC/UKQCD: first result (connected+leading disconnected) extrapolated to physical point
  - Mainz: connected contribution

- important check: consistency with Glasgow Consensus?
  → would need $\approx 3 \times$ larger $a_{\mu}^{\text{lbl}}$ than Glasgow Consensus to explain $a_{\mu}$ discrepancy
  → preliminary lattice results suggest this is unlikely

- lattice calculations of the pion transition form factor $\pi^0 \rightarrow \gamma\gamma$
  → pion pole contribution to $a_{\mu}^{\text{lbl}}$
  → constrain long-distance tail to $a_{\mu}^{\text{lbl}}$ lattice calculation
Conclusions - light-by-light

- two collaborations working on lattice calculations
  - RBC/UKQCD: first result (connected+leading disconnected) extrapolated to physical point
  - Mainz: connected contribution
- important check: consistency with Glasgow Consensus?
  → would need \( \approx 3 \times \) larger \( a_{\mu}^{\text{lbl}} \) than Glasgow Consensus to explain \( a_{\mu} \) discrepancy
  → preliminary lattice results suggest this is unlikely

- lattice calculations of the pion transition form factor \( \pi^0 \rightarrow \gamma \gamma \)
  - pion pole contribution to \( a_{\mu}^{\text{lbl}} \)
  - constrain long-distance tail to \( a_{\mu}^{\text{lbl}} \) lattice calculation
Outline

Hadronic Vacuum Polarisation
- Introduction
- light quark contribution
- strange and charm quark contribution
- disconnected contribution
- Isospin Breaking corrections to the HVP
- Summary and Prospects

Hadronic light-by-light scattering
- Introduction
- Lattice Calculations
- Summary and Prospects

Final remarks
Final remarks

- $a_\mu$ measured and calculated very precisely
  - test of the Standard Model
  - new experiment running at Fermilab, see [Dikai Li, Fri 09:45]
  - largest uncertainty in Standard Model prediction from hadronic contributions

- huge effort in the lattice community to calculate hadronic contributions from first principles
  - 15 parallel talks at Lattice 2019!

- work in progress on g-2 Theory Whitepaper from the Muon g-2 Theory Initiative,
  several workshops since 2017, next workshop: September 9 - 13, 2019 at INT
Final remarks

- $a_\mu$ measured and calculated very precisely
  → test of the Standard Model
  → new experiment running at Fermilab, see [Dikai Li, Fri 09:45]
  → largest uncertainty in Standard Model prediction from hadronic contributions

- huge effort in the lattice community to calculate hadronic contributions from first principles
  → 15 parallel talks at Lattice 2019!

- work in progress on g-2 Theory Whitepaper from the Muon g-2 Theory Initiative,
  several workshops since 2017, next workshop: September 9 - 13, 2019 at INT

- hadronic vacuum polarisation contribution to $a_\mu$
  - first lattice calculations of $a_{\mu}^{\text{hvp}}$ with $\lesssim 1\%$ precision available within $O(\text{year})$
  - $\lesssim 0.2\%$ within a few years
Thank you

Thanks for sending me material and/or discussions:

N. Asmussen, C. Aubin, T. Blum, C. DeTar, D. Giusti, C. Lehner, L. Lellouch,
A. Meyer, B. Toth, G. von Hippel

VG has received funding from the European Research Council (ERC) under the
European Union’s Horizon 2020 research and innovation programme under grant agreement No 757646.
Outline

Hadronic Vacuum Polarisation
  • Introduction
  • light quark contribution
  • strange and charm quark contribution
  • disconnected contribution
  • Isospin Breaking corrections to the HVP
  • Summary and Prospects

Hadronic light-by-light scattering
  • Introduction
  • Lattice Calculations
  • Summary and Prospects

Final remarks
Backup
## Results - total HVP

<table>
<thead>
<tr>
<th>collaboration</th>
<th>$a_{\mu}^{\text{hvp}} \times 10^{10}$</th>
<th>action</th>
<th>arXiv</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mainz 19</td>
<td>720.0(12.4)(9.9)</td>
<td>clover</td>
<td>1904.03120</td>
</tr>
<tr>
<td>PACS-CS 19</td>
<td>737(9)(^{13}_{-18})</td>
<td>StoutWilson</td>
<td>1902.00885</td>
</tr>
<tr>
<td>RBC/UKQCD 18</td>
<td>715.4(16.3)(9.2)</td>
<td>DWF</td>
<td>1801.07224</td>
</tr>
<tr>
<td>BMW 18</td>
<td>711.1(7.5)(17.4)</td>
<td>Stout4S</td>
<td>1711.04980</td>
</tr>
<tr>
<td>ETMC 18</td>
<td>682(19)</td>
<td>tm</td>
<td>1808.00887, 1901.10462</td>
</tr>
<tr>
<td>Flab/HPQCD/MILC 19</td>
<td>691(8)(1)(13)</td>
<td>HISQ</td>
<td>1902.04223</td>
</tr>
</tbody>
</table>

Vera Gülpers (University of Edinburgh)
Lattice 2019
June 21, 2019
# Results - light quark HVP

<table>
<thead>
<tr>
<th>collaboration</th>
<th>$a_{\mu,\ell}^{\text{hvp}} \times 10^{10}$</th>
<th>arXiv</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mainz 19</td>
<td>674(12)(5)</td>
<td>1904.03120</td>
</tr>
<tr>
<td>PACS-CS 19</td>
<td>673(9)(11)</td>
<td>1902.00885</td>
</tr>
<tr>
<td>RBC/UKQCD 18</td>
<td>649.7(14.2)(4.9)</td>
<td>1801.07224</td>
</tr>
<tr>
<td>BMW 18</td>
<td>647.6(7.5)(17.7)</td>
<td>1711.04980</td>
</tr>
<tr>
<td>ETMC 18</td>
<td>619.0(14.7)(10.0)</td>
<td>1808.00887</td>
</tr>
<tr>
<td>Flab/HPQCD/MILC 19</td>
<td>630.1(4.4)(7.0)</td>
<td>1902.04223</td>
</tr>
<tr>
<td>Aubin et al 19</td>
<td>650(20)(8)</td>
<td>1905.09307</td>
</tr>
</tbody>
</table>
## Results - strange and charm quark HVP

<table>
<thead>
<tr>
<th>collaboration</th>
<th>$a_{\mu,s}^{\text{hvp}} \times 10^{10}$</th>
<th>$a_{\mu,c}^{\text{hvp}} \times 10^{10}$</th>
<th>arXiv</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mainz 19</td>
<td>54.5(2.4)(0.6)</td>
<td>14.66(0.45)(0.06)</td>
<td>1904.03120</td>
</tr>
<tr>
<td>PACS-CS 19</td>
<td>52.1(2)(5)</td>
<td>11.7(2)(1.6)</td>
<td>1902.00885</td>
</tr>
<tr>
<td>RBC/UKQCD 18</td>
<td>53.2(0.4)(0.3)</td>
<td>14.3(0.0)(0.7)</td>
<td>1801.07224</td>
</tr>
<tr>
<td>BMW 18</td>
<td>53.7(0.0)(0.4)</td>
<td>14.7(0.0)(0.1)</td>
<td>1711.04980</td>
</tr>
<tr>
<td>ETMC 17</td>
<td>53.1(16)(2.0)</td>
<td>14.75(42)(37)</td>
<td>1707.03019</td>
</tr>
<tr>
<td>HPQCD14</td>
<td>53.41(59)</td>
<td>1.42(39)</td>
<td>1403.1778</td>
</tr>
</tbody>
</table>

## Results - disconnected HVP

<table>
<thead>
<tr>
<th>collaboration</th>
<th>$a_{\mu,\text{disc}}^{\text{hvp}} \times 10^{10}$</th>
<th>arXiv</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mainz 19</td>
<td>$-23.2(2.2)(4.5)$</td>
<td>1904.03120</td>
</tr>
<tr>
<td>RBC/UKQCD 18</td>
<td>$-11.2(3.3)(2.3)$</td>
<td>1801.07224</td>
</tr>
<tr>
<td>BMW 18</td>
<td>$-12.8(1.0)(1.6)$</td>
<td>1711.04980</td>
</tr>
</tbody>
</table>
Hadronic Vacuum Polarisation from time moments

- using time moments of the vector two-point function [B. Chakraborty et al, Phys.Rev. D89 (2014) no.11, 114501]
- Taylor expansion for HVP function

\[ \Pi(Q^2) = \Pi_0 + \sum_{j=1}^{\infty} \Pi_j Q^{2j} \]

- coefficients given by time moments of the vector correlator \( C(t) \)

\[ \Pi_j = (-1)^{j+1} \frac{G_{2j+2}}{(2j+2)!} \quad G_{2n} = \int_{-\infty}^{\infty} dt \ t^{2n} C(t) \]

- Taylor coefficients can be used to construct, e.g., Padé approximants for \( \Pi(Q^2) \)

\[ \Pi_{[N,M]}(Q^2) = \Pi(0) + \sum_{i=1}^{N} a_i Q^{2i} \sum_{k=1}^{M} b_k Q^{2k} \]

- higher moments probe larger times in Euclidean vector correlator

\[ \rightarrow \quad \text{can be used to systematically compare results from different collaborations} \]
comparison time moments

- compare light connected time moments from different collaborations
FV effects for light quark HVP


![Graph showing FV effects for light quark HVP](image)

FIG. 19: Values of $\Delta F_{V,E}a_{\mu}^{HVP}(ud)$ (see Eq. (34)), evaluated in the continuum limit according to our "dual + $\pi\pi$" representation at the physical pion point (red circles) and at a larger pion mass equal to $M_{\pi} = 300$ MeV (blue squares). The dotted line corresponds to the predictions of ChPT at NLO [47, 60].

► RBC/UKQCD  [C. Lehner, Mon 14:20]

<table>
<thead>
<tr>
<th>lattice</th>
<th>$a_{\mu}^{hvp}(L = 6.22 \text{ fm}) - a_{\mu}^{hvp}(L = 4.66 \text{ fm})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NLO ChiPT</td>
<td>$21.6(6.3) \times 10^{-10}$</td>
</tr>
<tr>
<td>GS</td>
<td>$12.2 \times 10^{-10}$</td>
</tr>
<tr>
<td></td>
<td>$20(3) \times 10^{-10}$</td>
</tr>
</tbody>
</table>
The zero-mode of the photon field

- zero-mode of the photon field
  shift symmetry of the photon action $A_\mu(x) \rightarrow A_\mu(x) + c_\mu$
  → cannot be constrained by gauge fixing

- different prescriptions of QED:

  - $\text{QED}_{\text{TL}}$: remove the zero-mode of the photon field, i.e. $\tilde{A}_\mu(k = 0) = 0$

  - $\text{QED}_L$: remove all the spatial zero-modes, i.e. $\tilde{A}_\mu(k_0, \vec{k} = 0) = 0$
    [S. Uno, M. Hayakawa, Prog. Theor. Phys. 120, 413 (2008)]
    \[
    \Delta_{\mu\nu}(x - y) = \frac{1}{V} \sum_{\vec{k}, \vec{k} \neq 0} \frac{e^{i\vec{k} \cdot (x - y)}}{\hat{k}^2}
    \]

  - $\text{QED}_m$: use a massive photon and take $m_\gamma \rightarrow 0$

  - $\text{QED}_C$: $C^*$ boundary conditions in spatial direction, i.e. fields are periodic up to charge conjugation
    [B. Luchini et al., JHEP 02 (2016) 076]

- for detailed discussion on different prescriptions of QED see e.g. [A. Patella 1702.03857]
Window Method

- combining lattice with \( R \)-ratio data

\[ a_\mu = a_\mu^{SD} + a_\mu^W + a_\mu^{LD} \]

\[ a_\mu^{SD} = \sum_t w_t C(t) [1 - \theta(t, t_0, \Delta)] \]

\[ a_\mu^W = \sum_t w_t C(t) [\theta(t, t_0, \Delta) - \theta(t, t_1, \Delta)] \]

\[ a_\mu^{LD} = \sum_t w_t C(t) \theta(t, t_1, \Delta) \]
long-distance isovector correlator in infinite volume given by time-like pion form factor $F_\pi(\omega)$

$$C_{I=1}^l(t) = \int_0^\infty d\omega \ \omega^2 \rho(\omega^2) e^{-\omega t}$$

$$\rho(\omega^2) = \frac{1}{48\pi^2} \left(1 - \frac{4m_\pi^2}{\omega^2}\right)^{3/2} |F_\pi(\omega)|^2$$

- e.g. use Gounaris-Sakurai Parameterisation of $F_\pi(\omega)$; two parameters $m_\rho$, $\Gamma_\rho$

- Lüscher: infinite volume phase shifts $\leftrightarrow$ FV energy levels
  

  $$\delta_{11}(k_n) + \phi\left(\frac{k_nL}{2\pi}\right) = n\pi$$

  $$\tan(\phi(z)) = -\frac{\pi^{3/2}z}{Z_{00}(1; z^2)}$$


  $$|F_\pi(\omega_n)|^2 = \left([z\phi'(z)]_{z=\frac{k_nL}{2\pi}} + k_n \frac{\partial \delta_{11}(k_n)}{\partial k_n}\right) \frac{3\pi\omega_n^2}{2k_n^5} |A_n|^2$$

and use $|F_\pi(\omega_n)|^2$ to obtain $|A_n|^2$ and

$$C(t) = \sum_n |A_n|^2 e^{-\omega_n t}$$
strong IB and QED separation prescription

- **ETMC** [D. Giusti et al, arXiv:1901.10462]
  
  \[ \text{impose that } m_{ud}, m_s \text{ and } m_c \text{ and the strong coupling constant } \alpha_s \text{ match at } \overline{\text{MS}}(2 \text{ GeV}) \text{ in QCD+QED and pure QCD} \]
  

  
  \( \text{tune } (u,d,s) \text{ masses to reproduce experimental } \pi^+, K^+ \text{ and } K_0 \text{ mass} \)

\[
\begin{align*}
    m_{\pi^+}^{\exp} &= \left[ m_\pi^0 + \alpha m_{\pi^+}^{\text{QED}} + \Delta m_d m_{\pi^+}^{\Delta m_d} + \Delta m_u m_{\pi^+}^{\Delta m_u} \right] \\
    m_{K^+}^{\exp} &= \left[ m_K^0 + \alpha m_{K^+}^{\text{QED}} + \Delta m_u m_{K^+}^{\Delta m_u} + \Delta m_s m_{K^+}^{\Delta m_s} \right] \\
    m_{K_0}^{\exp} &= \left[ m_K^0 + \alpha m_{K^0}^{\text{QED}} + \Delta m_d m_{K^0}^{\Delta m_d} + \Delta m_s m_{K^0}^{\Delta m_s} \right]
\end{align*}
\]

disconnected light-by-light diagrams

- leading disconnected

- $SU(3)^2$-suppressed

- $SU(3)$-suppressed

- $SU(3)^4$-suppressed
Hadronic light-by-light

**hadronic light-by-light matrix element**

\[
\langle \mu(p')|j_\rho^\gamma|\mu(p)\rangle = -(ie)^6 \int \frac{d^4q_1}{(2\pi)^4} \int \frac{d^4q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 (q_1 + q_2 - k)^2}
\]

\[
\times \bar{u}(p') \gamma^\mu \frac{(ip' - iq_1 - m)}{(p' - q_1)^2 + m^2} \gamma^\nu \frac{(ip' - iq_1 - iq_2 - m)}{(p' - q_1 - q_2)^2 + m^2} \gamma^\lambda u(p)
\]

\[
\times \Pi_{\mu\nu\lambda\rho}(q_1, q_2, q_3 - q_1 - q_2)
\]

**Hadronic tensor**

\[
\Pi_{\mu\nu\lambda\rho}(q_1, q_2, q_3) = \int d^4x_1 d^4x_2 d^4x_3 e^{-i(q_1 x_1 + q_2 x_2 + q_3 x_3)} \langle j_\mu^\gamma(x_1) j_\nu^\gamma(x_2) j_\lambda^\gamma(x_3) j_\rho^\gamma(0) \rangle_{QCD}
\]
light-by-light: finite volume photons

- position space calculation naively: volume sum over 6 photon vertices
- approach proposed in [T. Blum et al, Phys.Rev. D93 (2016) no.1, 014503]
- position space sampling
  - stochastic evaluation of sum over $r$ → importance sampling
  - pairs of point sources at random $-r/2$ and $r/2$
    → sample important region for small $|r|$ more frequently
  - compute all pairs for $|r| < r_{\text{max}}$
  - construct hadronic part of the diagram from the two point-sources at $-r/2$ and $r/2$
- exact photon propagators
- finite volume, e.g. QED$_L$
  → power law finite volume effects
light-by-light: infinite volume photons

- approach proposed by Mainz [J. Green et al, arXiv:1510.08384], [N. Asmussen et al, 1609.08454]

\[ a_\mu^{\text{HIBI}} = \hat{F}_M(0) = \frac{m e^6}{3} \int d^4x d^4y \bar{L}_{[\rho, \sigma]; \mu \nu \lambda}(x, y) i \hat{\Pi}_{\rho; \mu \nu \lambda \sigma}(x, y) \]

- hadronic part

\[ \hat{\Pi}_{\rho; \mu \nu \lambda \sigma} = \int dz^4 i z^\rho \left[ j^\gamma_\mu(x) j^\gamma_\nu(y) j^\gamma_\lambda(0) j^\gamma_\sigma(z) \right] \]

→ calculate using sequential propagators

- QED Kernel function \( \bar{L} \) (averaged over direction of muon momentum)
  - calculate (semi-) analytical in the continuum and infinite volume
  - integrand \( \bar{L} \hat{\Pi} \) Lorentz scalar → depends only on \( x^2, y^2, x \cdot y \)
  - pre-calculate \( \bar{L} \) and store on a \(|x|, |y|, \cos \beta \) Grid, with \( \cos \beta = x \cdot y / |x||y| \)

- remaining finite volume effects are \( \mathcal{O}(e^{-mL}) \)

- similar approach developed by RBC/UKQCD [T. Blum et al, Phys. Rev. D96, 034515 (2017)]

- suitable subtraction for QED Kernel can reduce discretisation effects [T. Blum et al, Phys. Rev. D96, 034515 (2017)], [N. Asmussen, g-2 workshop Mainz], e.g.

\[ \overline{L}^{(2)}(x, y) = \overline{L}(x, y) - \overline{L}(0, y) - \overline{L}(x, 0) \]
light-by-light from QCD+QED


\[
\langle \text{QCD+QED} \rangle \quad - 
\langle \text{QCD+QED} \rangle \quad = \quad 3 \times 
\langle \text{QCD+QED} \rangle 
\quad + \quad \ldots
\]

- using combined QCD+(quenched) QED gauge ensembles
- large subtraction of terms
- leading remaining term is (three times) hadronic light-by-light scattering
The anomalous magnetic moment of the electron

- new physics contribution to $a_\ell$ factor $m_e^2/m_\mu^2 = 2 \times 10^{-5}$ smaller

$$a_{NP}^\ell \propto \frac{m_\ell^2}{M_{NP}^2}$$


$$a_e = 11596521.8073(28) \times 10^{-10}$$

- new $\alpha$ measurement [R. Parker et al, Science 360 (2018) 191]

$$\alpha^{-1} = 137.035999046(27)$$

as input for Standard Model prediction

- see, e.g. [H. Davoudiasl, W. Marciano, Phys. Rev. D98, 075011 (2018)]

$$\Delta a_e = -87(28)^{\text{exp}}(23) \alpha(2)^{\text{SM}} \times 10^{-14}$$

2.4$\sigma$ discrepancy with opposite sign to $\Delta a_\mu$